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Real ruled degree four toric surfaces in projective 3-space*

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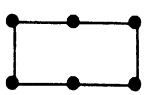
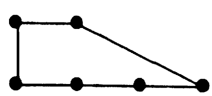
1. Introduction

In this notes a rational, ruled with a directrix line surfaces of order four in a real projective 3-space $\mathbb{P}^3(\mathbb{R})$ are studied and classified. In short we denote them by $X_{3,1}$. It turns out that those surfaces have a singular curve of degree three. We describe all possibilities for singular curves and pinch points on those curves.

One can get any surface $X'_{1,3}$ as the linear projection (from a line) of another surface $X'_{1,3} \subset \mathbb{P}^5(\mathbb{R})$. Geometrically the surface $X'_{1,3}$ can be obtained as follows. Take a line d_1, projective 3-subspace $W \subset \mathbb{P}^5$ ($W \cap d_1 = \emptyset$), rational normal cubic curve $d_3 \subset W$ and fix some isomorphism $f\colon d_1 \to d_3$ then $X'_{1,3}$ is the union of all lines between points $x \in d_1$ and $f(x) \in d_3$. This is so called rational ruled surface of type $X_{3,1}$ in the notation of the paper [1] (for more information about those surfaces see pp. 523–527 in the book [6]). Notice that if we choose other line d_1 and cubic d_3 then we obtain a new surface which is projectively equivalent to $X'_{1,3}$. So for classification of surfaces $X_{1,3}$ we need to classify all possible positions of the projection line to respect the surface $X'_{1,3}$. One can prove by counting parameters that there is 3-parameter family of projectively non-equivalent surfaces of type $X_{1,3}$ in $\mathbb{P}^3(\mathbb{R})$. It is possible to give explicitly formulas for those parameters but we will not develop this point here.

One can prove that rational ruled surfaces of order four in $\mathbb{P}^3(\mathbb{R})$ are of two types, namely $X'_{1,3}$, $S(2,1)$. The surfaces of type $S(2,1)$ can be parameterized by the points of $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$ using polynomials of bidegree (2,1). The classification of surfaces $S(2,1)$ is presented in [8] and this paper can be considered as an extension of results and methods obtained in the article [8] for the surfaces of type $X'_{1,3}$. It is worth pointing out that the surfaces $S(2,1)$, $X'_{1,3}$ are toric varieties. The toric structure of $S(2,1)$ and $X'_{1,3}$ correspond to polygons:

$S(2,1)$  $X_{1,3}$ 

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We refer the reader for the basic theory of toric varieties to [5]. It is very useful for our studies and geometric design to notice that both surfaces are uniquely defined by the position of six control points with weights in $\mathbb{P}^3(\mathbb{R})$.

The paper is organized into three sections as follows. In Section 2 we introduce notation and definitions in the form they will be used later on. The collection of main results and proofs are presented in Section 3.

2. Notations and definitions

We will work over the real field $\mathbb{R}$. We denote by $\mathbb{P}^n(\mathbb{R})$ projective n-space. We will omit the field $\mathbb{R}$ in the notations below. A point $x \in \mathbb{P}^n$ is represented in terms of homogeneous coordinates $x = (x_0 : x_1 : \ldots : x_n)$. Denote by $\langle a, b, \ldots, c\rangle$ the smallest projective space which contains subsets $a, b, \ldots, c \subset \mathbb{P}^n$. We write $X \approx Y$ if X is isomorphic to Y. The variety defined by homogeneous polynomials $f_1, f_2, \ldots, f_k$ is denoted as $\{f_1, f_2, \ldots, f_k\} := \{x \in \mathbb{P}^n \mid f_1(x) = 0, f_2(x) = 0, \ldots, f_k(x) = 0\}$. Let $S_f = \{x = (x_0, x_1, x_2, x_3) \in \mathbb{P}^3 \mid f(x) = 0\}$ Denote by

$$\mathrm{Sing}(S_f) = \left\{x \in S_f \;\middle|\; \frac{\partial f}{\partial x_i} = 0, \quad \text{for all } i = 0, \ldots, 3\right\}$$

the singular variety of S. Also denote the variety of triple points by

$$\mathrm{Sing}_3(S_f) = \left\{x \in \mathrm{Sing}(S_f) \;\middle|\; \frac{\partial^2 f}{\partial x_i \partial x_j} = 0, \quad \text{for all } i, j = 0, \ldots, 3\right\}.$$

We shall say that a curve $C \subset S_f$ is a *double* curve if $C \subset \mathrm{Sing}(S_f)$ and C contains a finite number of triple points. Similarly C is a *triple* curve if $C \subset \mathrm{Sing}_3(S_f)$. A *pinch* point $p \in S_f$ is a point on a double curve C where the rank of *Hessian*

$$Hf := \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{0 \leqslant i, j \leqslant 3}$$

is equal or less than one. Geometrically this means that there is only one tangent plane for the pinch point p. Here by definition the tangent plane is the plane of lines which are the limits of the chordal lines $\langle p, x\rangle$, $x \in S_f$ when x approaches to p. Recall that for a general double point on a general double curve C there are two tangent planes and only a finite number of points on a general double curve can be pinch points. We emphasize that there is a special double curve C such that on any point of C there exists only one tangent plane (see the definition below). There is also the third definition of a pinch point. Let us denote by $\sigma\colon S' \to S_f$ the desingularization map of S_f and $B := \sigma^{-1}(C)$ then the map $\sigma\colon B \to C$ is two-to-one and pinch points are exactly ramification points for this map (i.e. p is a pinch point if $\sigma^{-1}(p)$ is one point). A *cuspidal* curve C is a double curve on a surface S such that every point on C is a pinch point. If $\sigma\colon S' \to S_f$ is the desingularization map of S and $B := \sigma^{-1}(C)$ then the map $\sigma\colon B \to C$ is one-to-one.

This is the distinguished property between a *cuspidal* curve and *two-fold* double curve. A double curve is *two-fold* (or it has multiplicity two) if tangent sheets along this curve are the same but have different curvature. A surface formed by a single infinite system of straight lines is called a *ruled surface*; the lines are called the *generators* of the surface. A curve on a ruled surface meeting every generator in one point will be called a *directrix*.

3. Projection from $\mathbb{P}^5$ to $\mathbb{P}^3$

Consider the surface $X'_{1,3} \subset \mathbb{P}^5$ defined as follows:

$$X'_{3,1} := \left\{ \operatorname{rank} \begin{pmatrix} x_0 & x_2 & x_3 & x_4 \\ x_1 & x_3 & x_4 & x_5 \end{pmatrix} = 1 \right\}. \tag{1}$$

This is a normal rational ruled surface of degree four with a directrix line $\{x_2, x_3, x_4, x_5\}$ and a directrix cubic $(0 : 0 : t^3 : t^2 : t : 1) \subset \{x_0, x_1\} \approx \mathbb{P}^3$ (see [1]). By the definition the surface $X'_{1,3}$ is the linear projection of the normal surface $X'_{1,3}$ from $\mathbb{P}^5$ to $\mathbb{P}^3$ (see [3] §54).

PROPOSITION 1. *Suppose $C \subset X'_{1,3}$ is a curve of degree two then $C = l_1 \cup l_2$, where l_1, l_2 are lines on $X'_{1,3}$.*

Proof. A smooth irreducible curve C of degree two belongs to a plane P. Consider the intersection of the surface $X'_{1,3}$ and projective 4-space W, where $W = \langle P, g \rangle$ (g is a generator of the ruled surface $X'_{1,3}$). One can see that $X'_{1,3} \cap W = C \cup g \cup l'$, where l' is a line on $X'_{1,3}$. Notice that any generator of $X'_{1,3}$ meets W in one point therefore a curve $C \cup l'$ is a directrix of $X'_{1,3}$. Hence if we choose W such that it has not contain a directrix line then a conic C should be a directrix. But from this we get that degree of $X'_{1,3}$ is three. This contradiction prove that a curve C should be reducible.

Take any line $l \subset \mathbb{P}^5$ and a projective 3-space $W \subset \mathbb{P}^5$ such that $l \cap W = \emptyset$. Then consider a linear projection

$$\pi_{l,W} \colon \mathbb{P}^5 \setminus l \to W \approx \mathbb{P}^3. \tag{2}$$

By the definition this means that $\pi_{l,W}(p) = q$ where $q = W \cap \langle l, p \rangle$. Recall that $\langle l, p \rangle \approx \mathbb{P}^2$ hence projective 2-space $\langle l, p \rangle$ and projective 3-space W has exactly one common point q. Notice that the choice of W is not very important because $\pi_{l,W} = \phi \circ \pi_{l,W'}$ here $\phi \colon W \to W'$ is a projective isomorphism, i.e. projections essentially are dependent on the choose of the line but not on projective 3-space W. For this reasons we usually omit the second argument and hope that the meaning of W will be claire from the context or not important.

PROPOSITION 2. *Assume $\pi_l \colon X'_{1,3} \to X_{1,3}$ is a projection and L is a singular line for $X_{1,3}$. Then one of the following properties holds*:

(a) *L is a double line if and only if $\pi_l^{-1}(L) = d_1 \cup g'$, here d is a directrix line and g' is a generator.*

(b) *L is a triple line if and only if $\pi_l^{-1}(L) = d_1 \cup g' \cup g''$ or $\pi_l^{-1}(L) = d_3$, here d_1 is a directrix line, d_3 is a smooth directrix cubic g', g'' are generators of $X'_{1,3}$.*

Proof. (a) If L is a double line on $X_{1,3}$ then deg $\pi_l^{-1}(L) = 2$. This means that $\pi_l^{-1}(L) = l_1 \cup l_2$ (by the Proposition 1). Take any plane $H \subset \mathbb{P}^3, L \subset H$ then $H \cap X_{1,3} = 2L \cup l' \cup l''$. Therefore the line L is a directrix. Hence we have $\pi_l^{-1}(L) = d_1 \cup g_3$ here d is a directrix line on $X'_{1,3}$. Notice that in this case the line of projection l meets exactly one plane $P(\lambda) = \langle d, l(\lambda)\rangle$ for $\lambda \in \mathbb{P}^1$, here $l(\lambda)$ is a generator of $X'_{1,3}$.

(b) If L is a triple line then deg $\pi_l^{-1}(L) = 3$. If we consider the intersection $H \cap X_{1,3} = 3L \cup l$ ($L \subset H$ is a plane) we see that L is a directrix line. Therefore a curve $\pi_l^{-1}(L)$ is also a directrix. Assume $\pi_l^{-1}(L)$ is not a smooth curve then by Proposition 1 $\pi_l^{-1}(L) = d_1 \cup g' \cup g''$ here d_1 is a directrix line, g', g'' are generators of $X'_{1,3}$. If $\pi_l^{-1}(L) = d_3$ is smooth then it should be a smooth directrix cubic curve on $X'_{1,3}$ (this will be if and only if the projection line l belongs to projective 3-space U, $d_3 \subset U$).

PROPOSITION 3. *Let $X_{1,3} \subset \mathbb{P}^3$ be a rational ruled surface of degree four then a singular curve B of $X'_{1,3}$ has degree three.*

Proof. Take any plane $H \subset \mathbb{P}^3$ such that $g \subset H$, here g is a generator of $X'_{1,3}$, then $H \cap X_{1,3} = g \cup c_3$ (c_3 is a plane cubic). Notice that any generator of $X_{1,3}$ meets exactly in one point the plane H therefore the cubic c_3 is a directrix and any point on this cubic corresponds to a generator of $X'_{1,3}$. Since the generator g and cubic c_3 have three common points p, p_1, p_2 one of them p corresponds to the generator g and through rest two points p_1, p_2 passes another generators g_1, g_2 of $X'_{1,3}$. One can see that there are two tangent planes for the surface $X'_{1,3}$ in the points $p_i, i = 1, 2$, namely H, $\langle g, g_i\rangle$. Therefore a singular curve B meets any generator of $X'_{1,3}$ in two points. Hence a curve B has degree more or equal to three. The intersection of $X'_{1,3}$ and any general plane P is a plane quartic curve which has at most three singular points therefore deg $B = 3$. Note that there are possible degeneration i.e. the singular curve can be reducible or even a line of triple points.

If deg $X_{1,3} \leqslant 3$ then the complete real projective classification obtained in [2], see also [8]. If deg $X_{1,3} = 4$ there are three possibilities for singularities of $X_{1,3}$.

A. $\mathrm{Sing}(X_{1,3}) = C$ is smooth cubic curve in $\mathbb{P}^3$. The points on this cubic are double points for $X_{1,3}$. There are four pinch points on C. It can be that all four of them are real or two pinch points are real and two imaginary or four pinch points are not real. (The proof is similar to the proof of Proposition 13 in [8] (see also a remark on the page 25)).

B. $\mathrm{Sing}(S) = L \cup C$ where L is a line of double points on S, C is a smooth conic of double points which intersect L in one point. There is *one* real pinch point on the line L and *no* pinch points on the conic C. (The proof is similar to the proof of Propositions 9,10 in [8] (see also a remark on the page 23)).

C. $\mathrm{Sing}(S) = L$ is the line of triple points. This is the most degenerated case. Also we say that the point $p \in L$ is the pinch point for S if there is a plane $H \ni p$ such that $H \cap S$ is degree four reducible curve which reducible components are a line and a smooth conic.

Geometrically this means that there are two (or less) tangent planes at point p. Recall that for general triple point there are three different tangent planes. It turns out that on triple line L there are four (real or complex) pinch points. (The proof is similar to the proof of Propositions 4,5 in [8] (see also a remark on the page 17)).

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Realūs tiesialinijiniai toriniai ketvirtojo laipsnio paviršiai projekcinėje trimatėje erdvėje

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Darbe nagrinėjami ir klasifikuojami racionalūs realūs tiesialinijiniai ketvirtojo laipsnio su direktrisine tiese paviršiai. Pasirodo, kad tokie paviršiai turi trečios eilės kreivę ypatingųjų taškų. Straipsnyje yra suklasifikuotos visos tokios galimos ypatingos kreivės ir pinčo taškai jose. Tai turėtų būti pakankama norint gauti pilną projekcinę tokių realių paviršių klasifikaciją.