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Approximation of analytic functions by discrete shifts of the Hurwitz zeta-function in short intervals

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Abstract: In the paper, we prove theorems in short intervals on approximation of analytic functions by shifts $\zeta(s+ikh,\alpha)$, $h>0$, $k\in\mathbb{N}_0$, of the Hurwitz zeta-function. Two cases are discussed. In the first case, it is assumed that the set $\{(\log(m+\alpha)\colon m\in\mathbb{N}_0), 2\pi/h\}$ is linearly independent over $\mathbb{Q}$, and it is obtained that the set of the above shifts approximating every analytic function defined on the strip $\mathcal{D}=\{s\in\mathbb{C}\colon 1/2<\sigma<1\}$ has a positive lower density (or density) in the interval of length H, $h^{-1}(Nh)^{27/82}\le H\le(Nh)^{1/2}h^{-1}$. The second case is devoted to arbitrary $0<\alpha<1$, $\alpha\ne 1/2$. It is obtained that there exists a closed non-empty set $\mathcal{F}_{h,\alpha}$ of analytic functions defined on $\mathcal{D}$ such that, for every $f\in\mathcal{F}_{h,\alpha}$, the set of shifts $\zeta(s+ikh,\alpha)$ approximating the function f also has a positive lower density (or density) in the above interval. These results extend the known theorems proved for the interval of length N.

Keywords: approximation of analytic functions; Hurwitz zeta-function; universality; weak convergence

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1 Introduction

We preserve standard notations $\mathbb{P}$ for the set of all prime numbers, $\mathbb{N}$ of all positive integers, $\mathbb{N}_0$ of all non-negative integers, $\mathbb{R}$ of all real numbers, and $\mathbb{C}$ of all complex numbers. By $s=\sigma+it$, $\sigma,t\in\mathbb{R}$, we denote the complex variable.

In the paper, we consider approximation of a class of analytic functions by shifts of one and the same function defined by a Dirichlet series. We notice that approximation of analytic functions is an old and important problem of the function theory. There are various types of approximation of analytic functions, however, the main attention was devoted to approximation by polynomials, or more general rational functions. A type of approximation depends on the definition region of the considered analytic functions $f(s)$, or used metric, purposes of approximation, and, of course, on the function f, for results and references, see [1]. The most convenient and popular approximation by polynomials is uniform approximation on compact sets. This problem was completely solved by S. N. Mergelyan in 1952. He proved that a function continuous on compact set $K\subset\mathbb{C}$ and analytic inside of K is approximated by polynomials if and only if the set K has a connected complement [2]. This result covers partial ones given by C. Runge [3], M. A. Lavrent'ev [4] and M. V. Keldysh [5].

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We focus on the approximation of analytic functions defined in the strip $\mathcal{D} = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ by shifts of the Hurwitz zeta-function. Let $0 < \alpha \leq 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ was defined in [6] by Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1,$$

and by meromorphic continuation elsewhere with the unique simple pole at the point $s = 1$ and $\operatorname*{Res}_{s=1} \zeta(s, \alpha) = 1$. Recall that the first approximation results by zeta-functions was obtained by S. M. Voronin [7] for the Riemann zeta-function $\zeta(s) = \zeta(s, 1)$. Roughly speaking, he proved that every non-vanishing analytic function in the strip $\mathcal{D}$ is approximated uniformly on discs of $\mathcal{D}$ by shifts $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$. This approximation property of $\zeta(s)$ is called universality because a large class of analytic functions is approximated by shifts of one and the same function. Here is the main difference with approximation of analytic functions by polynomials, where for every given analytic function a new polynomial is constructed. Many results on universality of zeta-functions were obtained, see original works [8]–[14], monographs [15] and [16] and a survey paper [17]. Universality property is used for investigations of the functional independence of zeta-functions [18]–[22], estimation of the number of their zeros [23], and moments [24]. In physics, universality is useful for modelling the chaotic behavior of particles [25], and estimation of quantum mechanics integrals [26]. By the way, Voronin had a plan to rewrite quantum mechanics in terms of universality of zeta-functions.

Universality of the Hurwitz zeta-function $\zeta(s, \alpha)$ is more complicated than of $\zeta(s)$, and depends on arithmetic of the parameter α. Moreover, the function $\zeta(s)$, differently from $\zeta(s, \alpha)$, $\alpha \neq 1$, $\alpha \neq 1/2$, has the Euler product over primes

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Voronin's original proof of this universality theorem is based on approximation of $\zeta(s)$ by finite products of Euler's product, and application of a Riemann type rearrangement theorem in Hilbert space [27]. B. Bagchi in [9] proposed a new way for the proof of universality for zeta-functions involving probabilistic limit theorems in the space of analytic functions. In the both above methods, the linear independence over $\mathbb{Q}$ of the set $\{\log\ p : \ p \in \mathbb{P}\}$ occupies an important place.

Proof of universality of $\zeta(s, \alpha)$ with transcendental α (α is not a root of any non-trivial polynomial with rational coefficients) is the simplest one. In this case, the set $\{\log(m + \alpha) : m \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$, and a certain analogy with $\zeta(s)$ can be used.

If α is a rational number, say $\alpha = a/q$, $0 < a < q$, $(a, q) = 1$, then a proof of universality for $\zeta(s, \alpha)$ reduces to joint universality of Dirichlet L-functions. Suppose that $\chi : \mathbb{N} \to \mathbb{C}$ is a Dirichlet character modulo q, i.e., $\chi(m + q) = \chi(m)$ for all $m \in \mathbb{N}$, $\chi(m_1 m_2) = \chi(m_1)\chi(m_2)$ for all $m_1, m_2 \in \mathbb{N}$, and $\chi(m) \neq 0$ for $(m, q) = 1$, and $\chi(m) = 0$ for $(m, q) > 1$. The Dirichlet L-function $L(s, \chi)$ with character χ is defined, for $\sigma > 1$, by the series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and admits analytic continuation to the whole complex plane if the character χ is not principal, i.e.,

$$\chi(m) = \chi_0(m) = \begin{cases} 1 & \text{if } (m, q) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$L(s, \chi_0) = \zeta(s) \prod_{\substack{p \in \mathbb{P} \\ p | q}} \left(1 - \frac{1}{p^s}\right).$$

Thus, $L(s,\chi_0)$ has a simple pole at the point $s=1$, and

$$\operatorname*{Res}_{s=1} L(s,\chi_0) = \prod_{\substack{p\in\mathbb{P}\\ p|q}} \left(1-\frac{1}{p}\right).$$

Let $\varphi(q)$ stand for the Euler totient function, and $\overline{\chi}$ denote the conjugate of χ. The functions $\zeta(s,a/q)$ and $L(s,\chi)$ are connected by the equality

$$\zeta\left(s,\frac{a}{q}\right) = \frac{q^s}{\varphi(q)} \sum_{\chi(\mathrm{mod}\, q)} \overline{\chi}(a)L(s,\chi),$$

where the sum runs over all Dirichlet characters modulo q. Then application of the joint universality for Dirichlet L-functions leads to universality by $\zeta(s,a/q)$.

Denote by $\mathcal{K}$ a collection of all compact sets of the strip D with connected complements, and by $H(K)$, $K\in\mathcal{K}$, the class of continuous functions on K that are analytic in the interior of K. Moreover, let meas$A$ stand for the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then the universality of $\zeta(s,\alpha)$ with transcendental and rational α is described by the following theorem [8], [9], [15], [28].

Theorem 1.1. *Let the parameter α be transcendental or rational $\neq 1/2$ or $\neq 1$. Then, for every $K\in\mathcal{K}$, $f(s)\in H(K)$ and $\varepsilon>0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]: \sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}>0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]: \sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon>0$.

If $\alpha=1/2$, or $\alpha=1$, the statement of Theorem 1.1 remains true provided $f(s)\neq 0$ on K.

For algebraic irrational α, a big progress was made in [29]. The authors of [29] indicated an explicitly given interval $[T,2T]$ containing a value τ such that $\zeta(s+i\tau,\alpha)$ approximate $f(s)$ for all but finitely many α of certain degree. This remark is the first effective universality theorem for the Hurwitz zeta-function. We observe that Theorem 1.1 is not effective in the sense that any approximating shifts, or at least interval containing τ are not known.

In Theorem 1.1, a density of approximating shifts is studied in the interval of length T. More precise theorems in analytic number theory deal with so-called short intervals. We recall one result on density of non-trivial zeros of $\zeta(s)$. Let $N(\widehat{\sigma},T)$ denote the number of non-trivial zeros in the region $\{s\in\mathbb{C}:\sigma\geq\widehat{\sigma}, |t|\leq T\}$. Then, in [19], it was proved that, for $H=T^\delta$, $\delta>27/82$ and $1/2<\widehat{\sigma}<1$,

$$N(\widehat{\sigma},T+H)-N(\widehat{\sigma},T)=O\left(\frac{H}{\widehat{\sigma}-1/2}\right)$$

with implied constant independent on $\widehat{\sigma}$.

The first universality in short intervals was obtained for the Riemann zeta-function in [30]. In [31], a conditional universality theorem in short intervals for zeta-function of certain cusp forms was given. The theorem of [30] was improved in [32]. Recently, the best universality result in short intervals for $\zeta(s)$, under Riemann hypothesis, was given in [33]. We state a such theorem for $\zeta(s,\alpha)$.

Theorem 1.2 ([34], Theorem 4). *Let α be transcendental, and $T^{27/82}\leq H\leq T^{1/2}$. Then, for every $K\in\mathcal{K}$, $f(s)\in H(K)$ and $\varepsilon>0$,*

$$\liminf_{T\to\infty} \frac{1}{H}\operatorname{meas}\left\{\tau\in[T,T+H]: \sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}>0.$$

Moreover, "lim inf", *for all but at most countably many $\varepsilon>0$, can be replaced by* "lim".

Theorems 1.1 and 1.2 are of continuous type: τ in shifts $\zeta(s+i\tau,\alpha)$ can take arbitrary real values. Also, there exist discrete universality theorems for $\zeta(s,\alpha)$: in that case, analytic functions are approximated by shifts $\zeta(s+i\tau,\alpha)$ when τ runs over a certain discrete set.

Denote by $\#A$ the cardinality of the set A, and, for $N\in\mathbb{N}_0$, define

$$M_N(\cdots)=\frac{1}{N+1}\#\{0\le k\le N:\ \dots\},$$

where in place of dots a condition satisfied by k is to be written.

The first discrete universality theorem for $\zeta(s,\alpha)$ was proved by B. Bagchi in [9].

Theorem 1.3 ([9], Corollary 5.3.7). *Let α be a rational number. Then, for every $K\in\mathcal{K}$, $f(s)\in H(\mathcal{D})$, $h>0$ and $\varepsilon>0$,*

$$\liminf_{N\to\infty} M_N\left(\sup_{s\in K}|\zeta(s+ikh,\alpha)-f(s)|<\varepsilon\right)>0. \tag{1.1}$$

The discrete case with transcendental α is more complicated than that with rational α. Let, for $h>0$,

$$L(\alpha,h,\pi)=\left\{(\log(m+\alpha):m\in\mathbb{N}_0),\frac{2\pi}{h}\right\}.$$

Theorem 1.4 ([35], Theorem 1.3; [36], Theorem 4). *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Then, for every $K\in\mathcal{K}$, $f(s)\in H(K)$ and $\varepsilon>0$, the equality* (1.1) *holds. Moreover, "*lim inf*" in* (1.1) *can be replaced by "*lim*" for all but at most countably many $\varepsilon>0$.*

It was observed in [35], one can take, for example, $\alpha=1/\pi$ and rational h.

In [37] and [38], approximation of analytic functions by generalized discrete shifts $\zeta(s+ih\varphi(k),\alpha)$ with certain function $\varphi(k)$ also was studied. The paper [37] contains a theorem on approximation by composition $F(\zeta(s+ih\varphi(k),\alpha))$ with a certain continuous operator $F: H(\mathcal{D})\to H(\mathcal{D})$.

The purpose of this paper is discrete universality theorems in short intervals for the function $\zeta(s,\alpha)$. We will prove the following theorems. For this, we will use the notation

$$M_{N,H}(\cdots)=\frac{1}{H+1}\#\{N\le k\le N+H:\ \dots\}$$

with a condition for k in place of dots.

Theorem 1.5. *Let the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$, and $h^{-1}(Nh)^{27/82}\le H\le(Nh)^{1/2}h^{-1}$. Then, for every $K\in\mathcal{K}$, $f(s)\in H(K)$ and $\varepsilon>0$,*

$$\liminf_{N\to\infty} M_{N,H}\left(\sup_{s\in K}|\zeta(s+ikh,\alpha)-f(s)|<\varepsilon\right)>0.$$

Moreover, the inferior limit can be replaced by a limit for all but at most countably many $\varepsilon>0$.

The next theorem demonstrates good approximation properties of the function $\zeta(s,\alpha)$ independently of the arithmetic nature of α.

Theorem 1.6. *Suppose that the parameter $0<\alpha<1$, $\alpha\ne1/2$, and $h>0$ are arbitrary, and $h^{-1}(Nh)^{27/82}\le H\le(Nh)^{1/2}h^{-1}$. Then there exists a closed non-empty set $\mathcal{F}_{\alpha,h}\subset H(\mathcal{D})$ such that, for every compact set $K\subset\mathcal{D}$, $f(s)\in\mathcal{F}_{\alpha,h}$ and $\varepsilon>0$,*

$$\liminf_{N\to\infty} M_{N,H}\left(\sup_{s\in K}|\zeta(s+ikh,\alpha)-f(s)|<\varepsilon\right)>0.$$

Moreover, the limit

$$\lim_{N\to\infty} M_{N,H}\left(\sup_{s\in K}|\zeta(s+ikh,\alpha)-f(s)|<\varepsilon\right)$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The interval $h^{-1}(Nh)^{27/82} \le H \le (Nh)^{1/2}h^{-1}$ comes from the contemporary mean square estimate for $\zeta(s,\alpha)$ in short intervals, see Lemmas 2.1 and 2.3 bellow. The main problem consists from the decreasing of the lower bound for H. Our conjecture that $H \ge N^{\varepsilon}$, $\forall \varepsilon > 0$ is a very big problem.

For the proofs, we will apply the Bagchi probabilistic method based on limit theorems for weakly convergent probability measures in the space of analytic functions. In our case, the mean square estimates for $\zeta(s,\alpha)$ in short intervals play a crucial role for application of this method.

Theorems 1.5 and 1.6 will be proved in Section 5. Section 2 is devoted to mean square estimates in short intervals for $\zeta(s,\alpha)$. In Section 3, we will study the approximation in the mean of the function $\zeta(s,\alpha)$ by a certain absolutely convergent Dirichlet series. Section 4 is devoted to probabilistic discrete limit theorems in short intervals in the space $H(\mathcal{D})$ for $\zeta(s,\alpha)$.

2 Mean value results

Let $\theta > 1/2$ be a fixed number, we will define it more precisely later. For $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, set

$$a_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n}\right)^{\theta}\right\},$$

where $\exp\{a\} = e^a$. In this section, we will approximate the function $\zeta(s,\alpha)$ by the Dirichlet series

$$\zeta_n(s,\alpha) = \sum_{m=0}^{\infty} \frac{a_n(m,\alpha)}{(m+\alpha)^s}.$$

The exponential decreasing with respect to m of $a_n(m,\alpha)$ shows that the latter series is absolutely convergent in any half-plane $\sigma > \sigma_0$ with finite σ_0.

The Mellin formula

$$e^{-a} = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \Gamma(z)a^{-z}\,\mathrm{d}z, \quad a > 0,$$

where $\Gamma(z)$ is the Euler gamma-function, implies the representation

$$\zeta_n(s,\alpha) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,\alpha)w_n(z)\,\mathrm{d}z \tag{2.1}$$

with $w_n(z) = \theta^{-1}\Gamma(\theta^{-1}z)n^z$.

Now, we state a mean square estimate for the Hurwitz zeta-function in short intervals [39].

Lemma 2.1 ([39], Theorem 2). *Let* $0 < \alpha < 1$, $\alpha \ne 1/2$, $1/2 < \sigma \le 7/12$ *be fixed, and* $T^{27/82} \le H \le T^{\sigma}$. *Then, uniformly in H,*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it,\alpha)|^2\,\mathrm{d}t \ll_{\sigma,\alpha} H.$$

Recall that the notation $x \ll_{\varepsilon} y$, $x \in \mathbb{C}$, $y > 0$, indicates that there is a constant $c = c(\varepsilon) > 0$ such that $|x| \le cy$.

In order to pass from the continuous mean square for $\zeta(s,\alpha)$ to the discrete, we will use the following result.

Lemma 2.2 ([40], Lemma 1.4). *Let $\delta > 0$, $T_0, T \geq \delta$, $\mathcal{T} \neq \emptyset$ be a finite set lying in the interval $[T_0 + \delta/2, T_0 + T - \delta/2]$, and*

$$\mathcal{N}_\delta(\tau) = \sum_{\substack{t \in \mathcal{T} \\ |\tau - t| < \delta}} 1, \quad \tau \in \mathcal{T}.$$

Suppose that a complex-valued function $\mathcal{F}(t)$ is continuous on $[T_0, T_0 + T]$, and has a continuous derivative $\mathcal{F}'(t)$ on $(T_0, T_0 + T)$. Then the inequality

$$\sum_{t \in \mathcal{T}} \mathcal{N}_\delta^{-1}(t) |\mathcal{F}(t)|^2 \leq \frac{1}{\delta} \int_{T_0}^{T_0+T} |\mathcal{F}(t)|^2 \, \mathrm{d}t + \left(\int_{T_0}^{T_0+T} |\mathcal{F}(t)|^2 \, \mathrm{d}t \int_{T_0}^{T_0+T} |\mathcal{F}'(t)|^2 \, \mathrm{d}t \right)^{1/2}$$

holds.

Lemma 2.3. *Suppose that $0 < \alpha < 1$, $\alpha \neq 1/2$, $1/2 + \varepsilon \leq \sigma \leq 7/12$, $\varepsilon > 0$, $h > 0$, $h^{-1}(Nh)^{27/82} \leq H \leq (Nh)^{1/2} h^{-1}$ and $|t| \leq \log^2 Nh$. Then*

$$\sum_{k=N}^{N+H} |\zeta(\sigma + it + ikh, \alpha)|^2 \ll_{\sigma,\alpha,h} H(1 + |t|).$$

Proof. Limitations for σ and t are conditioned by an application of Lemma 2.1, and by needs for estimation of

$$\sum_{k=N}^{N+H} \left| \zeta\left(\frac{1}{2} + \varepsilon + ikh + i\tau, \alpha \right) \right|^2$$

with $|\tau| \leq \log^2 Nh$ in the proof of Proposition 3.1.

In Lemma 2.2, take $\delta = 1$, $T_0 = N - 1/2$, $T = H + 1$. Then we have

$$\mathcal{N}_\delta(k) = 1.$$

This, an application of Lemma 2.2 with $\mathcal{T} = \{k \in \mathbb{N} : k \in [N, N + H]\}$, and $\mathcal{F}(k) = \zeta(\sigma + it + ikh, \alpha)$ gives

$$\sum_{k=N}^{N+H} |\zeta(\sigma + it + ikh, \alpha)|^2 \ll_h \int_{N-1}^{N+H} |\zeta(\sigma + it + ih\tau, \alpha)|^2 \, \mathrm{d}\tau$$
$$+ \left(\int_{N-1}^{N+H} |\zeta(\sigma + it + ih\tau, \alpha)|^2 \, \mathrm{d}\tau \int_{N-1}^{N+H} |\zeta'(\sigma + it + ih\tau, \alpha)|^2 \, \mathrm{d}\tau \right)^{1/2}. \tag{2.2}$$

In view of Lemma 2.1, shifting $t + h\tau$ to τ, for large N

$$\int_{N-1}^{N+H} |\zeta(\sigma + it + ih\tau, \alpha)|^2 \, \mathrm{d}\tau = \frac{1}{h} \int_{(N-1)h+t}^{(N+H)h+t} |\zeta(\sigma + i\tau, \alpha)|^2 \, \mathrm{d}\tau$$
$$\ll_{\sigma,\alpha,h} \int_{(N-1)h-|t|}^{(N+H)h+|t|} |\zeta(\sigma + i\tau, \alpha)|^2 \, \mathrm{d}\tau. \tag{2.3}$$

Since $H \geq (Nh)^{27/82} h^{-1}$, it follows that $Hh + |t| \geq (Nh)^{27/82}$. Moreover, for $|t| \leq \log^2 N$, $Hh + |t| \leq (Nh)^{1/2} + \log^2 Nh \leq (Nh)^\sigma$. This, (2.3) and Lemma 2.1 yield

$$\int_{N-1}^{N+H} |\zeta(\sigma + ih\tau + it, \alpha)|^2 \, \mathrm{d}\tau \ll_{\sigma,\alpha,h} Hh + |t| \ll_{\sigma,\alpha,h} H(1 + |t|). \tag{2.4}$$

The latter estimate together with Cauchy integral formula imply the bound

$$\int_{N-1}^{N+H} |\zeta'(\sigma + ih\tau + it, \alpha)|^2 \, d\tau \ll_{\sigma,\alpha,h} H(1 + |t|).$$

This, (2.4) and (2.2) prove the lemma. □

3 Approximation in the mean

Recall a metric in the space $H(\mathcal{D})$ which induces its topology. It is known that there is a sequence $\{K_m : m \in \mathbb{N}\} \subset \mathcal{D}$ of compact subsets such that

$$\mathcal{D} = \bigcup_{m=1}^{\infty} K_m,$$

$K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and the inclusion $K \subset K_m$ holds with some $m \in \mathbb{N}$ for any compact set $K \subset \mathcal{D}$. Then,

$$\rho(g_1, g_2) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\sup_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|}, \quad g_1, g_2 \in H(\mathcal{D}),$$

is the desired metric.

The main result of the section is the following statement.

Proposition 3.1. *Suppose that* $0 < \alpha < 1$, $\alpha \neq 1/2$, *and* $h > 0$ *are fixed numbers, and* $h^{-1}(Nh)^{27/82} \leq H \leq (Nh)^{1/2} h^{-1}$. *Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{H+1} \sum_{k=N}^{N+H} \rho\big(\zeta(s + ikh, \alpha), \zeta_n(s + ikh, \alpha)\big) = 0.$$

Proof. The definition of the metric ρ shows that the proposition follows from the equality

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{H+1} \sum_{k=N}^{N+H} \sup_{s \in K} \big|\zeta(s + ikh, \alpha) - \zeta_n(s + ikh, \alpha)\big| = 0 \tag{3.1}$$

with compact sets $K \subset \mathcal{D}$. Thus, fix a compact set $K \subset \mathcal{D}$. Then there exists $\varepsilon > 0$ such that all $s = \sigma + it \subset K$ lie in the strip $1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon$. In (2.1), take $\theta = 1/2 + \varepsilon$, and move the line of integration to $(\theta_1 - i\infty, \theta_1 + i\infty)$ with $\theta_1 = 1/2 + \varepsilon - \sigma$. Clearly, $\theta_1 < 0$, and $\theta_1 > -1/2 - \varepsilon$. Therefore, the integrand in (2.1) has a simple pole at the point $z = 0$, and a simple pole at the point $z = 1 - s$. This remark, the residue theorem and (2.1) give the equality

$$\zeta_n(s, \alpha) - \zeta(s, \alpha) = \frac{1}{2\pi i} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} \zeta(s + z, \alpha) w_n(z) \, dz + w_n(1 - s)$$

which is valid for all $s \in K$. From this, we get

$$\begin{aligned} \zeta_n(s + ikh, \alpha) - \zeta(s + ikh, \alpha) \ll & \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + \varepsilon + it + ikh + i\tau, \alpha\right)\right| \sup_{s \in K} \left| w_n\left(\frac{1}{2} + \varepsilon - \sigma + i\tau\right)\right| d\tau \\ & + |w_n(1 - s - ikh)|. \end{aligned}$$

Now, shifting $t+\tau$ to τ yields

$$\sup_{s\in K}|\zeta_n(s+ikh,\alpha)-\zeta(s+ikh,\alpha)| \ll \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+i\tau,\alpha\right)\right|\sup_{s\in K}\left|w_n\left(\frac{1}{2}+\varepsilon-s+i\tau\right)\right|\mathrm{d}\tau + \sup_{s\in K}|w_n(1-s-ikh)|. \tag{3.2}$$

Truncate the integration in the above integral. For the function $\zeta(s,\alpha)$ the estimate

$$\zeta(\sigma+it,\alpha) \ll_{\sigma,\alpha} \sqrt{t}, \quad \alpha\geq\frac{1}{2},\ |t|\geq 2, \tag{3.3}$$

is known [41]. Moreover,

$$\Gamma(\sigma+it)\ll\exp\{-c|t|\}, \quad c>0, \tag{3.4}$$

uniformly in σ in any finite interval. Hence, for $s\in K$,

$$w_n\left(\frac{1}{2}+\varepsilon-s+i\tau\right)\ll_\varepsilon n^{1/2+\varepsilon-\sigma}\exp\left\{-\frac{c}{\theta}|\tau-t|\right\}\ll_{\varepsilon,K} n^{-\varepsilon}\exp\{-c_1|\tau|\}, \quad c_1>0. \tag{3.5}$$

Therefore, this and (3.3) yield

$$\begin{aligned}&\left(\int_{-\infty}^{-\log^2 Nh}+\int_{\log^2 Nh}^{\infty}\right)\left|\zeta\left(\frac{1}{2}+\varepsilon+it+ikh+i\tau,\alpha\right)\right|\sup_{s\in K}\left|w_n\left(\frac{1}{2}+\varepsilon-\sigma+i\tau\right)\right|\mathrm{d}\tau\\ &\ll_{\varepsilon,\alpha,K} n^{-\varepsilon}\left(\int_{-\infty}^{-\log^2 Nh}+\int_{\log^2 Nh}^{\infty}\right)\left(\sqrt{kh}+\sqrt{|\tau|}\right)\exp\{-c_1|\tau|\}\,\mathrm{d}\tau\\ &\ll_{\alpha,K,h} n^{-\varepsilon}\left(\sqrt{kh}+1\right)\exp\left\{-c_2\log^2 Nh\right\}, \quad c_2>0.\end{aligned}$$

Thus,

$$\begin{aligned}J &\overset{\text{def}}{=} \frac{1}{H+1}\sum_{k=N}^{N+H}\sup_{s\in K}|\zeta(s+ikh,\alpha)-\zeta_n(s+ikh,\alpha)|\\ &\ll_{\alpha,K,h}\int_{-\log^2 Nh}^{\log^2 Nh}\frac{1}{H+1}\sum_{k=N}^{N+H}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+i\tau,\alpha\right)\right|\sup_{s\in K}\left|w_n\left(\frac{1}{2}+\varepsilon-s+i\tau\right)\right|\mathrm{d}\tau\\ &\quad+\frac{1}{H+1}\sum_{k=N}^{N+H}\sup_{s\in K}|w_n(1-s-ikh)|+n^{-\varepsilon}\exp\left\{-c_2\log^2 Nh\right\}\frac{1}{H+1}\sum_{k=N}^{N+H}(\sqrt{kh}+1)\\ &\overset{\text{def}}{=} I+S_1+S_2.\end{aligned} \tag{3.6}$$

Clearly,

$$S_2=o(1), \quad N\to\infty. \tag{3.7}$$

By the Cauchy-Schwarz inequality,

$$\frac{1}{H+1}\sum_{k=N}^{N+H}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+i\tau,\alpha\right)\right|\leq\left(\frac{1}{H+1}\sum_{k=N}^{N+H}\left|\zeta\left(\frac{1}{2}+\varepsilon+ikh+i\tau,\alpha\right)\right|^2\right)^{1/2},$$

and the application of Lemma 2.3 gives

$$\frac{1}{H+1}\sum_{k=N}^{N+H}\left|\zeta\Big(\frac{1}{2}+\varepsilon+ikh+i\tau,\alpha\Big)\right| \ll_{\varepsilon,\alpha,h,K} (1+|\tau|)^{1/2} \ll_{\alpha,h,K} (1+|\tau|)^{1/2}$$

because ε depends on the set K. This together with (3.5) shows that

$$I \ll_{\alpha,h,K} n^{-\varepsilon} \int\limits_{-\log^2 Nh}^{\log^2 Nh} (1+|\tau|)^{1/2} \exp\{-c_1|\tau|\}\,\mathrm{d}\tau \ll_{\alpha,h,K} n^{-\varepsilon}. \tag{3.8}$$

Again, in view of (3.4), we find, for $s \in K$,

$$w_n(1-s-ikh) \ll_K \exp\Big\{-\frac{c}{\theta}|t-kh|\Big\} \ll_{K,h} \exp\{-c_3kh\}, \quad c_3>0.$$

Therefore,

$$S_1 \ll_{K,h} \frac{1}{H}\sum_{k=N}^{N+H} \exp\{-c_3kh\} \ll_{K,h} \exp\{-c_3Nh\}.$$

This and (3.6)–(3.8) imply the estimate

$$J \ll_{\alpha,h,K} n^{-\varepsilon} + \exp\{-c_3Nh\} + o(1), \quad N\to\infty,$$

and this proves (3.1). The proposition is proven. □

4 Limit theorems on weak convergence

In this section, we will prove limit theorems on weakly convergent probability measures in the space $H(\mathcal{D})$. Let $\mathcal{B}(\mathfrak{X})$ stand for the Borel σ-field of the topological space $\mathfrak{X}$, and P, P_n, $n\in\mathbb{N}$, be probability measures on $(\mathfrak{X},\mathcal{B}(\mathfrak{X}))$. By the definition, P_n converges weakly to P as $n\to\infty$ ($P_n \underset{n\to\infty}{\xrightarrow{\mathrm{w}}} P$) if, for every real bounded continuous function x on $\mathfrak{X}$,

$$\int\limits_{\mathfrak{X}} x\,\mathrm{d}P_n \underset{n\to\infty}{\to} \int\limits_{\mathfrak{X}} x\,\mathrm{d}P.$$

We will consider the weak convergence for the measure

$$P_{N,H,h,\alpha}(A) = M_{N,H}\big(\zeta(s+ikh,\alpha)\in A\big), \quad A\in\mathcal{B}(H(\mathcal{D})).$$

We start with a limit lemma in a simpler space. Define the Cartesian product

$$\mathbb{T} = \prod_{m\in\mathbb{N}_0}\{s\in\mathbb{C}: |s|=1\}.$$

The set $\mathbb{T}$, as a product of compact sets, is compact. Therefore, with the operation of pointwise multiplication and product topology, $\mathbb{T}$ is a compact topological group. Hence, on $(\mathbb{T},\mathcal{B}(\mathbb{T}))$, the probability Haar measure μ_H exists. The measure μ_H is invariant with respect to shifts by elements $\mathfrak{t}=(\mathfrak{t}_m: m\in\mathbb{N})$. This means that, for every $A\in\mathcal{B}(\mathbb{T})$, and $\mathfrak{t}\in\mathbb{T}$,

$$\mu_H(A)=\mu_H(\mathfrak{t}A)=\mu_H(A\mathfrak{t}).$$

For $A\in\mathcal{B}(\mathbb{T})$, set

$$Q_{N,H,h,\alpha}(A) = M_{N,H}\big(\big((m+\alpha)^{-ikh}: m\in\mathbb{N}_0\big)\in A\big),$$

and consider the weak convergence for $Q_{N,H,h,\alpha}$ as $N \to \infty$.

Lemma 4.1. *Suppose that* $0 < \alpha < 1$, $\alpha \neq 1/2$, *and* $h > 0$ *are arbitrary numbers, and* $H = o(N)$*as* $N \to \infty$. *Then, on* $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, *there exists a probability measure* $Q_{h,\alpha}$ *such that* $Q_{N,H,h,\alpha} \underset{N\to\infty}{\xrightarrow{\mathrm{w}}} Q_{h,\alpha}$.

Proof. A direct proof of weak convergence of probability measures using the definition is difficult and even impossible. There are so-called continuity theorems for some transforms of probability measures whose convergence implies the weak convergence of the corresponding measures. Since the set $\mathbb{T}$ is a group, it is convenient to utilise the Fourier transforms [42].

It is well known that characters of the group $\mathbb{T}$ are of the form

$$\chi(\mathfrak{t}) = \prod_{m\in\mathbb{N}}^{*} \mathfrak{t}_m^{l_m},$$

where the star shows that only a finite number of $l_m \in \mathbb{Z}$ are not zeros. Therefore, the Fourier transform $F_{N,H,h,\alpha}(\underline{l})$, $\underline{l} = (l_m : m \in \mathbb{N})$, of $Q_{N,H,h,\alpha}$ is given by

$$F_{N,H,h,\alpha}(\underline{l}) = \int_{\mathbb{T}} \left(\prod_{m\in\mathbb{N}_0}^{*} \mathfrak{t}^{l_m} \right) \mathrm{d}Q_{N,H,h,\alpha}.$$

Hence, the definition of $Q_{N,H,h,\alpha}$ yields

$$\begin{aligned} F_{N,H,h,\alpha}(\underline{l}) &= \frac{1}{H+1} \sum_{k=N}^{N+H} \prod_{m\in\mathbb{N}_0}^{*} (m+\alpha)^{-ikl_m h} \\ &= \frac{1}{H+1} \sum_{k=N}^{N+H} \exp\left\{ -ikh \sum_{m\in\mathbb{N}_0}^{*} l_m \log(m+\alpha) \right\}. \end{aligned} \tag{4.1}$$

Let, for brevity,

$$A(\underline{l}, \alpha) = \sum_{m\in\mathbb{N}_0}^{*} l_m \log(m+\alpha).$$

If

$$hA(\underline{l}, \alpha) = 2\pi r, \quad r \in \mathbb{Z}, \tag{4.2}$$

then

$$\exp\{-ikhA(\underline{l}, \alpha)\} = 1.$$

Thus, for $\underline{l}$ satisfying (4.2),

$$F_{N,H,h,\alpha}(\underline{l}) = 1. \tag{4.3}$$

If there is not $r \in \mathbb{Z}$ satisfying (4.2), then

$$\exp\{-ihA(\underline{l}, \alpha)\} \neq 1.$$

Thus, in view of (4.1),

$$F_{N,H,h,\alpha}(\underline{l}) = \frac{\exp\{-iNA(\underline{l}, \alpha)\} - \exp\{-i(N+H+1)A(\underline{l}, \alpha)\}}{(H+1)(1 - \exp\{-ihA(\underline{l}, \alpha)\})}.$$

Therefore, in this case,

$$\lim_{N\to\infty} F_{N,H,h,\alpha}(\underline{l}) = 0. \tag{4.4}$$

This and (4.3) show that

$$\lim_{N\to\infty} F_{N,H,h,\alpha}(\underline{l}) = \begin{cases} 1 & \text{if } hA(\underline{l},\alpha) = 2\pi r \text{ with } r \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$Q_{N,H,h,\alpha} \underset{N\to\infty}{\xrightarrow{w}} Q_{h,\alpha},$$

where the probability measure $Q_{h,\alpha}$ is defined by the Fourier transform

$$F_{h,\alpha}(\underline{l}) = \begin{cases} 1 & \text{if } hA(\underline{l},\alpha) = 2\pi r \text{ with } r \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

□

Corollary 4.1. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$, and $H = o(N)$ as $N \to \infty$. Then $Q_{H,N,h,\alpha} \underset{N\to\infty}{\xrightarrow{w}} \mu_H$.*

Proof. We have

$$hA(\underline{l},\alpha) + 2\pi r = h\Big(A(\underline{l},\alpha) + \frac{2\pi r}{h}\Big).$$

Since the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$,

$$A(\underline{l},\alpha) + \frac{2\pi r}{h} \neq 0$$

for $\underline{l} \neq \underline{0}$. Hence, for $\underline{l} \neq \underline{0}$,

$$\exp\{-ihA(\underline{l},\alpha)\} \neq 1,$$

and, by the same lines as above, we find that equality (4.4) is true. Obviously, for $\underline{l} = \underline{0}$,

$$F_{N,H,h,\alpha}(\underline{l}) = 1.$$

Thus, we obtain that

$$\lim_{N\to\infty} F_{N,H,h,\alpha}(\underline{l}) = \begin{cases} 1 & \text{if } \underline{l} = \underline{0}, \\ 0 & \text{if } \underline{l} \neq \underline{0}. \end{cases}$$

This shows that μ_H is the limit measure of $Q_{N,H,h,\alpha}$. □

Lemma 4.1 is applied for weak convergence of the probability measure defined by means of the function $\zeta_n(s,\alpha)$. Let, for $A \in \mathcal{B}(H(\mathcal{D}))$,

$$\widehat{P}_{N,H,h,\alpha}(A) = M_{N,H}\big(\zeta_n(s+ikh,\alpha) \in A\big).$$

Consider the mapping $u_{n,\alpha} : \mathbb{T} \to H(\mathcal{D})$ given by

$$u_{n,\alpha}(\mathfrak{t}) = \zeta_n(s,\mathfrak{t},\alpha),$$

where

$$\zeta_n(s,\mathfrak{t},\alpha) = \sum_{m=0}^{\infty} \frac{\mathfrak{t}(m) w_n(m,\alpha)}{(m+\alpha)^s}.$$

Since $|\mathfrak{t}(m)| = 1$, the series for $\zeta_n(s,\mathfrak{t},\alpha)$, as for $\zeta_n(s,\alpha)$, is absolutely convergent in any half-plane $\sigma > \sigma_0$. Hence, $u_{n,\alpha}$ is continuous, therefore, $(\mathcal{B}(\mathbb{T}), \mathcal{B}(H(\mathcal{D})))$-measurable.

On $(H(\mathcal{D}), \mathcal{B}(H(\mathcal{D})))$, define the measure

$$\widehat{P}_{n,h,\alpha}(A) = Q_{h,\alpha} u_{n,\alpha}^{-1}(A) = Q_{h,\alpha}\left(u_{n,\alpha}^{-1}A\right), \quad A \in \mathcal{B}(H(\mathcal{D})).$$

Lemma 4.2. *Suppose that* $0 < \alpha < 1$, $\alpha \neq 1/2$, *and* $h > 0$ *are fixed arbitrary numbers, and* $H = o(N)$ *as* $N \to \infty$. *Then* $\widehat{P}_{N,H,h,\alpha} \xrightarrow[N\to\infty]{\mathrm{w}} \widehat{P}_{n,h,\alpha}$.

Proof. By the definitions of $\widehat{P}_{N,H,h,\alpha}$, $Q_{N,H,h,\alpha}$ and $u_{n,\alpha}$, we have

$$\begin{aligned}\widehat{P}_{N,H,h,\alpha} &= M_{N,H}\left(u_{n,\alpha}\left((m+\alpha)^{-ikh} : m \in \mathbb{N}_0\right) \in A\right)\\ &= M_{N,H}\left(\left((m+\alpha)^{-ikh} : m \in \mathbb{N}_0\right) \in u_{n,\alpha}^{-1}A\right)\\ &= Q_{N,H,h,\alpha}\left(u_{n,\alpha}^{-1}A\right)\end{aligned}$$

for all $A \in \mathcal{B}(H(\mathcal{D}))$. Therefore $\widehat{P}_{N,H,h,\alpha} = Q_{N,H,h,\alpha}u_{n,\alpha}^{-1}$. Hence, the continuity of $u_{n,\alpha}$, Lemma 4.1 and Theorem 5.1 of [43] on preservation of weak convergence under continuous mappings yield the relation

$$\widehat{P}_{N,H,h,\alpha} \xrightarrow[N\to\infty]{\mathrm{w}} Q_{h,\alpha}u_{n,\alpha}^{-1} = \widehat{P}_{n,h,\alpha}.$$

□

Corollary 4.2. *Suppose that the set* $L(\alpha, h, \pi)$ *is linearly independent over* $\mathbb{Q}$, *and* $H = o(N)$. *Then* $\widehat{P}_{N,H,h,\alpha} \xrightarrow[N\to\infty]{\mathrm{w}} \mu_H u_{n,\alpha}^{-1}$.

Proof. The corollary is a direct result of Lemma 4.2 and Corollary 4.1. □

Next, we will study the measure $\widehat{P}_{n,h,\alpha}$. We will prove that, for every $\varepsilon > 0$, there is a compact subset $K(\varepsilon) \subset H(\mathcal{D})$ such that

$$\widehat{P}_{n,h,\alpha}(K(\varepsilon)) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. The latter property of $\widehat{P}_{n,h,\alpha}$ is called a tightness.

Lemma 4.3. *The measure* $\widehat{P}_{n,h,\alpha}$ *is tight.*

Proof. Let K_m be a compact set from the definition of the metric ρ. Denote by $\mathcal{L}_m$ a simple closed contour lying in $\mathcal{D}$ and enclosing the set K_m. Then, by the Cauchy integral formula, for $s \in K_m$,

$$\zeta(s + ikh, \alpha) = \frac{1}{2\pi i}\int\limits_{\mathcal{L}_m} \frac{\zeta(z + ikh, \alpha)}{z - s}\,\mathrm{d}z.$$

Hence, for suitably chosen $\mathcal{L}_m$,

$$\sum_{k=N}^{N+H} \sup_{s\in K_m} |\zeta(s + ikh, \alpha)| \ll_{K_m} \int\limits_{\mathcal{L}_m} |\mathrm{d}z| \sum_{k=N}^{N+H} |\zeta(\mathrm{Re}z + \mathrm{iIm}z + ikh, \alpha)|$$

and, in view of Lemma 2.3,

$$\sum_{k=N}^{N+H} |\zeta(\mathrm{Re}z + \mathrm{iIm}z + ikh, \alpha)| \ll \left((H+1)\sum_{k=N}^{N+H} |\zeta(\mathrm{Re}z + \mathrm{iIm}z + ikh, \alpha)|^2\right)^{1/2} \ll_{K_m,h,\alpha} H.$$

Thus, we have

$$\sum_{k=N}^{N+H} \sup_{s\in K_m} |\zeta(s+ikh,\alpha)| \ll_{K_m,h,\alpha} H.$$

Now, the equality (3.1) gives

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{H+1} \sum_{k=N}^{N+H} \sup_{s\in K_m} |\zeta_n(s+ikh,\alpha)| \\
&\quad\le \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{H+1} \sum_{k=N}^{N+H} \sup_{s\in K_m} |\zeta(s+ikh,\alpha)| \\
&\qquad + \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{H+1} \sum_{k=N}^{N+H} \sup_{s\in K_m} |\zeta_n(s+ikh,\alpha)-\zeta(s+ikh,\alpha)| \\
&\quad\le R_{K_m,h,\alpha} < \infty.
\end{aligned} \tag{4.5}$$

On a certain probability space $(\Omega, \mathcal{B}, \nu)$, define the random variable $\xi_{N,H,h}$ by the distribution

$$\nu\{\xi_{N,H,h} = kh\} = \frac{1}{H+1}, \quad k = N, N+1, \ldots, N+H.$$

Let the $H(\mathcal{D})$-valued random element $X_{N,H,n} = X_{H,N,n,h,\alpha}(s)$ be given by

$$X_{N,H,n} = \zeta_n(s+i\xi_{N,H,h},\alpha),$$

and $X_n = X_{n,h,\alpha}(s)$ have the distribution $\widehat{P}_{n,h,\alpha}$. Then, denoting by $\xrightarrow{\mathcal{D}}$ the convergence in distribution, we have, by Lemma 4.2,

$$X_{N,H,n} \underset{N\to\infty}{\xrightarrow{\mathcal{D}}} X_n. \tag{4.6}$$

This implies the relation

$$\sup_{s\in K_m} |X_{N,H,n}(s)| \underset{N\to\infty}{\xrightarrow{\mathcal{D}}} \sup_{s\in K_m} |X_n(s)|. \tag{4.7}$$

For a fixed $\varepsilon > 0$, set $M_m = 2^m \varepsilon^{-1} R_{K_m,h,\alpha}$. Then, taking into account (4.7) and (4.5) gives

$$\begin{aligned}
\nu\left\{ \sup_{s\in K_m} |X_n(s)| \ge M_m \right\} &= \lim_{N\to\infty} \nu\left\{ \sup_{s\in K_m} |X_{N,H,n}(s)| \ge M_m \right\} \\
&\le \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{(H+1)M_m} \sum_{k=N}^{N+H} \sup_{s\in K_m} |\zeta_n(s+ikh,\alpha)| \le \frac{\varepsilon}{2^m}.
\end{aligned} \tag{4.8}$$

Let

$$K(\varepsilon) = \left\{ g \in H(\mathcal{D}) \colon \sup_{s\in K_m} |g(s)| \le M_m, m \in \mathbb{N} \right\}.$$

Then the set $K(\varepsilon)$ is a compact set in $H(\mathcal{D})$. Moreover, in view of (4.8),

$$\nu\{X_n \in K(\varepsilon)\} = 1 - \nu\{X_n \notin K(\varepsilon)\} \ge 1 - \varepsilon \sum_{m=1}^{\infty} \frac{1}{2^m} = 1 - \varepsilon$$

for all $m \in \mathbb{N}$. Thus, $\widehat{P}_{n,h,\alpha}$ is tight. □

Now, we are in position to prove weak convergence for $P_{N,H,h,\alpha}$. For this we will apply the following general result on convergence in distribution.

Lemma 4.4 ([43], Theorem 4.2). *Suppose that $(\mathfrak{X}, d)$ is a separable metric space, ξ_{nm} and η_n, $m, n \in \mathbb{N}$, are $\mathfrak{X}$-valued random elements defined on the same probability space $(\Omega, \mathcal{B}, \nu)$, and the relations*

$$\xi_{nm} \xrightarrow[n\to\infty]{\mathcal{D}} \xi_m, \qquad \xi_m \xrightarrow[m\to\infty]{\mathcal{D}} \xi$$

hold. If, for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \nu\{d(\xi_{nm}, \eta_n) \geq \varepsilon\} = 0,$$

then $\eta_n \xrightarrow[n\to\infty]{\mathcal{D}} \xi$.

Proposition 4.1. *Suppose that $0 < \alpha < 1$, $\alpha \neq 1/2$, and $h0$ are fixed numbers, and $h^{-1}(Nh)^{27/82} \leq H \leq (Nh)^{1/2}h^{-1}$. Then, on $(H(\mathcal{D}), \mathcal{B}(H(\mathcal{D})))$, there exists a probability measure $P_{h,\alpha}$ such that $P_{N,H,h,\alpha} \xrightarrow[N\to\infty]{\mathrm{w}} P_{h,\alpha}$.*

Proof. Let $\xi_{N,H,h}$ be the same random variable as in proof of Lemma 4.3. Define one more $H(\mathcal{D})$-valued random element

$$Y_{N,H} = Y_{N,H,h,\alpha}(s) = \zeta(s + i\xi_{N,H,h}, \alpha).$$

By Lemma 4.3, the measure $\widehat{P}_{n,h,\alpha}$ is tight, therefore, by the Prokhorov theorem ([43], Theorem 6.1), it is relatively compact. Hence, then there are a subsequence $\widehat{P}_{n_r,h,\alpha}$ and a probability measure $\widehat{P}_{h,\alpha}$ on $(H(\mathcal{D}), \mathcal{B}(H(\mathcal{D})))$ such that $\widehat{P}_{n_r,h,\alpha} \xrightarrow[r\to\infty]{\mathrm{w}} \widehat{P}_{h,\alpha}$. Since $\widehat{P}_{n_r,h,\alpha}$ is the distribution of $X_{n_r,h,\alpha}$, this implies the relation

$$X_{n_r} = X_{n_r,h,\alpha} \xrightarrow[r\to\infty]{\mathcal{D}} \widehat{P}_{h,\alpha}. \tag{4.9}$$

For every $\varepsilon > 0$, Proposition 3.1 implies

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \nu\big\{\rho(Y_{N,H}, X_{N,H,h}) \geq \varepsilon\big\} \\
&= \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{H+1} M_{N,H}\big(\rho(\zeta(s+ikh,\alpha), \zeta_n(s+ikh,\alpha)) \geq \varepsilon\big) \\
&\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(H+1)} \sum_{k=N}^{N+H} \rho(\zeta(s+ikh,\alpha), \zeta_n(s+ikh,\alpha)) = 0.
\end{aligned}$$

This equality, and the relations (4.6), (4.9) show that the hypotheses of Lemma 4.4 are satisfied. Thus we have the relation

$$Y_{N,H} \xrightarrow[N\to\infty]{\mathcal{D}} \widehat{P}_{h,\alpha},$$

and this proves the proposition with $P_{h,\alpha} = \widehat{P}_{h,\alpha}$. □

Define, for $s \in \mathcal{D}$,

$$\zeta(s, \alpha, \mathfrak{t}) = \sum_{m=0}^{\infty} \frac{\mathfrak{t}(m)}{(m+\alpha)^s}, \quad \mathfrak{t} \in \mathbb{T}.$$

Then, $\zeta(s,\alpha,\mathfrak{t})$ is an $H(\mathcal{D})$-valued random element on the probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu_H)$, see Section 5.5.2 of [28]. We notice, that the series for $\zeta(s,\alpha,\mathfrak{t})$, for almost all $\mathfrak{t}$ with respect to the measure μ_H, is uniformly convergent on compact sets of the strip $\mathcal{D}$. Denote by $P_{\zeta,\alpha}$ the distribution of $\zeta(s,\alpha,\mathfrak{t})$, i.e.,

$$P_{\zeta,\alpha}(A) = \mu_H\{t \in \mathbb{T} : \zeta(s,\alpha,\mathfrak{t}) \in A\}, \quad A \in \mathcal{B}(H(\mathcal{D})).$$

Corollary 4.3. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$, and $h^{-1}(Nh)^{27/82} \leq H \leq (Nh)^{1/2}h^{-1}$. Then $P_{N,H,h,\alpha} \xrightarrow[N\to\infty]{w} P_{\zeta,\alpha}$.*

Proof. For $A \in \mathcal{B}(H(\mathcal{D}))$, set

$$P_{N,h,\alpha}(A) = M_N(\zeta(s+ikh,\alpha) \in A).$$

Then, by the proof of Theorem 2.1 from [35], we have that

$$P_{N,h,\alpha} \xrightarrow[N\to\infty]{w} P_{\zeta,\alpha}, \quad \text{and} \quad \widehat{P}_{n,h,\alpha} \xrightarrow[n\to\infty]{w} P_{\zeta,\alpha}.$$

Thus, in the proof of Proposition 4.1, the measure $\widehat{P}_{n,\alpha}$ coincides with $P_{\zeta,\alpha}$. □

5 Proofs of approximation results

We use the classical scheme of obtaining universality theorems from limit theorems in the space of analytic functions. This method also involves the mentioned in Introduction Mergelyan theorem on approximation of analytic functions by polynomials. Moreover, the support of the limit measures is an important ingredient of the proof.

Lemma 5.1 ([35], Theorem 3.1). *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\zeta,\alpha}$ is the whole of $H(\mathcal{D})$.*

Proof of Theorem 1.5. By the Mergelyan theorem, there is a polynomial $p(s)$ satisfying

$$\sup_{s\in K}|f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{5.1}$$

Consider the set

$$\mathcal{G}_\varepsilon = \left\{ g \in H(\mathcal{D}) : \sup_{s\in K}|g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

In view of Lemma 5.1, the polynomial $p(s)$ is an element of the support of the measure $P_{\zeta,\alpha}$. Therefore, the set $\mathcal{G}_\varepsilon$ is an open neighbourhood of an element of the support of $P_{\zeta,\alpha}$. Hence, by a support property,

$$P_{\zeta,\alpha}(\mathcal{G}_\varepsilon) > 0. \tag{5.2}$$

Using Corollary 4.3 and the equivalent of weak convergence in terms of open sets, we find that

$$\liminf_{N\to\infty} P_{N,H,h,\alpha}(\mathcal{G}_\varepsilon) \geq P_{\zeta,\alpha}(\mathcal{G}_\varepsilon).$$

The inequality (5.1) shows that if $g(s) \in \mathcal{G}_\varepsilon$, then $g(s)$ also lies in the set

$$G_\varepsilon = \left\{ g \in H(\mathcal{D}) : \sup_{s\in K}|g(s) - f(s)| < \varepsilon \right\}.$$

Thus, by (5.2),

$$P_{\zeta,\alpha}(G_\varepsilon) > 0.$$

This inequality, Corollary 4.3 in terms of weak convergence using open sets yield

$$\liminf_{N\to\infty} P_{N,H,h,\alpha}(G_\varepsilon) \geq P_{\zeta,\alpha}(G_\varepsilon) > 0,$$

and the definitions of G_ε and $P_{n,H,h,\alpha}$ prove the first statement of the theorem.

For the proof of the second statement of the theorem, we will apply the equivalent of weak convergence in terms of continuity sets, i.e., sets G_ε whose boundary ∂G_ε satisfies $P_{\zeta,\alpha}(\partial G_\varepsilon) = 0$. By the definition of G_ε, we see that the boundary ∂G_ε lies in the set

$$\left\{ g \in H(\mathcal{D}) \colon \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\}.$$

This implies that $\partial G_{\varepsilon_1} \cap \partial G_{\varepsilon_2} = \emptyset$ for $\varepsilon_1 \neq \varepsilon_2$. So, $P_{\zeta,\alpha}(\partial G_\varepsilon) > 0$ for at most countably many values of ε. In other words, the set G_ε is a continuity set of $P_{\zeta,\alpha}$ for all but at most countably many $\varepsilon > 0$. This remark and Corollary 4.3 in terms of weak convergence using continuity sets give

$$\lim_{N \to \infty} P_{N,H,h,\alpha}\left(G_\varepsilon\right) = P_{\zeta,\alpha}\left(G_\varepsilon\right) > 0$$

for all but at most countably many $\varepsilon > 0$, and the second statement of the theorem is proved. □

Proof of Theorem 1.6. Let

$$\mathcal{F}_{\alpha,h} = \mathrm{supp} P_{h,\alpha}.$$

Then, clearly, $\mathcal{F}_{\alpha,h}$ is a closed non-empty set. Since $f(s) \in \mathcal{F}_{\alpha,h}$, the set G_ε is an open neighbourhood of an element of the support of the measure $P_{h,\alpha}$. Thus,

$$P_{h,\alpha}\left(G_\varepsilon\right) > 0,$$

and, by Proposition 4.1,

$$\liminf_{N \to \infty} P_{N,H,h,\alpha}\left(G_\varepsilon\right) \geq P_{h,\alpha}\left(G_\varepsilon\right) > 0.$$

This proves the first statement of the theorem.

The second statement of the theorem, by the same arguments as in the proof of Theorem 1.5, follows from Proposition 4.1 and the equality

$$\liminf_{N \to \infty} P_{N,H,h,\alpha}\left(G_\varepsilon\right) = P_{h,\alpha}\left(G_\varepsilon\right)$$

which is valid for all but at most countably many $\varepsilon > 0$. □

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