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ROLE OF PARABOLIC VISCOSITY IN NUMERICAL ANALYSIS OF DERIVATIVE NONLINEAR EVOLUTION EQUATIONS

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Abstract

We consider the difference schemes applied to a derivative nonlinear system of evolution equations. For the boundary-value problem with initial conditions

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x} + f(x,u) + g(x,u)\frac{\partial u}{\partial x}, \quad (t,x) \in (0,T] \times (0,1),$$
$$u(t,0) = u(t,1) = 0, \quad t \in [0,T],$$
$$u(0,x) = u^{(0)}(x), \quad x \in (0,1)$$

we use the Crank-Nicolson discretization. A is complex and B – real diagonal matrixes; u, f and g are complex vector-functions. The analysis shows that proposed schemes are uniquely solvable, convergent and stable in a grid norm W_2^2 if all (diagonal) elements in $Re(A)$ are positive. This is true without any restriction on the ratio of time and space grid steps.

INTRODUCTION

In recent years there has been a growing interest in nonlinear evolution equations. Such well-known equations (as well as their systems) as nonlinear Schrödinger equation (NLS), nonlinear reaction-diffusion equation (NLRD) and the nonlinear Kuramoto-Tsuzuki equation (NLKT) appear in many models of mathematical physics. For example, one often finds NLS in nonlinear optics [1, 2], plasma physics [3]. NLRD systems are used in investigating a wide class of nonlinear processes [4, 6]. NLKT describes the behavior of two-component systems in a neighborhood of a bifurcation point [4, 5]. In some models there is necessary to study the effects born by higher order perturbations and the derivative nonlinear (DN) equations appear.

We deal with the difference schemes applied to a derivative nonlinear system of evolution equations. For the boundary-value problem with initial conditions

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial u}{\partial x} + f(x,u) + g(x,u)\frac{\partial u}{\partial x}, \qquad (t,x) \in Q, \tag{1}$$
$$u(t,0) = u(t,1) = 0, \qquad t \in [0,T], \tag{2}$$
$$u(0,x) = u^{(0)}(x), \qquad x \in \Omega \tag{3}$$

we use the Crank-Nicolson discretization. Here $u = (u_1, u_2, \dots, u_n)$ is a complex vector-function, $\Omega = (0, 1)$ and $Q = (0, T] \times \Omega$. Diagonal matrices A and B contain complex and real coefficients, respectively.

We consider the "truncation" of corresponding Cauchy problem to a bounded domain. Such approach is often used solving the problem numerically.

DIFFERENCE SCHEME

Denote the diagonal elements of matrices A and B by a_{jj} and b_{jj}, respectively. By introducing new functions $y_j = u_j \exp(b_{jj}x/2a_{jj})$ one may neglect the first order partial derivatives $\partial u_j/\partial x$ in the linear part of system (1). Also note that since the matrix A is diagonal, there is no essential difference between the study of system (1) and the study of one equation. For simplicity we assume that the nonlinear functions $f(x, u)$ and $g(x, u)$ do not depend on x. Therefore we further consider one equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(u) + g(u)\frac{\partial u}{\partial x}, \qquad (t, x) \in Q. \tag{4}$$

(4) is a DN Schrödinger equation if $Re(a) = 0$. In the case $Im(a) = 0$ (4) represents a DN reaction-diffusion equation and, finally, when both $Re(a), Im(a) \neq 0$, it stands for a DN Kuramoto-Tsuzuki type equation.

We assume that:

a) the partial derivatives of f and g with respect to u are continuous, and

$$|f(u)|, \quad |g(u)|, \quad \left|\frac{\partial f(u)}{\partial u}\right|, \quad \left|\frac{\partial g(u)}{\partial u}\right| \leq \varphi(|u|),$$

where φ is a continuous nondecreasing function,

b) $f(0) = g(0) = 0$,

c) $Re(a) \geq \delta > 0$.

Conditions a) and b) are satisfied for the models [1–6]. The condition c) means the positivity of the heat conduction coefficient.

Using the notation of [7], we introduce the uniform grids ω_τ and Ω_h with steps τ and h for the variables t and x, respectively. We relate the problem (4),(2), (3) with the following Crank–Nicolson type symmetric difference scheme:

$$v_t = a\,\overset{\circ}{v}_{\bar{x}x} + f\left(\overset{\circ}{v}\right) + g\left(\overset{\circ}{v}\right)\overset{\circ}{v}_{\overset{\circ}{x}}, \qquad (t, x) \in \omega_\tau \times \Omega_h, \tag{5}$$

$$v(t, 0) = v(t, 1) = 0, \qquad t \in \bar{\omega}_\tau, \tag{6}$$

$$v(0, x) = u^{(0)}(x), \qquad x \in \bar{\Omega}_h, \tag{7}$$

where $\hat{v} = v(t + \tau, x)$, $\overset{\circ}{v} = (\hat{v} + v)/2$, $v_{\overset{\circ}{x}} = (v(t, x + h) - v(t, x - h))/2h$, $v_t = (\hat{v} - v)/\tau$, $v_{\bar{x}x} = (v(t, x + h) - 2v(t, x) + v(t, x - h))/h^2$.

The scheme (5) – (7) is implicit and nonlinear. To calculate a solution on the upper layer $t = t_{j+1}$ one can apply the iterative method:

$$\frac{v^{s+1} - v}{\tau} - a\frac{v^{s+1}_{\bar{x}x} + v_{\bar{x}x}}{2} = f\left(\frac{v^s + v}{2}\right) + g\left(\frac{v^s + v}{2}\right)\frac{v^s_{\overset{\circ}{x}} + v_{\overset{\circ}{x}}}{2}, \qquad x \in \Omega_h,$$

$$v^{s+1}(0) = v^{s+1}(1) = 0,$$
$$v^0 = v.$$

The next iteration v^{s+1} can be found by the matrix sweep method, for example.

Using the grid analogues of a new type *a priori* estimates we justify (5) – (7) difference scheme. It appears to be convergent and stable without any restrictions on the ratio of time and space grid steps. Note only, that proving the boundedness of numerical problem solution, the usual estimate for transition to the upper layer doesn't work here, as the the mathematical induction, based on this estimate, fails (see further). The main difficulty concerns the treatment of the DN terms in the equation. For justification of difference schemes without gradient-dependent nonlinearity see [8]. To overcome this, we modified the above method, estimating a sum of vanishing geometric progression (see proof of Lema 1). For details, consider the auxiliary linear difference scheme

$$v_t - a\,\overset{\circ}{v}_{\bar{x}x} = r, \qquad (t,x) \in \omega_\tau \times \Omega_h, \tag{8}$$
$$v(t,0) = v(t,1) = 0, \qquad t \in \bar{\omega}_\tau, \tag{9}$$
$$v(0,x) = v^{(0)}(x), \qquad x \in \bar{\Omega}_h, \tag{10}$$

with the right-hand side $r(t) \in \overset{\circ}{W}{}_2^1(\Omega_h)$, $t \in \omega_\tau$, and an initial data $v^{(0)}(x) \in \overset{\circ}{W}{}_2^2(\Omega_h)$.

Lemma 1. *Suppose the hypothesis* c) *is satisfied,* $r(t) \in \overset{\circ}{W}{}_2^1(\Omega_h)$, $t \in \omega_\tau$ *and* $v^{(0)}(x) \in \overset{\circ}{W}{}_2^2(\Omega_h)$. *Then with all* $t_j \in \bar{\omega}_\tau$, j *- the layer number, for the solution of the problem* (8) – (10) *the following estimates hold:*

$$\|v(t_j)\|_{L_2(\Omega_h)} \le \|v(0)\|_{L_2(\Omega_h)} + \tau \sum_{s=0}^{j-1} \|r(t_s)\|_{L_2(\Omega_h)}, \tag{11}$$

$$\|v_{\bar{x}}(t_j)\|_{L_2(\Omega_h^+)} \le \|v_{\bar{x}}(0)\|_{L_2(\Omega_h^+)} + \frac{1}{\sqrt{2\delta}} \sqrt{\tau \sum_{s=0}^{j-1} \|r(t_s)\|^2_{L_2(\Omega_h)}}, \tag{12}$$

$$\|v_{\bar{x}x}(t_j)\|_{L_2(\Omega_h)} \le \|v_{\bar{x}x}(0)\|_{L_2(\Omega_h)} + \frac{1}{\sqrt{2\delta}} \sqrt{\tau \sum_{s=0}^{j-1} \|r_{\bar{x}}(t_s)\|^2_{L_2(\Omega_h^+)}}, \tag{13}$$

where $\|\ \|$ *are grid analogues of the corresponding continuous spaces norms* (*see* [7]).

Proof. We apply the Fourier separation of variables method. As a basis, consider the functions

$$\mu_k(x) = \sqrt{2}\sin(k\pi x), \qquad k = 1, 2, \ldots, N-1, \qquad h = \frac{1}{N}.$$

It is known [9, p. 285] that $\{\mu_k(x)\}$ is an orthonormal and complete function system in $L_2(\Omega_h)$; the systems of difference derivatives

$$\mu_{k\bar{x}}(x) = \sqrt{\lambda_k}\sqrt{2}\cos(k\pi[x - h/2]), \qquad k = 1, 2, \ldots, N-1,$$

and

$$\mu_{k\bar{x}x}(x) = \lambda_k \mu_k(x), \qquad \lambda_k = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right), \qquad k = 1, 2, \ldots, N-1,$$

appear to be orthogonal in $L_2(\Omega_h^+)$ and $L_2(\Omega_h)$, respectively.

For grid functions introduce the inner product

$$(y,\, v)_{\Omega_h} = h \sum_{x\in\Omega_h} u(x)v^*(x),$$

where v^* denotes the complex conjugate of v.

We look for the solution of problem (8)–(10) of the form

$$v(t,x) = \sum_{k=1}^{N-1} v_k(t)\mu_k(x), \tag{14}$$

where $v_k(t) = (v(t,x),\, \mu_k(t))_{\Omega_h}$ are the Fourier coefficients. By Eq. (8) we have the relations

$$\hat{v} = \alpha_k v_k + \tau\beta_k r_k, \tag{15}$$

where $r_k = (r,\, \mu_k)_{\Omega_h}$, $\beta_k = \left(1 + \frac{\tau a\lambda_k}{2}\right)^{-1}$, $\alpha_k = \left(1 - \frac{\tau a\lambda_k}{2}\right)\beta_k$. By the condition c) it follows that

$$|\alpha_k|, |\beta_k| \le 1, \qquad k = 1, 2, \ldots, N-1. \tag{16}$$

From (14), (15), using the Parseval identity, estimate (16), and Minkowski inequality we obtain

$$\|\hat{v}\|_{L_2(\Omega_h)} \le \|v\|_{L_2(\Omega_h)} + \tau\|r\|_{L_2(\Omega_h)}.$$

This completes the proof of estimate (11).

We now pass to the estimate of the first order difference derivatives $v_{\bar{x}}$. Since $\{\mu_{k_{\bar{x}}}(x)\}$ is an orthogonal system, by (15) it follows that

$$\|v_{\bar{x}}(t_j)\|_{L_2(\Omega_h^+)} = \sqrt{\sum_{k=1}^{N-1} \lambda_k \left|v_k\left(t_j\right)\right|^2} =$$

$$\sqrt{\sum_{k=1}^{N-1} \lambda_k \left|\alpha_k^j v_k(0) + \tau\beta_k\left(\alpha_k^{j-1} r_k(0) + \alpha_k^{j-2} r_k\left(t_1\right) + \cdots + r_k\left(t_{j-1}\right)\right)\right|^2}.$$

Apply Minkowski and Cauchy inequalities. We get

$$\|v_{\bar{x}}(t_j)\|_{L_2(\Omega_h^+)} \le \|v_{\bar{x}}(0)\|_{L_2(\Omega_h^+)} + \sqrt{\tau \sum_{k=1}^{N-1}\sum_{s=0}^{j-1} \left|r_k\left(t_s\right)\right|^2 \tau\lambda_k \left|\beta_k\right|^2 \sum_{s=0}^{j-1} \left|\alpha_k\right|^{2s}}. \tag{17}$$

A simple estimate of the sum of a geometrical progression shows that

$$\tau\lambda_k \left|\beta_k\right|^2 \sum_{s=0}^{j-1} \left|\alpha_k\right|^{2s} \le \frac{\tau\lambda_k \left|\beta_k\right|^2}{1 - \left|\alpha_k\right|^2} \le \frac{1}{2\delta}.$$

Now estimate (12) can be obtained from (17). In a similar way we can deduce (13). Lemma is proved.

Remark 1. Note that appearance of derivative type nonlinearities in (4) requires a lower norm on the right-hand side r in the estimates (12) and (13). This enable us further to treat the problem in a way similar to the usual nonlinear evolution equation (see [8]). Also, taking $j = 1$ in (12), one could the estimate for transition to the upper layer

$$\|\hat{v}_{\bar{x}}\|_{L_2(\Omega_h^+)} \leq \|v_{\bar{x}}\|_{L_2(\Omega_h^+)} + c\sqrt{\tau}\|r\left(t_s\right)\|_{L_2(\Omega_h)},$$

which is not sufficient to apply mathematical induction principle.

Now we state the results of our analysis:

Theorem 1. (Convergence) *Suppose the hypotheses* a) – c) *are held, and the solution of the problem* (4),(2),(3) *is smooth enough. Then there exist constants* $\tau_0, h_0 > 0$ *such that, for* $\tau \leq \tau_0$, $h \leq h_0$, *there exists a unique solution of the problem* (5) – (7), *converging to the solution of the problem* (4),(2),(3) *and the following estimate holds:*

$$\|u - v\|_{W_2^1(\Omega_h)} \leq c\left(\tau^2 + h^2\right), \qquad t \in \bar{\omega}_\tau.$$

By the imbedding theorem $W_2^1 \to C$ the convergence in C follows.

If $f(u)$ and $g(u)$ are polynomials then one can prove the convergence of the difference method in W_2^2:

$$\|u - v\|_{W_2^2(\Omega_h)} \leq c\left(\tau^2 + h\right), \qquad t \in \bar{\omega}_\tau.$$

In this case the scheme is convergent in C^1, too.

Theorem 2. (Stability) *Let* v_1 *and* v_2 *be the solutions of the problem* (5) – (7) *with initial data* $u_1^{(0)}$ *and* $u_2^{(0)}$. *Suppose the hypotheses of* Theorem 1 *are satisfied. Then there exist constants* $\tau_0, h_0 > 0$ *such that, for* $\tau \leq \tau_0$, $h \leq h_0$, *the following estimate holds:*

$$\|v_1(t) - v_2(t)\|_{W_2^1(\Omega_h)} \leq \|v_1(0) - v_2(0)\|_{W_2^1(\Omega_h)}, \qquad t \in \bar{\omega}_\tau.$$

Remark 3. If $f(u)$ and $g(u)$ are polynomials then the difference scheme (5) – (7) is stable in W_2^2.

Remark 4. In a similar way one can prove the convergence and stability of the difference schemes of the form

$$\begin{aligned} &v_t = av_{\bar{x}x}^{(\sigma)} + F(v, \hat{v}) + G(v, \hat{v})D_x(v, \hat{v}), \qquad (t, x) \in \omega_\tau \times \Omega_h, \\ &v(t, 0) = v(t, 1) = 0, \qquad t \in \bar{\omega}_\tau, \\ &v(0, x) = u^{(0)}(x), \qquad x \in \bar{\Omega}_h, \end{aligned}$$

where F, G, D_x approximate f, g, $\dfrac{\partial u}{\partial x}$, respectively, $v^{(\sigma)} = \sigma\hat{v} + (1-\sigma)v$, $\frac{1}{2} \leq \sigma \leq 1$.

CONCLUSIONS AND COMMENTS

We have justified Crank–Nicolson type finite difference schemes for evolution equation systems. There is no any restriction on the ratio of time and space grid steps

τ and h. Note that for $Re(a) = 0$ (as in (4)), i.e. derivative nonlinear Schrödinger equations these schemes remain questionable. Parameter $Re(a) = \delta > 0$ can be viewed as artificial (parabolic) viscosity in this case.

(5) – (7) scheme was practically examined on a computer for the problem

$$\frac{\partial u}{\partial t} = (\delta + i)\frac{\partial^2 u}{\partial x^2} + id|u|^2 u + \alpha |u|^2 \frac{\partial u}{\partial x} + \beta u \frac{\partial |u|^2}{\partial x}, \qquad (t,x) \in Q. \tag{18}$$

$$u(t,0) = u(t,1) = 0, \qquad t \in [0,T], \tag{19}$$

$$u(0,x) = u^{(0)}(x), \qquad x \in \Omega. \tag{20}$$

No negative signs were observed, even if $\delta = 0$ provided L_2 norm of $u^{(0)}$ is sufficiently small. Note that for $\delta = 0$ (18) – (20) problem has an energy conservation law in L_2 norm. A few conservative difference schemes were built in this case. We investigate them together with well-posedness of (18) – (20) boundary problem under assumption that L_2 norm of (20) initial data is small enough. In the case of DN Schrödinger equation ($\delta = 0$) for some initial functions the blow-up of solution was observed, i.e. introducing the small parabolic viscosity $\delta > 0$ is one of possible ways to regularize mathematical model.

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