

On the distributions of additive functions

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1. Introduction

We shall consider the distribution functions $\nu_x(f_x(m) < u)$, where $\{f_x(m), x \geq 2\}$ is some set of strongly additive functions and

$$\nu_x(f_x(m) < u) = \frac{1}{[x]} \#\{m \leq x : f_x(m) < u\}.$$

In the papers [2–4] the weak convergence of these functions to the Poisson law was investigated. In this paper we examine the conditions for the weak convergence of $\nu_x(f_x(m) < u)$ to some distribution function $F(u)$. We investigate only those additive functions $f_x(m)$, for which $f_x(p) \in \{0, 1\}$ for primes p .

2. The main result

Theorem. *Let $\{f_x(m), x \geq 2\}$ be a set of strongly additive functions, $f_x(p) \in \{0, 1\}$ for each prime number p . The distribution function $\nu_x(f_x(m) < u)$ converge weakly as $x \rightarrow \infty$ if and only if the finite limits*

$$\lim_{x \rightarrow \infty} \sum_{\substack{p_1 \leq x \\ f_x(p_1)=1}} \frac{1}{p_1} \cdots \sum_{\substack{p_{l-1} \leq x \\ p_{l-1} \neq p_1 \cdot p_2 \cdots p_{l-2} \\ f_x(p_{l-1})=1}} \frac{1}{p_{l-1}} \sum_{\substack{p_l \leq x/p_1 p_2 \cdots p_{l-1} \\ p_l \neq p_1 \cdot p_2 \cdots p_{l-1} \\ f_x(p_l)=1}} \frac{1}{p_l} = g_l$$

exist for each natural number l .

Moreover, in this case the limiting distribution has characteristic function

$$1 + \sum_{l=1}^{\infty} \frac{g_l}{l!} (e^{it} - 1)^l.$$

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3. The proof of necessity

Let $F(u)$ is the limit distribution function for $\nu_x(f_x(m) < u)$. This function is a distribution function of some discrete random variable with jumps at non-negative integer numbers. Assume that $\varphi_k = F(k + 0) - F(k)$. Then from the weak convergence we have

$$\lim_{x \rightarrow \infty} \nu_x(f_x(m) = k) = \varphi_k \tag{1}$$

for each non-negative integer k .

It is clear that $\varphi_{\hat{k}} > 0$ for some \hat{k} . Using the Halász's inequality (see [1])

$$\nu_x(h(m) = a) \leq c_1 \left(\sum_{\substack{p \leq x \\ h(p) \neq 0}} \frac{1}{p} \right)^{-1/2},$$

which holds for every additive function $h(m)$ and every real number a , we obtain

$$\sum_{\substack{p \leq x \\ f_x(p) = 1}} \frac{1}{p} \leq \frac{4c_1^2}{\varphi_{\hat{k}}^2}$$

for sufficiently large x .

Hence we get

$$\limsup_{x \rightarrow \infty} \sum_{p \leq x}^* \frac{1}{p} \leq c_2, \tag{2}$$

where c_2 may depend on the limit distribution function $F(u)$, and the sign $*$ means that the summation is taken over primes p for which $f_x(p) = 1$.

Let

$$\beta_x(l) = \frac{1}{x} \sum_{m \leq x} f_x(m)(f_x(m) - 1) \dots (f_x(m) - l + 1)$$

for a natural number l .

An easy computation shows that

$$\begin{aligned} \beta_x(l) &= \frac{1}{x} \sum_{m \leq x} \sum_{p_1 | m}^* \sum_{\substack{p_2 | m \\ p_2 \neq p_1}}^* \dots \sum_{\substack{p_l | m \\ p_l \neq p_1 \cdot p_2 \dots p_{l-1}}}^* 1 \\ &= \frac{1}{x} \sum_{p_1 \leq x}^* \sum_{\substack{p_2 \leq x \\ p_2 \neq p_1}}^* \dots \sum_{\substack{p_{l-1} \leq x \\ p_{l-1} \neq p_1 \cdot p_2 \dots p_{l-2}}}^* \sum_{\substack{p_l \leq x/p_1 p_2 \dots p_{l-1} \\ p_l \neq p_1 \cdot p_2 \dots p_{l-1}}}^* \left[\frac{x}{p_1 p_2 \dots p_l} \right]. \end{aligned} \tag{3}$$

According to the inequality (2) we have

$$\limsup_{x \rightarrow \infty} \beta_x(l) \leq \limsup_{x \rightarrow \infty} \left(\sum_{p \leq x}^* \frac{1}{p} \right)^l \leq c_l^l \tag{4}$$

for each l .

Suppose now that l is a fixed natural number and $K \geq l + 10$. We have

$$\begin{aligned} \beta_x(l) &= \frac{1}{x} \sum_{\substack{m \leq x \\ f_x(m) \leq K}} f_x(m)(f_x(m) - 1) \dots (f_x(m) - l + 1) \\ &\quad + \frac{1}{x} \sum_{\substack{m \leq x \\ f_x(m) > K}} f_x(m)(f_x(m) - 1) \dots (f_x(m) - l + 1) \frac{f_x(m) - l}{f_x(m) - l} \\ &= \sum_{k=l}^K k(k-1) \dots (k-l+1) \frac{1}{x} \sum_{\substack{m \leq x \\ f_x(m)=k}} 1 + \frac{B}{K-l} \beta_x(l+1). \end{aligned} \tag{5}$$

It follows from (1) and (4) that

$$\limsup_{x \rightarrow \infty} \beta_x(l) = \limsup_{K \rightarrow \infty} \sum_{k=l}^K k(k-1) \dots (k-l+1) \varphi_k.$$

According to estimate (4) we see that sequence

$$g_{lK} = \sum_{k=l}^K k(k-1) \dots (k-l+1) \varphi_k$$

is increasing and bounded. Therefore the limit

$$g_l = \lim_{K \rightarrow \infty} g_{lK} = \sum_{k=l}^{\infty} k(k-1) \dots (k-l+1) \varphi_k$$

exist for each fixed natural l .

Hence from (4) and (5) we have

$$\lim_{x \rightarrow \infty} \beta_x(l) = g_l. \tag{6}$$

On the other hand (3) shows that

$$\begin{aligned} \beta_x(l) &= \sum_{p_1 \leq x}^* \frac{1}{p_1} \sum_{\substack{p_2 \leq x \\ p_2 \neq p_1}}^* \frac{1}{p_2} \dots \sum_{\substack{p_{l-1} \leq x \\ p_{l-1} \neq p_1, p_2, \dots, p_{l-2}}}^* \frac{1}{p_{l-1}} \sum_{\substack{p_l \leq x / p_1 p_2 \dots p_{l-1} \\ p_l \neq p_1, p_2, \dots, p_{l-1}}}^* \frac{1}{p_l} \\ &\quad + B l! \nu_x(\omega(m) = l). \end{aligned} \tag{7}$$

Since

$$\nu_x(\omega(m) = l) \sim \frac{(\ln \ln x)^{l-1}}{(l-1)! \ln x}$$

for each fixed natural l (see, for example, [5] Ch.II), we conclude from (6) that the conditions of our theorem are satisfied.

4. The proof of sufficiency

Suppose now that the limit

$$g_l = \lim_{x \rightarrow \infty} \sum_{p_1 \leq x}^* \frac{1}{p_1} \sum_{\substack{p_2 \leq x \\ p_2 \neq p_1}}^* \frac{1}{p_2} \dots \sum_{\substack{p_{l-1} \leq x \\ p_{l-1} \neq p_1, p_2, \dots, p_{l-2}}}^* \frac{1}{p_{l-1}} \sum_{\substack{p_l \leq x / p_1 p_2 \dots p_{l-1} \\ p_l \neq p_1, p_2, \dots, p_{l-1}}}^* \frac{1}{p_l} \tag{8}$$

exist for each fixed natural l .

Applying (7) we can assert that

$$\lim_{x \rightarrow \infty} \hat{\beta}_x(l) = \lim_{x \rightarrow \infty} \beta_x(l) = g_l, \tag{9}$$

where l is a natural number and

$$\hat{\beta}_x(l) = \frac{1}{[x]} \sum_{m \leq x} f_x(m)(f_x(m) - 1) \dots (f_x(m) - l + 1).$$

Let $\psi_x(t)$ is the characteristic function of $\nu_x(f_x(m) < u)$. It is clear that

$$\psi_x(t) = \frac{1}{[x]} \sum_{m \leq x} e^{it f_x(m)}$$

for $x \geq 2, t \in \mathbb{R}$.

If r and n are natural numbers, then

$$|e^{itr} - 1 - \sum_{j=1}^{n-1} \binom{r}{j} (e^{it} - 1)^j| \leq \binom{r}{n} |e^{it} - 1|^n.$$

Hence

$$\psi_x(t) = 1 + \sum_{l=1}^L \frac{(e^{it} - 1)^l}{l!} \hat{\beta}_x(l) + \frac{B}{(L+1)!} |e^{it} - 1|^{L+1} \hat{\beta}_x(L),$$

where $L \in \mathbb{N}$.

We obtain from (8) for the case $l = 1$

$$\lim_{x \rightarrow \infty} \sum_{p \leq x}^* \frac{1}{p} = g_1.$$

Repeated application of (8) enables us to write

$$g_l \leq \lim_{x \rightarrow \infty} \left(\sum_{p \leq x}^* \frac{1}{p} \right)^l = g_1^l$$

for each natural number l .

Therefore from (9) we have

$$\psi_x(t) = 1 + \sum_{l=1}^L \frac{(e^{it} - 1)^l}{l!} \hat{\beta}_x(l) + \frac{B}{(L+1)!} |e^{it} - 1|^{L+1} g_1^{L+1}, \quad (10)$$

where $t \in \mathbb{R}$, $x \geq 2$ and $L \in \mathbb{N}$.

Inequality $g_l \leq g_1^l$, $l \in \mathbb{N}$, shows that the series

$$\sum_{l=1}^{\infty} \frac{(e^{it} - 1)^l}{l!} g_l$$

converges uniformly to some continuous function $\psi(t)$.

By (10) we obtain

$$\lim_{x \rightarrow \infty} \psi_x(t) = \psi(t)$$

for each real number t .

Since $\psi(t)$ is continuous, it follows from the last equality that distribution functions $\nu_x(f_x(m) < u)$ converge weakly to some distribution function $F(u)$, which has its characteristic function $\psi(t)$. This completes the proof.

5. Concluding remarks

It can be seen, that the limit distribution function $F(u)$ from our theorem has some special properties. We can obtain after some calculations the following statements.

1. $F(u)$ is distribution function of some discret random variable with jumps at non-negative numbers.
2. There exist the factorial moments

$$g_n = \int_{-\infty}^{+\infty} u(u-1) \dots (u-n+1) dF(u)$$

for each natural n .

3. The factorial moments g_n satisfy the inequalities

$$g_n \leq g_{n-k} g_k, \quad k = 0, 1, \dots, n-1.$$

4. If $g_l = 0$ for some $l \geq 2$, then $g_1 \leq \ln l$.

I am sure that $F(u)$ has more special properties. I think only a few of distribution functions can occur as weak limits in our theorem.

References

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Apie adityviųjų funkcijų skirstinius

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Darbe gautos būtinos ir pakankamos sąlygos skirstinių

$$\nu_x(f_x(m) < u) = \frac{1}{[x]} \#\{m \leq x, f_x(m) < u\}$$

silpnam konvergavimui. Čia $f_x(m)$ yra šeima ($x \geq 2$) stipriai adityviųjų funkcijų, kurioms $f_x(p) \in \{0, 1\}$ visiems pirminiams p .