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NONLINEARITIES IN THE NEURAL NETWORK AND THEIR SIGNIFICANCE (NEGATIVE NON-LINEAR FEEDBACK)

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One of the most challenging mysteries of the living organism is the functional structure of the nervous system. The currently prevailing opinion maintains that the functional organization of the nervous system is based on neural networks, transneuronal connections and specific properties of neurons and their contacts. There is a paradigm that its functional significance is to identify the organism's environment, i. e., create and improve an informational model of the environment and, making decisions according to it, control the organism's action as well as realize goal-oriented programs.

Neurobiological experiments clearly show that neurons as non-linear summators (logical and algebraic) possess certain linear properties [1]. In addition, there is substantial evidence that in biological networks there is strong feedback (by way of axon collaterals, interneurons and, etc.), both between nearby neurons and between bigger structural units of the nervous system (ganglia, nuclei, cortical fields, etc.) [2]. It has been emphasized that it is these feedback collaterals that grow and form new synaptic contacts during the life of an organism [3]. It has been pointed out that this feedback is nonlinear, and its significance for the functional properties of the neuronal network has been considered [4].

Physicochemical and biological models of dynamical systems, their phase portraits clearly demonstrate the significance of non-linear feedback for a system and even for its functions. It is known that non-linear positive feedback in dynamical systems may induce autogeneration, hystereses, bifurcations. All these effects are observed in neurophysiological experiments and may be explained by some theories [5].

More interesting and less investigated effects arise from non-linear negative feedback. The dynamical systems theory shows that positive feedback leads to unnecessary parasitic autogeneration, whereas negative feedback helps to stabilize the system. One of the best known example of a system which uses negative feedback is the regulator, automatic regulation and control theory has been created [6]. Such systems and their functional organization are of great interest to biologists because regulation is one of the most prominent features of living organisms. Nonlinear feedback (interaction) is fundamental to some mathematical models (Jocob-Monod) of cell morphogenesis-differentiation, which explain possible bifurcations in embryogenesis [7]. It has been suggested that the nervous system is the main system which determines the organism's regulator-like properties. Therefore, it can be assumed that it is the negative neuronal feedback that forms these regulator- like properties.

These properties of non-linear negative feedback, in the context of the purpose of the nervous system, call attention to a statement by the creator of biometrics, mathematician R. Fischer, in which he proposed that dynamical systems with non-linear feedback may identify an object, or create a model of the object. This means that non-linear neuronal feedback may be one of the most important mechanisms in brain functional organization and functioning. Therefore, it makes sense to formulate a purpose and find a way to synthesize appropriate functions, i. e., a neuronal structure which would generate the needed phase portrait. One should explore the potential of negative neuronal feedback and create a basic memory-endowed network, a continuous neuronal factorial bifurcator, which would be able to remember the permutation that arranges the positive continuous components of an input vector in increasing order.

There are some neural network models which non-linear negative feedback endows with new specific features: to separate and pass on only the highest value out of several parallel inputs (the maximum filter); to differentially form selective neuronal structures according to their thresholds and neurons selective to the intensity of the input signal [8].

Therefore, understanding of the synthesis of neuronal structures could explain not only neurobiological facts but also would help create more effective technology for information processing.

1. THE FUNCTIONAL CHARACTERISTICS OF THE ANALOGOUS NEURON

Neuromorphological studies show that the structural and functional element of the nervous system, the neuron, has a multitude of synaptic contacts with other neurons and one long process, the axon. The axon branches and impinges on neurons and other cells, making its synapses. This is how neuronal structures and neural networks are formed. Neurons come in different shapes but in most cases they may be divided into "stellate" and "pyramidal" neurons. For the sake of simplicity we assume that our neurons (quasineurons) are summators with many functional inputs and one functional output (Fig. 1, a).

It is known from neurophysiological studies that neurons generate spikes, thus expressing their level of excitation. A non-excited or inhibited neuron is silent, whereas its excitation makes it generate neuronal spikes of different frequencies, the frequency being indicative of the level of the excitation. Due to the fact that spikes last for a certain time and are subject to refractory effects, neurons have their maximal firing frequency, X_m. Generally, this frequency does not reach 1000 spikes/sec, although some small interneurons may fire at as many as $\sim$ 1500 spikes/sec. It has been suggested that the firing frequencies may be summed with a positive (+) and a negative (-) signs, and also with different summations weights. Therefore, the quasineuron is considered to be a summator of continuous (analogous) inputs (spike frequencies). It is able to weigh every synaptic input by a synaptic weight S, which may take on any value. Since synapses may be excitatory and inhibitory, the weights of excitatory synapses are often considered positive $(+S)$, whereas the weights of inhibitory synapses are negative $(-S)$.

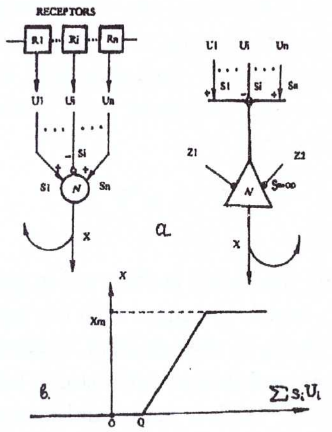

Fig. 1. Neurons schemes (a) and functional characteristics (b).

Therefore, the quasineuron's static functional characteristic may be described by a nonlinear equation,

$$X = \begin{cases} 0, & \text{if } 0 \geq \sum_{i=1}^{n} S_i \mathrm{U}_i, \\ \sum_{i=1}^{n} S_i \mathrm{U}_i, & \text{if } 0 \leq \sum_{i=1}^{n} S_i \mathrm{U}_i \leq X_m, \\ X_m, & \text{if } \sum_{i=1}^{n} S_i \mathrm{U}_i \geq X_m, \end{cases}$$

whose graphical representation is given in Fig. 1, b. It is easy to notice that this is a "diode-like" nonlinearity with saturation at X_m (in the case when the sum of inputs exceeds the maximal frequency that the neuron may reach).

In some cases it is important to take into account the absolute threshold of the neuron, Q. Then the neuron's static function shifts to the right by the value of Q, or, alternatively, the X-axis moves to the left. In many cases it makes sense to consider the threshold to be zero, and introduce another inhibitory input from a neuron (pacemaker), which generates a stable maximal frequency X_m, and whose action at the synapse with a certain weight S will ultimately determine the threshold $Q = sX_m$.

It is easy to find the condition under which the neuron is "unsaturable". This is the case when all the inhibitory inputs are silent, i. e., equal to zero, and all the excitatory inputs j are carrying the X_m frequencies. In this state of maximal excitation the neuron cannot reach and only approaches the saturated X_m value. Then the following inequality must be true:

$$\sum_{j}^{m} S_j \leq 1.$$

This means that the larger the number of excitatory synapses on a neuron, the smaller the weight of every synapse. If all the weights of the excitatory synapses are equal and their number is m, then $s < 1/m$. If we assume that neurons often react only to the difference between the inputs, i. e., they do not react when all the input frequencies are equal, we come to the conclusion that the sums of the excitatory S_j and inhibitory S_i synaptic weights are equal, and their sum (taking into account the signs) is zero. That indicates that

$$\sum_j S_j \cong \sum_i S_i$$

In some cases very strong inhibition is observed in pyramidal neurons. This effect has been ascribed to some somatic synapses with big inhibitory synaptic weights Z_1, Z_2 (neuromorphologists relate it to the action of "basket" neurons). Such inhibitory synapses may realize logical prohibition operations, universal logical Pirs's arrows (Dager) or Shaffer's functions. Such a pyramidal neuron sums up its input signals and produces an output, which in this case is a logical operation. The pyramidal neuron becomes an algebraic/logical functional device.

Considering dynamical properties of a neuron, it makes sense to characterize it as a first-order summator with a time constant T, describing the functioning of the neuron by a first-order differential equation

$$T\frac{dX}{dt} = \sum_{i=1}^{n} S_i \mathrm{U}_i - X.$$

In some cases in addition to the synaptic weight S_i, every synapse may also be characterized by its time constant T_i.

Therefore, from the functional point of view, every neuron may be characterized as an inertial algebraic summator with a time constant T, n non-negative inputs (frequencies) with their synaptic weights S_i, and the neuron's non-linear ("diode-like") characteristic N. Its output is also a spike frequency X, which may take on only non-negative continuous values which do not exceed X_m. In some cases the neuron may also be a logical summator.

The non-linearities of a neuron may be compensated for by an additional parallel neuron which has exactly the same absolute values of its synaptic weights, but the signs of these weights are reversed. Such a pair of neurons satisfies the condition of "non-saturability" and becomes a simple linear summator. Its diagram and functional properties are depicted in Fig. 2.

2. A NEURON WITH FEEDBACK

Generally, feedback may radically change the functional properties of a neuron. The feedback through an inhibitory synapse does not qualitatively change a neuron's function and only decreases its steepness. In contrast, the feedback through an excitatory synapse makes this function steeper, and, because of the saturation effect, the neuron becomes a "yes-no" switch, or a hysteresis effect emerges, or it may even

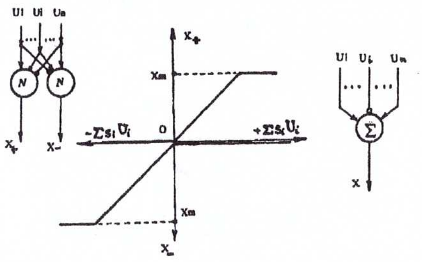

Fig. 2. Pair of neurons as simple linear summator and functional characteristics.

become a binary element with memory and an autogenerator (pacemaker) of the maximal frequency X_m (Fig. 3). It is easy to show that the static transfer function of a neuron with linear feedback is

$$X = \frac{S_1}{1 \pm S_0} \mathrm{U},$$

where the positive sign in the denominator is the inhibitory feedback synapse, and the negative sign is the excitatory synapse. It can be seen that, in the latter case, when S_0 approaches 1, the steepness of the neuron's function approaches infinity and, when it becomes 1, the neuron becomes a "yes-no" switch. If the synaptic weight further increases, the steepness becomes negative and a hysteresis emerges. If the threshold Q is taken into consideration, the neuron becomes a two-state memory element, or a pacemaker.

Such feedback also changes the time constant:

$$T = \frac{T_1}{1 \pm S_0}.$$

It can be seen that in the case of the excitatory feedback (when the synaptic weight approaches 1), the neuron becomes an integrator (T approaches infinity),

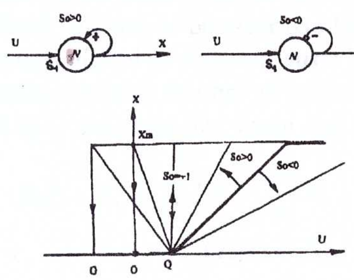

Fig. 3. Neuron with feedback.

whereas an increase in the inhibitory synaptic weight, in contrast, improves the neuron's dynamical function (T decreases).

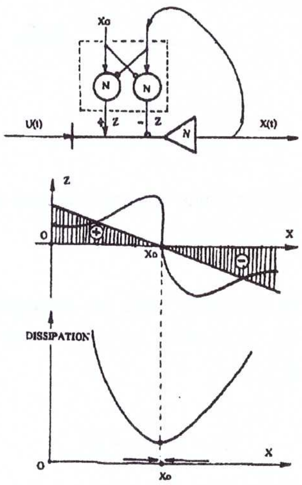

Fig. 4. Neuronal regulator and functional characteristics.

Here we point out to the properties of an inertial (T large) pyramidal neuron with feedback through two complementary non-inertial ($T < T_0$) neurons which together carry out the difference $Z = X_0 - X = N\{X_0 - X\} + N\{X - X_0\}$. Here X_0 is a constant, $X(t)$ is the neuron's reaction, and N is a nonlinearity indicating the polarity of the difference. In the case of negative feedback the first neuron acts through an excitatory synapse, and the second one through an inhibitory synapse (Fig. 4). This scheme models a classical neuronal regulator which stabilizes X, i. e., it tries to maintain $X(t) = X_0$ constant. The solution of the function $Z = F(X)$, intersecting the X-axis at a negative angle at point X_0, shows the pyramidal neuron's stable state in a "potential pit". By reversing the signs of the synaptic connections we could get a dynamical system with the opposite effect, i. e., a non-stable "potential hill" state. In the latter case, $Z(X)$ would intersect the X-axis at a positive angle. That would correspond to positive feedback.

The described neuronal structure gives us insight into a one-dimentional neuronal network with more complicated dynamical characteristics, where the feedback non-linear function $Z = H(X)$ has many real solutions (Fig. 5, a)

$$\begin{cases} T\dfrac{dX}{dt} = S_i\mathrm{U}(t) - X + S_0 Z, \\ Z = H(X). \end{cases}$$

Some of these roots, at which the X-axis is crossed at a negative angle, will form stable states ("potential pits") of this dynamical system. The others, at which the function intersects the X-axis at a positive angle, will form unstable states

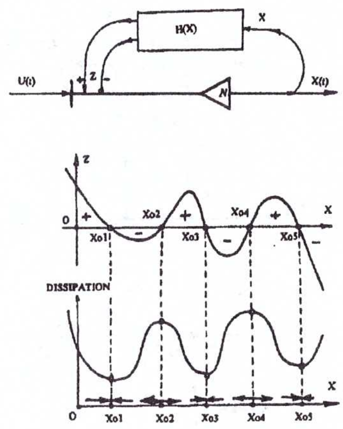

Fig. 5. One-dimentional network with negative non-linear feedback.

("potential hills"). In such a way, one can synthesize a dynamical system with a desired phase relief, portrait, dissipation, or Liapunoff's function.

Take, for instance, $H(X) = S_0(X_01-X)(X-X_02)(X_03-X)(X-X_04)(X_05-X)$. This fifth order polynomial forms three "potential pits" and two "potential hills" in the feedback (Fig.5, b).

A neuronal structure, realizing a third-order polynomial feedback and having two "potential pits" and "a hill" in between at desired values of X, can be made of three neuronal pairs, calculating differences. If their outputs are fed into the appropriate neuronal structures passing on the minimal value, and one of which gives excitatory and the other inhibitory feedback effects, one gets a function made of broken lines, which approximates a third-order polynomial (Fig. 6).

Now let us move on to the problem of the synthesis of a multidimentional dynamical neural structure.

3. A MULTIDIMENTIONAL NEURAL NET STRUCTURE WITH NONLINEAR FEEDBACK

The main feature of neuronal structures is parallel information processing of signal vectors. Therefore, it is important to understand the possibility of synthesizing a neuronal structure with desired properties and required multidimentional phase portrait. This can be realized by using a few or many simultaneously functioning pyramidal neurons with appropriate nonlinear negative feedback connections. The negative feedback keeps in check the basic structure elements, pyramidal neurons in this case, not allowing them to reach the saturation limit and, when they get to a certain point of excitation, pushes the system to a level of excitation which is less than X_m. When the level of excitation is low, positive feedback may come into action, too.

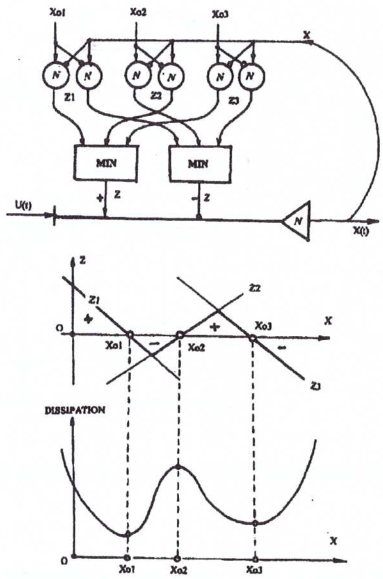

Fig. 6. One-dimentional neuronal structure realizing non-linear third-order dynamical system with two "potential pits" and "a hill".

Consider the most simple case. Suppose we have a two-dimensional structure with two inputs (U1 and U2) to two pyramidal neurons N, which in turn have two outputs (X_1 and X_2) with such interneuronal feedback that it creates a desired phase relief (portrait). For instance, this system may have two "potential pits" positioned symmetrically with respect to the line $X_1 = X_2$, in the sectors $X_1 > X_2$ and $X_2 > X_1$, and on neither of the coordinate axes (Fig. 7). It would be an "on-off" switch, which could remember the state of the vector U by which component of the vector was bigger. This property emerges in the interneuronal network composed of two parts functioning in parallel; the first part realizes the nonlinear algebraic equation $Z+ = +S_0 * N[X_m - (X_1 + X_2)]$, $Z- = -S_0 * N[(X_1 + X_2) - X_m]$, and the second one realizes the disjunctive (connected by the analog logical operation OR, or $\cup$) expression $Z = S_0 * \{N[N(X_1 - X_2) - 1/2 * X_m] \cup N[1/2X_m - N(X_1 - X_2)] \cup N[1/2X_m - N(X_2 - X_1)] \cup N[N(X_2 - X_1) - 1/2X_m]\}$.

These in nonlinear equations, embedded in neural networks, not unlikely as in the case of the regulator, "push" the state of pyramidal neuron excitation towards one of the points of intersection between the lines $X_1 + X_2 = X_m$, $X_2 = X_1 + 1/2 * X_m$, $X_1 = X_2 + 1/2 * X_m$, i. e., towards one of the two possible states: either $X_1 > X_2$, or $X_2 > X_1$. Such a nine-neuron dynamical system with nonlinear feedback has a phase portrait with two "potential pits".

Likewise, one can synthesize a three-dimensional, four-dimensional, and, in the general case, n-dimensional switch, which would remember one of the $n!$ symmetric states of an n-dimensional input vector. Such a structure would be made of n pyramidal neurons, and $3n+2$ interneurons, realizing $n+1$ intersecting hyperplanes.

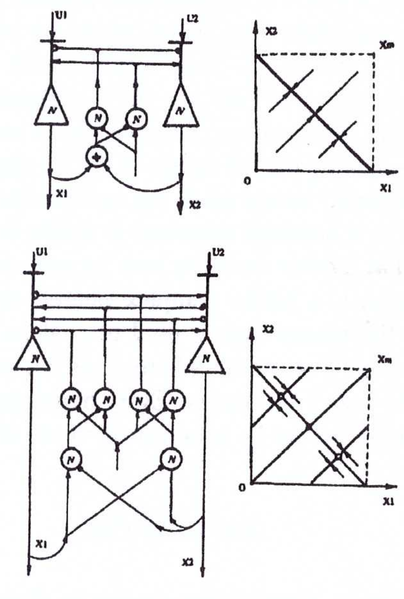

Fig. 7. Two-dimentional neural net with two "potential pits".

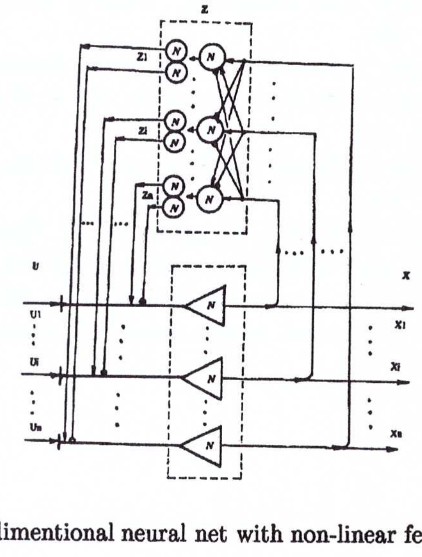

Fig. 8. Multidimentional neural net with non-linear feedback.

One hyperplane would divide the hypercube of the phase space by a diagonal hyperplane perpendicular to the hyperline "all equal", i. e., $X_1 = X_2 = \cdots = X_i = \cdots = X_n$. All the other $n*(n-1)$ hyperplanes, parallel to the hyperline "all equal" and moved to every coordinate axis, which would be away from them by $k * X_m$, $(k < 1)$ in the positive direction. That would create $n!$ absolutely symmetrical intersection points, $n!$ "potential pits", in the n-dimensional space, every of which would indicate a certain permutation of the vector U components (arrangement in increasing order). Depending on the values of the U components, the interneurons (acting by way of feedback) would push the system into one of these "pits". The general diagram of such a neuronal structure is shown in Fig. 8.

It is easy to see that similar methods may be used to synthesize n-dimensional dynamical structures with a rather complex phase portraits. We can call them factorial switches. If the binary logic is used to analyze the states of a neural net, then n neurons can have $2**n$ states, whereas the factorial logic of analog neurons can see as many as $M = n!*2**n$ states. Every hyperquandrant of the phasic space can have $n!$ stable states. It can be attained by virtue of the feedback nonlinearity of analog interneurons.

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