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NEW STATIONARITY CONDITIONS FOR IARCH PROCESSES

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Abstract

We prove the existence of a stationary IARCH (integrated autoregressive conditionally heteroskedastic) process for a class of models with polynomially decaying coefficients.

Keywords: ARCH processes; stationarity conditions; IARCH problem

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1. Introduction

The most popular model for financial data r_t is the so-called GARCH(p, q) (generalized autoregressive conditionally heteroscedastic) model, introduced in [2]:

$$r_t = z_t\sigma_t, \qquad \sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i r_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \quad t \in \mathbb{Z}, \tag{1.1}$$

where $(z_t \mid t \in \mathbb{Z})$ is a family of independent and identically distributed (i.i.d.) random variables with $\mathbb{E}z_t = 0$ and $\mathbb{E}z_t^2 = 1$, $\sigma_t \geq 0$ is $\mathcal{F}_{t-1}$-measurable for all $t \in \mathbb{Z}$ (with $\mathcal{F}_t = \sigma(z_s \mid s \leq t)$), $\alpha_0 > 0$, and α_i, $\beta_j \geq 0$ for $i, j \geq 1$. The GARCH(0, q) model is called ARCH(q) and was introduced in [5] a bit earlier. Of course, the equalities in (1.1) are in fact equations with respect to r_t, and it is not clear if they have a strictly stationary solution, i.e. if the GARCH process exists.

If there exists a GARH process then (see, e.g., [8]) $\sum_{j=1}^{p} \beta_j < 1$ and

$$r_t^2 = z_t^2\Big(a_0 + \sum_{i\geq 1} a_i r_{t-i}^2\Big), \quad t \in \mathbb{Z}, \tag{1.2}$$

where the coefficients a_i are obtained from the equations

$$a_0 = \Big(1 - \sum_{j=1}^{p} \beta_j\Big)^{-1}, \qquad \sum_{i\geq 1} a_i u^i = \Big(1 - \sum_{j=1}^{p} \beta_j u^j\Big)^{-1} \sum_{i=1}^{q} \alpha_i u^i \quad \text{for } |u| \leq 1.$$

Moreover, the process $(r_t \mid t \in \mathbb{Z})$ is strictly stationary and adapted: r_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{Z}$. It is therefore natural to consider the general ARCH(∞) model, defined by (1.2) without any restraints on the coefficients (a_i) (except non-negativity), and ask when such equations are consistent – i.e. have an adapted strictly stationary solution.

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For GARCH models one condition is well known: if $\sum_{i=1}^{q}\alpha_i+\sum_{j=1}^{p}\beta_j<1$ then a GARCH process exists, it is unique, and square-integrable. [3] proved that the same holds if $\sum_{i=1}^{q}\alpha_i+\sum_{j=1}^{p}\beta_j=1$ (provided z_t^2 is not degenerate at 1); however, in this case the GARCH process is not square-integrable. We believe that the analogous statement is true for the general ARCH(∞) model. It is easily shown that the condition $\sum_{i\geq 1}a_i<1$ is necessary and sufficient for the existence of a strictly stationary square-integrable solution to (1.2). However, the case where $\sum_{i\geq 1}a_i=1$, is not so obvious. In this case, we call any adapted strictly stationary solution to (1.2) an IARCH process (integrated ARCH, following the analogous situation in ARMA processes). Below, we prove the existence of an IARCH process under some mild assumptions on the coefficients (a_i).

To formulate our result, let us make some changes in notation. First, we denote r_t^2 and z_t^2 by $\tilde{X}_t$ and $\tilde{\varepsilon}_t$, respectively. Then (1.2) becomes

$$\tilde{X}_t=\tilde{\varepsilon}_t\left(a_0+\sum_{i\geq 1}a_i\tilde{X}_{t-i}\right),\qquad t\in\mathbb{Z}. \tag{1.3}$$

Of course, $(\tilde{\varepsilon}_t \mid t\in\mathbb{Z})$ are i.i.d. with $\mathbb{E}\tilde{\varepsilon}_t=1$. If equations (1.3) have an adapted strictly stationary solution $(\tilde{X}_t \mid t\in\mathbb{Z})$, equations (1.2) have a solution $r_t=\sqrt{\tilde{X}_t}\operatorname{sign}(z_t)$, i.e. an IARCH process exists. Next, we define $X_n=\tilde{X}_{-n}/a_0$ and $\varepsilon_n=\tilde{\varepsilon}_{-n}$ for $n\geq 1$. Then (a_i) is a sequence of non-negative real numbers, (ε_i) a sequence of independent copies of some non-negative random variable ε, and $\sum_{i\geq 1}a_i=\mathbb{E}\varepsilon=1$. Moreover, (1.3) transforms into the system of equations that we call a *non-homogeneous IARCH equation*:

$$X_n=\varepsilon_n\left(1+\sum_{i\geq 1}a_iX_{n+i}\right),\qquad n\geq 1. \tag{1.4}$$

By a *solution* to it we mean any strictly stationary sequence (X_n) of non-negative random variables such that, almost surely, all equalities (1.4) hold. We call the equation *consistent* if it has a solution. The *non-homogeneous IARCH problem* is to find conditions on (a_i) and the distribution of ε that guarantee consistency of (1.4).

It is known from the general theory of ARCH processes (see [8]) that if the IARCH equation is consistent then it has a *minimal solution* (X_n^*) with

$$X_n^*=\varepsilon_n\left(1+\sum_{k\geq 1}\sum_{i_1,\ldots,i_k\geq 1}a_{i_1}\cdots a_{i_k}\varepsilon_{n+i_1}\varepsilon_{n+i_1+i_2}\cdots\varepsilon_{n+i_1+\cdots+i_k}\right). \tag{1.5}$$

Then the family $(\tilde{X}_t \mid t<0)$ defined by $\tilde{X}_t=a_0X_{-t}^*$ satisfies equations (1.3) for $t<0$, and can be extended by recursion to a complete solution of (1.3) that is adapted and strictly stationary. Therefore the consistency of (1.4) means that an IARCH process exists.

Equation (1.3) with $a_0=0$ is essentially different. We write it in the form

$$Z_n=\varepsilon_n\sum_{i\geq 1}a_iZ_{n+i},\qquad n\geq 1, \tag{1.6}$$

and call it the *homogeneous IARCH equation*. Its *solution* is a strictly stationary sequence (Z_n) of non-negative random variables such that, almost surely, all equations (1.6) hold. Equation (1.6) always has a trivial solution $Z_n=0$, and therefore the *homogeneous IARCH problem* is to find out if non-trivial solutions exist.

The homogeneous IARCH problem is related to the problem of uniqueness of a solution to IARCH equation. A solution (X_n) to (1.4) is called *causal* if, for some measurable function $f\colon \mathbb{R}_+^\infty \to \mathbb{R}_+$, $X_n = f(\varepsilon_n, \varepsilon_{n+1}, \ldots)$, $n \geq 1$. Causal solutions to (1.6) are defined in a similar way. The minimal solution (1.5) is a causal solution with the following property: for any solution (X_n) and all $n \geq 1$, almost surely $X_n^* \leq X_n$. Hence, if the IARCH equation is consistent and (Z_n) is a non-trivial causal solution to (1.6) then (X_n) with $X_n = X_n^* + Z_n$ is another solution to (1.4). Conversely, if the minimal solution (X_n^*) is not unique and (X_n) is another causal solution, then (Z_n) with $Z_n = X_n - X_n^*$ is a solution to (1.6).

However, the homogeneous IARCH problem is larger than the uniqueness problem: the homogeneous IARCH equation can possibly also have a non-trivial solution in the case where equation (1.4) is inconsistent. Moreover, it is not clear how equations (1.4) and (1.6) are related, if non-causal solutions are also considered.

We refer to the set of problems described above as the *IARCH problem*. It has a long history. The first result concerning IARCH processes was obtained in [3], which proved that the IARCH equation is consistent in the case where ε is non-degenerate and the generating function $a(u) = \sum_{i\geq 1} a_i u^i$ of the sequence (a_i) is rational (equals the quotient of two polynomials). [1] considered the model with $a(u) = 1 - (1-u)^d$ (where $0 < d < 1$), but their proof of consistency was incorrect; see the discussion in [11]. [9] proved the consistency of (1.4) in the case where the a_i decay exponentially. [4] gave a simple consistency proof under the following condition: for some $0 < r < 1$,

$$\sum_{i\geq 1} a_i^r \mathbb{E}\varepsilon^r < 1. \tag{1.7}$$

There are also some results concerning the homogeneous IARCH problem. [3] proved the uniqueness of the solution in the case where $a(u)$ is a rational function, and [10] generalized that result to the case where coefficients decay exponentially. [4] showed, under condition (1.7), that there exists a unique causal solution (X_n) with $\mathbb{E}X_n^r < \infty$. [6] is the only case that considered equation (1.6) itself, unrelated to the corresponding non-homogeneous equation. They obtained necessary and sufficient conditions for the existence of a square-integrable solution (Z_n) to (1.6).

Our main result in this paper is the following theorem.

Theorem 1.1. *If ε is non-degenerate and, for some $\alpha > 1$, $\sum_{i>l} a_i = O(l^{-\alpha})$ as $l \to \infty$, then* (1.4) *has a solution (X_n) with $\mathbb{E}X_n^r < \infty$ for all $r \in (0; 1)$.*

Since $l^\alpha \sum_{i>l} a_i \leq \sum_{i>l} i^\alpha a_i \leq \sum_{i\geq 1} i^\alpha a_i$, we also get the following statement as a corollary of Theorem 1.1.

Corollary 1.1. *If ε is non-degenerate and, for some $\alpha > 1$, $\sum_{i\geq 1} i^\alpha a_i < \infty$, then* (1.4) *has a solution (X_n) with $\mathbb{E}X_n^r < \infty$ for all $r \in (0; 1)$.*

Since $\mathbb{E}\varepsilon = 1$, the assumption that ε is non-degenerate excludes only one case, where $\varepsilon = 1$ almost surely. It is easily seen that in that case (1.4) is inconsistent; therefore, the assumption is absolutely necessary. The statement about moments of a solution is the strongest that could be expected, because the IARCH equation has no solution in L^1: taking expectations of both sides in (1.4) yields (at least for causal solutions with $\mathbb{E}X_n = \mu$) $\mu = 1 + \sum_{i\geq 1} a_i \mu = 1 + \mu$.

Of course, Theorem 1.1 does not solve the non-homogeneous IARCH problem – it does not even cover completely the result of [4]. To see this, consider the sequences (a_i) such that $a_i \sim c i^{-1-\alpha}$ as $i \to \infty$, with some c, $\alpha > 0$ (α should be positive, because (a_i) is summable). If $\alpha > 1$ then our Theorem 1.1 applies and we have the answer: (1.4) is always consistent, and no assumptions about ε are needed (so our result in this case is better than that of [4], since (1.7)

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requires $\mathbb{E}\varepsilon^r$ to be smaller than $\big(\sum_{i\ge1} a_i^r\big)^{-1}$). The case $0<\alpha\le 1$, however, is still unclear: our theorem does not apply here, while the result of [4] shows that (1.4) is consistent for some ε. So, there are two possibilities, in our opinion. First, the IARCH equation is always consistent, provided ε is non-degenerate (until now, we still do not have any counterexample). Second, starting from the point $\alpha=1$ we can get both consistent and inconsistent IARCH equations, depending on the distribution of ε. In the second case, the necessary and sufficient conditions for consistency would be very different from (1.7) (again, in our opinion). We intend to clarify the situation in the near future.

The corollary generalizes our unpublished result in [7], which corresponds to the case $\alpha=2$. There, we used our subadditive ergodic theorem for double sequences as the main tool for cracking the problem. In the present paper we use a similar but somewhat sharper tool, which is detailed in Section 2 (see Proposition 2.1). Section 3 contains the proof of Theorem 1.1.

We end this introduction by describing some notation used in both sections. By a *sequence* we mean a family with index set $\mathbb{N}=\{1, 2, \dots\}$, and $E^\infty=E^{\mathbb{N}}$ is the set of all sequences of elements of the set E. If J is a finite set then the number of its elements is denoted by $|J|$.

We assume that all random variables are defined on some probability space $(\Omega, \mathbb{P})$, $\mathbb{E}$ stands for the corresponding expectation operator, and $\mathbb{E}^Y$ for conditional expectation given a random variable Y. We write $\mathcal{L}(Y)$ for the distribution of a random variable Y, $\mathcal{L}(Y\mid W)$ for its conditional distribution given an event W, and $\mathbf{1}_W$ for the indicator of the event W. For short, we write L^r instead of $L^r(\Omega, \mathbb{P})$.

We set $I=\{i\mid a_i>0\}$ and use the letter i for elements of various sets I^k, $k\ge 1$. For $i=(i_1, \dots, i_k)\in I^k$, we define $|i|=i_1+\cdots+i_k$ and $\xi(i)=a_{i_1}\cdots a_{i_k}\varepsilon_{i_1}\varepsilon_{i_1+i_2}\cdots\varepsilon_{i_1+\cdots+i_k}$, and set $\eta_{k,n}=\sum_{i\in I^k,|i|=n}\xi(i)$. Note also that $\eta_{k,n}=0$ for $k>n$.

If Y is any random variable of the form $Y=f(\varepsilon_1, \varepsilon_2, \dots)$, where $f\colon \mathbb{R}^\infty\to\mathbb{R}$ is a measurable function, we define, for $j\ge 1$, $Y^{(j)}=f(\varepsilon_{j+1}, \varepsilon_{j+2}, \dots)$. Obviously, $Y^{(j)}$ is equidistributed with Y and independent of $(\varepsilon_i\mid i\le j)$.

Finally, ι stands for the random variable taking values $i\in I$ with probabilities a_i, (ι_k) is a sequence of independent copies of ι, and $S_k=\iota_1+\cdots+\iota_k$. Clearly, for any $i\in I^k$,

$$\mathbb{E}\xi(i)=a_{i_1}\cdots a_{i_k}=\mathbb{P}(\iota_1=i_1, \dots, \iota_k=i_k),$$

and therefore $\mathbb{E}\eta_{k,n}=\mathbb{P}(S_k=n)$. Note also that, under the assumptions of Theorem 1.1, $\mathbb{E}\iota<\infty$.

2. The main tool

The main result in this section is Proposition 2.1. It is preceded by two lemmas about some special functions on $\mathbb{R}_+^m$, where $\mathbb{R}_+=[0;\infty)$ and $m\ge 2$ (we will use them in the case where $m=2$). Here is some additional notation that is used only in this section: for $x=(x_1, \dots, x_m)\in\mathbb{R}_+^m$, we define $|x|=x_1+\cdots+x_m$ and $x^*=x/|x|$ (provided $|x|\ne 0$); we write $e=(1, \dots, 1)\in\mathbb{R}_+^m$ and $H_u=\{x\in\mathbb{R}_+^m\mid |x|=u\}$.

Let us introduce the functions of interest. If $b=(b_1, \dots, b_m)$ is any vector with positive components, for $x\in\mathbb{R}_+^m$ define

$$f_b(x)=|x|\log|x|-\sum_{i=1}^m x_i\log x_i+\sum_{i=1}^m x_i\log b_i \tag{2.1}$$

(assuming $0\log 0=0$). Here is the first result we need.

Lemma 2.1. *The largest value of f_b on H_u equals $u\log|b|$ and is attained at the unique point ub^*.*

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Proof. If $u = 0$, the statement of the theorem is obvious, because the set H_u only contains the point 0. If $u > 0$ then, for all $x \in H_u$,

$$\begin{aligned} f_b(x) &= u \log u - u \sum_{i=1}^{m} x_i^* \log x_i + u \sum_{i=1}^{m} x_i^* \log b_i \\ &= -u \left(\sum_{i=1}^{m} x_i^* \log x_i^* - \sum_{i=1}^{m} x_i^* \log b_i \right) = u \log|b| - u D_{\mathrm{KL}}(x^* \parallel b^*), \end{aligned}$$

where $D_{\mathrm{KL}}(x^* \parallel b^*)$ is the Kullback–Leibler divergence from b^* to x^* (both considered as probability distributions on $\{1, \ldots, m\}$). The assertion of the lemma now follows from the well-known fact that $D_{\mathrm{KL}}(x^* \parallel b^*) \geq 0$, with equality only in the case where $x^* = b^*$, i.e. $x = ub^*$. □

In the proof of the lemma we established a useful formula: for $x \neq 0$,

$$f_b(x) = |x| \log|b| - |x| D_{\mathrm{KL}}(x^* \parallel b^*).$$

It yields that $f_b(cx) = c f_b(x)$ for all $c \geq 0$ and $x \in \mathbb{R}_+^m$; if $c > 0$ and $x \neq 0$ the equality follows from the fact that $(cx)^* = x^*$ and in the remaining cases it is verified directly.

Our second lemma explains why the function f_e is so important. Recall the definition of the multinomial coefficients: if $p_1, \ldots, p_m$ are non-negative integers, $p = (p_1, \ldots, p_m)$, and $q = p_1 + \cdots + p_m$, then

$$\binom{q}{p} = \frac{q!}{p_1! \cdots p_m!}.$$

Lemma 2.2. *Let n and $p_1, \ldots, p_m$ be non-negative integers, depending on some natural parameter k (which is ommitted from the notation for the sake of readability), and $q = p_1 + \cdots + p_m$. If $n \to \infty$ and $p_i/n \to x_i$ as $k \to \infty$ for some finite x_i (and all $i = 1, \ldots, m$), then*

$$\frac{1}{n} \log \binom{q}{p} \to f_e(x) \quad \text{as } k \to \infty,$$

where $p = (p_1, \ldots, p_m)$ and $x = (x_1, \ldots, x_m)$.

Proof. We consider two separate cases.

Step 1: The case where $n = q$. By Stirling's formula, $\log s! = s \log s - s + o(s)$ as $s \to \infty$. Hence $\log p_i! = p_i \log p_i - p_i + o(p_i) = p_i \log p_i - p_i + o(q)$ as $N \ni k \to \infty$ for all infinite $N \subset \mathbb{N}$ such that $p_i \to \infty$ as $N \ni k \to \infty$. On the other hand, the assumption $q \to \infty$ yields that the relation obtained also holds for all N such that $p_i = O(1)$ as $N \ni k \to \infty$. Any sequence of non-negative numbers contains a subsequence that either tends to ∞ or is bounded; therefore, for all i, $\log p_i! = p_i \log p_i - p_i + o(q)$ as $k \to \infty$. Moreover, by Stirling's formula, $\log q! = q \log q - q + o(q)$ as $k \to \infty$. Hence

$$\log \binom{q}{p} = q \log q - q - \sum_{i=1}^{m} (p_i \log p_i - p_i) + o(q) = f_e(p) + o(q)$$

and

$$\frac{1}{q} \log \binom{q}{p} = \frac{1}{q} f_e(p) + o(1) = f_e(p/q) + o(1) \to f_e(x) \quad \text{as } k \to \infty.$$

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Step 2: The general case. First let $|x| > 0$. If $p_i/n \to x_i$, then $q/n \to |x|$ (and therefore $q \to \infty$) and $p_i/q \to x_i/|x| = x_i^*$ as $k \to \infty$. By Step 1,

$$\frac{1}{n}\log\binom{q}{p} = \frac{q}{n}\frac{1}{q}\log\binom{q}{p} \to |x| f_e(x^*) = f_e(x) \quad \text{as } k \to \infty.$$

Now consider the remaining case, where $x = 0$ and therefore $f_e(x) = 0$. Let N be an arbitrary infinite set of natural numbers. If $q \to \infty$ as $N \ni k \to \infty$, by Step 1 we get

$$\frac{1}{n}\log\binom{q}{p} = \frac{q}{n}\frac{1}{q}\log\binom{q}{p} = o(1)O(1) \to 0 \quad \text{as } N \ni k \to \infty.$$

If $q = O(1)$ as $N \ni k \to \infty$, then

$$\frac{1}{n}\log\binom{q}{p} = \frac{1}{n}O(1) \to 0 \quad \text{as } N \ni k \to \infty.$$

The same argument as in Step 1 then yields

$$\frac{1}{n}\log\binom{q}{p} \to 0 = f_e(x) \quad \text{as } k \to \infty. \qquad \square$$

Now we are ready to prove the main result of this section.

Proposition 2.1. *If the random variable ε is non-degenerate then, for all $r, \gamma \in (0; 1)$, there exist a $\delta > 0$ and a random variable C with the following properties:*

(i) $\mathbb{E}C^r < \infty$;

(ii) *for all $n \geq 1$ and $k \geq \gamma n$, $\eta_{k,n} \leq C\mathrm{e}^{-\delta n}$.*

Proof. We write $\pi_2 = \mathbb{P}(\varepsilon > 1)$ and $\pi_1 = 1 - \pi_2$. Since $\mathbb{E}\varepsilon = 1$ and ε is non-degenerate, both these numbers are positive. Let $Q_n = \sum_{i=1}^n \mathbf{1}_{\{\varepsilon_i > 1\}}$ and $\tilde{\eta}_{k,n} = \mathbb{E}^{Q_n}\eta_{k,n}$. Clearly, Q_n is distributed according to the binomial law:

$$\mathbb{P}(Q_n = q) = \binom{n}{q}\pi_1^{n-q}\pi_2^q \quad \text{for } q = 0, \ldots, n.$$

Let us find out how $\tilde{\eta}_{k,n}$ is computed.

Write $z_1 = (1/\pi_1)\mathbb{E}\varepsilon\mathbf{1}_{\{\varepsilon \leq 1\}}$ and $z_2 = (1/\pi_2)\mathbb{E}\varepsilon\mathbf{1}_{\{\varepsilon > 1\}}$, and fix any $0 \leq q \leq n$ and any partition (S, T) of the set $\{1, \ldots, n\}$ with $|S| = k$. Then

$$\begin{aligned}
&\mathbb{E}\prod_{j\in S}\varepsilon_j\mathbf{1}_{\{Q_n = q\}} \\
&\quad = \sum_{p=0}^{q}\sum_{\substack{S'\subset S\\|S'|=p}}\sum_{\substack{T'\subset T\\|T'|=q-p}}\mathbb{E}\prod_{j\in S}\varepsilon_j\prod_{j\in S'}\mathbf{1}_{\{\varepsilon_j>1\}}\prod_{j\in S\setminus S'}\mathbf{1}_{\{\varepsilon_j\leq 1\}}\prod_{j\in T'}\mathbf{1}_{\{\varepsilon_j>1\}}\prod_{j\in T\setminus T'}\mathbf{1}_{\{\varepsilon_j\leq 1\}} \\
&\quad = \sum_{p=0}^{q}\sum_{\substack{S'\subset S\\|S'|=p}}\sum_{\substack{T'\subset T\\|T'|=q-p}}(\pi_1 z_1)^{k-p}(\pi_2 z_2)^p\pi_1^{n-k-q+p}\pi_2^{q-p} \\
&\quad = \pi_1^{n-q}\pi_2^q\sum_{p=0}^{q}\binom{k}{p}\binom{n-k}{q-p}z_1^{k-p}z_2^p.
\end{aligned}$$

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Denoting the sum on the right-hand side by $h_{k,n}(q)$ we get, on the set $\{Q_n = q\}$,

$$\mathbb{E}^{Q_n} \prod_{j\in S} \varepsilon_j = \binom{n}{q}^{-1} h_{k,n}(q),$$

which yields

$$\tilde{\eta}_{k,n} = \sum_{i\in I^k, |i|=n} a_{i_1} \cdots a_{i_k} \mathbb{E}^{Q_n} \prod_{j\in\{i_1, i_1+i_2, \ldots, i_1+\cdots+i_k\}} \varepsilon_j = \mathbb{P}(S_k = n) \binom{n}{q}^{-1} h_{k,n}(q).$$

Since $\binom{m}{l} = 0$ for $l < 0$ and for $l > m$, the actual domain of index p in the sum defining $h_{kn}(q)$ is defined by the inequalities

$$0 \le p \le k, \qquad 0 \le q - p \le n - k. \tag{2.2}$$

Notice also that

$$0 < z_1 < 1 < z_2, \qquad \pi_1 + \pi_2 = 1, \qquad \pi_1 z_1 + \pi_2 z_2 = 1. \tag{2.3}$$

Now we fix $r, \gamma \in (0; 1)$ and define $C_n = \max_{k\ge\gamma n} \eta_{k,n}$, $\tilde{C}_n = \max_{k\ge\gamma n} \tilde{\eta}_{k,n}$, and $\bar{C}_n(q) = \max_{k\ge\gamma n} h_{k,n}(q)$. Clearly,

$$\tilde{C}_n \le \binom{n}{Q_n}^{-1} \bar{C}_n(Q_n).$$

The remaining proof is performed in four steps.

Step 1. We prove that $\overline{\lim}_{n\to\infty} \left(\log \mathbb{E}\tilde{C}_n^r\right)/n \le c$ with $c = \max_{(x,y)\in E} h(x, y)$, where the set $E \subset \mathbb{R}^2 \times \mathbb{R}^2$ is defined by inequalities $0 \le x_1 \le y_1$, $0 \le x_2 \le y_2$, $x_1 + x_2 \ge \gamma$, $y_1 + y_2 = 1$, and, for $(x, y) \in E$, $h(x, y) = (1 - r) f_\pi(y) + r(f_{\pi z}(x) + f_\pi(y - x))$ (here, $\pi = (\pi_1, \pi_2)$, $\pi z = (\pi_1 z_1, \pi_2 z_2)$, and the functions f_b are defined by (2.1)).

Find q_n such that

$$\binom{n}{q_n}^{-r} \bar{C}_n(q_n)^r \mathbb{P}(Q_n = q_n) = \max_{0\le q\le n} \binom{n}{q}^{-r} \bar{C}_n(q)^r \mathbb{P}(Q_n = q);$$

then

$$\begin{aligned}
\mathbb{E}\tilde{C}_n^r \le \mathbb{E}\binom{n}{Q_n}^{-r} \bar{C}_n(Q_n)^r &= \sum_{q=0}^{n} \binom{n}{q}^{-r} \bar{C}_n(q)^r \mathbb{P}(Q_n = q) \\
&\le (n+1)\binom{n}{q_n}^{-r} \bar{C}_n(q_n)^r \mathbb{P}(Q_n = q_n) \\
&= (n+1)\binom{n}{q_n}^{1-r} \bar{C}_n(q_n)^r \pi_1^{n-q_n} \pi_2^{q_n}
\end{aligned}$$

and it suffices to show that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \left((1 - r) \log \binom{n}{q_n} + r \log \bar{C}_n(q_n) + (n - q_n) \log \pi_1 + q_n \log \pi_2 \right) \le c.$$

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Let N_0 be an infinite subset of $\mathbb{N}$ such that in the latter formula $\overline{\lim}_{n\to\infty}$ can be replaced by $\lim_{N_0\ni n\to\infty}$. Let N be an arbitrary infinite subset of N_0. Then it suffices to prove the analogous inequality, where $\overline{\lim}_{n\to\infty}$ is replaced by $\lim_{N\ni n\to\infty}$. Without loss of generality we can assume that $q_n/n \to y_2$ as $N \ni n \to \infty$, where $y_2 \in [0; 1]$ is some fixed number. Write $y_1 = 1 - y_2$ and $y = (y_1, y_2)$. Then, by Lemma 2.2,

$$\frac{1}{n}\log\binom{n}{q_n} \to f_e(y) \quad \text{as } N \ni n \to \infty.$$

Moreover,

$$\log \bar{C}_n(q_n) = \log \bar{h}_{k_n,n}(q_n) \le \log\left((n+1)\binom{k_n}{p_n}\binom{n-k_n}{q_n-p_n} z_1^{k_n-p_n} z_2^{p_n}\right)$$

for some sequences (k_n) and (p_n) with $k_n \ge \gamma n$. Without loss of generality, we can assume that $p_n/n \to x_2$ and $(k_n - p_n)/n \to x_1$ as $N \ni n \to \infty$, with some non-negative x_1, x_2. Writing $x = (x_1, x_2)$, then inequalities (2.2) and $k_n \ge \gamma n$ imply $(x, y) \in E$. Moreover, by Lemma 2.2,

$$\lim_{N\ni n\to\infty} \frac{1}{n}\log \bar{C}_n(q_n) \le f_e(x) + f_e(y-x) + \sum_{i=1}^{2} x_i \log z_i.$$

Therefore the limit of interest does not exceed

$$\begin{aligned}
&(1-r)f_e(y) + r\left(f_e(x) + f_e(y-x) + \sum_{i=1}^{2} x_i \log z_i\right) + \sum_{i=1}^{2} y_i \log \pi_i \\
&\quad = (1-r)\left(f_e(y) + \sum_{i=1}^{2} y_i \log \pi_i\right) + r\left(f_e(x) + f_e(y-x) + \sum_{i=1}^{2} x_i \log z_i + \sum_{i=1}^{2} y_i \log \pi_i\right) \\
&\quad = (1-r)f_\pi(y) + r(f_{\pi z}(x) + f_\pi(y-x)) \le c.
\end{aligned}$$

Step 2. We prove that $c < 0$. The equalities in (2.3) mean that $|\pi| = |\pi z| = 1$. By Lemma 2.1, for all $(x, y) \in E$,

$$f_\pi(y) \le 0, \qquad f_{\pi z}(x) \le 0, \qquad f_\pi(y-x) \le 0, \tag{2.4}$$

and therefore $h(x, y) \le 0$. Hence $c \le 0$. Suppose $c = 0$. Then, by continuity, $h(x, y) = 0$ for some $(x, y) \in E$, and at that point all three of the inequalities in (2.4) become equalities. Since $|y| = 1$, by Lemma 2.1 we get $y = \pi$, $x = |x|\pi z$, and $y - x = |y - x|\pi = (1 - |x|)\pi$, which yields $|x|\pi z = x = |x|\pi$. We get a contradiction, because $\pi z \ne \pi$ and $(x, y) \in E$ implies $|x| \ge \gamma$.

Step 3. We prove that $\overline{\lim}_{n\to\infty} \left(\log \mathbb{E}C_n^r\right)/n < 0$. By the concavity of the function x^r and by the fact that $\eta_{k,n} = 0$ for $k > n$,

$$\begin{aligned}
\mathbb{E}C_n^r \le \mathbb{E}\left(\sum_{k\ge\gamma n} \eta_{k,n}\right)^r &= \mathbb{E}\mathbb{E}^{Q_n}\left(\sum_{k\ge\gamma n} \eta_{k,n}\right)^r \\
&\le \mathbb{E}\left(\mathbb{E}^{Q_n}\sum_{k\ge\gamma n} \eta_{k,n}\right)^r \\
&= \mathbb{E}\left(\sum_{k\ge\gamma n} \tilde{\eta}_{k,n}\right)^r \le \mathbb{E}\left(n \max_{k\ge\gamma n} \tilde{\eta}_{k,n}\right)^r = n^r \mathbb{E}\tilde{C}_n^r.
\end{aligned}$$

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Then, by the results of Steps 1 and 2,

$$\overline{\lim_{n\to\infty}} \frac{\log \mathbb{E}C_n^r}{n} \le \overline{\lim_{n\to\infty}} \frac{r\log n + \log \mathbb{E}\tilde{C}_n^r}{n} = \overline{\lim_{n\to\infty}} \frac{\log \mathbb{E}\tilde{C}_n^r}{n} < 0.$$

Step 4. By the Step 3 result, there exists a $t > 1$ such that $\sum_{n\ge 1} t^n \mathbb{E}C_n^r < \infty$. Then, by the well-known inequality $(\sum_i x_i)^r \le \sum_i x_i^r$, valid for any family (x_i) of non-negative numbers, we get

$$\mathbb{E}\left(\sum_{n\ge 1} C_n t^n\right)^r \le \mathbb{E}\sum_{n\ge 1} C_n^r t^{nr} \le \mathbb{E}\sum_{n\ge 1} C_n^r t^n < \infty.$$

Writing $C = \sum_{n\ge 1} C_n t^n$, then $\mathbb{E}C^r < \infty$. Writing $t = \mathrm{e}^{\delta}$, we get, for all $n \ge 1$ and $k \ge \gamma n$, $C_n \mathrm{e}^{\delta n} \le C$ and $\eta_{k,n} \le C_n \le C\mathrm{e}^{-\delta n}$. □

The assertion of the proposition can be reformulated as follows: for any $r \in (0; 1)$ and $u > 1$ there exists a $\delta > 0$ and a random variable C with $\mathbb{E}C^r < \infty$ such that $\eta_{k,n} \le C\mathrm{e}^{-\delta k}$ for all $k \ge 1$ and $n \le uk$. Indeed, if we write $\gamma = u^{-1}$ and find δ and C as in the theorem, then for $k \ge 1$ and $n \le uk$ we get $\eta_{k,n} \le C\mathrm{e}^{-\delta n} \le C\mathrm{e}^{-\delta k}$ (because $\eta_{k,n} = 0$ for $n < k$).

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is preceded by the following lemma. We believe that it is well known, but we could not find its proof in the literature, so we have added our own proof for completeness.

Lemma 3.1. *If ζ is a non-negative random variable with $\mathbb{E}\zeta < \infty$, and ζ_l is another random variable with $\mathcal{L}(\zeta_l) = \mathcal{L}(\zeta \mid \zeta \le l)$, then $l \log \mathbb{E}\mathrm{e}^{\zeta_l/l} \xrightarrow[l\to\infty]{} \mathbb{E}\zeta$.*

Proof. Since $\mathbb{P}(\zeta \le l) \to 1$, we get $\mathbb{E}\mathrm{e}^{\zeta_l/l} \sim \mathbb{E}\mathbf{1}_{\{\zeta\le l\}}\mathrm{e}^{\zeta/l}$ as $l \to \infty$. For all l, almost surely, $\mathbf{1}_{\{\zeta\le l\}}\mathrm{e}^{\zeta/l} \to 1$ and $\mathbf{1}_{\{\zeta\le l\}}\mathrm{e}^{\zeta/l} \le \mathrm{e}$; therefore $\mathbb{E}\mathbf{1}_{\{\zeta\le l\}}\mathrm{e}^{\zeta/l} \to 1$ as $l \to \infty$. Hence $\mathbb{E}\mathrm{e}^{\zeta_l/l} \to 1$ and

$$l \log \mathbb{E}\mathrm{e}^{\zeta_l/l} \sim l(\mathbb{E}\mathrm{e}^{\zeta_l/l} - 1) = \mathbb{E}l(\mathrm{e}^{\zeta_l/l} - 1) \sim \mathbb{E}l(\mathrm{e}^{\zeta/l} - 1)\mathbf{1}_{\{\zeta\le l\}}.$$

It remains to apply the dominated convergence theorem once again: almost surely,

$$l(\mathrm{e}^{\zeta/l} - 1)\mathbf{1}_{\{\zeta\le l\}} \to \zeta, \qquad 0 \le l(\mathrm{e}^{\zeta/l} - 1)\mathbf{1}_{\{\zeta\le l\}} \le \sup_{\theta\in(0;\,1)} (\zeta \mathrm{e}^{\theta\zeta/l})\mathbf{1}_{\{\zeta\le l\}} \le \mathrm{e}\zeta,$$

and $\mathbb{E}\mathrm{e}\zeta < \infty$; therefore $\mathbb{E}l(\mathrm{e}^{\zeta/l} - 1)\mathbf{1}_{\{\zeta\le l\}} \to \mathbb{E}\zeta$ as $l \to \infty$. □

Proof of Theorem 1.1. We first give an outline of the proof. Since $L^{r_2} \subset L^{r_1}$ for $0 < r_1 < r_2$, we need to prove the second assertion of the theorem only for $r < 1$ sufficiently close to 1. We fix $r \in (1/\alpha; 1)$, because in that case

$$\sum_{i\ge 1} a_i^r < \infty. \tag{3.1}$$

Indeed, the assumption of the theorem implies that $a_i = O(i^{-\alpha})$ (a very rough estimate, but sufficient for our purposes); then $a_i^r = O(i^{-\alpha r})$ and (3.1) follows, because $\alpha r > 1$. We will need inequality (3.1) at the end of the proof.

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The minimal solution to the IARCH equation is given by (1.5); we need to prove that it belongs to L^r. Of course, this is true if and only if $\sum_{k\geq 1}\sum_{i\in I^k}\xi(i)\in L^r$. For any fixed k,

$$\mathbb{E}\sum_{i\in I^k}\xi(i)=\sum_{i_1,\dots,i_k\in I}a_{i_1}\cdots a_{i_k}=\left(\sum_{i\geq 1}a_i\right)^k=1;$$

therefore $\sum_{i\in I^k}\xi(i)$ belongs to L^1 and it suffices to prove that $\sum_{k\geq k_0}\sum_{i\in I^k}\xi(i)\in L^r$ for some k_0. The idea of the proof is to cover the set $I^*=\bigcup_{k\geq k_0}I^k$ by a finite number of its subsets I^*_l and show that each partial sum $\sum_{i\in I^*_l}\xi(i)$ belongs either to L^r or to L^1. Different subsets I^*_l may intersect. Let us describe the appropriate cover.

Fix $q\in(1/\alpha;1)$ and an integer $s\geq 1$ with

$$\frac{s+1}{s\alpha}<q, \tag{3.2}$$

and let $k_0=s(s+1)$. For $k\geq k_0$ and $i=(i_1,\dots,i_k)\in I^k$ let $p(i)$ denote the number of indices j with $i_j>k^q$: $p(i)=|J(i)|$, where $J(i)=\{j\mid i_j>k^q\}$. Then the first set of our cover will be

$$\bigcup_{k\geq k_0}\{i\in I^k\mid p(i)\geq s\}. \tag{3.3}$$

Next, fix $u>\mathbb{E}\iota$ and set $\delta'=1/(s+1)$. If $p(i)=p<s$ then $J(i)=\{j_1,\dots,j_p\}$ with $j_1<\dots<j_p$. Write $j_0=0$, $j_{p+1}=k+1$, and, for $0\leq m\leq p$, $l_m(i)=\sum_{j_m<j<j_{m+1}}1$ and $n_m(i)=\sum_{j_m<j<j_{m+1}}i_j$. Of course, $n_m(i)\leq k^q l_m(i)$ for all m, and $\sum_{m=0}^p l_m(i)=k-p$. The latter equality implies that, for $k\geq k_0$ and some $0\leq m\leq p$,

$$l_m(i)\geq\frac{k-p}{p+1}>\frac{k}{s}-1\geq\delta' k.$$

Let $m(i)$ be the first such m. Now, set $I_k(m,p)=\{i\in I^k\mid p(i)=p,\ m(i)=m\}$ and

$$I'_k(m,p)=\{i\in I_k(m,p)\mid n_m(i)>2ul_m(i)\},$$

$$I''_k(m,p)=\{i\in I_k(m,p)\mid n_m(i)\leq 2ul_m(i)\}.$$

Then our cover also contains the sets

$$\bigcup_{k\geq k_0}I'_k(m,p),\qquad 0\leq m\leq p<s, \tag{3.4}$$

$$\bigcup_{k\geq k_0}I''_k(m,p),\qquad 0\leq m\leq p<s. \tag{3.5}$$

Below we prove that the sums with index sets (3.3) and (3.4) belong to L^1, while the sums with index sets (3.5) belong to L^r.

Step 1. We first prove that $\sum_{k\geq k_0}\sum_{i\in I^k,p(i)\geq s}\mathbb{E}\xi(i)<\infty$. For any fixed k, the inner sum equals the probability that at least s random variables from $\iota_1,\dots,\iota_k$ are greater than k^q. That probability does not exceed

$$\binom{k}{s}(\mathbb{P}(\iota>k^q))^s=O(k^s)O(k^{-q\alpha s})\quad\text{as }k\to\infty.$$

Moreover, by (3.2), $q\alpha s>1+s$. Hence the sequence of inner sums is summable.

Step 2. We next prove that, for any $p < s$ and $0 \le m \le p$, $\sum_{k \ge k_0} \sum_{i \in I'_k(m,\, p)} \mathbb{E}\xi(i) < \infty$. For $k \ge k_0$, $p < s$, and $0 \le m \le p$ write

$$T'_k(m, p) = \sum_{i \in I'_k(m,p)} \xi(i), \qquad T''_k(m, p) = \sum_{i \in I''_k(m,p)} \xi(i).$$

Then

$$T'_k(m, p) = \sum_{\substack{0 \le l_0, \dots, l_{m-1} \le \delta' k < l_m \\ l_{m+1}, \dots, l_p \ge 0 \\ l_0 + \cdots + l_p = k - p}} \sum_{\substack{n_0, \dots, n_p \ge 0 \\ n_m > 2u l_m}} \sum_{j_1, \dots, j_p > k^q} \tilde{\eta}_{k, l_0, n_0} a_{j_1} \varepsilon_{n_0 + j_1} \tilde{\eta}^{(n_0 + j_1)}_{k, l_1, n_1} \times \cdots$$

$$\cdots \times a_{j_p} \varepsilon_{n_0 + j_1 + \cdots + n_{p-1} + j_p} \tilde{\eta}^{(n_0 + j_1 + \cdots + n_{p-1} + j_p)}_{k, l_p, n_p},$$

where, for $l, n \ge 0$,

$$\tilde{\eta}_{k,l,n} = \sum_{\substack{i \in I^l \\ i_1, \dots, i_l \le k^q \\ i_1 + \cdots + i_l = n}} \xi(i)$$

and we use the notation $Y^{(j)}$ described in the introduction. Note also that in the case $l = 0$ we assume the existence of the unique 'empty family' $(i_1, \dots, i_l)$ with $\sum_{j=1}^{l} i_j = 0$ and $\xi(i_1, \dots, i_l) = 1$; therefore $\tilde{\eta}_{k,0,0} = 1$ and $\tilde{\eta}_{k,0,n} = 0$ for $n > 0$. Since $Y^{(j)}$ is equidistributed with Y and independent of $(\varepsilon_i \mid i \le j)$, we get

$$\mathbb{E}T'_k(m, p)$$
$$\le \sum_{\substack{0 \le l_0, \dots, l_{m-1} \le \delta' k < l_m \\ l_{m+1}, \dots, l_p \ge 0 \\ l_0 + \cdots + l_p = k - p}} \sum_{\substack{n_0, \dots, n_p \ge 0 \\ n_m > 2u l_m}} \sum_{j_1, \dots, j_p > k^q} \mathbb{E}\tilde{\eta}_{k, l_0, n_0} a_{j_1} \mathbb{E}\tilde{\eta}_{k, l_1, n_1} \cdots a_{j_p} \mathbb{E}\tilde{\eta}_{k, l_p, n_p}.$$

Let $\tilde{\iota}_k$ be the random variable with $\mathcal{L}(\tilde{\iota}_k) = \mathcal{L}(\iota \mid \iota \le k^q)$, $(\tilde{\iota}_{k,j})$ a sequence of independent copies of $\tilde{\iota}_k$, and $\tilde{S}_{k,l} = \tilde{\iota}_{k,1} + \cdots + \tilde{\iota}_{k,l}$. Then

$$\mathbb{P}(\tilde{S}_{k,l} > 2ul) = \mathbb{P}\big(\mathrm{e}^{\tilde{S}_{k,l}/k^q} > \mathrm{e}^{2ul/k^q}\big) \le \mathrm{e}^{-2ul/k^q} \mathbb{E}\mathrm{e}^{\tilde{S}_{k,l}/k^q} = \mathrm{e}^{l\Lambda_k},$$

where $\Lambda_k = \log \mathbb{E}\mathrm{e}^{\tilde{\iota}_k/k^q} - 2u/k^q$. By Lemma 3.1, $k^q \log \mathbb{E}\mathrm{e}^{\tilde{\iota}_k/k^q} \to \mathbb{E}\iota < u$ as $k \to \infty$. Therefore, if k is large enough, say $k \ge k'_0 \ge k_0$, then $\Lambda_k \le u/k^q - 2u/k^q = -u/k^q$ and hence, for all l, $\mathbb{P}(\tilde{S}_{k,l} > 2ul) \le \mathrm{e}^{-ul/k^q}$. This yields, for $k \ge k'_0$,

$$\sum_{n_m > 2ul_m} \mathbb{E}\tilde{\eta}_{k, l_m, n_m} \le \mathbb{P}\big(\iota_1, \dots, \iota_{l_m} \le k^q,\ \iota_1 + \cdots + \iota_{l_m} > 2ul_m\big)$$
$$\le \mathbb{P}(\tilde{S}_{k, l_m} > 2ul_m) \le \mathrm{e}^{-ul_m/k^q} \le \mathrm{e}^{-u\delta' k^{1-q}}$$

(the last inequality holds for $l_m > \delta' k$). Since $\sum_{n \ge 0} \mathbb{E}\tilde{\eta}_{k,l,n} \le 1$ for all l, we get (for $k \ge k'_0$)

$$\mathbb{E}T'_k(m, p) \le \left(\sum_{l=0}^{k} 1 \right)^p \left(\sum_{j > k^q} a_j \right)^p \mathrm{e}^{-u\delta' k^{1-q}} = O\big(k^{p(1-q\alpha)}\big) \mathrm{e}^{-u\delta' k^{1-q}},$$

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which means that $\sum_{k\geq k_0'}\mathbb{E}T_k'(m,p)<\infty$. Of course, then $\sum_{k\geq k_0}\mathbb{E}T_k'(m,p)<\infty$ as well, because $\mathbb{E}T_k'(m,p)\leq 1$ for all k.

Step 3. We now prove that, for any $p<s$ and $0\leq m\leq p$, $\sum_{k\geq k_0}\sum_{i\in I_k''(m,\,p)}\xi(i)$ belongs to L^r. Since $\big(\sum_k x_k\big)^r\leq\sum_k x_k^r$ for any countable family of non-negative numbers (x_k), it suffices to show that $(\mathbb{E}T_k''(m,p)^r)$ is summable, where $T_k''(m,p)$ is defined in Step 2. Similarly to before,

$$T_k''(m,p)\leq\sum_{\substack{0\leq l_0,\dots,l_{m-1}\leq\delta'k<l_m\\ l_{m+1},\dots,l_p\geq 0\\ l_0+\cdots+l_p=k-p}}\ \sum_{\substack{n_0,\dots,n_p\geq 0\\ n_m\leq 2ul_m}}\ \sum_{j_1,\dots,j_p>k^q}\tilde\eta_{k,l_0,n_0}a_{j_1}\varepsilon_{n_0+j_1}\tilde\eta_{k,l_1,n_1}^{(n_0+j_1)}\times\cdots$$
$$\cdots\times a_{j_p}\varepsilon_{n_0+j_1+\cdots+n_{p-1}+j_p}\tilde\eta_{k,l_p,n_p}^{(n_0+j_1+\cdots+n_{p-1}+j_p)};$$

therefore $T_k''(m,p)^r$ does not exceed the same sum of the rth powers of variables. Since all the multipliers are independent and $Y^{(j)}$ is equidistributed with Y, we get that $\mathbb{E}T_k''(m,p)^r$ does not exceed

$$\sum_{\substack{0\leq l_0,\dots,l_{m-1}\leq\delta'k<l_m\\ l_{m+1},\dots,l_p\geq 0\\ l_0+\cdots+l_p=k-p}}\ \sum_{\substack{n_0,\dots,n_p\geq 0\\ n_m\leq 2ul_m}}\ \sum_{j_1,\dots,j_p>k^q}\mathbb{E}\tilde\eta^r_{k,l_0,n_0}a^r_{j_1}\mathbb{E}\varepsilon^r\mathbb{E}\tilde\eta^r_{k,l_1,n_1}\cdots a^r_{j_p}\mathbb{E}\varepsilon^r\mathbb{E}\tilde\eta^r_{k,l_p,n_p}.$$

Note that $\tilde\eta_{k,l,n}=0$ for $n\notin[l;k^ql]$, and $\mathbb{E}\tilde\eta^r_{k,l,n}\leq(\mathbb{E}\tilde\eta_{k,l,n})^r\leq 1$ for all k, l, n. Moreover, $\mathbb{E}\varepsilon^r\leq(\mathbb{E}\varepsilon)^r\leq 1$. Next, we use Proposition 2.1 and find a random variable C with $\mathbb{E}C^r<\infty$ and δ such that $\eta_{l,n}\leq C\mathrm{e}^{-\delta n}$ for all $l\geq 1$ and $n\leq 2ul$. Then

$$\begin{aligned}\sum_{\substack{l_m>\delta'k\\ n_m\leq 2ul_m}}\mathbb{E}\tilde\eta^r_{k,l_m,n_m}&\leq\sum_{\substack{l_m>\delta'k\\ l_m\leq n_m\leq 2ul_m}}\mathbb{E}\eta^r_{l_m,n_m}\\ &\leq\sum_{\substack{l_m>\delta'k\\ l_m\leq n_m\leq 2ul_m}}\mathbb{E}(C\mathrm{e}^{-\delta n_m})^r\\ &\leq\mathbb{E}C^r\sum_{\substack{l_m>\delta'k\\ l_m\leq n_m\leq 2ul_m}}\mathrm{e}^{-\delta rn_m}\\ &\leq 2uk\mathbb{E}C^r\sum_{l>\delta'k}\mathrm{e}^{-\delta rl}\leq\frac{2uk\mathbb{E}C^r}{1-\mathrm{e}^{-\delta r}}\mathrm{e}^{-\delta\delta'rk}.\end{aligned}$$

Hence

$$\mathbb{E}T_k''(m,p)^r\leq\left(\sum_{j>k^q}a_j^r\sum_{l=0}^k\sum_{n\leq k^ql}1\right)^p\frac{2uk\mathbb{E}C^r}{1-\mathrm{e}^{-\delta r}}\mathrm{e}^{-\delta\delta'rk}=O(k^{(q+2)p+1})\mathrm{e}^{-\delta\delta'rk},$$

which means that $\sum_{k\geq k_0}\mathbb{E}T_k''(m,p)^r<\infty$. □

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