LMD mokslo darbai, 21–23
1998, Vilnius

A note on the multiplicative dependence of consecutive integers

A. Dubickas* (VU)

Let $n \geqslant 2$ be a positive integer and let $k(n)$ be the minimal $k \in \mathbb{N}$ such that the integers $n, n+1, n+2, \ldots, n+k$ are multiplicatively dependent over $\mathbb{Q}$. It is clear that $k(n) \leqslant n^2 - n$, since n and n^2 are multiplicatively dependent.

The examples of the multiplicative dependence relations were considered by E.M. Nikišin [4] and V.K. Ryzhov. He noticed the identity

$$\bigl((n+2)(n-1)^2\bigr)\bigl((n-2)(n+1)^2\bigr)(n^3) = (n^3-4n)(n^3-n)^2$$

(see [3, Ch. III, problem 7]) which implies that the inequality $k(n) \leqslant 8n^{1/3}$ holds for infinitely many n. The upper bounds for $k(n)$ were also considered by V.K. Ryzhov in [5] and by J. Turk [6], [7]. The best result so far comes from the work of J. Turk [7]. He proved that there exists an effective constant c such that the inequality

$$k(n) < \exp\left(c\sqrt{\log n \ \log\log n}\right) \tag{1}$$

holds for infinitely many n.

On the other hand, A.A. Karatsuba (see [3], [4]) and J. Turk [7] obtained the lower bounds for $k(n)$. The proof of the best lower bound so far [7]

$$k(n) > c_1 \frac{\log n \ \log\log n}{\log\log\log n}$$

involves Gel'fond–Baker's method.

In this note we will look more closely at the constant c in the inequality (1).

Theorem. *Suppose that* $\varepsilon > 0$. *Then the inequality*

$$k(n) < \exp\left(\sqrt{(2+\varepsilon)\log n \log\log n}\right) \tag{2}$$

holds for infinitely many n.

We will follow [5] and [7] in our proof of the theorem. Let $\Phi(x, y)$ be the number of positive integers $\leqslant x$ and free of prime factors $> y$. Various estimates for $\Phi(x, y)$ were

*Partially supported by Grant from Lithuanian Foundation of Studies and Science.

obtained, e.g., by N.G. de Bruijn [1], A.I. Vinogradov [8]. We will use the lower bound due to A.S. Fainleib [2]: if $y > \log x$ and

$$t = \frac{\log x}{\log y} > t_0,$$

then

$$\Phi(x, y) > x \exp\big(-t\log t - t\log\log t\big). \tag{3}$$

Let us assume that for some small positive ε the inequality opposite to (2) holds for all $n \geqslant N$, where N is a large enough positive integer. We define integers k, M, y, J by

$$\begin{aligned} k &= \left[\exp\left(\sqrt{(2+\varepsilon)\log N\ \log\log N}\right)\right], \\ M &= \left[N\exp\left((\log N)^{2/3}\right)\right], \\ y &= \left[\exp\left(\sqrt{\frac{1}{2}\log N\ \log\log N}\right)\right], \\ J &= 1 + \left[(M-N+1)/k\right]. \end{aligned}$$

Since the inequality $k(n) \geqslant k$ holds for $n \geqslant N$, the integers $N+jk, N+jk+1, \dots, N+jk+k-1$ are multiplicatively independent for a fixed non-negative integer j. Suppose that A_j integers among $N+jk, \dots, N+jk+k-1$ are free of prime factors $> y$. Then $A_j \leqslant \pi(y)$ (see Lemma 1 in [5] or the respective statement in [7]). Counting the number of positive integers $\leqslant N+Jk-1$ and free of prime factors $> y$ we now obtain

$$\Phi(N+Jk-1, y) \leqslant N - 1 + \sum_{j=0}^{J-1} A_j \leqslant N - 1 + J\pi(y).$$

It is obvious that $\pi(y) \leqslant y$, $N + Jk - 1 > M$ and $J < M/k$. Thus,

$$\Phi(M, y) < N + My/k. \tag{4}$$

Taking into account our choice of k, M, y we see that the right-hand side of the inequality (4) is less than

$$M\exp\left(-\left(\frac{1}{\sqrt{2}} + \frac{\varepsilon}{3}\right)\sqrt{\log N\ \log\log N}\right).$$

On the other hand, for large N we have

$$t = \frac{\log M}{\log y} \sim \sqrt{\frac{2\log N}{\log\log N}}.$$

Hence, we can bound

$$t\log t + t\log\log t < \Big(\frac{1}{\sqrt{2}} + \frac{\varepsilon}{4}\Big)\sqrt{\log N\ \log\log N}.$$

Now (3) implies that the left-hand side of (4) is greater than

$$M\ \exp\bigg(- \Big(\frac{1}{\sqrt{2}} + \frac{\varepsilon}{4}\Big)\sqrt{\log N\ \log\log N}\bigg).$$

This contradicts our assumption and completes the proof.

REFERENCES

[1] N. G. de Bruijn, On the number of positive integers $\leqslant x$ and free of prime factors $> y$. II, *Indag. Math.*, **28** (1966), 239–247.

[2] A. S. Fainleib, On the lower estimate of numbers with small prime divisors, *Dokl. Akad. Nauk Uzbek. SSR*, **7** (1967), 3–5.

[3] A. A. Karatsuba, *The Principles of Analytic Number Theory* (in Russian), Nauka, Moscow, 1983.

[4] E. M. Nikišin, On logarithms of natural numbers, *Izv. Akad. Nauk SSSR*, **43** (1979), 1319–1327.

[5] V. K. Ryzhov, Logarithms of natural numbers, *Russian Math. Surveys*, **41** (1986), 239–240.

[6] J. Turk, *Multiplicative properties of neighbouring integers*, 1979, Thesis, Leiden.

[7] J. Turk, Multiplicative properties of integers in short intervals, *Indag. Math.*, **42** (1980), 429–436.

[8] A. I. Vinogradov, On numbers with small prime divisors, *Dokl. Akad. Nauk SSSR*, **109** (1956), 683–686.

Pastaba apie iš eilės einančių natūraliųjų skaičių multiplikatyvų priklausomumą

A. Dubickas (VU)

Tegu $k(n)$ yra mažiausias $k \in \mathbb{N}$, toks kad skaičiai $n, n+1, n+2, \ldots, n+k$ yra multiplikatyviai priklausomi virš $\mathbb{Q}$. Straipsnyje įrodoma, kad egzistuoja ba galo daug natūraliųjų n, kuriems teisinga nelygybė

$$k(n) < \exp\Big(\sqrt{(2+\varepsilon)\log\log\log n}\Big).$$