

Discrete limit theorems for trigonometric polynomials

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1. Įvadas

Let, as usual, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all real numbers and complex numbers, respectively, and let

$$p_n(t) = \sum_{j \leq n} a_j e^{i\lambda_j t}, \quad a_j \in \mathbb{C}, \quad \lambda_j \in \mathbb{R},$$

be a trigonometrical polynomial. It is known [1], that the polynomial $p_n(t)$ has a limit distribution law, i.e., the probability measure

$$\frac{1}{T} \text{meas} \{t \in [0, T], p_n(t) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}), \quad (1)$$

converges weakly to some measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \rightarrow \infty$. Here $\text{meas}\{A\}$ is the Lebesgue measure of the set A , and $\mathcal{B}(S)$ stands for the class of Borel sets of the space S .

Instead of the measure (1) we may consider its discrete version. Let $h > 0$ be a fixed number. Define on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ the probability measure

$$P_N(A) = \frac{1}{N+1} \#(0 \leq m \leq N : p_n(mh) \in A)$$

and consider its weak convergence as $N \rightarrow \infty$. Here $\#A$ denotes the number of elements of the set A .

Theorem 1. *On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P such that the measure P_N converges weakly to P as $N \rightarrow \infty$.*

The latter theorem is a discrete version of a limit theorem for the measure (1).

Proof of Theorem 1. Clearly, instead of the measure P_N we can consider the measure

$$Q_N(A) = \frac{1}{N+1} \#(0 \leq m \leq N : (\text{Re } p_n(mh), \text{Im } p_n(mh)) \in A), \quad A \in \mathcal{B}(\mathbb{R}^2).$$

To prove the weak convergence of this measure it suffices to study the asymptotic behaviour of its characteristic function

$$\begin{aligned}\varphi_N(\tau_1, \tau_2) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\tau_1 x_1 + i\tau_2 x_2} dQ_N \\ &= \frac{1}{N+1} \sum_{m=0}^N e^{i\tau_1 \operatorname{Re} p_n(mh) + i\tau_2 \operatorname{Im} p_n(mh)}.\end{aligned}\quad (2)$$

It is well known that

$$\operatorname{Re} p_n(t) = \sum_{j=1}^n |a_j| \cos(\lambda_j t + \eta_j),$$

$$\operatorname{Im} p_n(t) = \sum_{j=1}^n |a_j| \sin(\lambda_j t + \eta_j).$$

Here $\eta_j = \arg a_j$. Let $J_m(x)$ stand for the Bessel functions. Since [1]

$$e^{ix \sin \theta} = \sum_{m=-\infty}^{\infty} J_m(x) e^{im\theta},$$

$$e^{ix \cos \theta} = \sum_{m=-\infty}^{\infty} i^m J_m(x) e^{im\theta},$$

we have that

$$\begin{aligned}e^{i\tau_1 \operatorname{Re} p_n(mh)} &= \sum_{k_1, \dots, k_n = -\infty}^{\infty} J_{k_1}(|a_1|\tau_1) \dots J_{k_n}(|a_n|\tau_1) \\ &\quad \times \exp \left\{ i \left(\frac{\pi}{2} \sum_{j=1}^n k_j + mh \sum_{j=1}^n k_j \lambda_j + \sum_{j=1}^n k_j \eta_j \right) \right\} \\ &= \sum_{k_1, \dots, k_n = -\infty}^{\infty} J_{k_1}(|a_1|\tau_1) \dots J_{k_n}(|a_n|\tau_1) \\ &\quad \times \exp \left\{ i \left(\sum_{j=1}^n k_j \left(\frac{\pi}{2} + \eta_j \right) + mh \sum_{j=1}^n k_j \lambda_j \right) \right\},\end{aligned}\quad (3)$$

$$\begin{aligned}e^{i\tau_2 \operatorname{Im} p_n(mh)} &= \sum_{l_1, \dots, l_n = -\infty}^{\infty} J_{l_1}(|a_1|\tau_2) \dots J_{l_n}(|a_n|\tau_2) \\ &\quad \times \exp \left\{ i \left(\sum_{j=1}^n l_j \eta_j + mh \sum_{j=1}^n l_j \lambda_j \right) \right\}.\end{aligned}\quad (4)$$

Let k be an integer, and let

$$\begin{aligned} \varphi(\tau_1, \tau_2) = & \sum_{\substack{k_1, \dots, k_n = -\infty \\ (k_1 + l_1)\lambda_1 + \dots + (k_n + l_n)\lambda_n = \frac{2k\pi}{h}}}^{\infty} \sum_{\substack{l_1, \dots, l_n = -\infty \\ (k_1 + l_1)\lambda_1 + \dots + (k_n + l_n)\lambda_n = \frac{2k\pi}{h}}}^{\infty} J_{k_1}(|a_1|\tau_1) \dots J_{k_n}(|a_n|\tau_1) \\ & \times J_{l_1}(|a_1|\tau_2) \dots J_{l_n}(|a_n|\tau_2) \\ & \times \exp \left\{ i \left(\frac{\pi}{2} \sum_{j=1}^n k_j + \sum_{j=1}^n (k_j + l_j) \eta_j \right) \right\}. \end{aligned} \quad (5)$$

Then we find from (2)–(4) that

$$\begin{aligned} \varphi_N(\tau_1, \tau_2) = & \varphi_n(\tau_1, \tau_2) \\ & + \sum_{\substack{k_1, \dots, k_n = -\infty \\ (k_1 + l_1)\lambda_1 + \dots + (k_n + l_n)\lambda_n \neq \frac{2k\pi}{h}}}^{\infty} \sum_{\substack{l_1, \dots, l_n = -\infty \\ (k_1 + l_1)\lambda_1 + \dots + (k_n + l_n)\lambda_n \neq \frac{2k\pi}{h}}}^{\infty} J_{k_1}(|a_1|\tau_1) \dots J_{k_n}(|a_n|\tau_1) \\ & \times J_{l_1}(|a_1|\tau_2) \dots J_{l_n}(|a_n|\tau_2) \exp \left\{ i \left(\frac{\pi}{2} \sum_{j=1}^n k_j + \sum_{j=1}^n (k_j + l_j) \eta_j \right) \right\} \\ & \times \frac{1}{N+1} \sum_{m=0}^N \exp \left\{ imh \sum_{j=1}^n (k_j + l_j) \lambda_j \right\}. \end{aligned} \quad (6)$$

Obviously, if $(k_1 + l_1)\lambda_1 + \dots + (k_n + l_n)\lambda_n \neq \frac{2\pi k}{h}$,

$$\sum_{m=0}^N \exp \left\{ imh \sum_{j=1}^n (k_j + l_j) \lambda_j \right\} = \frac{1 - \exp \left\{ i(N+1)h \sum_{j=1}^n (k_j + l_j) \lambda_j \right\}}{1 - \exp \left\{ ih \sum_{j=1}^n (k_j + l_j) \lambda_j \right\}}.$$

For the Bessel functions the estimate

$$J_k(z) = \frac{Bc_1^{|k|}}{|k|!}, \quad |z| \leq c_1,$$

is valid. Here $c_1, c_2, \dots$ are positive constants. Thus, taking an arbitrary $\varepsilon > 0$, we can find $K = K(\varepsilon)$ such that

$$\begin{aligned} & \left| \sum_{\substack{k_1, \dots, k_n = -\infty \\ (|k_1| + |l_1| + \dots + |k_n| + |l_n|) > K(\varepsilon)}}^{\infty} \sum_{\substack{l_1, \dots, l_n = -\infty \\ (|k_1| + |l_1| + \dots + |k_n| + |l_n|) > K(\varepsilon)}}^{\infty} J_{k_1}(|a_1|\tau_1) \dots J_{k_n}(|a_n|\tau_1) \right. \\ & \quad \left. \times J_{l_1}(|a_1|\tau_2) \dots J_{l_n}(|a_n|\tau_2) \right| \end{aligned}$$

$$\begin{aligned}
& \times \exp \left\{ i \left(\frac{\pi}{2} \sum_{j=1}^n k_j + \sum_{j=1}^n (k_j + l_j) \eta_j \right) \right\} \\
& \times \frac{1}{N+1} \frac{1 - \exp \left\{ i(N+1)h \sum_{j=1}^n (k_j + l_j) \lambda_j \right\}}{1 - \exp \left\{ ih \sum_{j=1}^n (k_j + l_j) \lambda_j \right\}} \Bigg| < \frac{\varepsilon}{2}
\end{aligned} \quad (7)$$

for all $|\tau_1| \leq c_2, |\tau_2| \leq c_3$. Now we choose $N_0 = N_0(\varepsilon)$ such that for the remainder part of the sum in the equality (5) the inequality

$$\begin{aligned}
& \left| \sum_{\substack{k_1, \dots, k_n = -\infty \\ (|k_1| + |l_1| + \dots + |k_n| + |l_n|) \leq K(\varepsilon)}} \sum_{l_1, \dots, l_n = -\infty}^{\infty} J_{k_1}(|a_1|\tau_1) \dots J_{k_n}(|a_n|\tau_1) \right. \\
& \quad \times J_{l_1}(|a_1|\tau_2) \dots J_{l_n}(|a_n|\tau_2) \\
& \quad \times \exp \left\{ i \left(\frac{\pi}{2} \sum_{j=1}^n k_j + \sum_{j=1}^n (k_j + l_j) \eta_j \right) \right\} \\
& \quad \times \frac{1}{N+1} \frac{1 - \exp \left\{ i(N+1)h \sum_{j=1}^n (k_j + l_j) \lambda_j \right\}}{1 - \exp \left\{ ih \sum_{j=1}^n (k_j + l_j) \lambda_j \right\}} \Bigg| < \frac{\varepsilon}{2}
\end{aligned} \quad (8)$$

should be satisfied for $N \geq N_0$. Consequently, we obtained that $\varphi_N(\tau_1, \tau_2)$ converges to $\varphi(\tau_1, \tau_2)$ as $N \rightarrow \infty$ uniformly on τ_1 and τ_2 in each finite interval. Hence in view of the well known continuity theorem, see, for example [2] we obtain the weak convergence of the measure Q_N to the measure Q defined by the characteristic function $\varphi(\tau_1, \tau_2)$. The theorem is proved.

Now let G be a region on the complex plane, and let $H(G)$ denote the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Define a probability measure

$$Q_N(A) = \frac{1}{N+1} \# \{0 \leq m \leq N : p_n(s + imh) \in A\}, \quad A \in \mathcal{B}(H(G)).$$

Theorem 2. *There exists a probability measure Q on $(H(G), \mathcal{B}(H(G)))$ such that the measure Q_N converges weakly to Q as $N \rightarrow \infty$.*

We begin the proof of Theorems 2 with the following lemma. Let γ be the unit circle on the complex plane, and let

$$\Omega_n = \prod_{l=1}^n \gamma_l,$$

where $\gamma_l = \gamma$ for all $l = 1, \dots, n$. Consider on $(\Omega_n, \mathcal{B}(\Omega_n))$ the probability measure

$$Q'_N(A) = \frac{1}{N+1} \# (0 \leq m \leq N : (\lambda_1^{imh}, \dots, \lambda_n^{imh}) \in A).$$

Lemma 1. *There exists a probability measure Q' on $(\Omega_n, \mathcal{B}(\Omega_n))$ such that the measure Q'_N converges weakly to Q' as $N \rightarrow \infty$.*

Proof. We have that the Fourier transforms $g_N(k_1, \dots, k_n)$ of the measure Q'_N (for the definition see [1]) is

$$g_N(k_1, \dots, k_n) = \frac{1}{N+1} \sum_{m=0}^N e^{imh \sum_{l=1}^n k_l \lambda_l}.$$

Therefore, we have that

$$g_N(k_1, \dots, k_n) = \begin{cases} 1 & \text{if } h \sum_{l=1}^n k_l \lambda_l = 2\pi r, r \in \mathbb{Z}, \\ \frac{1}{N+1} \frac{1 - e^{i(N+1)h \sum_{l=1}^n k_l \lambda_l}}{1 - e^{ih \sum_{l=1}^n k_l \lambda_l}} & \text{if } h \sum_{l=1}^n k_l \lambda_l \neq 2\pi r. \end{cases}$$

Consequently,

$$\lim_{N \rightarrow \infty} g_N(k_1, \dots, k_n) = \begin{cases} 1 & \text{if } \sum_{l=1}^n k_l \lambda_l = \frac{2\pi r}{h}, \\ 0 & \text{if } \sum_{l=1}^n k_l \lambda_l \neq \frac{2\pi r}{h}. \end{cases}$$

Hence, by Theorem 1.3.19 from [1] the lemma follows.

Proof of Theorem 2. Define a function $h : \Omega_n \rightarrow H(G)$ by the formula

$$h(x_1, \dots, x_n) = \sum_{l=1}^n a_l e^{\lambda_l s} x_l^{\lambda_l m h}, (x_1, \dots, x_n) \in \Omega_n.$$

Clearly, the equality

$$\sum_{l=1}^n a_l e^{\lambda_l (s + imh)} = h(e^{i\lambda_1 m h}, \dots, e^{i\lambda_n m h})$$

holds, and the function h is continuous. Therefore from Lemma 2 and Theorem 5.1 from [2] we obtain that the measure $Q_N h^{-1}$ converges weakly to $Q' h^{-1}$ as $N \rightarrow \infty$.

Theorems 1 and 2 will be applied to prove discrete limit theorems for Dirichlet series.

References

- [1] A. Laurinćikas, *Limit Theorems for the Riemann Zeta-function*, Kluwer, Dordrecht, Boston, London (1996).
- [2] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York (1968).

Diskrečioji ribinė teorema trigonometriniam polinomams

R. Kačinskaitė

Straipsnyje įrodomos dvi diskrečios ribinės teoremos trigonometriniam polinomams tikimybinių matų silpno konvergavimo prasme.