

On the divisor sums in arithmetical progressions

E. Stankus (VU)

Let $\sigma(n)$ denote the sum of the divisors of the positive integer n and let $D(x)$ be the number of positive integers n for which $\sigma(n) \leq x$, i.e.

$$D(x) = \sum_{\sigma(n) \leq x} 1.$$

In the papers [1]–[4] the asymptotic behaviour of the sum $D(x)$ as $x \rightarrow \infty$ was considered. In [3] and [2] the existence of the limit

$$\lim_{x \rightarrow \infty} \frac{D(x)}{x} = d,$$

where

$$d = \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \sum_{j=1}^{\infty} \frac{1}{1 + p + \dots + p^j}\right),$$

is proved. P. T. Bateman [1] has proved the equality

$$D(x) = dx + O_\varepsilon \left(x \exp \left\{ - (1 - \varepsilon) \sqrt{\frac{1}{2} \log x \log \log x} \right\} \right) \quad (1)$$

for any positive number ε . In [4] for the remainder term of equality (1), the estimation

$$O \left(x \exp \left\{ - \left(1 + \frac{c_1 \log \log \log x}{\log \log x}\right) \sqrt{\frac{1}{2} \log x \log \log x} \right\} \right), \quad c_1 > 0$$

is obtained.

In this paper the similar result for

$$D(x; k, l) := \sum_{\substack{\sigma(n) \leq x \\ n \equiv l \pmod{k}}} 1; \quad (l, k) = 1, \quad k \geq 2$$

is proved. Besides, the numerical results of frequencies

$$\nu_x(k, l) = \frac{1}{x} D(x; k, l)$$

for $k = 5$ are presented (see Table 1). The data indicate the stability of the frequencies as $x \rightarrow \infty$ with the same value for different l , $(l, 5) = 1$.

Table 1

x	500	1000	2000	5000	10000	20000
$\nu_x(5, 0)$	0,114	0,114	0,115	0,115	0,115	0,115
$\nu_x(5, 1)$	0,136	0,140	0,140	0,139	0,139	0,139
$\nu_x(5, 2)$	0,140	0,141	0,137	0,139	0,138	0,139
$\nu_x(5, 3)$	0,140	0,135	0,139	0,138	0,139	0,139
$\nu_x(5, 4)$	0,140	0,135	0,140	0,138	0,139	0,139
$D(x)$	0,668	0,668	0,6725	0,671	0,6719	0,6727

THEOREM. For any integers $k \geq 2$ and l , $(l, k) = 1$, the asymptotic equality

$$\nu_x(k, l) = d_k + O\left(k \exp\left\{-\left(1 + \frac{c \log \log \log x}{\log \log x}\right) \sqrt{\frac{1}{2} \log x \log \log x}\right\}\right),$$

where $c > 0$,

$$d_k = \frac{d}{k} \prod_{p|k} \left(1 + \frac{1}{p-1}\right) \left(1 + \sum_{j=1}^{\infty} \frac{1}{1+p+\dots+p^j}\right)^{-1},$$

as $x \rightarrow \infty$, is true.

The proof of the theorem is analytic and similar to that of analogous result for the Euler's function (see [5]). In this paper we only give the main ideas of the proof.

We consider the function

$$F(s) = \sum_{\substack{\sigma(n) \leq x \\ n \equiv l \pmod{k}}} \sigma(n)^{-s}, \quad s = \sigma + it, \quad \sigma > 1.$$

The analytic properties of $F(s)$ may be deduced from the properties of the functions

$$\Phi(s, \chi) = \sum_{n=1}^{\infty} \chi(n) \sigma(n)^{-s},$$

since

$$F(s) = \frac{1}{\varphi(k)} \sum_{\chi} \bar{\chi}(l) \Phi(s, \chi).$$

Here $\chi = \chi(n)$ is a Dirichlet character mod k and $\varphi(k)$ is the Euler totient function.

For the function $\Phi(s, \chi)$ we have

$$\Phi(s, \chi) = \prod_p \left(1 + \sum_{j=1}^{\infty} \chi^j(p) (p^j + p^{j-1} + \dots + p + 1)^{-s}\right) = L(s, \chi) \cdot G(s, \chi),$$

where

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

is a Diriclet L -function and

$$G(s, \chi) = \prod_p \left(1 - \sum_{j=1}^{\infty} \chi^j(p) \left(p^{-s}(1+p+\dots+p^{j-1})^{-s} - (1+p+\dots+p^j)^{-s} \right) \right) =: \prod_p (1 - \omega_p(s)).$$

As in the Lemma 1 of the paper [4] for any $\chi \pmod{k}$ and for $\sigma > 0$, we obtain

$$|\omega_p(s)| \leq c_2 |s| p^{-1-\sigma}.$$

Hence the function $G(s, \chi)$ is analytic in the half-plane $\sigma > 0$ for every $\chi \pmod{k}$. Consequently the function $\Phi(s, \chi)$ for $\chi \neq \chi_0$ is also analytic on the half-plane $\sigma > 0$. For the principal character $\chi = \chi_0$ we have the analyticity of $\Phi(s, \chi_0)$ in the half-plane $\sigma > 0$ except the point $s = 1$. The residue at the simple pole $s = 1$ is $\varphi(k)d_k$.

Let $\tau = k|t|$, τ_0 be a fixed number (sufficiently large). Denote

$$\eta(\tau) = \frac{\log \log \tau}{\log \tau}, \quad \tau \geq \tau_0.$$

Similarly as in [5] we can deduce the estimates:

$$F(s) = O\left(\exp\left\{\frac{c_3 \log \tau}{\log \log \tau}\right\}\right), \quad 1 - \eta(\tau) \leq \sigma \leq 1, \quad \tau \geq \tau_0;$$

$$F(s) = O\left((\log \tau)^{c_4}\right), \quad 1 \leq \sigma \leq 2, \quad \tau \geq 2.$$

Here c_3 and c_4 are positive constants. To prove the theorem, the contour integration may be applied:

$$E(x) := \frac{1}{x} \int_1^x D(u; k, l) du = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(s) x^s \frac{ds}{s(s+1)}, \quad b > 1.$$

The integration over contour K ,

$$K = \{s : s = a + i\tau \text{ for } |\tau| \leq \tau_0; s = 1 - \eta(\tau) + i\tau \text{ if } |\tau| > \tau_0\},$$

where $a = 1 - \eta(\tau_0)$, gives the assertion of the theorem for $E(x)$. Then by applying the standard arguments we deduce the equality of the theorem.

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Skaičių iš aritmetinės progresijos daliklių sumos

E. Stankus

Pažymėkime $\sigma(n)$ skaičiaus n daliklių sumą ir tegu

$$D(x; k, l) := \sum_{\substack{\sigma(n) \leq x \\ n \equiv l \pmod{k}}} 1; \quad (l, k) = 1, \quad k \geq 2.$$

Darbe įrodoma, jog bet kuriems sveikiesiems $k \geq 2$ ir l , $(l, k) = 1$, kai $x \rightarrow \infty$, galioja asimptotinė lygybė

$$D(x; k, l) = d_k x + O\left(xk \exp\left\{-\left(1 + \frac{c \log \log \log x}{\log \log x}\right) \sqrt{\frac{1}{2} \log x \log \log x}\right\}\right),$$

kurioje c – teigiama konstanta,

$$d_k = \frac{1}{k} \prod_{p \nmid k} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{j=1}^{\infty} \frac{1}{1 + p + \dots + p^j}\right).$$