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ASYMPTOTIC BEHAVIOUR OF A MODEL OF AN AGE-SEX-STRUCTURED POPULATION DYNAMICS

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Abstract

This paper deals with a model of an age-sex-structured population consisting of male, single (nonfertilized) female and fertilized female subclasses taking into account random mating of sexes (during the coupling only), and females' pregnancy. We analyze the special case of this model where death moduli can be decomposed into the sum of two terms: the first depends on age only and represents death by natural causes, while the second is a function of total population and represents environmental effects. For this model the separable solutions are presented and their asymptotic behaviour is demonstrated.

NOTATION

τ_1, τ_2 and τ_3 denote the age of males, females and embryos, respectively;

t is time;

$\rho(t)$ is the total population size;

$u_1(t,\tau_1)$, $u_2(t,\tau_2)$ and $u_3(t,\tau_1,\tau_2,\tau_3)$ are age densities of number of males, single and fertilized female, respectively;

$p(t,\tau_1,\tau_2,\rho)$ is the females fertilization rate;

$\nu_1(t,\tau_1,\rho)$, $\nu_2(t,\tau_2,\rho)$ and $\nu_3(t,\tau_1,\tau_2,\tau_3,\rho)$ are the death rates of males, single and fertilized females, respectively;

$X_2(u_3)(t,\tau_2)$ gives the female supply rate due to conceiving and deliveries;

$\sigma_1 = (\tau_{11},\tau_{12}]$, $0<\tau_{11}<\tau_{12}<\infty$ is the male sexual activity interval,

$\sigma_3 = (0,T]$, $0<T<\infty$ is the female gestation interval,

$\sigma_2(\tau_3) = (\tau_{21}+\tau_3, \tau_{22}+\tau_3]$, $0<\tau_{21}<\tau_{22}<\infty$,

$\sigma_2(0)$ and $\sigma_2(T)$ are the female fertilization and reproductivity intervals, respectively;

$n_1(t)$ is the size of male subclass with age from σ_1;

$b_1(t,\tau_1,\tau_2)$ and $b_2(t,\tau_1,\tau_2)$ represent the expected numbers of offspring produced at time t by a fertilized female with characteristics (τ_1,τ_2,T) and having the sex of male and female, respectively;

$u_1^0(\tau_1)$, $u_2^0(\tau_2)$, $u_3^0(\tau_1,\tau_2,\tau_3)$ denote initial distributions;

$\sigma = \sigma_1 \times \sigma_2(T)$, $d\sigma = d\tau_1\, d\tau_2$;

$\tau_2^0 = 0$, $\tau_2^1 = \tau_{21}$, $\tau_2^2 = \min(\tau_{21}+T, \tau_{22})$, $\tau_2^3 = \max(\tau_{21}+T, \tau_{22})$, $\tau_2^4 = \tau_{22}+T$, $\tau_2^5 = \infty$;

$I = (0,\infty)$, $I_4 = (\tau_2^4, \infty)$, $I_s = (\tau_2^s, \tau_2^{s+1}]$, $s = \overline{0,3}$;

$Q_1 = I \times I$, $Q_2 = I \times (I \setminus \bigcup_{s=1}^{4} \tau_2^s)$, $Q_3 = I \times \sigma_1 \times \sigma_2(\tau_3) \times \sigma_3$;

$[u_2|_{\tau_2=\tau_2^s}]$ is a jump of the function u_2 at the plane $\tau_2 = \tau_2^s$;

$L_j = \partial/\partial t + \partial/\partial\tau_j$, $j = 1, 2$, $L_3 = L_2 + \partial/\partial\tau_3$,

$D_1 = \sqrt{2}\tilde{D}_1$, $D_2 = \sqrt{2}\tilde{D}_2$, $D_3 = \sqrt{3}\tilde{D}_3$, where $\tilde{D}_i, i = 1,2,3$ is the directional derivative along the positive direction of characteristics of the operator L_i;

$X_1(u_3) \equiv 0$, $X_3(u_3) \equiv 0$,

$R(u_1,u_2,u_3)(t) = \int_I u_1 d\tau_1 + \int_I u_2 d\tau_2 + \int_{\sigma_3} d\tau_3 \int_{\sigma_2(\tau_3)} d\tau_2 \int_{\sigma_1} u_3 d\tau_1.$

PROBLEM FORMULATION

In the recent paper [4] we introduced a model for an age-sex-structured population consisting of male, single (nonfertilized) female and fertilized female subpopulations taking into account a random coupling of sexes (for the period of mating only) and females' pregnancy. The existence and uniqueness theorem for this model for the unlimited population (all the vital rates do not depend on environmental factors) has also been proved in this paper, while the asymptotic behaviour of the general solution of this model for the unlimited population has been presented in [5]. In the present paper we consider the same model as that in [5], but for the limited population, and analyze the case where only the death rates depend on environmental factors, i. e.

$$\nu_i(t,\tau_i,\rho) = \nu_i^n(\tau_i) + \nu(t,\rho), \quad i = 1,2,$$
$$\nu_3(t,\tau_1,\tau_2,\tau_3,\rho) = \nu_3^n(\tau_1,\tau_2,\tau_3) + \nu(t,\rho),$$

while the other vital functions b_1, b_2, p do not depend on time t and population size ρ.

The model to be considered below consists of the system

$$L_i u_i + \nu_i u_i - X_i(u_3) = 0 \text{ in } Q_i, \quad i = 1,2,3,$$
$$X_2(u_3) = -\begin{cases} 0, & \tau_2 \notin \sigma_2(0) \\ \int_{\sigma_1} u_3|_{\tau_3=0}\, d\tau_1, & \tau_2 \in \sigma_2(0) \end{cases} + \begin{cases} 0, & \tau_2 \notin \sigma_2(T) \\ \int_{\sigma_1} u_3|_{\tau_3=T} d\tau_1, & \tau_2 \in \sigma_2(T), \end{cases} \tag{1}$$
$$\rho = R(u_1,u_2,u_3)$$

supplemented by the conditions

$$
\begin{gathered}
u_i|_{t=0} = u_i^0, \quad i = 1, 2, 3, \\
u_j|_{\tau_j=0} = \int_\sigma b_j u_3|_{\tau_3=T} d\sigma, \quad j = 1, 2, \\
u_3|_{\tau_3=0} = p u_1 u_2 / n_1, \quad n_1 = \int_{\sigma_1} u_1 d\tau_1, \\
[u|_{\tau_2=\tau_2^s}] = 0, \quad s = \overline{1,4}.
\end{gathered} \tag{2}
$$

Here u_1, u_2, u_3 are unknown functions, while the other functions are assumed to be given. As it follows from the foregoing all the functions in (1) – (2) are positive valued, otherwise they have no biological significance. In addition we require the following compatibility conditions

$$
\begin{gathered}
u_j^0 = \int_\sigma b_j|_{t=0} u_3^0|_{\tau_3=0} d\sigma, \quad j = 1, 2, \\
u_3^0|_{\tau_3=0} = p|_{t=0} u_1^0 u_2^0 / \int_{\sigma_1} u_1^0 d\tau_1, \quad [u_2^0|_{\tau_2=\tau_2^s}] = 0, \quad s = \overline{1,4}.
\end{gathered} \tag{3}
$$

SEPARABLE SOLUTIONS OF (1) – (3) AND THEIR ASYMPTOTICS

A particular solutions of (1) – (3) will be considered and their asymptotic behaviour will be demonstrated. Let $u_1^0(\tau_1)$, $u_2^0(\tau_2)$, $u_3^0(\tau_1, \tau_2, \tau_3)$ and $\rho(t)$ be the unknown functions and assume that ρ^0 is a positive constant.

Substituting

$$
u = u^0 \rho(t)/\rho^0, \quad u = (u_1, u_2, u_3), \quad u^0 = (u_1^0, u_2^0, u_2^0) \tag{4}
$$

into (1) – (3) yields the following two problems

$$
\begin{gathered}
\partial u_1^0/\partial \tau_1 + (\lambda + \nu_1^n) u_1^0 = 0 \ \text{ in } I, \\
\partial u_2^0/\partial \tau_2 + (\lambda + \tilde{\nu}_2) u_2^0 - \tilde{X}(u_3^0) = 0 \ \text{ in } I \setminus \bigcup_{s=1}^{4} \tau_2^s, \\
\tilde{X}(u_3^0) = \begin{cases} 0, \ \tau_2 \notin \sigma_2(T) \\ \int_{\sigma_1} u_3^0|_{\tau_3=T} d\tau_1, \ \tau_2 \in \sigma_2(T), \end{cases} \\
D^* u_3^0 + (\lambda + \nu_3^n) u_3^0 = 0 \ \text{ in } \sigma_1 \times \sigma_2(\tau_3) \times \sigma_3, \\
\tilde{\nu}_2 = \nu_2^n + \begin{cases} 0, \ \tau_2 \notin \sigma_2(0) \\ n_1^{-1} \int_{\sigma_1} u_1^0 p d\tau_1, \ n_1 = \int_{\sigma_1} u_1^0 d\tau_1, \ \tau_2 \in \sigma_2(0), \end{cases} \\
\rho^0 = R(u_1, u_2, u_3), \\
u_j^0(0) = \int_\sigma b_j u_3^0|_{\tau_3=T} d\sigma, \ j = 1, 2, \ u_3^0|_{\tau_3=0} = p u_1^0 u_2^0 / n_1, \ [u_2^0|_{\tau_2=\tau_2^s}] = 0, \ s = \overline{1,4},
\end{gathered} \tag{5}
$$

$$d\rho/dt = (\lambda - \nu)\rho, \quad \rho(0) = \rho^0 \tag{6}$$

with λ a constant and $D^* = \sqrt{2}\tilde{D}^*$, $\tilde{D}^*$ being the directional derivative along the characteristics of the operator $\partial/\partial\tau_2 + \partial/\partial\tau_3$. Following [1–3] solution (4) will be called the product (separable or persistent) solution of (1) – (3). In the case where $\nu(t,\rho) = \nu(\rho)$ Eq. (6) was investigated by Langlais and Milner [3]. System (5) is the eigenvalue problem, nonlinear in the general case, and has been investigated in [5]. According to [5], in the general case there may exist some real eigenvalues of problem (5), but in the case, where all the vital rates of the fertilized female do not depend on age of the mated male, there exists the unique real eigenvalue λ_0. In the last case result of Langlais and Milner [3] allows us to formulate the following assertion.

THEOREM. *Assume that:*

1) $p \in C(\overline{\sigma}_2(0))$, $\nu_1, \nu_2 \in C(\overline{I})$, $\nu_3 \in C(\overline{\sigma}_2(\tau_3) \times \overline{\sigma}_3)$, $b_1, b_2 \in L_1(\sigma_2(T))$,

2) $\nu(\rho) \in C(\overline{I}) \cap C^1(I)$.

Then problem (1) – (3) *admits the non-negative product solution* (4) *and:*

a) $\lim_{t\to\infty} u = 0$ *if* $\lambda_0 < \nu(\rho)$ *for all* $\rho \in (0, \rho^0]$;

b) $\lim_{t\to\infty} u = \infty$ *if* $\lambda_0 > 0$ *and* $\lambda_0 > \nu(\rho)$ *for all* $\rho \in [\rho^0, \infty)$;

c) $u = u^0$ *for all* $t > 0$ *if* $\lambda_0 = \nu(\rho^0)$;

d) $\lim_{t\to\infty} u = u^0\rho_*/\rho^0$, $0 < \rho_* < \infty$ *if either* $\lambda_0 > 0$ *and* $\lambda_0 = \nu(\rho_*)$, $\nu'(\rho_*) > 0$, *or* $\lambda_0 = 0$ *and* $\nu(\rho_*) = 0$, $\nu'(\rho_*) > 0$, *where prime denotes differentiation.*

Observe that in the product solution (4) the distribution of ages remains unchanged as time increases.

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