

Approximation of characteristic functions in R^k spaces

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Let the series $\sum_{j=1}^{\infty} \xi_j$ of independent k -dimensional random variables ξ_1, ξ_2, \dots weakly converges to a random variable (r.v.) η , which has a nondegenerate covariance matrix $V = Cov(\eta)$. We are assuming, that $E\xi_j = 0$ ($j = 1, 2, \dots$) and that moments of required order are existing. Let $B^2 = V^{-1}$ and B is positively definite symmetric matrix. $L(\zeta)$ denotes the distribution of a r.v. ζ and $\widehat{L}(\zeta)$ denotes the characteristic function of this r.v. Let us denote $\widehat{G}_j(t) := \widehat{L}(\xi_j)$ ($j = 1, 2, \dots$), $\widehat{Q}(t) := \widehat{L}(B\eta)$ and

$$l_s := \sup_{\|t\|=1} \frac{\sum_{j=1}^{\infty} E|\langle t, \xi_j \rangle|^s}{\left(\sum_{j=1}^{\infty} E\langle t, \xi_j \rangle^2\right)^{\frac{s}{2}}} \quad (s = 3, 4).$$

We have this estimation of the characteristic function $\widehat{Q}(t)$ of the r.v. η :

THEOREM 1. *Let $l_3 < \infty$. For any t under condition $\|t\| \leq dl_3^{-1/3}$, $d \in (0, \sqrt{2})$*

$$\left| \widehat{Q}(Bt) - \exp\left\{-\frac{1}{2}\|t\|^2\right\} \right| \leq \left(d \cdot a(d) + \frac{1}{6}\right) l_3 \|t\|^3 \exp\{-b(l_3, d)\|t\|^2\},$$

here

$$a(d) := \frac{1}{4} \sum_{r=2}^{\infty} \frac{1}{r} \left(\frac{d^2}{2}\right)^{r-2}, \quad b(l_3; d) := \frac{1}{2} - d \left(da(d) + \frac{1}{6}\right) l_3^{\frac{2}{3}}.$$

Proof. Because $Cov(B\eta) = I$, where I is the identity matrix, we have

$$E|\langle t, B\xi_j \rangle|^s \leq \sum_{j=1}^{\infty} E|\langle t, B\xi_j \rangle|^s \leq l_s \|t\|^s \quad (j = 1, 2, \dots).$$

From this we conclude the inequalities:

$$|\widehat{G}_j(Bt) - 1| \leq \frac{1}{2} E\langle t, B\xi_j \rangle^2 \leq \frac{1}{2} (E|\langle t, B\xi_j \rangle|^3)^{\frac{2}{3}} \leq \frac{1}{2} l_3^{\frac{2}{3}} \|t\|^2 \leq \frac{1}{2} d^2$$

($j = 1, 2, \dots$). So there exists the logarithm $\ln \widehat{G}_j(Bt)$ ($j = 1, 2, \dots$), and we are getting

$$\left| \ln \widehat{Q}(t) + \frac{1}{2}\|t\|^2 \right| = \left| \sum_{j=1}^{\infty} \ln \widehat{G}_j(Bt) + \frac{1}{2}\|t\|^2 \right|$$

$$\begin{aligned}
 &= \left| \sum_{j=1}^{\infty} \left[\ln(1 - (1 - \widehat{G}_j(Bt))) + \frac{1}{2} E\langle t, B\xi_j \rangle^2 \right] \right| \\
 &= \left| \sum_{j=1}^{\infty} \left[- \sum_{r=2}^{\infty} \frac{1}{r} (1 - \widehat{G}_j(Bt))^r + \widehat{G}_j(Bt) - 1 + \frac{1}{2} E\langle t, B\xi_j \rangle^2 \right] \right| \\
 &\leq \sum_{j=1}^{\infty} \left[\frac{1}{4} (E\langle t, B\xi_j \rangle^2)^2 \sum_{r=2}^{\infty} \frac{1}{r} \left(\frac{d^2}{2}\right)^{r-2} + \frac{1}{6} E|\langle t, B\xi_j \rangle|^3 \right] \\
 &\leq (d) \sum_{j=1}^{\infty} (E|\langle t, B\xi_j \rangle|^3)^{\frac{4}{3}} + \frac{1}{6} l_3 \|t\|^3 \leq a(d) l_3^{\frac{4}{3}} \|t\|^4 + \frac{1}{6} l_3 \|t\|^3 \\
 &\leq \left(d \cdot a(d) + \frac{1}{6} \right) l_3 \|t\|^3 \leq d \left(d \cdot a(d) + \frac{1}{6} \right) l_3^{\frac{2}{3}} \|t\|^2.
 \end{aligned}$$

By using the inequality $|e^x - 1| \leq |x|e^{|x|}$ we conclude

$$\begin{aligned}
 \left| \widehat{Q}(t) - \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \right| &= \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \left| \exp \left\{ \ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 \right\} - 1 \right| \\
 &\leq \left(d \cdot a(d) + \frac{1}{6} \right) l_3 \|t\|^3 \exp \left\{ -\frac{1}{2} \|t\|^2 + d \left(d \cdot a(d) + \frac{1}{6} \right) l_3^{\frac{2}{3}} \|t\|^2 \right\} \\
 &= \left(d \cdot a(d) + \frac{1}{6} \right) l_3 \|t\|^3 \exp\{-b(l_3; d) \|t\|^2\}.
 \end{aligned}$$

In the case $L(\xi_j) = L(A^{j-1}\xi)$, ($j = 1, 2, \dots$) with linear operator A under conditions $m := \inf_{\|x\|=1} \|Ax\| > 0$, $\|A\| < 1$ we have the estimation:

$$l_s \leq \frac{1 - m}{1 - \|A\|} (1 - m)^{\frac{s-2}{2}} \tilde{l}_s,$$

here

$$\tilde{l}_s = \sup_{\|t\|=1} \frac{E|\langle t, \xi \rangle|^s}{(E\langle t, \xi \rangle^2)^{s/2}}.$$

This means, that we have the rate of approximation $\sqrt{1 - m}$ in the Theorem 1, if $m \nearrow 1$.

THEOREM 2. Let $l_4 \leq 1$. For any t under condition $\|t\| \leq 2^{-1} l_4^{-1/4}$

$$\begin{aligned}
 &\left| \widehat{Q}(t) - \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \left(1 + \frac{i^3}{6} \sum_{j=1}^{\infty} E\langle t, B\xi_j \rangle^3 \right) \right| \\
 &\leq 0, 174 l_4 \|t\|^4 \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} + [0, 018 l_4^2 \|t\|^8 + \frac{1}{36} l_3^2 \|t\|^6] \exp\{-0, 383 \|t\|^2\}.
 \end{aligned}$$

Proof. In this case we use inequality

$$|\widehat{G}_j(t) - 1| \leq \frac{1}{2} (E\langle t, B\xi_j \rangle^4)^{\frac{1}{2}} \leq \frac{1}{2} l_4^{\frac{1}{2}} \|t\|^2 \leq \frac{1}{8}$$

for getting

$$\begin{aligned} \left| \ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 \right| &\leq \frac{1}{4} \sum_{r=2}^{\infty} r^{-1} 8^{2-r} \sum_{j=1}^{\infty} (E\langle t, B\xi_j \rangle^2)^2 + \frac{1}{6} l_3 \|t\|^3 \\ &\leq 0, 1325 l_4 \|t\|^4 + \frac{1}{6} \|t\|^3 \leq 0, 117 l_4^{\frac{1}{2}} \|t\|^2. \end{aligned}$$

Also we have

$$\begin{aligned} &\left| \ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 - \frac{i^3}{6} \sum_{j=1}^{\infty} E\langle t, B\xi_j \rangle^3 \right| \\ &= \left| \sum_{j=1}^{\infty} \left[\ln(1 - (1 - \widehat{G}_j(Bt))) + \frac{1}{2} E\langle t, B\xi_j \rangle^2 - \frac{i^3}{6} E\langle t, B\xi_j \rangle^3 \right] \right| \\ &= \left| \sum_{j=1}^{\infty} \left[- \sum_{r=2}^{\infty} r^{-1} (1 - \widehat{G}_j(Bt))^r + \widehat{G}_j(Bt) - 1 + \frac{1}{2} E\langle t, B\xi_j \rangle^2 - \frac{i^3}{6} E\langle t, B\xi_j \rangle^3 \right] \right| \\ &\leq \sum_{j=1}^{\infty} \left[\frac{1}{4} (E\langle t, B\xi_j \rangle^2)^2 \sum_{r=2}^{\infty} r^{-1} 8^{2-r} + \frac{1}{24} E\langle t, B\xi_j \rangle^4 \right] \leq 0, 174 l_4 \|t\|^4. \end{aligned}$$

The next inequality's complete proof:

$$\begin{aligned} &\left| \widehat{Q}(t) - \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \left(1 + \frac{i^3}{6} \sum_{j=1}^{\infty} E\langle t, B\xi_j \rangle^3 \right) \right| \\ &\leq \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \left| \exp \left\{ \ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 \right\} - \left(1 + \frac{i^3}{6} \sum_{j=1}^{\infty} E\langle t, B\xi_j \rangle^3 \right) \right| \\ &\leq \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \left| \exp \left\{ \ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 \right\} - \left(1 + \left(\ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 \right) \right) \right| \\ &\quad + \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \left| \ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 - \frac{i^3}{6} \sum_{j=1}^{\infty} E\langle t, B\xi_j \rangle^3 \right| \\ &\leq \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \left(\frac{1}{2} \left| \ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 \right|^2 \exp \left\{ \left| \ln \widehat{Q}(t) + \frac{1}{2} \|t\|^2 \right| \right\} + 0, 174 l_4 \|t\|^4 \right) \\ &\leq \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} \left(\frac{1}{2} (0, 1325 l_4 \|t\|^4 + \frac{1}{6} l_3 \|t\|^3)^2 \exp \{0, 117 l_4^{\frac{1}{2}} \|t\|^2\} + 0, 174 l_4 \|t\|^4 \right) \\ &\leq 0, 174 l_4 \|t\|^4 \exp \left\{ -\frac{1}{2} \|t\|^2 \right\} + \left[0, 018 l_4^2 \|t\|^8 + \frac{1}{36} l_3^2 \|t\|^6 \right] \exp \{-0, 383 \|t\|^2\}. \end{aligned}$$

In the case $L(\xi_j) = L(A^{j-1}\xi)$ ($j = 1, 2, \dots$) with linear operator A under conditions: $m > 0$, $\|A\| < 1$ we have the rate of approximation $1 - m$ in the theorem 2, if $m \nearrow 1$.

REFERENCES

- [1] R. N. Bhattacharya and R. Ranga Rao, *Normal Approximation and Asymptotic Expansions*, Wiley, 1976.