LMD mokslo darbai, III tomas, 24–30
© Matematikos ir informatikos institutas, 1999

On zeros of the Lerch zeta-function. III

Ramūnas GARUNKŠTIS* (VU)
e-mail:ramunas.garunkstis@maf.vu.lt

1. Introduction

Let $s = \sigma + it$ be a complex variable. The Lerch zeta-function $L(\lambda, \alpha, s)$, for $\sigma > 1$, is defined by the following Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

where λ, α are real numbers, $0 < \alpha \leqslant 1$, and by analytic continuation othervise (see [5], [6]). Further we suppose that $0 < \lambda < 1$.

In [4] A. Laurinčikas for the function

$$Z(s, \lambda) = \sum_{m=1}^{\infty} e^{2\pi i \lambda m} m^{-s} = e^{2\pi i \lambda} L(\lambda, 1, s), \qquad \sigma > 1,$$

$\lambda = a/q$, $(a, q) = 1$, $0 < a < q$, obtained the following zero-distribution rezults.

Theorem A. *Suppose that q is a prime number. Then there exists a constant $c = c(\lambda)$ such that for sufficiently large T the function $Z(s, \lambda)$ has more than cT zeros in the region*

$$\sigma > 1, \quad |t| < T.$$

Theorem B. *Suppose there exist at least two primitive characters modulo q. Then for any σ_1, σ_2, $1/2 < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\lambda, \sigma_1, \sigma_2) > 0$ such that for sufficiently large T the function $Z(s, \lambda)$ has more than cT zeros in the region*

$$\sigma_1 < \sigma < \sigma_2, \quad |t| < T.$$

By $A_T(\lambda, \alpha; a, b)$ we will denote the following assertion: For any σ_1, σ_2, $a < \sigma_1 < \sigma_2 < b$, there exists a constant $c = c(\lambda, \alpha, \sigma_1, \sigma_2) > 0$ such that for sufficiently large T the function $L(\lambda, \alpha, s)$ has more than cT zeros in the rectangle

$$\sigma_1 < \sigma < \sigma_2, \qquad |t| < T.$$

*Partially supported by Lithuanian State Studies and Science Foundation.

In [1] for the Lerch zeta function the following results were obtained.

Theorem C. *Let α be a nonrational number. Then there exists $\delta = \delta(\lambda, \alpha)$, $0 < \delta < \alpha$, such that the assertion $A_T(\lambda, \alpha; 1, 1+\delta)$ is true.*

If α is a transcendental number, then we can take $\delta = 0.6\alpha$.

Theorem D. *Let α be a transcendental number. Then the assertion $A_T(\lambda, \alpha; 1/2, 1)$ is true.*

Let $N(\lambda, \alpha, \sigma, T)$ denote the number of zeros of $L(\lambda, \alpha, s)$ in the region $\{s|\operatorname{Re} s > \sigma, 0 < \operatorname{Im} s \leqslant T\}$. In [2] it is proved that

$$L(\lambda, \alpha, s) \neq 0, \qquad \text{for} \quad \sigma \geqslant 1+\alpha. \tag{1}$$

In this note we investigate the uper bounds for the number of zeros of the Lerch zeta function. Let B_η denote a number bounded by a constant depending on η.

Theorem 1. *Let $1/2 \leqslant \sigma_0 \leqslant 1+\alpha$, then*

$$\int_{\sigma_0}^{1+\alpha} N(\lambda, \alpha; \sigma, T)\, \mathrm{d}\sigma = \sigma_0 T \log\alpha + \int_0^T \log\left|L(\lambda, \alpha, \sigma_0 + it)\right| \mathrm{d}t + B\log T, \quad T \to \infty.$$

By $\zeta(s, \alpha)$ we denote the Hurwitz zeta-function, i.e.

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \qquad \sigma > 1,$$

where $0 < \alpha \leqslant 1$.

Theorem 2. *Let $\sigma > 1/2$, then for any fixed $1/2 < \sigma_1 < \sigma$ and $T \to \infty$, we have*

$$N(\lambda, \alpha; \sigma, T) \leqslant \frac{\log(\alpha^{2\sigma_1}\zeta(2\sigma_1, \alpha))}{2(\sigma - \sigma_1)} T + R(\sigma_1, T),$$

where

$$R(\sigma_1, T) = \begin{cases} B_{\sigma_1} T^{2-2\sigma_1} & \text{for} \quad \dfrac{1}{2} < \sigma_1 < 1, \\ B_{\sigma_1} \log T & \text{for} \quad \sigma_1 \geqslant 1. \end{cases}$$

2. Lemmas

We will use a lemma of Littlewood [7, §9.9]. Let $\varphi(s)$ is a meromorphic function on the rectangle D with vertices $\alpha+i0$, $\beta+i0$, $\beta+iT$, $\alpha+iT$, and let $\varphi(s)$ be regular and nonvanishing on the line $\sigma=\beta$. Then $\varphi(s)$ be regular in some neighbourhood of the line $T=\beta$. In this neighbourhood we define a function $F(s)=\log\varphi(s)$, by choosing some branch of the $\log\varphi(s)$. On other points of the rectangle we define $F(s)$ by continuation of the $\log(\beta+it)$ left from $\beta+it$ to $\sigma+it$. If a zero is reached, we use

$$F(s)=\lim_{\varepsilon\to+0}F(\sigma+it+i\varepsilon).$$

Let $\nu(\sigma',T)$ mean a difference between a number of zeros and a number of poles of $\varphi(s)$ in a rectangle $\sigma'<\sigma\leqslant\beta$, $0<t\leqslant T$. Then we have the following lemma.

Lemma 1.

$$\int F(s)\,\mathrm{d}s=-2\pi i\int_{\alpha}^{\beta}\nu(\sigma,T)\,\mathrm{d}\sigma$$

where the integral on the left we take around the contour of D.

Lemma 2. *For any σ_0, $\sigma>\sigma_0$, we have*

$$L(\lambda,\alpha,s)=B_\lambda|t|^k.$$

where $k=k(\sigma_0)$.

Lemma is proved in [2].

Lemma 3. *Let $\sigma>1/2$. Then, for $T\to\infty$,*

$$\int_0^T|L(\lambda,\alpha,s)|^2\,\mathrm{d}t=\zeta(2\sigma,\alpha)T+r(\sigma,T),$$

where

$$r(\sigma,T)=\begin{cases}B_\sigma T^{2-2\sigma}, & \text{for } \dfrac{1}{2}<\sigma<1,\\ B_\sigma\log T, & \text{for } \sigma=1,\\ B_\sigma, & \text{for } \sigma>1.\end{cases}$$

For the proof see [4].

3. Proofs of theorems

Proof of Theorem 1. Let $\sigma_1 > 1+\alpha$ be arbitrary large number and be not an ordinate of the zero of the $L(\lambda,\alpha,s)$. Then from Lemma 1 and (1) we have for $1/2 \leqslant \sigma_0 \leqslant 1+\alpha$ that

$$2\pi \int_{\sigma_0}^{1+\alpha} N(\sigma,T;\lambda,\alpha)\,\mathrm{d}\sigma = \int_0^T \log\left|L(\lambda,\alpha,\sigma_0+it)\right|\,\mathrm{d}t$$
$$-\int_0^T \log\left|L(\lambda,\alpha,\sigma_1+it)\right|\,\mathrm{d}t + \int_{\sigma_0}^{\sigma_1} \arg L(\sigma+iT)\,\mathrm{d}\sigma + K(\sigma_0,\sigma_1)$$
$$= I_1 + I_2 + I_3 + K(\sigma_0,\sigma_1)$$

where $K(\sigma_0,\sigma_1)$ does not depend on T.

First we evaluate the integral I_2.

$$I_2 = \int_0^T \log\left|\alpha^{-\sigma_1-it}\left(1+\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(\frac{m+\alpha}{\alpha})^{\sigma_1+it}}\right)\right|\,\mathrm{d}t$$
$$= -\sigma_1 T\log\alpha + \int_0^T \log\left|1+\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(\frac{m+\alpha}{\alpha})^{\sigma_1+it}}\right|\,\mathrm{d}t = I_{21}+I_{22}. \tag{2}$$

It is clear, that there exists $\sigma' > 1+\alpha$, such that a modulo of the sum in I_{22} is less than 1 for $\sigma_1 > \sigma'$ and $t \in \mathbb{R}$. For such σ_1 by the Maclaurin formula we obtain

$$I_{22} = \int_0^T \operatorname{Re}\left(\sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}}{n}\left(\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(\frac{m+\alpha}{\alpha})^{\sigma_1+it}}\right)^n\right]\right)\mathrm{d}t$$
$$= \operatorname{Re}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{e^{2\pi i\lambda(m_1+m_2+\ldots+m_n)}}{\left(\frac{(m_1+\alpha)(m_2+\alpha)\ldots(m_n+\alpha)}{\alpha^n}\right)^{\sigma_1}}$$
$$\times\int_0^T\left(\frac{\alpha}{(m_1+\alpha)(m_2+\alpha)\ldots(m_n+\alpha)}\right)^{it}\mathrm{d}t$$
$$= B\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty}\left(\frac{\alpha}{m+\alpha}\right)^{\sigma_1}\right)^n. \tag{3}$$

We can choose σ_1 big enough such that

$$\sum_{m=1}^{\infty}\left(\frac{\alpha}{m+\alpha}\right)^{\sigma_1} < 1.$$

From this and (3) we have that

$$I_{22} = B,$$

and from (2)

$$I_2 = -\sigma_1 T \log \alpha + B.$$

Now it remains to estimate the I_3. We define two functions

$$\Phi(s) = \mathrm{e}^{iT \log \alpha} L(\lambda, \alpha, s), \qquad \Phi_1(s) = \mathrm{e}^{iT \log \alpha} L(1-\lambda, \alpha, s)$$

Then

$$I_3 = \int_{\sigma_0}^{\sigma_1} \arg \Phi(\sigma + it)\, \mathrm{d}\sigma - (\sigma_1 - \sigma_0) T \log \alpha. \tag{4}$$

It is easily seen that the leading terms of the Dirichlet series for $\Phi(s)$ and $\Phi_1(s)$ are positive at $s = \sigma_1 + iT$. Denote by q the number of zeros of $\operatorname{Re}\Phi(s)$ on the interval $J = (\sigma_0 + iT, \sigma_1 + iT)$, and divide J into at most $q+1$ subintervals in each of which $\operatorname{Re}\Phi(s)$ is of constant sign. Then the variation of $\arg \Phi(s)$ does not exceed π in each subinterval, and we obtain

$$\left| \arg \Phi(s) \Big|_{\sigma_1 + iT}^{\sigma_0 + iT} \right| \leqslant (q+1)\pi. \tag{5}$$

To estimate q we set

$$f(z) = \frac{1}{2}\left(\Phi(z + iT) + \overline{\Phi_1(\overline{z} + iT)}\right).$$

First we note that $f(z)$ is an entire function, and if $z = \sigma$ is real, then

$$f(\sigma) = \operatorname{Re}\Phi(\sigma + iT). \tag{6}$$

Let $n(\varrho)$ stand for the number of zeros of $f(z)$ in the disc $|z - \sigma_1| \leqslant \varrho$, and let $r = 2(\sigma_1 - \sigma_0)$, $r_1 = r/2$. Then, clearly,

$$\int_0^r \frac{n(\varrho)}{\varrho}\, \mathrm{d}\varrho \geqslant n(r_1) \int_{r_1}^r \frac{\mathrm{d}s}{\varrho} = n(r_1) \log 2,$$

and the well-known Jensen theorem yield

$$n(r_1) \leqslant \frac{1}{2\pi \log 2} \int_0^{2\pi} \log \left| f(r\mathrm{e}^{i\theta} + \sigma_1) \right| \mathrm{d}\theta - \frac{1}{\log 2} \log \left| f(\sigma_1) \right|. \tag{7}$$

By (6)

$$f(\sigma_1) = \mathrm{Re}\left(\frac{1}{\alpha^{\sigma_1}} + \frac{1}{\alpha^{\sigma_1}} \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m}}{(\frac{\alpha+m}{\alpha})^{\sigma_1+iT}}\right) \geqslant \frac{1}{\alpha^{\sigma_1}} - \frac{1}{\alpha^{\sigma_1}} \sum_{m=1}^{\infty} \frac{1}{(\frac{\alpha+m}{\alpha})^{\sigma_1}}.$$

For sufficiently large σ_1 this is $\geqslant 1/(2\alpha)^{\sigma_1}$, say. Hence and from (7), using Lemma 2, we obtain that

$$n(r_1) = B \log T. \tag{8}$$

By (6), the number of zeros of $\mathrm{Re}\,\Phi(s)$ on J is equal to the same number of zeros of $f(z)$ on (σ_0, σ_1). By the definition (σ_0, σ_1) is contained in the disc $|z - \sigma_1| \leqslant r_1$. This, (8), (5) and (4) show that

$$I_3 = -(\sigma_1 - \sigma_0)T \log \alpha + B \log T.$$

The theorem is proved for T is not an ordinate of the zero of the $L(\lambda, \alpha, s)$. For others T theorem is true in view of a continuity.

Proof of Theorem 2. Using the concavity of the logarithm, from Lemma 3, we have

$$\int_0^T \log\left|L(\lambda, \alpha, \sigma + it)\right| \,\mathrm{d}t = \frac{1}{2}\int_0^T \log\left|L(\lambda, \alpha, \sigma + it)\right|^2 \,\mathrm{d}t$$

$$\leqslant \frac{1}{2}T \log\left(\frac{1}{T}\int_0^T \left|L(\lambda, \alpha, \sigma + it)\right|^2 \,\mathrm{d}t\right) = \frac{1}{2}T \log\left(\zeta(2\sigma, \alpha) + \frac{r(\sigma, T)}{T}\right).$$

Then in view of Theorem 1 we obtain

$$N(\lambda, \alpha; \sigma, T) \leqslant \frac{1}{\sigma - \sigma_1}\int_{\sigma_1}^{\sigma} N(\lambda, \alpha; \sigma, T) \,\mathrm{d}\sigma \leqslant \frac{1}{\sigma - \sigma_1}\int_{\sigma_1}^{1+\alpha} N(\lambda, \alpha; \sigma, T) \,\mathrm{d}\sigma$$

$$= \frac{1}{\sigma - \sigma_1}\left(\frac{T}{2}\log\left(\zeta(2\sigma_1, \alpha) + \frac{r(\sigma_1, T)}{T}\right) + \sigma_1 T \log \alpha + B \log T\right).$$

From this the theorem follows.

References

[1] R. Garunkštis, On zeros of the Lerch zeta-function. II. *Preprint 98–23*.

[2] R. Garunkštis and A. Laurinčikas, On zeros of the Lerch zeta function, in: *Proceedings of the Conf. "Number Theory and its Applications"*, Kyoto (to appear (1999)).

[3] A. Laurinčikas, On zeros of some Dirichlet series, *Liet. Matem. Rink.*, **24**(40), 116–126 (1984) (in Russian).

[4] A. Laurinčikas, On the mean square of the Lerch zeta function, *LMD mokslo darbai: Specialus "Liet. Matem. Rink." priedas*, III tomas, (spaudoje).
[5] M. Lerch, Note sur la fonction $K(w, x, s) = \sum_{n\geqslant 0} \exp\{2\pi i x\}(n \cdot w)^{-s}$, *Acta Math.*, **11**, 19–24 (1987).
[6] M. Mikolás, New proof and extention of the functional equality of Lerch's zeta-function, *Ann. Univ. Sci. Budapest. Sec. Math.*, **14**, 111–116 (1971).
[7] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford (1951).

Apie Lercho dzeta funkcijos nulius. III

R. Garunkštis

Straipsnyje nagrinejami Lercho dzeta funkcijos nulių skaičiaus įverčiai iš viršaus.