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A note on functions of exponential type

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Let $0 < \theta_0 \leqslant \pi$. A function $f(s)$, $s = \sigma + it$, analytic in the closed angular region $|\arg s| \leqslant \theta_0$ is said to be of exponential type if

$$\limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r} < \infty$$

uniformly in θ, $|\theta| \leqslant \theta_0$.

Elements of the theory of functions of exponential type are applied in the investigation of the universality of Dirichlet series, see [1], Chapter 6. For example, the following result is very useful. Let $\mathbb{C}$ be the complex plane, $\mathcal{B}(\mathbb{C})$ denote the class of Borel sets of the space $\mathbb{C}$, and let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Suppose that

$$\varrho(s) = \int\limits_{\mathbb{C}} \mathrm{e}^{-sz} \, \mathrm{d}\mu(z)$$

and

$$\sum_p |\varrho(\log p)| < \infty.$$

Then, see Section 6.6.5 of [1],

$$\int\limits_{\mathbb{C}} z^r \, \mathrm{d}\mu(z) = 0, \qquad r = 0, 1, 2, \ldots.$$

To study the universality, for example, of the function

$$\zeta^2(s)\zeta(2s-1),$$

where, as usual, $\zeta(s)$ denotes the Riemann zeta-function, we need an assertion similar to stated above for a complex measure μ_p depending on prime numbers p. More precisely, let μ be a

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complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with support contained in the strip $D_1 = \{s \in \mathbb{C} : 3/4 < \sigma < 1\}$ and let the function $h : \mathbb{C} \to \mathbb{C}$ be given by the formula

$$h(s) = \frac{s+1}{2}, \qquad s \in \mathbb{C}.$$

Denote by

$$\mu_1(A) = \mu h^{-1}(A) = \mu(Ah^{-1}), \qquad A \in \mathbb{C}.$$

Then we have that $\mu_p = 2\mu + a_p\mu_1$, $|a_p| = 1$, is a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in the strip D_1. Define

$$\varrho_p(s) = \int_{\mathbb{C}} \mathrm{e}^{-sz} \, \mathrm{d}\,\mu_p(z).$$

Clearly, the function $\varrho_p(s)$ is a function of exponential type for each prime p.

Theorem 1. *Suppose that*

$$\sum_p \left|\varrho_p(\log p)\right| < \infty.$$

Then

$$\int_{\mathbb{C}} z^r \, \mathrm{d}\,\mu(z) = 0, \qquad r = 0, 1, 2, \ldots.$$

Proof. Define a number $\alpha > 0$ by

$$\sup_p \limsup_{t \to \infty} \frac{\log|\varrho_p(\pm it)|}{t} \leqslant \alpha,$$

and choose a number $\beta > 0$ satisfying the inequality $\alpha\beta < \pi$. Consider the set

$$A = \bigcup_{\substack{p \\ a_p \neq -1}} A_p,$$

where

$$A_p = \left\{ m \in \mathbb{N} : \exists r \in \left(\left(m - \frac{1}{4}\right)\beta, \left(m + \frac{1}{4}\right)\beta\right], \text{ and } |\varrho_p(r)| \leqslant \mathrm{e}^{-r} \right\}.$$

Then we have

$$\sum_p \left|\varrho_p(\log p)\right| \geqslant \sum_{m \notin A} \sum_m{}' \left|\varrho_p(\log p)\right| \geqslant \sum_{m \notin A} \sum_m{}' \frac{1}{p}, \tag{1}$$

where $\sum'_m$ denotes a sum over all prime numbers such that

$$\Big(m-\frac{1}{4}\Big)\beta < \log p \leqslant \Big(m+\frac{1}{4}\Big)\beta.$$

Using the well-known formula

$$\sum_{p\leqslant x}\frac{1}{p} = \log\log x + \gamma_0 + \mathrm{e}^{-c_1\sqrt{\log x}},$$

where γ_0 is Euler's constant and $c_1 > 0$, we find that

$$\begin{aligned}\sum_m{}'\frac{1}{p} &= \sum_{p\leqslant\exp\{m+1/4)\beta\}}\frac{1}{p} - \sum_{p\leqslant\exp\{m-1/4)\beta\}}\frac{1}{p}\\ &= \log\frac{m+1/4}{m-1/4} + B\exp\Big\{-c_2\Big(\Big(m-\frac{1}{4}\Big)\beta\Big)^{1/2}\Big\} = \frac{1}{2m}+\frac{B}{m^2}.\end{aligned}$$

Here B is a number bounded by a constant. Hence and from (1) it follows that

$$\sum_{m\notin A}\Big(\frac{1}{2m}+\frac{B}{m^2}\Big) \leqslant \sum_p \big|\varrho_p(\log p)\big| < \infty,$$

and therefore

$$\sum_{m\notin A}\frac{1}{m} < \infty. \tag{2}$$

Let $A = \{a_1, a_2, \ldots\}$ with $a_1 < a_2 < \ldots$. Then (2) implies

$$\lim_{m\to\infty}\frac{a_m}{m} = 1. \tag{3}$$

By the definition of the set A there exist a sequence $\{\lambda_m\}$ and a prime number p_0 such that

$$\Big(a_m-\frac{1}{4}\Big)\beta < \lambda_m \leqslant \Big(a_m+\frac{1}{4}\Big)\beta$$

and

$$\big|\varrho_{p_0}(\lambda_m)\big| \leqslant \mathrm{e}^{-\lambda_m}.$$

By (3)

$$\lim_{m\to\infty}\frac{\lambda_m}{m} = \beta,$$

and

$$\limsup_{m\to\infty} \frac{\log|\varrho_{p_0}(\lambda_m)|}{\lambda_m} \leqslant -1. \tag{4}$$

Now we will apply one version of the Bernstein theorem, Theorem 6.4.12 from [1]. Let $f(s)$ be an entire function of exponential type, and let $\{\lambda_m\}$ be a sequence of complex numbers. Let α, β and δ be positive real numbers such that

a) $\limsup_{t\to\infty} \frac{\log|f(\pm it)|}{t} \leqslant \alpha$;

b) $|\lambda_m - \lambda_n| \geqslant \delta|m-n|$;

c) $\lim_{m\to\infty} \frac{\lambda_m}{m} = \beta$;

d) $\alpha\beta < \pi$.

Then

$$\limsup_{m\to\infty} \frac{\log|f(\lambda_m)|}{|\lambda_m|} = \limsup_{r\to\infty} \frac{\log|f(r)|}{r}.$$

Consequently, by (4),

$$\limsup_{r\to\infty} \frac{\log|\varrho_{p_0}(r)|}{r} \leqslant -1. \tag{5}$$

Now we will use Lemma 6.4.10 from [1]. Let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in the half-plane $\sigma > \sigma_0$. Morevoer, let

$$f(s) = \int_{\mathbb{C}} e^{sz}\, d\mu(z), \qquad s \in \mathbb{C},$$

and $f(s) \not\equiv 0$. Then

$$\limsup_{r\to\infty} \frac{\log|f(r)|}{r} > \sigma_0.$$

Hence and from (5) we obtain that

$$\varrho_{p_0}(s) \equiv 0.$$

Differentiating the latter equation $r = 0, 1, 2, \ldots$ times, and then putting $s = 0$, we find

$$\int_{\mathbb{C}} z^r\, d\mu_{p_0}(z) = 0. \tag{6}$$

Taking $r = 0$, we have

$$\int_{\mathbb{C}} \mathrm{d}\,\mu_{p_0}(z) = 2\int_{\mathbb{C}} \mathrm{d}\,\mu(z) + a_{p_0}\int_{\mathbb{C}} \mathrm{d}\,\mu_1(z) = (2 + a_{p_0})\int_{\mathbb{C}} \mathrm{d}\,\mu(z) = 0.$$

Thus, the equality

$$\int_{\mathbb{C}} z^r \,\mathrm{d}\,\mu(z) = 0 \tag{7}$$

is true for $r = 0$. If $r = 1$, then (6) yields

$$\begin{aligned} 0 &= 2\int_{\mathbb{C}} z \,\mathrm{d}\,\mu(z) + a_{p_0}\int_{\mathbb{C}} z \,\mathrm{d}\,\mu_1(z) \\ &= 2\int_{\mathbb{C}} z \,\mathrm{d}\,\mu(z) + a_{p_0}\int_{\mathbb{C}} (2z - 1)\,\mathrm{d}\,\mu(z) \\ &= (2 + 2a_{p_0})\int_{\mathbb{C}} z \,\mathrm{d}\,\mu(z) - a_{p_0}\int_{\mathbb{C}} \mathrm{d}\,\mu(z). \end{aligned}$$

Hence, in view of (7), we have that (7) is true for $r = 1$. The further proof is obtained in a standard induction way.

Theorem 1 is applied to obtain in a standard way the universality of the function $\zeta^2(s)\zeta(2s - 1)$.

Theorem 2. *Let K be a compact subset of the strip D_1 with connected complement. Let $f(s)$ be a nonvanishing continuous function on K which is analytic in the interior of K. Then for every $\varepsilon > 0$*

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathrm{meas}\Big\{\tau \in [0, T], \sup_{s\in K}\big|\zeta^2(s + i\tau)\zeta(2(s + i\tau) - 1) - f(s)\big| < \varepsilon\Big\} > 0.$$

Here $\mathrm{meas}\{A\}$ denotes the Lebesgue measure of the set A.

References

[1] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht (1996).

Pastaba apie eksponentinio tipo funkcijas

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Straipsnyje nagrinėjama eksponentinio tipo funkcija, priklausanti nuo pirminių skaičių. Gauta tos funkcijos viena savybė, naudojama universalumo teorijoje.