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UNIVERSITY THEOREMS FOR THE PERIODIC HURWITZ ZETA-FUNCTION

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VILNIAUS UNIVERSITETAS

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Introduction

In the thesis, the value-distribution of periodic Hurwitz zeta-functions $\zeta(s, \alpha; \mathfrak{a}), s = \sigma + it$, is investigated. The main attention is devoted to the universality of $\zeta(s, \alpha; \mathfrak{a})$, i.e., to the approximation of a wide class of analytic functions by shifts $\zeta(s + i\tau, \alpha; \mathfrak{a})$, with $\tau \in \mathbb{R}$.

First, we recall the definition of the periodic zeta-function. Let $\alpha, 0 < \alpha \leq 1$, be a fixed parameter, and $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0\}, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$, i.e., $a_{m+q} = a_m$ for all $m \in \mathbb{N}_0$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}. \tag{1}$$

From the periodicity of the sequence, it follows that, for all $m \in \mathbb{N}_0$,

$$|a_m| \leq max(|a_0|, |a_1|, ..., |a_{q-1}|).$$

Thus, we have that the sequence $\mathfrak{a}$ is bounded. Therefore, the series (1) is convergent absolutely for $\sigma > 1$. Hence, by the well known property of Dirichlet series, the function $\zeta(s, \alpha; \mathfrak{a})$ is analytic in the half-plane $\sigma > 1$.

For analytic continuation to the remained part of the complex plane, the classical Hurwitz zeta-function is used. Let α be the same parameter as above. Then the Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and is analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s, \alpha)$ was introduced by A. Hurwitz in [12]. The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s, \alpha; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\Big(s, \frac{l + \alpha}{q}\Big). \tag{2}$$

The latter equality together with mentioned above properties of the Hurwitz zeta-function gives an analytic continuation for the function $\zeta(s, \alpha; \mathfrak{a})$ to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\hat{a} \stackrel{def}{=} \frac{1}{q} \sum_{l=0}^{q-1} a_l.$$

If $\hat{a} = 0$, then the function $\zeta(s, \alpha; \mathfrak{a})$ is entire one.

The periodic Hurwitz zeta-function was introduced in [18].

By the definitions of the functions $\zeta(s, \alpha)$ and $\zeta(s, \alpha; \mathfrak{a})$, we have that $\zeta(s, \alpha) = \zeta(s, \alpha; \mathfrak{a})$ with $\mathfrak{a} = \{a_m : a_m \equiv 1\}$. Thus, the periodic Hurwitz zeta-function is a generalisation of the classical Hurwitz zeta-function.

The function $\zeta(s, \alpha; \mathfrak{a})$ is connected to an another classical zeta-function - the Lerch zeta-function. Let $\lambda \in \mathbb{R}$, and let α be the same parameter as above. The Lerch zeta-function $L(\lambda, \alpha, s)$ is defined, for $\sigma > 1$, by Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}.$$

If $\lambda \in \mathbb{R}$, then $L(\lambda, \alpha, s)$ becomes the Hurwitz zeta-function. For $\lambda \notin \mathbb{R}$, the function $L(\lambda, \alpha, s)$ continues analytically to an entire function. The function $L(\lambda, \alpha, s)$ was introduced independently in [38] and [39]. The theory of the Lerch zeta-function is given in [26].

If parameter λ is rational, then the coefficients $e^{2\pi i \lambda m}$ are periodic. Therefore, the periodic Hurwitz zeta-function is a generalization of the classical Lerch zeta-function $L(\lambda, \alpha, s)$ with rational parameter λ.

Aims and problems

The aim of the thesis are continuous and discrete universality theorems for periodic Hurwitz zeta-functions. The problems of the thesis are the following:

1. An extension of a continuous universality theorem for the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental parameter α.

2. Extensions of a discrete universality theorem for the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental parameter α.

3. Continuous and discrete universality theorems for composite functions of the periodic Hurwitz zeta-function.

4. Estimation of the number of zeros of the periodic Hurwitz zeta-function.

Actuality

Universality of zeta-functions has a great influence to approximation of analytic functions. Zeta-functions usually can be approximated with a certain accuracy by Dirichlet polynomials that are finite trigonometric sums. Therefore, universality theorems for zeta-functions allow to reduce an approximation of complicated analytic functions to the approximation by finite trigonometric sums that are comparatively simple functions. The described procedure indicate a way for estimation of complicated analytic functions. For example, this way was applied by physicists for estimation of integrals over complicated analytic curves [3]. Therefore, universality theorems for zeta-functions have a big practical use, and this, of course, requires to extend investigations of universality for new classes of zeta-functions. Since periodic Hurwitz zeta-functions extend the classes of Hurwitz and Lerch zeta-functions, their universality theorems are the significant impact to applications of approximation theory.

Universality theorems for periodic Hurwitz zeta-functions have also a series of theoretical applications. They are used for the proof of the functional independence of a wide class of zeta-functions which is closely related to one of Hilbert's hypothesis on the algebraic-differential independence of certain Dirichlet series. Moreover, universality theorems contain an information on the zero-distribution of zeta-functions without Euler product over primes. Therefore, we are able to obtain estimates for the number of zeros for periodic Hurwitz zeta-functions.

In general, universality of zeta-functions is one of popular directions of analytic number theory. Lithuanian school of analytic number theory is the well-known in virtue of universality results. These facts also support the actuality of the subject of the thesis.

Methods

The proof of universality theorems for periodic Hurwitz zeta-functions is based on probabilistic limit theorems on weakly convergent probability measures in the space of analytic functions. This approach includes the Fourier transform method, the Prokhorov theory on tightness and relative compactness of families of probability measures, as well as some elements of ergodic theory. Moreover, the Mergelyan theorem on the approximation of analytic functions by polynomials plays an important role in the proofs. For estimation of the number of zeros of certain analytic functions, the classical Rouché theorem is applied.

Novelty

All results of the thesis are new. Some of universality theorems for the function $\zeta(s, \alpha; \mathfrak{a})$ extend the class of values of parameter α. Theorems on the number of zeros of the function $\zeta(s, \alpha; \mathfrak{a})$ were not considered.

History of the problem and main results

The problem of universality of zeta-functions was opened by S. M. Voronin. In [52], he discovered the universality of the Riemann zeta-function. We remind that $\zeta(s) = \zeta(s, 1; \{1\})$. The Voronin theorem has the following form.

Theorem A. *Let $0 < r < \frac{1}{4}$. Suppose that the function $f(s)$ is continuous and non-vanishing in the disc $|s| \leq r$, and analytic in $|s| < r$. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\leq r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Thus, by Theorem A, a wide class of analytic functions is approximated with desired accuracy by shifts of the same function $\zeta(s)$. This is the sense of the Voronin universality.

We note that the first universal object in analysis was constructed by M. Fekete, see [46], [10]. He proved that there exists a real power series

$$\sum_{m=1}^{\infty} a_m x^m \tag{3}$$

such that, for every continuous function $g(x), g(0) = 0$, there exists an increasing sequence $\{n_k\}$ of positive integers with a property that

$$\lim_{k\to\infty} \sum_{m\leq n_k} a_m x^m = g(x)$$

uniformly in $x \in [-1, 1]$. However, the series (3) was not given explicitly, only its existence was proved.

After a Fekete's result, various authors, among them G. D. Birkhoff, J. Marcinkiewicz, who obtained universal objects in analysis, however, all these objects, as the series (3), were not given explicitly. Thus, the Riemann zeta-function is the first explicitly given universal object in analysis.

Theorem A is a deep result of analytic number theory, therefore, it turned attention of number theorists. Variuos authors found a more general form for Theorem A. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ be the right-hand side of the critical strip. The following notation is convenient for statements of universality theorems for zeta-functions. Denote by $\mathcal{K}$ the class of compact sets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Moreover, let $\operatorname{meas} A$ be the Lebesgue measure of a measurable set $A \in \mathbb{R}$. Then the modern version of the Voronin theorem has the following form, see, for example, [17], [51].

Theorem B. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} \left| \zeta(s + i\tau) - f(s) \right| < \epsilon \right\} > 0.$$

The inequality of Theorem B means that the set of shifts $\zeta(s + i\tau)$ approximating a given function $f(s) \in H_0(K)$ with accuracy $\epsilon > 0$ has a positive lower density. From this, it follows that the latter set is infinite. Moreover, analytic functions are uniformly approximated not only on discs, as in Theorem A, but on general compact sets with connected complements. For example, a compact ring does not belong to the class $\mathcal{K}$.

After Voronin's work [52], it was observed that some other zeta and L-functions are also universal in the Voronin sense. Among them, Dirichlet L-functions [1], [40], [53], [15], [20], [54], [55], [9], Dedekind zeta-functions of number fields [47], [48], zeta-functions attached to normalized Hecke-eigen forms [31], [32], [33], [34], [28], L-functions of the Selberg class [50],[44] and other zeta and L-functions. The above mentioned zeta and $L-$functions have a common feature - they have Euler's type product over primes. For example, the Riemann zeta-function has, for $\sigma > 1$, the following Euler product expansion

$$\zeta(s) = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1}$$

over primes p. It is a reason to thing that the existence of Euler's product allows to approximate only non-vanishing analytic functions.

There is an another group of zeta-functions having no Euler product but universal in a bit different sense. The object of the thesis, the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$, in general, has no Euler's product. The simplest zeta-function without Euler's product is the Hurwitz zeta-function $\zeta(s, \alpha)$ which has Euler's product only in the cases $\zeta(s, 1) = \zeta(s)$ and

$$\zeta\Big(s, \frac{1}{2}\Big) = (2^s - 1)\zeta(s).$$

In the case of the functions $\zeta(s, \alpha)$ and $\zeta(s, \alpha; \mathfrak{a})$, the class $H_0(K)$, $K \in \mathcal{K}$, is replaced by the class $H(K), K \in \mathcal{K}$, of continuous functions on K that are analytic in the interior of K. The simplest case is of transcendental parameter α. We recall that α is a transcendental number if there is no polynomial $p(s) \not\equiv 0$ with rational coefficients such that $p(\alpha) = 0$. The first universality result for the periodic Hurwitz zeta-function was obtained in [14] and is of the following form.

Theorem C. *Suppose that the parameter α is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \sup_{s\in K} \Big|\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)\Big| < \epsilon\Big\} > 0.$$

In [9], Theorem C was proved for the sequence $\mathfrak{a}$ with $a_m \equiv 1$.

In the thesis, the requirement of Theorem C on the transcendence of the parameter α is replaced by weaker one. Define the set

$$L(\alpha) = \{\log(m + \alpha) : m \in \mathbb{N}_0\}.$$

Theorem 1.1. *Suppose that the set $L(\alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \sup_{s\in K} \Big|\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)\Big| < \varepsilon\Big\} > 0.$$

It is easy to see that if α is a transcendental number, then the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Suppose, on the contrary, that there exist $k_1, ..., k_r \in \mathbb{Z} \setminus \{0\}$ and $m_1, ..., m_r \in \mathbb{N}$ such that

$$k_1 \log(m_1 + \alpha) + ... + k_r \log(m_r + \alpha) = 0.$$

Hence, we find

$$(m_1 + \alpha)^{k_1} + ... + (m_r + \alpha)^{k_r} = 1.$$

From this, using the Newton expansions, we obtain that there exists a polynomial $p(s)$ with integer coefficients such that $p(\alpha) = 0$. However, this equality contradicts the transcendence of α.

On the other hand, we do not know any non-transcendental $\alpha, 0 < \alpha \leq 1$, with a linearly independent set $L(\alpha)$, however, by the famous Cassels theorem [4], this is theoretically possible. We remind that α is an algebraic number if there exists a polynomial $p(s) \not\equiv 0$ with rational coefficients such that $p(\alpha) = 0$. For example, $\frac{1}{\sqrt{2}}$ is an algebraic number because it is a root of the polynomial $2s^2 = 1$. All rational numbers are also algebraic. The Cassels theorem asserts that at least 51 percent of elements of the set $L(\alpha)$ with algebraic irrational α are linearly independent over $\mathbb{Q}$. Thus, it can happen that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ with algebraic irrational α.

The case of rational α is not easily treated. We recall that $\text{rad}(m)$ denotes the product of all distinct prime divisors of a positive integer m, i.e.,

$$\text{rad}(m) = \prod_{p|m} p.$$

The condition $\text{rad}(q)$ divides b means that every prime divisor of q divides b. We note that the latter condition is equivalent to the requirement that $(bl + a, bq) = 1$ for all $l = 0, ..., q-1$.

Theorem 1.2. *Suppose that* $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a, b) = 1, b \neq 2$ *and that* $\text{rad}(q)$ *divides* b. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} \left| \zeta(s + i\tau, \frac{a}{b}; \mathfrak{a}) - f(s) \right| < \varepsilon \right\} > 0.$$

Theorem 1.2 for the Hurwitz zeta-function (in this case $a_m \equiv 1, q = 1$ and $\alpha \neq 1, \frac{1}{2}$) was obtained in [1], [9], [54].

Theorems B, 1.1 and 1.2 are of continuous type because τ in shifts $\zeta(s + i\tau)$ and $\zeta(s + i\tau, \alpha; \mathfrak{a})$ can take arbitrary real values. If τ in the above shifts takes values from a certain discrete set, then we have discrete universality theorems. Discrete universality theorems for zeta-functions were proposed by A. Reich in [47]. He proved discrete universality theorems for Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$ of algebraic number fields $\mathbb{K}$. For $\sigma > 1$, the function $\zeta_{\mathbb{K}}(s)$ is defined by the Dirichlet series

$$\zeta_{\mathbb{K}}(s) = \sum_{I \subset O_{\mathbb{K}}} \frac{1}{(N(I))^s},$$

where I runs over the non-zero ideals of the ring of integers $O_{\mathbb{K}}$ of $\mathbb{K}$, and $N(I)$ denotes the norm of I. Moreover, $\zeta_{\mathbb{K}}(s)$ is meromorphically continued to the whole complex plane with unique simple pole at the point $s = 1$.

Reich considered the case when τ takes values from the arithmetical progressions $\{kh : k \in \mathbb{N}_0\}$ with fixed difference $h > 0$. A modern version of the Reich theorem is of the following form. We denote by $\#A$ the cardinality of the set A, and suppose that $N \in \mathbb{N}_0$.

Theorem D. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \le k \le N : \sup_{s\in K}\Big|\zeta_{\mathbb{K}}(s+ikh) - f(s)\Big| < \epsilon\Big\} > 0.$$

We note that in the case $\mathbb{K} = \mathbb{Q}$, Theorem D becomes the discrete analogue of Theorem B for Riemann zeta-function.

The first discrete theorem for the Hurwitz zeta-function $\zeta(s,\alpha)$ with rational parameter α was obtained in [1]. The case of transcendental α is more complicated, and it is required that the number $\exp\left\{\frac{2\pi}{h}\right\}$ would be rational. A similar situation is known also for the periodic Hurwitz zeta-function [27].

Theorem E. *Suppose that α is a transcendental number, and $h > 0$ is such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \le k \le N : \sup_{s\in K}\Big|\zeta(s+ikh,\alpha;\mathfrak{a}) - f(s)\Big| < \epsilon\Big\} > 0.$$

For example, the assertion of Theorem E is valid with transcendental $\alpha = e^{-1}$ and $h = 2\pi(\log 2)^{-1}$.

In the thesis, the following extension of Theorem E is considered. For this, a new hypothesis on the parameter α and h is applied. Let

$$L(\alpha,h,\pi) = \left\{\Big(\log(m+\alpha) : m \in \mathbb{N}_0\Big), \frac{2\pi}{h}\right\}.$$

Theorem 1.3. *Suppose that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\epsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \le k \le N : \sup_{s\in K}\Big|\zeta(s+ikh,\alpha;\mathfrak{a}) - f(s)\Big| < \epsilon\Big\} > 0.$$

We observe that Theorem 1.3 implies a similar theorem of [27] for the Hurwitz zeta-function.

The linear independence of the sets over $\mathbb{Q}$ is related in a certain sense to a very important but complicated problem of algebraic independence over $\mathbb{Q}$. We remind that the numbers $\alpha_1,...,\alpha_r$ are algebraically independent over $\mathbb{Q}$ if there is no any polynomial $p(s_1,...,s_r) \not\equiv 0$ with rational coefficients such that $p(\alpha_1,...,\alpha_r) = 0$. It is not difficult to see that if the numbers α and $\exp\left\{\frac{2\pi}{h}\right\}$ are algebraically independent over $\mathbb{Q}$, then the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Actually, suppose, on the contrary, that the set $L(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Then there exist $k_1,...,k_r,k \in \mathbb{Z}\setminus\{0\}$ and $m_1,...,m_r \in \mathbb{N}_0$ such that

$$k_1\log(m_1+\alpha) + ... + k_r\log(m_r+k) + \frac{2k\pi}{h} = 0.$$

Hence,

$$(m_1+\alpha)^{k_1}...(m_r+\alpha)^{k_r}\Big(\exp\left\{\frac{2\pi}{h}\right\}\Big)^k = 1.$$

Therefore, there exists a polynomial $p(s_1, s_2) \not\equiv 0$ with integer coefficients such that $p\big(\alpha, \exp\left\{\frac{2\pi}{h}\right\}\big) = 0$, and this gives a contradiction to the algebraic independence of the numbers α and $\exp\left\{\frac{2\pi}{h}\right\}$. By the famous Nesterenko theorem [45], the numbers π and e^{π} are algebraically independent over $\mathbb{Q}$. Therefore, for example, we can take $\alpha = \frac{1}{\pi}$ and rational h in Theorem 1.3.

A discrete analogue of Theorem 1.2 is of the following form.

Theorem 1.4. *Suppose that* $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a, b) = 1, b \neq 2$ *and that* $\operatorname{rad}(q)$ *divides* b. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then, for every* $\epsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} \Big|\zeta(s + ikh, \frac{a}{b}; \mathfrak{a}) - f(s)\Big| < \epsilon\Big\} > 0.$$

For the proof of Theorems 1.2 and 1.4, the representation of the function $\zeta(s, \frac{a}{b}; \mathfrak{a})$ by Dirichlet L-functions is applied. Let χ be a Dirichlet character modulo k , i.e., $\chi : \mathbb{N} \to \mathbb{C}$ is a periodic function with period k ($\chi(m + k) = \chi(m)$ for all $m \in \mathbb{N}$), completely multiplicative ($\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{N}$), $\chi(m) = 0$ for $(m, k) > 1$, and $\chi(m) \neq 0$ for $(m, n) = 1$. The corresponding Dirichlet L-function $L(s, \chi)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

There exist $\phi(k)$ distinct Dirichlet characters modulo k, where $\phi(k)$ is the Euler totient function: $\phi(k) = \#\{1 \le m \le k : (m, k) = 1\}$. A character χ_0 modulo k is called principal if $\chi_0(m) = 1$ for all $(m, k) = 1$. The function $L(s, \chi_0)$ can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue

$$\prod_{p|k} \Big(1 - \frac{1}{p}\Big),$$

where p denotes a prime number. If $\chi \neq \chi_0$, then the function $L(s, \chi)$ is analytically continued to an entire function, i.e., it is analytic in every finite region of the complex plane. The function $L(s, \chi)$, for $\sigma > 1$, has a representation by the Euler product over primes

$$L(s, \chi) = \prod_{p} \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1}.$$

Dirichlet L-functions are closely connected to Hurwitz zeta-functions with rational parameter, namely,

$$L(s, \chi) = \frac{1}{k^s} \sum_{l=1}^{k} \chi(l) \zeta\Big(s, \frac{l}{k}\Big),$$

and, for $a, b \in \mathbb{N}, (a, b) = 1, a < b$,

$$\zeta\Big(s, \frac{a}{b}\Big) = \frac{b^s}{\phi(b)} \sum_{\chi (\bmod\ b)} \overline{\chi}(a) L(s, \chi), \qquad (4)$$

where the last sum runs over all Dirichlet characters modulo b. Equality (4) is very useful for investigations of the periodic Hurwitz zeta-function with rational parameter.

In Chapter 2 of the thesis, the uniform distribution modulo 1 of sequences of real numbers is applied for the investigation of discrete universality for periodic Hurwitz zeta-function. We recall that a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called uniformly distributed modulo 1 if, for each interval $I = [a, b) \subset [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I\Big(\{x_k\}\Big) = b - a,$$

where $\{x_k\}$ denotes the fractional part of x_k and χ_I is the indicator function of the interval I, i.e.,

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{if } x \notin I. \end{cases}$$

An idea of application of the uniform distribution modulo 1 was proposed in [8], and the following theorem was proved for Riemann zeta-function.

Theorem F. *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$, and suppose that $0 < \beta < 1$ and $h > 0$ are fixed numbers. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} \Big|\zeta(s + ik^\beta h) - f(s)\Big| < \varepsilon\Big\} > 0.$$

We note that Theorem F is the first result in the theory of discrete universality of zeta-functions that uses, in place of the arithmetical progression $\{kh : k \in \mathbb{N}\}, h > 0$, a more general discrete set $\{k^\beta h : k \in \mathbb{N}_0\}, h > 0$.

In [30], Theorem F was extended for a collection of Dirichlet L-functions. Let $\mathbb{P}$ be the set of all prime numbers, $h_1 > 0, ..., h_r > 0$, and

$$L(h_1, ..., h_r; \mathbb{P}) = \Big\{(h_1 \log p : p \in \mathbb{P}), ..., (h_r \log p : p \in \mathbb{P})\Big\}.$$

Theorem G. *Suppose that $\chi_1, ..., \chi_r$ are arbitrary Dirichlet characters, $\beta \in (0, 1)$ is a fixed number, and the set $L(h_1, ..., h_r; \mathbb{P})$ is linearly independent over $\mathbb{Q}$. For $j = 1, ..., r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{1\le j\le r} \sup_{s\in K_j} \Big|L(s + ik^\beta h_j, \chi_j) - f_j(s)\Big| < \varepsilon\Big\} > 0.$$

The proofs of Theorems F and G are based on the fact that the sequence $\{k^\beta a : k \in \mathbb{N}_0\}$, with $0 < \beta < 1$ and $a \in \mathbb{R} \setminus \{0\}$ is uniformly distributed modulo 1.

An analogue of Theorem F for the Hurwitz zeta-function was obtained in [25].

Theorem H. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and $\beta, 0 < \beta < 1$, is a fixed number. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} \Big|\zeta(s + ik^\beta h, \alpha) - f(s)\Big| < \varepsilon\Big\} > 0.$$

In the thesis, the following extension of Theorem H is proved.

Theorem 2.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $\beta_1, 0 < \beta_1 < 1$, and $\beta_2 > 0$ are fixed numbers. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$ and $h > 0$*

$$\liminf_{N\to\infty} \frac{1}{N-1} \#\Big\{2 \le k \le N : \sup_{s\in K} \Big|\zeta(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}) - f(s)\Big| < \varepsilon\Big\} > 0.$$

For the proof of Theorem 2.1, the uniform distribution modulo 1 of the sequence $\{ak^{\beta_1}\log^{\beta_2} k : k = 2, 3, ...\}$ with $\beta_1, 0 < \beta_1 < 1, \beta_2 > 0$ and every $a \in \mathbb{R} \setminus \{0\}$ is applied.

In Chapter 3 of the thesis, generalizations of universality theorems for periodic Hurwitz zeta-functions are considered. More precisely, universality of composite functions $F(\zeta(s, \alpha; \mathfrak{a}))$ for some operators $F : H(D) \to H(D)$, where $H(D)$ is the space of analytic functions on D, is proved.

We note that the first universality theorems for composite functions were obtained in [19] and [21]. We remind some of them. We use the notation

$$S = \{g \in H(D) : g(s) \ne 0 \text{ or } g(s) \equiv 0\}.$$

Theorem I. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that, for any open set $G \subset H(D)$, the set $(F^{-1}G) \cap S$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \sup_{s\in K} \Big|F\Big(\zeta(s + i\tau)\Big) - f(s)\Big| < \varepsilon\Big\} > 0.$$

Let $a_1, ..., a_r$ be a distinct complex numbers, and $F : H(D) \to H(D)$ be an operator. Define

$$H_{a_1,...,a_r;F}(D) = \{g \in H(D) : g(s) \ne a_j, j = 1, ..., r\} \cup \{F(0)\}.$$

Then the following statement is known [21].

Theorem J. *Suppose that $F : H(D) \to H(D)$ is a continuous operator such that $F(S) \supset H_{a_1,...,a_r;F}(D)$. If $r = 1$, let $K \in \mathcal{K}$, and $f(s) \in H(K)$ and $f(s) \ne a_1$ on K. If $r \ge 2$, let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_{a_1,...,a_r;F}(D)$. Then the same assertion as in Theorem I is true.*

From Theorem J, the universality of some elementary functions, for example, $\sin(\zeta(s))$, follows.

The universality of composite functions $F(\zeta(s, \alpha))$ was considered in [23] and [22]. For example, in [23], the following universality theorem was obtained.

Theorem K. *Suppose that the number α is transcendental, and that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \sup_{s\in K} \Big|F\Big(\zeta(s + i\tau, \alpha)\Big) - f(s)\Big| < \varepsilon\Big\} > 0.$$

A discrete analogue of Theorem K and other discrete universality theorems for $F\big(\zeta(s, \alpha)\big)$ were obtained in [37]. Now, we state universality theorems for composite functions $F\big(\zeta(s, \alpha; \mathfrak{a})\big)$ obtained in the thesis. The first theorem is an analogue of Theorem I.

Theorem 3.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left|F\Big(\zeta(s+i\tau,\alpha;\mathfrak{a})\Big) - f(s)\right| < \varepsilon\right\} > 0.$$

We observe that the hypothesis of Theorem 3.1 that the set $F^{-1}G$ is non-empty is very general, however, on the other hand, it is difficult to check this hypothesis. In the next theorem, the hypothesis of Theorem 3.1 on the set $F^{-1}G$ is replaced by a stronger but simpler one. Thus, we have an analogue of Theorem K for the periodic Hurwitz zeta-function.

Theorem 3.2. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 3.1 is true.*

It is easily seen that, for every polynomial $p(s)$, there exist an another polynomial $q(s)$ such that, for all $r \in \mathbb{N}$ and $c_1, ..., c_r \in \mathbb{C} \setminus \{0\}$, the equality $c_1 q'(s) + ... + c_r q^{(r)}(s) = p(s)$ holds. Therefore, by Theorem 3.2, the function

$$c_1\zeta'(s,\alpha;\mathfrak{a}) + ... + c_r\zeta^{(r)}(s,\alpha;\mathfrak{a})$$

is universal in the sense of Theorem 3.2.

The continuity requirement for the operator F is Theorem 3.2 can be replaced by an analogue of the Lipschitz condition in the space of analytic functions. More precisely, the following theorem is true.

Theorem 3.3. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that the operator F : $H(D) \to H(D)$ is such that, for each polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is not empty, and, for each $K \in \mathcal{K}$, there exists positive constants c and β, and $K_1 \in \mathcal{K}$ that*

$$\sup_{s\in K} |F(g_1(s)) - F(g_2(s))| \le \sup_{s\in K_1} |g_1(s) - g_2(s)|^\beta$$

for all $g_1, g_2 \in H(D)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 3.1 is true.

It is not difficult to see that, in virtue of the Cauchy integral formula, the operator

$$F(g) = g^{(r)}, r \in \mathbb{N}, g \in H(D),$$

satisfies the hypothesis of Theorem 3.3 with $\beta = 1$.

Now we restrict the class of approximated analytic functions, and state an analogue of Theorem J for the periodic Hurwitz zeta-function obtained in Chapter 3 of the thesis. For different complex numbers $a_1, ..., a_r$, define the set

$$H_{a_1,...,a_r}(D) = \{g \in H(D) : g(s) \ne a_j, j = 1, ..., r\}.$$

Theorem 3.4. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator such that $F(H(D)) \supset H_{a_1,\dots,a_r}(D)$. For $r = 1$, let $K \in \mathcal{K}$ and $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. For $r \geq 2$, let $K \subset D$ be an arbitrary compact set, and $f(s) \in H_{a_1,\dots,a_r}(D)$. Then the same assertion as in Theorem 3.1 is true.*

We see that the set $H_{a_1,\dots,a_r}(D)$ differs a bit from that used in Theorem J. This difference arises from the fact that, in Theorem J, a condition for $F(S)$ is used, while, in Theorem 3.4, this condition is replaced by that for $F(H(D))$. More precisely, this is related to the approximation of analytic functions from the class $H_0(K), K \in \mathcal{K}$, while, in Theorem 1.1, the approximated functions belong to the class $H(D), K \in \mathcal{K}$.

Solving the equation

$$\sin(g) = f, g \in H(D),$$

we easily find that if $f \in H_{-1,1}(D)$, then, by Theorem 3.4 with $r = 2$, the function $f(s)$ can be approximated by shifts $\sin(\zeta(s + i\tau, \alpha; \mathfrak{a}))$. A similar statement is also true for the shifts $\cos(\zeta(s + i\tau, \alpha; \mathfrak{a}))$.

In the thesis, the following general theorem is obtained.

Theorem 3.5. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator. Let $K \subset D$ be an arbitrary compact subset, and $f(s) \in F(H(D))$. Then the same assertion as in Theorem 3.1 is true.*

Theorem 3.5 is the last theorem of Chapter 3.

The results of Chapter 3, under the hypothesis that the parameter α is transcendental, are obtained in [35]. Thus, the results of the thesis are more general that those of [35].

In Chapter 4, applications of universality theorems from previous chapters for estimates of the number of zeros of the function $\zeta(s, \alpha; \mathfrak{a})$ are given.

S. M. Voronin [54] applied the joint universality of Dirichlet L-functions to prove a lower estimate for the number of zeros of the Hurwitz zeta-function with rational parameter α. His theorem has the following form.

Theorem L. *Suppose that $\alpha = \frac{a}{b}$, $(a, b) = 1$, $0 < a < b$. Then, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c = c(\alpha, \sigma_1, \sigma_2) > 0$ such that, for sufficiently large T, the function $\zeta(s, \alpha)$ has more than cT zeros lying in the rectangle $\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, |t| < T\}$.*

In the thesis, we obtain generalizations of Theorem L. We say that, for a certain function $f(s)$, the assertion $A(\sigma_1, \sigma_2; c, T)$ is valid if, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c > 0$ such that, for sufficiently large T, the function $f(s)$ has more than cT zeros lying in the rectangle $\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}$.

Theorem 4.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then, for the function $\zeta(s, \alpha; \mathfrak{a})$, the assertion $A(\sigma_1, \sigma_2; c, T)$ is valid.*

If $a_m \equiv 1$, Theorem 4.1 extends Theorem L for the Hurwitz zeta-function.

Theorem 4.2. *Suppose that* $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a, b) = 1, b \neq 2$ *and that* $\mathrm{rad}(q)$ *divides* b. *Then, for the function* $\zeta\big(s, \frac{a}{b}; \mathfrak{a}\big)$, *the assertion* $A(\sigma_1, \sigma_2; c, T)$ *is valid.*

Other theorems of Chapter 4 are of discrete type. We say that, for a certain function $f(s)$, the assertion $B(\sigma_1, \sigma_2; c, \varphi, k_0, N)$ is valid if, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c > 0$ such that, for sufficiently large N, the function $f\big(s + i\varphi(k)\big)$ has a zero in the disc

$$\left| s - \frac{\sigma_1 + \sigma_2}{2} \right| \leq \frac{\sigma_2 - \sigma_1}{2}$$

for more than cN integers $k, k_0 \leq k \leq N$.

Theorem 4.3. *Suppose that the set* $L(\alpha, h, \pi)$ *is linearly independent over* $\mathbb{Q}$. *Then, for the function* $\zeta(s, \alpha; \mathfrak{a})$, *the assertion* $B(\sigma_1, \sigma_2; c, kh, 0, N)$ *is valid.*

The next theorem is a discrete analogue of Theorem 4.2, with rational α.

Theorem 4.4. *Suppose that* $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a, b) = 1, b \neq 2$ *and that* $\mathrm{rad}(q)$ *divides* b. *Then, for the function* $\zeta\big(s, \frac{a}{b}; \mathfrak{a}\big)$, *the assertion* $B(\sigma_1, \sigma_2; c, kh, 0, N)$ *is valid.*

The thesis ends by a generalization of Theorem 4.3.

Theorem 4.5. *Suppose that the set* $L(\alpha)$ *is linearly independent over* $\mathbb{Q}$, *and that* β_1, $0 < \beta_1 < 1$, *and* $\beta_2 > 0$ *are fixed numbers. Then, for the function* $\zeta(s, \alpha; \mathfrak{a})$, *the assertion* $B(\sigma_1, \sigma_2; c, k^{\beta_1} \log^{\beta_2} k, 2, N)$ *is valid.*

Approbation

The main results of the thesis were presented at the MMA (Mathematical modelling and Analysis) International Conferences ((MMA 2015, May 26–29, 2015, Sigulda, Latvia), (MMA 2016, June 1–4, 2016, Tartu, Estionia), (MMA 2017, May 30–June 2, 2017, Druskininkai, Lithuania), (MMA 2018, May 29–June 1, 2018, Sigulda, Latvia)), XV International Conference Algebra, Number Theory and Discrete Geometry: modern problems and application (May 28–31, 2018, Tula, Russia), at the Conferences of the Lithuanian Mathematical Society (2017, 2018), as well as at the Number Theory Seminars of Vilnius University.

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4. D. Mochov and A. Laurinčikas, Universality of the periodic Hurwitz zeta-function with rational parameter: 25-th Intern. Conf. May 28-31, 2018, Tula, Russia.

5. D. Mochov and A. Laurinčikas, On discrete universality of the periodic Hurwitz zeta-function with rational parameter, Abstracts of MMA2018, May 29-June 1, 2018, Sigulda, Latvia, p.53.

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Chapter 1

Various cases of the parameter α

The periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a}), s=\sigma+it$, depends on the parameter $\alpha, 0<\alpha\leq 1$, and the arithmetic of this parameter is reflected in its analytic properties, including the universality. We divide the set $\{\alpha\in\mathbb{R}:\alpha, 0<\alpha\leq 1\}$ into three parts: of transcendental, rational and algebraic irrational numbers. A continuous universality theorem for $\zeta(s,\alpha;\mathfrak{a})$ with transcendental α was obtained by A. Javtokas and A. Laurinčikas in [14], therefore, it remains to discuss the case of algebraic parameter α. Unfortunately, we have no any idea to obtain the universality of $\zeta(s,\alpha;\mathfrak{a})$ with algebraic irrational α. In place of this, we prove an universality theorem for $\zeta(s,\alpha;\mathfrak{a})$ with α such that the set

$$L(\alpha)=\{\log(m+\alpha):m\in\mathbb{N}_0\}.$$

is linearly independent over $\mathbb{Q}$. As it was mentioned in the introduction, the Cassels theorems suggests a conjecture that there exist algebraic irrational numbers α with the set $L(\alpha)$ linearly independent over $\mathbb{Q}$. Moreover, using of the set $L(\alpha)$ extends the case of transcendental α.

In this chapter, we also consider the universality of $\zeta(s,\alpha;\mathfrak{a})$ with rational parameter α. This case is based on the joint universality of Dirichlet L-functions.

A discrete universality theorem of this chapter also extends the case of transcendental parameter α discussed in [27], and uses the linear independence over $\mathbb{Q}$ of the set

$$L(\alpha,h,\pi)=\left\{(\log(m+\alpha):m\in\mathbb{N}),\frac{2\pi}{h}\right\}.$$

A discrete universality theorem for $\zeta(s,\alpha;\mathfrak{a})$ with rational parameter α, as a continuous one, also is based on the joint universality theorem for Dirichlet L-functions.

For all proofs, we apply probabilistic limit theorems in the space of analytic functions.

1.1 A continuous universality theorem involving the set $L(\alpha)$

In this section, we prove the following theorem. We remind that $\mathcal{K}$ is the class of compact subsets of the strip $D=\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\}$ with connected complements, and $H(K), K\in\mathcal{K}$, is the class of continuous functions on K that are analytic in the interior of K.

Theorem 1.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} \Big|\zeta(s+i\tau,\alpha;\mathfrak{a}) - f(s)\Big| < \varepsilon\Big\} > 0.$$

For the proof of Theorem 1.1, we apply a probabilistic approach based on weakly convergent probability measures in the space of analytic functions.

Let X be a metric space, and let $\mathcal{B}(X)$ be its Borel σ-field. Suppose that $P_n, n \in \mathbb{N}$, and P are probability measures on $(X, \mathcal{B}(X))$. We recall that P_n as $n \to \infty$, converges weakly to P if, for every real continuous bounded function f on X,

$$\lim_{n\to\infty} \int_X f\,dP_n = \int_X f\,dP.$$

We start with the definition of one topological structure. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, i.e., γ is the unit circle on the complex plane. Define the set

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. The set Ω consists of all functions $f : \mathbb{N}_0 \to \gamma$. On Ω, the product topology and operation of pointwise multiplication can be defined. Since γ is a compact set, the set Ω becomes a compact topological Abelian group. Therefore, the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$ can be defined, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of an element $\omega \in \Omega$ to the circle $\gamma_m, m \in \mathbb{N}_0$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s,\alpha,\omega;\mathfrak{a})$ ($H(D)$ is the space of analytic functions on D endowed with the topology of uniform convergence on compacta) by the formula

$$\zeta(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}.$$

We observe that the latter series is uniformly convergent on compact subsets of the strip D [13]. Let P_ζ be the distribution of the random element $\zeta(s,\alpha,\omega;\mathfrak{a})$, i.e., P_ζ is a probability measure on $\big(H(D), \mathcal{B}(H(D)\big)$ defined by

$$P_\zeta(A) = m_H\{\omega \in \Omega : \zeta(s,\alpha,\omega;\mathfrak{a}) \in A\},$$

Then the following proposition is the main ingredient of the proof of Theorem 1.1.

Proposition 1.2. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then*

$$P_T(A) \stackrel{def}{=} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \zeta(s,\alpha,\omega;\mathfrak{a}) \in A\Big\}, A \in \mathcal{B}\big(H(D)\big),$$

converges weakly to the measure P_ζ as $T \to \infty$.

We divide the proof of Proposition 1.2 into several lemmas. First of all, we will prove a limit theorem on weakly convergent probabilities measures on Ω. For $A \in \mathcal{B}(\Omega)$, define

$$Q_T(A) = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\big) \in A\Big\}.$$

Lemma 1.3. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then Q_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. We will apply the Fourier transform method. It is well known that the characters χ of the group Ω are of the form

$$\chi(\omega) = \prod_{m=0}^{\infty} \omega^{k_m}(m),$$

where only a finite number of integers k_m are distinct from zero. Therefore, the Fourier transform $g_T(\underline{k}), \underline{k} = \{k_m \in \mathbb{Z} : m \in \mathbb{N}_0\}$, is defined by

$$g_T(\underline{k}) = \int_{\Omega} \prod_{m=0}^{\infty} \omega^{k_m}(m) dQ_T.$$

Thus, by the definition of Q_T,

$$g_T(\underline{k}) = \frac{1}{T} \int_0^T (m+\alpha)^{-ik_m\tau} d\tau,$$

where only a finite numbers of integers k_m are distinct from zero. Hence,

$$g_T(\underline{k}) = \frac{1}{T} \int_0^T \exp\Big\{-i\tau {\sum_{m=0}^{\infty}}' k_m \log(m+\alpha)\Big\} d\tau, \tag{1.1}$$

where $\sum'$ means that

$${\sum_{m=0}^{\infty}}' k_m \log(m+\alpha)$$

is a finite sum because only a finite number of integers are non-zeros. Obviously,

$$g_T(\underline{0}) = 1. \tag{1.2}$$

Now suppose that $\underline{k} \neq \underline{0}$. Since the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, we have in this case that

$${\sum_{m=0}^{\infty}}' k_m \log(m+\alpha) \neq 0.$$

Therefore, it follows from (1.1) after integration that

$$g_T(\underline{k}) = \frac{1 - \exp\Big\{-iT {\sum_{m=0}^{\infty}}' k_m \log(m+\alpha)\Big\}}{iT {\sum_{m=0}^{\infty}}' k_m \log(m+\alpha)}.$$

Hence, for $\underline{k} \neq \underline{0}$,

$$\lim_{T\to\infty} g_T(\underline{k}) = 0,$$

and together with (1.2) we have that

$$\lim_{T\to\infty} g_T(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Since the right-hand side of this equality is the Fourier transform of the Haar measure m_H, we obtain by a continuity theorem for probability measures on compact groups, see, for example, Theorem 1.4.2 of [11], the assertion of the lemma. □

Now, let $\theta > \frac{1}{2}$ be a fixed number, and, for $m \in \mathbb{N}_0, n \in \mathbb{N}$,

$$v_n(m) = \exp\Big\{ - \Big(\frac{m+\alpha}{n+\alpha}\Big)^{\theta} \Big\}.$$

Consider the series

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m v_n(m,\alpha)}{(m+\alpha)^s}.$$

Lemma 1.4. *The series for the function $\zeta_n(s,\alpha;\mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$.*

Proof. By the Mellin formula [17], for positive a and b,

$$\frac{1}{2\pi i} \int\limits_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-1} ds = e^{-a},$$

where $\Gamma(s)$ is the Euler gamma-function. Therefore, putting

$$l_n(s,\alpha) = \frac{s}{\theta}\Gamma\Big(\frac{s}{\theta}\Big)(n+\alpha)^s, n \in \mathbb{N},$$

we find that

$$\frac{1}{2\pi i} \int\limits_{\theta-i\infty}^{\theta+i\infty} \frac{l_n(s,\alpha)}{s(s+\alpha)^s} ds = \frac{1}{2\pi i} \int\limits_{\theta-i\infty}^{\theta+i\infty} \Gamma\Big(\frac{s}{\theta}\Big)\Big(\frac{m+\alpha}{n+\alpha}\Big)^{-s} d\Big(\frac{s}{\theta}\Big) = \exp\Big\{ - \Big(\frac{m+\alpha}{n+\alpha}\Big)^{\theta} \Big\} = v_n(m,\alpha). \quad (1.3)$$

Hence,

$$v_n(m,\alpha) \ll (m+\alpha)^{-\theta} \int\limits_{-\infty}^{+\infty} \big|l_n(\theta+it,\alpha)\big| dt \ll (m+\alpha)^{-\theta}.$$

Since, by periodicity, the coefficients a_m are bounded, and $\theta > \frac{1}{2}$, this proves the lemma. □

For $\omega \in \Omega$, define

$$\zeta(s, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) v_n(m,\alpha)}{(m+\alpha)^s}.$$

Since $\big|\omega(m)\big| = 1$, the latter series is also absolutely convergent for $\sigma > \frac{1}{2}$.

The next lemma consider the weak convergence for

$$P_{T,n}(A) \overset{def}{=} \frac{1}{T} \text{meas}\Big\{ \tau \in [0,T] : \zeta(s+i\tau, \alpha; \mathfrak{a}) \in A \Big\}$$

and

$$\hat{P}_{T,n}(A) \overset{def}{=} \frac{1}{T} \text{meas}\Big\{ \tau \in [0,T] : \zeta(s+i\tau, \alpha, \hat{\omega}; \mathfrak{a}) \in A \Big\},$$

where $A \in \mathcal{B}(H(D))$ and $\hat{\omega} \in \Omega$. For this, the following property of weak convergence of probability measures will be useful. Let X_1 and X_2 be two metric (or topological) spaces. The mapping $u : X_1 \to X_2$ is called $(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable if $u^{-1}\mathcal{B}(X_2) \subset \mathcal{B}(X_1)$. If u is a continuous mapping, then it is $(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable [2]. Suppose that u is $(\mathcal{B}(X_1), \mathcal{B}(X_2))$-measurable. Then, it is known [2] that every probability measure P on $(X_1, \mathcal{B}(X_1))$ induces the unique probability measure Pu^{-1} on $(X_2, \mathcal{B}(X_2))$ which is defined, for all $A \in \mathcal{B}(X_2)$, by the equality

$$Pu^{-1}(A) = P(u^{-1}A).$$

Here $u^{-1}A$ denotes the preimage of the set A. Also, the following lemma is valid [2].

Lemma 1.5. *Let $P_n, n \in \mathbb{N}$, and P be probability measures on $(X_1, \mathcal{B}(X_1))$, and let $u : X_1 \to X_2$ be a continuous mapping. Suppose that P_n converges weakly to P as $n \to \infty$. Then $P_n u^{-1}$ also converges weakly to Pu^{-1} as $n \to \infty$.*

Lemma 1.6. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,n}$ and $\hat{P}_{T,n}$ both converge weakly to the same probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$.*

Proof. Define the function $u_n : \Omega \to H(D)$ by the formula

$$u_n(\omega) = \zeta_n(s, \alpha, \omega; \mathfrak{a}), \omega \in \Omega.$$

The series for $\zeta_n(s, \alpha, \omega; \mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$. Therefore, it is uniformly convergent on compact subsets of the strip D, and uniformly in ω. Hence, we can deal with a partial sum of the above series. In view of properties of the product topology, we see that this partial sum is a continuous function. Thus, we obtain that the function u_n is continuous.

By the definition of $\zeta_n(s, \alpha, \omega; \mathfrak{a})$, we have that

$$u_n\big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\big) = \zeta_n(s, \alpha, \omega; \mathfrak{a}).$$

Therefore, for all $A \in \mathcal{B}(H(D))$,

$$P_{T,n}(A) = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\big) \in u^{-1}A\Big\} = Q_T(u^{-1}A),$$

where Q_T is from Lemma 1.3. This shows that $P_{T,n} = Q_T u^{-1}$. Since, by Lemma 1.3, Q_T converges weakly to m_H as $T \to \infty$, using of Lemma 1.5 shows that $P_{T,n}$ converges weakly to the probability measure $P_n = m_H u_n^{-1}$ as $T \to \infty$.

Now let the function $\hat{u} : \Omega \to H(D)$ be given by the formula

$$\hat{u}_n(\omega) = \zeta_n(s, \alpha, \hat{\omega}\omega; \mathfrak{a}), \omega \in \Omega.$$

Then, similarly as above, we find that $\hat{P}_n$ converges weakly to the probability measure $\hat{P}_n = m_H \hat{u}_n^{-1}$ as $T \to \infty$. We have to show that $P_n = \hat{P}_n$. For this, we take a function $u : \Omega \to \Omega$ given by

$u(\omega) = \hat{\omega}\omega, \omega \in \Omega$. Then we see that $\hat{u}_n = u_n(u)$. Since the Haar measure m_H is invariant with respect to translations by point of Ω, we obtain that

$$\hat{P}_n = m_H\big(u_n(u)\big)^{-1} = (m_H u^{-1})u_n^{-1} = m_H u_n^{-1} = P_n,$$

and the lemma is proved. □

The next part of the proof of Proposition 1.2 consists of the approximation in the mean of the function $\zeta(s,\alpha;\mathfrak{a})$ by $\zeta_n(s,\alpha;\mathfrak{a})$, and of the function $\zeta(s,\alpha,\omega;\mathfrak{a})$ by $\zeta_n(s,\alpha,\omega;\mathfrak{a})$. For this, we recall the metric of the space $H(D)$ which induces its topology of uniform convergence on compacta. It is well known, see, for example, [5], that there exists a sequence $\{K_l : l \in \mathbb{N}\}$ of compact subsets of the strip D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \in D$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. For $f, g \in H(D)$, define

$$\rho(f,g) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s\in K_l} |f(s)-g(s)|}{1+\sup_{s\in K_l} |f(s)-g(s)|}.$$

Then ρ is the desired metric on $H(D)$.

Lemma 1.7. *The equality*

$$\lim_{n\to\infty} \liminf_{T\to\infty} \frac{1}{T}\int_0^T \rho\big(\zeta(s+i\tau,\alpha;\mathfrak{a}), \zeta_n(s+i\tau,\alpha;\mathfrak{a})\big)d\tau = 0$$

holds for all $0<\alpha\leq 1$ *and* $\mathfrak{a}$.

Proof. In view of (1.3) and the definition of $\zeta(s,\alpha;\mathfrak{a})$, we find that, for $\sigma > \frac{1}{2}$,

$$\zeta(s,\alpha;\mathfrak{a}) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z,\alpha;\mathfrak{a}) l_n(z,\alpha)\frac{dz}{z}. \tag{1.4}$$

Let K be arbitrary compact subset of the strip D, and we suppose that $s \in K$. We take σ_1, $\frac{1}{2} < \sigma_1 < 1$, and $\sigma_1 < \sigma$. Then, moving the line of integration in (1.4) to the left, we obtain by the residue theorem that

$$\zeta_n(s,\alpha;\mathfrak{a}) - \zeta(s,\alpha;\mathfrak{a}) = \frac{1}{2\pi i}\int_{\sigma_1-\sigma-i\infty}^{\sigma_1-\sigma+i\infty} \zeta(s+z,\alpha;\mathfrak{a}) l_n(z,\alpha)\frac{dz}{z} + R_n(s,\alpha;\mathfrak{a}),$$

where

$$R_n(s,\alpha;\mathfrak{a}) = \operatorname*{Res}_{z=1-s} \zeta(s+z,\alpha;\mathfrak{a}) l_n(z,\alpha) z^{-1}. \tag{1.5}$$

If the function $\zeta(s,\alpha;\mathfrak{a})$ is entire, then $R_n(s,\alpha;\mathfrak{a}) = 0$. Let L be a simple closed contour of length $|L|$ lying in D and enclosing the set K. Then, denoting by δ the distance of the contour L from the set

K and applying the Cauchy integral formula, we obtain easily the estimate

$$\sup_{s\in K}\Big|\zeta(s+i\tau,\alpha;\mathfrak{a})-\zeta_n(s+i\tau,\alpha;\mathfrak{a})\Big|\le\frac{1}{2\pi\delta}\int\limits_L\Big|\zeta(z+i\tau,\alpha;\mathfrak{a})-\zeta_n(z+i\tau,\alpha;\mathfrak{a})\Big||dz|.$$

Therefore,

$$\frac{1}{T}\int\limits_0^T\sup_{s\in K}\big|\zeta(s+i\tau,\alpha;\mathfrak{a})-\zeta_n(s+i\tau,\alpha;\mathfrak{a})\big|d\tau\ll\frac{1}{T\delta}\int\limits_L|dz|\Big(\int\limits_0^T\big|\zeta(z+i\tau,\alpha;\mathfrak{a})-\zeta_n(z+i\tau,\alpha;\mathfrak{a})\big|d\tau\Big)\ll$$

$$\frac{|L|}{T\delta}\sup_{s\in L}\int\limits_0^T\big|\zeta(\sigma+it+i\tau,\alpha;\mathfrak{a})-\zeta_n(\sigma+it+i\tau,\alpha;\mathfrak{a})\big|d\tau. \tag{1.6}$$

Here and in the sequel $a\ll b, b>0$, means that there exists a constant $C>0$ such that $|a|\le Cb$. By (1.5), we have that

$$\zeta(\sigma+it+i\tau,\alpha;\mathfrak{a})-\zeta_n(\sigma+it+i\tau,\alpha;\mathfrak{a})\ll$$

$$\int\limits_{-\infty}^{+\infty}\big|\zeta(\sigma_1+it+i\tau+iu,\alpha;\mathfrak{a})\big|\big|l_n(\sigma_1-\sigma+iu,\alpha)\big|du+\big|R_n(\sigma+it+i\tau,\alpha;\mathfrak{a})\big|.$$

Thus, in view of (1.6),

$$\frac{1}{T}\int\limits_0^T\big|\zeta(\sigma+it+i\tau,\alpha;\mathfrak{a})-\zeta_n(\sigma+it+i\tau,\alpha;\mathfrak{a})\big|d\tau\ll$$

$$\int\limits_{-\infty}^{+\infty}l_n\big|(\sigma_1-\sigma+iu,\alpha)\big|\Big(\frac{1}{T}\int\limits_0^T\big|\zeta(\sigma_1+it+i\tau+iu,\alpha;\mathfrak{a})\big|d\tau\Big)du+\frac{1}{T}\int\limits_0^T\big|R_n(\sigma+it+i\tau,\alpha;\mathfrak{a})\big|d\tau. \tag{1.7}$$

If $s\in L$, then t is bounded by a constant depending on the set K. Therefore, using the estimate [13]

$$\int\limits_0^T\big|\zeta(\sigma+it,\alpha;\mathfrak{a})\big|^2\ll T$$

which is valid for $\sigma>\frac{1}{2}$, and the Cauchy-Schwarz inequality, we obtain that

$$\frac{1}{T}\int\limits_0^T\big|\zeta(\sigma_1+it+i\tau+iu,\alpha;\mathfrak{a})\big|du\ll\Big(\frac{1}{T}\int\limits_0^T\big|\zeta(\sigma_1+it+i\tau+iu,\alpha;\mathfrak{a})\big|^2d\tau\Big)^{\frac{1}{2}}\ll1+|u|. \tag{1.8}$$

Moreover, the well-known estimates for the gamma-function, see, for example, [26], give that

$$\frac{1}{T}\int\limits_0^T|R_n(\sigma+it+i\tau,\alpha;\mathfrak{a})|d\tau=o(1)$$

as $T\to\infty$. this and estimates (1.6)-(1.8) lead to

$$\frac{1}{T}\int\limits_0^T\sup_{s\in K}\big|\zeta(s+i\tau,\alpha;\mathfrak{a}))-\zeta_n(s+i\tau,\alpha;\mathfrak{a}))\big|d\tau\ll\sup_{s\in L}\int\limits_{-\infty}^{+\infty}\big|l_n(\sigma_1-\sigma+iu,\alpha)\big|(1+|u|)du+o(1) \tag{1.9}$$

as $T \to \infty$. However, by choice of σ_1, we have $\sigma_1 - \sigma < 0$. Therefore,

$$\lim_{n\to\infty} \sup_{s\in L} \int_{-\infty}^{+\infty} \big|l_n(\sigma_1 - \sigma + iu, \alpha)\big|(1+|u|)du = 0,$$

and, in view of (1.9),

$$\lim_{n\to\infty} \liminf_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} \big|\zeta(s+i\tau, \alpha; \mathfrak{a})) - \zeta_n(s+i\tau, \alpha; \mathfrak{a}))\big|d\tau = 0.$$

This and the definition of the metric ρ prove the lemma. □

We also need the analogue of Lemma 1.7 for the functions $\zeta(s, \alpha, \omega; \mathfrak{a})$ and $\zeta_n(s, \alpha, \omega; \mathfrak{a})$. For the proof of such lemma, we will apply some elements of the ergodic theory.

For real numbers τ, we put

$$a_\tau = \big\{(m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\big\},$$

and define the family of $\big\{\varphi_\tau : \tau \in \mathbb{R}\big\}$ of transformations on the torus Ω by

$$\varphi_\tau(\omega) = a_\tau \omega, \omega \in \Omega.$$

Then $\big\{\varphi_\tau : \tau \in \mathbb{R}\big\}$ is the one-parameter group of measurable measure preserving transformations on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. We say that a set $A \in \mathcal{B}(\Omega)$ is invariant with respect to the group $\big\{\varphi_\tau : \tau \in \mathbb{R}\big\}$ if the sets A and $A_\tau = \varphi(A)$ can differ one form another at most by a set of zero m_H measure. All invariant sets from a sub-σ-field of the field $\mathcal{B}(\Omega)$. The group $\big\{\varphi_\tau : \tau \in \mathbb{R}\big\}$ is called ergodic if its σ-field of invariant sets consists only from sets of m_H-measure 0 or 1.

Lemma 1.8. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then the group $\big\{\varphi_\tau : \tau \in \mathbb{R}\big\}$ is ergodic.*

Proof. The lemma is a one-dimensional case of Lemma 6 from [14]. However, for fullness, we give its proof.

In the proof of Lemma 1.3, we have seen that the characters χ of the group Ω are of the form

$$\chi(\omega) = \prod_{m=0}^{\infty} \omega^{km}(m), \tag{1.10}$$

where only a finite number of integers k_m are distinct from zero. Suppose that χ is a non-trivial character $\big(\chi(\omega) \not\equiv 1\big)$. Clearly, a_τ is an element of Ω, thus, in view of (1.10),

$$\chi(a_\tau) = \prod_{m=0}^{\infty} (m+\alpha)^{-i\tau k_m} = \exp\Big\{ -i\tau \sum_{m=0}^{\infty} k_m \log(m+\alpha)\Big\},$$

where, as above, only a finite number of integers k_m are non-zeros. Since the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$,

$$\sum_{m=0}^{\infty} k_m \log(m+\alpha) \neq 0$$

for all $k_m \neq 0$. Therefore, there exists $\tau_0 \in \mathbb{R} \setminus \{0\}$ such that

$$\chi(a_{\tau_0}) \neq 1. \tag{1.11}$$

Let A be an invariant set of the group $\{\varphi_\tau : \tau \in \mathbb{R}\}$, and let I_A be the indicator function of the set A. Then the invariant property implies, for every and almost all $\omega \in \Omega$ and fixed $\tau \in \mathbb{R}$, the equality

$$I_A\big(\varphi_\tau(\omega)\big) = I_A(\omega). \tag{1.12}$$

Haar measure m_H is invariant, i.e.,

$$m_H(A) = m_H(\omega A) = m_H(A\omega)$$

for all $\omega \in \Omega$. Therefore, in view of (1.12), for the Fourier transform $\hat{I}_A(\chi)$ of the function $I_A(\omega)$, we have

$$\hat{I}_A(\chi) = \int\limits_\Omega \chi(\omega) I_A(\omega) m_H(d\omega) = \int\limits_\Omega \chi\big(\varphi_{\tau_0}(\omega)\big) I_A\big(\varphi_{\tau_0}(\omega)\big) m_H(d\omega) =$$

$$\chi(a_{\tau_0}) \int\limits_\Omega \chi(\omega) I_A(\omega) m_H(d\omega) = \chi(a_{\tau_0}) \hat{I}_A(\chi).$$

This together with (1.11) shows that $\hat{L}_A(\chi) = 0$ for all non-trivial characters χ of Ω.

Now let χ_0 be the trivial character of Ω, i.e. $\chi_0(\omega) \equiv 1$. Suppose that $\hat{I}_A(\chi_0) = u$. Then the equalities $\hat{I}_A(\chi) = 0$ for $\chi \neq \chi_0$ and $\hat{I}_A(\chi_0) = u$, and the orthogonality of characters

$$\int\limits_\Omega \chi(\omega) m_H(d\omega) = \begin{cases} 1 & \text{if} \quad \chi = \chi_0, \\ 0 & \text{if} \quad \chi \neq \chi_0, \end{cases}$$

give

$$\hat{I}_A(\chi) = u \int\limits_\Omega \chi(\omega) m_H(d\omega) = u\hat{1}(\chi) = \hat{u}(\chi).$$

It is well known that the function $I_A(\omega)$ is uniquely determined by its Fourier transform. Therefore, $I_A(\omega) = u$ for almost all $\omega \in \Omega$. On the other hand, $I_A(\omega)$ is the indicator function of the set A. Thus, $u = 0$ or $u = 1$. Hence, it follows that $I_A(\omega) = 0$ or $I_A(\omega) = 1$ for almost all $\omega \in \Omega$. This means that $m_H(A) = 0$ or $m_H(A) = 1$, i.e., the group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is ergodic. □

For the statement of the classical Birkkoff-Klintchine theorem, we remind the definition of the ergodic process. Let $X(\tau, \omega)$ be a random process defined on a certain probability space. The process $X(\tau, \omega)$ is called strongly stationary if its finite-dimensional distributions are invariant with respect to the translations. The strongly stationary process $X(\tau, \omega)$ is called ergodic if its σ-field of invariant sets consists only of the sets of Q-measure 0 or 1, where Q is the probability measure defined by the family of finite-dimensional distributions of $X(\tau, \omega)$. Denote by $\mathbb{E}X$ the expectation of X.

Lemma 1.9. *(Birkhoff-Klintchine theorem). Suppose that $X(\tau, \omega)$ is an ergodic process, $\mathbb{E}|X(\tau, \omega)| < \infty$, with sample paths integrable almost surely in the Riemann sense over every finite integral. Then, for almost all ω,*

$$\lim_{T \to \infty} \frac{1}{T} \int\limits_0^T X(\tau, \omega) d\tau = \mathbb{E}X(0, \omega).$$

The proof of the lemma and other elements of the ergodic theory are given, for example, in [6].

We use Lemmas 1.8 and 1.9 for the proof of the following estimate.

Lemma 1.10. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ and $\sigma > \frac{1}{2}$. Then, for almost all $\omega \in \Omega$,*

$$\int_0^T \left|\zeta(\sigma + it, \alpha, \omega; \mathfrak{a})\right|^2 dt \ll T.$$

Proof. Let

$$\hat{\zeta}(\sigma, \alpha, \omega; \mathfrak{a}) = \Big| \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^\sigma} \Big|^2.$$

The random variables $\omega(m)$ are pairwise orthogonal, i.e.,

$$\int_\Omega \omega(m)\overline{\omega(n)} dm_H = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

Therefore,

$$\mathbb{E}\hat{\zeta}(\sigma, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} < \infty. \tag{1.13}$$

On the other hand, the definition of $\varphi_t(\omega)$ shows that

$$\hat{\zeta}(\sigma, \alpha, \varphi_t(\omega); \mathfrak{a}) = \left|\zeta(\sigma, \alpha, \varphi_t(\omega); \mathfrak{a})\right|^2 = \left|\zeta(\sigma + it, \alpha, \omega; \mathfrak{a})\right|^2.$$

In view of Lemma 1.8, the random process $\left|\zeta(\sigma + it, \alpha, \omega; \mathfrak{a})\right|^2$ is ergodic. Therefore, Lemma 1.9 and (1.13) give

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left|\zeta(\sigma + it, \alpha, \omega; \mathfrak{a})\right|^2 dt = \mathbb{E}\hat{\zeta}(\sigma, \alpha, \omega; \mathfrak{a}) < \infty$$

for almost all $\omega \in \Omega$, and lemma is proved. □

Now, we a ready to obtain the analogue of Lemma 1.7 for the functions $\zeta(s, \alpha, \omega; \mathfrak{a})$ and $\zeta_n(s, \alpha, \omega; \mathfrak{a})$.

Lemma 1.11. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then, for almost all $\omega \in \Omega$,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\zeta(s + i\tau, \alpha, \omega; \mathfrak{a}), \zeta_n(s + i\tau, \alpha, \omega; \mathfrak{a})\big) d\tau = 0$$

Proof. We repeat the proof of Lemma 1.7, and use Lemma 1.10 in place of the estimate

$$\int_0^T \left|\zeta(\sigma + it, \alpha; \mathfrak{a})\right|^2 dt \ll T, \qquad \frac{1}{2} < \sigma < 1.$$

□

Lemmas 1.7 and 1.11 allow to obtain limit theorems in the space $H(D)$ for the functions $\zeta(s, \alpha; \mathfrak{a})$ and $\zeta(s, \alpha, \omega; \mathfrak{a})$. However, we start with some results of probability theory. The family $\{P\}$ of probability measures on $\big(X, \mathcal{B}(X)\big)$ is called relatively compact if every sequence of elements $\{P\}$

contains a weakly convergent subsequence. The family $\{P\}$ is called tight if, for every $\varepsilon > 0$, there exists a compact set $K \subset X$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$.

The Prokhorov theorem connects the notions of relative compactness and tightness.

Lemma 1.12. *Suppose that the family $\{P\}$ is tight. Then it is relatively compact.*

Proof of the lemma is given in [2], Theorem 6.1.

The following statement will be also useful. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

Lemma 1.13. *Let (X, ρ) be separable metric space, and $Y_n, X_{1n}, X_{2n}, \ldots$ be a X-valued random elements defined on certain probability space with the measure μ. Suppose that, for each $k \in \mathbb{N}$,*

$$X_{kn} \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X_k,$$

and

$$X_k \underset{k\to\infty}{\xrightarrow{\mathcal{D}}} X.$$

If, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\big(\rho(X_{kn}, Y_n) \geq \varepsilon\big) = 0,$$

then also

$$Y_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X.$$

Proof of the lemma is given in [2], Theorem 4.2.

For $A \in \mathcal{B}\big(H(D)\big)$ and $\omega \in \Omega$, define

$$\hat{P}_T(A) = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \zeta(s + i\tau, \alpha, \omega; \mathfrak{a}) \in A\Big\}.$$

Lemma 1.14. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then P_T and $\hat{P}_T$ both converge weakly to the same probability measure P on $\big(H(D), \mathcal{B}(H(D))\big)$ as $T \to \infty$.*

Proof. Let P_n be the limit measure in Lemma 1.6. At first we will prove that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight. On a certain probability space with probability measure μ, define a random variable ξ that is uniformly distributed on the interval $[0, 1]$, and set

$$X_{T,n} = X_{T,n}(s) = \zeta_n(s + iT_\xi, \alpha; \mathfrak{a}).$$

Let X_n be an $H(D)$-valued random element with the distribution P_n. Then, by Lemma 1.6, we have the relation

$$X_{T,n} \underset{T\to\infty}{\xrightarrow{\mathcal{D}}} X_n. \tag{1.14}$$

The series for $\zeta_n(s, \alpha; \mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$. Therefore,

$$\lim_{T\to\infty} \int_0^T \big|\zeta_n(\sigma + it, \alpha; \mathfrak{a})\big|^2 dt = \sum_{m=0}^{\infty} \frac{|a_m|^2 v_n^2(m, \alpha)}{(m + \alpha)^{2\sigma}} \ll \sum_{m=0}^{\infty} \frac{|a_m|^2}{(m + \alpha)^{2\sigma}} < \infty.$$

This and a simple application of the Cauchy integral formula show that

$$\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K_l}\big|\zeta_n(s+i\tau,\alpha;\mathfrak{a}))\big|d\tau \ll R_l<\infty,$$

where K_l is a compact set from the definition of the metric $\rho, l\in\mathbb{N}$. Let $\varepsilon>0$ be an arbitrary number, and $M_l=M_l(\varepsilon)=R_l 2^l\varepsilon^{-1}$. Then we find that, for all $l\in\mathbb{N}$ and $n\in\mathbb{N}$,

$$\limsup_{T\to\infty}\mu\big(\sup_{s\in K_l}|X_{T,n}(s)|>M_l\big)=\limsup_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:\sup_{s\in K_l}|\zeta_n(s+i\tau,\alpha;\mathfrak{a})|>M_l\Big\}\ll$$

$$\limsup_{T\to\infty}\frac{1}{M_lT}\int_0^T\sup_{s\in K_l}|\zeta_n(s+i\tau,\alpha;\mathfrak{a})|d\tau\le\frac{R_l}{M_l}=\frac{\varepsilon}{2^l}.$$

Hence, in view of (1.14),

$$\mu\big(\sup_{s\in K_l}|X_n(s)|>M_l\big)\le\frac{\varepsilon}{2^l}\tag{1.15}$$

for all $l\in\mathbb{N}$ and $n\in\mathbb{N}$. Define the set

$$K_\varepsilon=\big\{g\in H(D):\sup_{s\in K_l}|g(s)|\le M_l(\varepsilon), l\in\mathbb{N}\big\}.$$

Then the set K_ε is a compact in the space $H(D)$, and, in virtue of (1.15),

$$\mu(X_n\in K_\varepsilon)\ge 1-\sum_{l=1}^{\infty}\frac{\varepsilon}{2^l}=1-\varepsilon$$

for all $n\in\mathbb{N}$. Thus, by the definition of X_n,

$$P_n(K_\varepsilon)\ge 1-\varepsilon$$

for all $n\in\mathbb{N}$, i.e., the family of probability measures $\{P_n:n\in\mathbb{N}\}$ is tight.

Now, by Lemma 1.12, the family $\{P_n:n\in\mathbb{N}\}$ is relatively compact. Therefore, every sequence of $\{P_n\}$ contains a subsequence $\{P_{n_r}\}$ such that P_{n_r} converges weakly to a certain probability measure P on $\big(H(D),\mathcal{B}(H(D))\big)$ as $r\to\infty$. In other words,

$$X_{n_r}\xrightarrow[r\to\infty]{\mathcal{D}}P.\tag{1.16}$$

Define one more $H(D)$-valued random element $Y_T=Y_T(s)$ by the formula

$$Y_T(s)=\zeta(s+iT\xi,\alpha;\mathfrak{a}).$$

Then, taking into account Lemma 1.7, find that, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\mu\big(\rho(X_{T,n},Y_T)\ge\varepsilon\big)=\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:\rho\big(\zeta(s+i\tau,\alpha;\mathfrak{a}),\zeta_n(s+i\tau,\alpha;\mathfrak{a})\big)\ge\varepsilon\Big\}\le$$

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T\varepsilon}\int_0^T\rho\big(\zeta(s+i\tau,\alpha;\mathfrak{a}),\zeta_n(s+i\tau,\alpha;\mathfrak{a})\big)d\tau=0.$$

This equality, (1.14) and (1.16) show that all hypotheses of Lemma 1.13 are satisfied. Therefore, the relation

$$Y_T \xrightarrow[T\to\infty]{\mathcal{D}} P \tag{1.17}$$

holds. In other words, we have that P_T converges weakly to P as $T \to \infty$. Moreover, the relation (1.17) gives that the limit measure P is independent of the choice of the subsequence $\{P_{n_r}\}$. Since the sequence $\{P_n\}$ is relatively compact, this remark implies the relation

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{1.18}$$

It remains to prove that $\hat{P}_T$ also converges weakly to P as $T \to \infty$. For this, we define the $H(D)$-valued random elements

$$\hat{X}_{T,n}(s) = \zeta_n(s + iT\xi, \alpha, \omega; \mathfrak{a})$$

and

$$\hat{X}_T(s) = \zeta(s + iT\xi, \alpha, \omega; \mathfrak{a}),$$

where $\omega \in \Omega$, and repeat the above arguments with using Lemma 1.11 in place of Lemma 1.7, and the relation (1.18). This leads to the assertion that $\hat{P}_T$ also converges weakly to the measure P as $T \to \infty$. The lemma is proved. □

We will use the following equivalent of weak convergence of probability measures. We remind that $A \in \mathcal{B}(X)$ is called a continuity set of the probability measure P on $\big(X, \mathcal{B}(X)\big)$ if $P(\partial A) = 0$, where ∂A denotes the boundary of the set A.

Lemma 1.15. *Let $P_n, n \in \mathbb{N}$, and P be probability measures on $\big(X, \mathcal{B}(X)\big)$. P_n converges weakly toP as $n \to \infty$ if and only if, for every continuity set A of the measure P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

Proof of the lemma is given in [2], Theorem 1.2.

Proof of Proposition 1.2. In view of Lemma 1.14, it suffices to show that the limit measure P in Lemma 1.14 coincides with P_ζ.

We fix an arbitrary continuity set A of the measure P, and, on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$, define the random variable η by the formula

$$\eta(\omega) = \begin{cases} 1 & \text{if} \quad \zeta(s, \alpha, \omega; \mathfrak{a}) \in A, \\ 0 & \text{if} \quad \zeta(s, \alpha, \omega; \mathfrak{a}) \notin A. \end{cases}$$

Lemmas 1.14 and 1.15 imply the relation

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \zeta(s + i\tau, \alpha, \omega; \mathfrak{a}) \in A\Big\} = P(A). \tag{1.19}$$

By definition of η, we have

$$\mathbb{E}\eta = \int_{\Omega} \eta dm_H = m_H\big\{\omega \in \Omega : \zeta(s, \alpha, \omega; \mathfrak{a}) \in A\big\} = P_\zeta(A). \tag{1.20}$$

In virtue of Lemma 1.8, the process $\eta\big(\varphi_\tau(\omega)\big)$ is ergodic. Therefore, by Lemma 1.9,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \eta\big(\varphi_\tau(\omega)\big) d\tau = \mathbb{E}\eta \tag{1.21}$$

for almost all $\omega \in \Omega$. The definitions of η and φ_τ show that

$$\frac{1}{T} \int_0^T \eta\big(\varphi_\tau(\omega)\big) d\tau = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \zeta(s + i\tau, \alpha, \omega; \mathfrak{a}) \in A\Big\}.$$

From this and (1.20), (1.21), we find that

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \zeta(s + i\tau, \alpha, \omega; \mathfrak{a}) \in A\Big\} = P_\zeta(A).$$

This and (1.19) prove that $P(A) = P_\zeta(A)$ for every continuity set A of the measure P. However, all continuity sets constitute a determining class [2]. Therefore, $P(A) = P_\zeta(A)$ for all $A \in \mathcal{B}\big(H(D)\big)$, i.e. $P = P_\zeta$. The proposition is proved. □

The proof of Theorem 1.1 also uses the support of the measure P_ζ. Let P be a probability measure on $\big(X, \mathcal{B}(X)\big)$, and the space X is separable. We recall that the minimal closed set $S_P \subset X$ such that $P(S_P) = 1$ is called the support of P. The set S_P consists of all $x \in X$ such that, for every open neighborhood G of x, the inequality $P(G) > 0$ is satisfied.

We have mentioned in Introduction (Theorem C) that in [14] the universality of the function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental α was obtained. However, the transcendence of the parameter α is only used for the proof of a limit theorem, while the assertion of the support of the measure P_ζ is independent on the arithmetic of α. Therefore, we have the following statement [14].

Proposition 1.3. *The support of the measure P_ζ is the whole space $H(D)$.*

Theorem 1.1 is a corollary of Propositions 1.2 and 1.3, and of the Mergelyan theorem on the approximation of analytic functions by polynomials. The Mergelyan theorem is very important in the theory of universality of zeta-functions, therefore, we state it as a separate lemma.

Lemma 1.16. *Let $K \subset \mathbb{C}$ be a compact set with connected complement, and let $f(s)$ be a continuous function on K that is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

The Mergelyan theorem was obtained in [41], see also [56].

We also remind the equivalent of weak convergence of probability measures in terms of open sets.

Lemma 1.17. *Suppose that $P_n, n \in \mathbb{N}$, and P are probability measures on the space $\big(X, \mathcal{B}(X)\big)$. Then P_n, as $n \to \infty$, converges weakly to P if and only if, for every open set $G \subset X$,*

$$\liminf_{n\to\infty} P_n(G) \geq P(G).$$

Proof of the lemma is given in [2], Theorem 2.1.

Proof of Theorem 1.1. By Lemma 1.16, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{1.22}$$

Define the set

$$G_\varepsilon = \Big\{ g \in H(D) : \sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2} \Big\}.$$

Then G_ε is an open neighbourhood of the polynomial $p(s)$ which, in view of Proposition 1.3, is an element of the support of the measure P_ζ. Therefore $P_\zeta(G_\varepsilon) > 0$. This, Proposition 1.2 and Lemma 1.17 show that

$$\liminf_{T\to\infty} P_T(G_\varepsilon) \geq P_\zeta(G_\varepsilon) > 0.$$

Hence, by the definitions of P_T and G_ε,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{a}) - p(s)| < \frac{\varepsilon}{2} \Big\} > 0. \tag{1.23}$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{a}) - p(s)| < \frac{\varepsilon}{2}.$$

Then, for such τ, by inequality (1.22),

$$\sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon.$$

Thus,

$$\Big\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{a}) - p(s)| < \frac{\varepsilon}{2} \Big\} \subset \Big\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon \Big\}.$$

This and (1.23) prove the inequality

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \Big\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon \Big\} > 0.$$

The theorem is proved. □

The result of Section 1.1 is published in [29].

1.2 Universality of the function $\zeta(s,\alpha;\mathfrak{a})$ with rational parameter α

Theorem 1.1 is true for all periodic sequences $\mathfrak{a}$. However, in the case of rational parameter α, the sequence $\mathfrak{a}$ has a certain influence for universality of $\zeta(s,\alpha;\mathfrak{a})$. We need some conditions connecting the parameter α and the minimal period q of the sequence $\mathfrak{a}$. In this section, we prove the following theorem.

Theorem 1.2. *Suppose that* $\alpha = \frac{a}{b}, a,b \in \mathbb{N}, a < b, (a,b) = 1, b \neq 2$ *and that* $\mathrm{rad}(q)$ *divides* b. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} \big|\zeta\big(s+i\tau, \frac{a}{b}; \mathfrak{a}\big) - f(s)\big| < \varepsilon\Big\} > 0.$$

We note that the condition $\mathrm{rad}(q)|b$ is technical and is conditioned by the used method. We believe that this condition can be removed. The proof of Theorem 1.2 is based on the joint value-distribution of Dirichlet L-functions.

Let χ be a Dirichlet character modulo d. We remind that χ is a completely multiplicative $\big(\chi(m,n) = \chi(m)\chi(n)$ for all $m,n \in \mathbb{N}\big)$, periodic with period d $\big(\chi(m+d) = \chi(m)$, for all $m \in \mathbb{N}\big)$ function defined on $\mathbb{N}$ and taking complex values, such that $\chi(m) = 0$ for $(m,d) > 1$ and $\chi(m) \neq 0$ for $(m,d) = 1$. The corresponding Dirichlet L-function $L(s,\chi)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

and can be meromorphically continued to the whole complex plane. The function $L(s,\chi_0)$ with the principal character χ_0 modulo d has the unique simple pole at the point $s = 1$ with residue

$$\prod_{p|d} \Big(1 - \frac{1}{p}\Big),$$

where p is a prime number. If $\chi \neq \chi_0$, then the function $L(s,\chi)$ is entire. There are $\varphi(d)$, where $\varphi(d)$ denotes the Euler totient function, of L-functions $L(s,\chi)$ with characters modulo d.

Suppose that $(a,b) = 1, a,b \in \mathbb{N}$. Then the connection between the Hurwitz zeta function and Dirichlet L-functions is well known, namely,

$$\zeta\Big(s, \frac{a}{b}\Big) = \frac{b^s}{\varphi(b)} \sum_{\chi \bmod b} \overline{\chi}(a) L(s,\chi), \tag{1.23}$$

where sum runs over all Dirichlet characters modulo b. Since

$$\zeta(s,\alpha;\mathfrak{a}) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\Big(s, \frac{\alpha+l}{q}\Big),$$

we have from (1.23) that if $\mathrm{rad}(q)$ divides b, thus, $(a+bl, bq) = 1$ for all $l = 0,1,...,q-1$, then

$$\zeta(s, \frac{a}{b}; \mathfrak{a}) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta\Big(s, \frac{a+bl}{bq}\Big) =$$

$$\frac{b^s}{\varphi(bq)}\sum_{l=0}^{q-1} a_l \sum_{\chi \bmod b} \overline{\chi}(a+bl)L(s,\chi). \tag{1.24}$$

Let, for brevity, $r = \varphi(bq)$. Then equality (1.24) can be written in the form

$$\zeta\Big(s, \frac{a}{b}; \mathfrak{a}\Big) = \frac{b^s}{r}\big(b_1 L(s,\chi_1) + ... + b_r(s,\chi_1)\big), \tag{1.25}$$

where

$$b_j = \sum_{l=o}^{j-1} a_l \overline{\chi}_j(a+bl), j = 1, ..., r.$$

If $\frac{a}{b} \neq \frac{1}{2}$ and $a < b$, then $b \geq 3$. Thus, $bq \geq 3$ and $r \geq 2$. Moreover, in the right-hand side of (1.25), at least two coefficients b_j are distinct from zero. Actually, if only one $b_j \neq 0$, say, $b_1 \neq 0$, then

$$\zeta\Big(s, \frac{a}{b}; \mathfrak{a}\Big) = \frac{b^s b_1}{r} L(s,\chi_1),$$

and we have that the right-hand side has the Euler product

$$\frac{b^s b_1}{r}\prod_{p|bq}\Big(1 - \frac{\chi_1(p)}{p^s}\Big)^{-1},$$

while the function $\zeta\big(s, \frac{a}{b}; \mathfrak{a}\big)$ has no such a product.

As in section 1.1, we need a limit theorem for $\zeta\big(s, \frac{a}{b}; \mathfrak{a}\big)$ in the space of analytic functions. In this case, it is convinient, in place of the strip D, to use the rectangle $D_V = \big\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V\big\}$, where $V > 0$ is an arbitrary fixed number. Thus, $H(D_V)$ is the space of analytic functions on D_V endowed with the topology of uniform convergence on compacta.

Define the set

$$\Omega_1 = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω_1, as Ω, is a compact topological Abelian group, therefore, on $\big(\Omega_1, \mathcal{B}(\Omega_1)\big)$,the probability Haar measure m_{1H} exists. This gives the probability space $\big(\Omega_1, \mathcal{B}(\Omega_1), m_{1H}\big)$. Denote by $\omega(p)$ the projection of an element $\omega_1 \in \Omega_1$ to the circle γ_p and extend $\omega_1(p)$ to the set $\mathbb{N}$ by the formula

$$\omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), m \in \mathbb{N}.$$

Now, on the probability space $\big(\Omega_1, \mathcal{B}(\Omega_1), m_{1H}\big)$, define two $H(D_V)$-valued random elements

$$f_1(s,\omega_1) = \frac{b^s \overline{\omega_1(p)}}{r}$$

and

$$f_2(s,\omega_1) = \sum_{j=1}^{r} b_j L(s,\omega_1,\chi_j),$$

where

$$L(s,\omega_1,\chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)\omega_1(m)}{m^s}, j = 1, ..., r.$$

Moreover, we put

$$\zeta\big(s, \frac{a}{b}, \omega_1; \mathfrak{a}\big) = f_1(s, \omega_1) f_2(s, \omega_1). \tag{1.26}$$

We note that the series $L(s, \omega_1, \chi_j)$ is uniformly convergent on compact subsets of D_V for almost all $\omega_1 \in \Omega$. Denote by $P_{1\zeta}$ the distribution of the random element $\zeta\big(s, \frac{a}{b}, \omega_1; \mathfrak{a}\big)$, i.e.,

$$P_{1\zeta}(A) = m_{1H}\Big\{\omega_1 \in \Omega_1 : \zeta\Big(s, \frac{a}{b}, \omega_1; \mathfrak{a}\Big) \in A\Big\}, A \in \mathcal{B}\big(H(D_V)\big).$$

For $A \in \mathcal{B}\big(H(D_V)\big)$, define

$$P_{1T}(A) = \frac{1}{T} \operatorname{meas}\Big\{\tau \in [a, b] : \zeta\Big(s + i\tau, \frac{a}{b}; \mathfrak{a}\Big) \in A\Big\}.$$

Proposition 1.4. *Under the hypothesis of Theorem 1.2, P_{1T} converges weakly to the measure $P_{1\zeta}$ as $T \to \infty$. Moreover, the support of $P_{1\zeta}$ is the whole $H(D_V)$.*

Proof of Proposition 1.4. The function

$$f_1(s) = \frac{b^s}{r}$$

is a Dirichlet polynomial. Therefore, it is well known [17] that

$$\frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : f_1(s + i\tau) \in A\Big\}, A \in \mathcal{B}\big(H(D_V)\big), \tag{1.27}$$

converges weakly to the distribution of the random element $f_1(s, \omega_1)$, as $T \to \infty$. Moreover, in [20], Lemma 1, it was proved that

$$\frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \big(L(s + i\tau, \chi_1), ..., L(s + i\tau, \chi_r)\big) \in A\Big\}, A \in \mathcal{B}\big(H^r(D_V)\big), \tag{1.28}$$

converges weakly to the distribution of the random element

$$\big(L(s, \omega_1, \chi_1), ..., L(s, \omega_1, \chi_r)\big) \tag{1.29}$$

as $T \to \infty$. The function $u : H^r(D_V) \to H(D_V)$ given by the formula

$$u(g_1, ..., g_r) = \sum_{j=1}^{r} b_j g_j, g_1, ..., g_r \in H(D_V),$$

is continuous, therefore, Lemma 1.5 and the weak convergence of the measure (1.28) show that

$$\frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : f_2(s + i\tau) \in A\Big\}, A \in \mathcal{B}\big(H(D_V)\big), \tag{1.30}$$

converges weakly to the measure $P_L u^{-1}$ as $T \to \infty$, where P_L is the distribution of the random element (1.29). By the definitions of P_L and u, we find that, for $A \in \mathcal{B}\big(H(D_V)\big)$,

$$P_L u^{-1}(A) = P_L(u^{-1}A) = m_{1H}\big\{\omega_1 \in \Omega_1 : \big(L(s, \omega_1, \chi_1), ..., L(s, \omega_1, \chi_r)\big) \in u^{-1}A\big\}$$

$$= m_{1H}\big\{\omega_1 \in \Omega_1 : u\big(L(s, \omega_1, \chi_1), ..., L(s, \omega_1, \chi_r)\big) \in A\big\} = m_{1H}\big\{f_2(s, \omega_2) \in A\big\}.$$

Now, using the above results on the measures (1.27) and (1.30), and a standard Cramér-Wold method developed in [26], we obtain that

$$\frac{1}{T}\,\text{meas}\Big\{\tau\in[0,T]:\big(f_1(s+i\tau),f_2(s+i\tau)\big)\Big\},A\in\mathcal{B}\big(H^2(D_V)\big),\tag{1.31}$$

converges weakly to the distribution of the random element $\big(f_1(s,\omega_1),f_2(s,\omega_2)\big)$ as $T\to\infty$. The function $u_1: H^r(D_V)\to H(D_V)$ defined by $u_1(g_1,g_2)=g_1g_2, g_1,g_2\in H(D_V)$, is continuous. Therefore, taking into account the weak convergence of (1.31) and Lemma 1.5, we obtain that

$$\frac{1}{T}\,\text{meas}\Big\{\tau\in[0,T]:f_1(s+i\tau)f_2(s+i\tau)\Big\},A\in\mathcal{B}\big(H(D_V)\big),$$

converges weakly to the distribution of the random element $f_1(s+i\tau)f_2(s+i\tau)$, as $T\to\infty$. In other words, we have that P_{1T} converges weakly to $P_{1\varsigma}$ as $T\to\infty$.

It remains to find the support of the measure $P_{1\varsigma}$. By lemma 13 of [20] stated for an arbitrary collection of non-equivalent Dirichlet characters (non-equivalent characters are not generated by the same primitive character), the support of the measure P_L is the set S_V^r, where

$$S_V=\big\{g\in H(D_V):g(s)\neq 0 \text{ or } g(s)\equiv 0\big\}.$$

We do not need the equivalence of characters because $\chi_1,...,\chi_r$ are different characters modulo qb. We will prove that the support S_{f_2} of the distribution P_{f_2} of the random element is the whole of $H(D_V)$. Really, let g be any element of $H(D_V)$, and G be its arbitrary open neighbourhood. Since the function u is continuous, the set $u^{-1}G$ is open as well. Suppose that

$$(u^{-1}G)\cap S_V^r\neq\emptyset.\tag{1.32}$$

Then $u^{-1}G$ is an open neighbourhood of a certain element of the support S_V^r of the measure P_L. Therefore,

$$P_L(u^{-1}G)\geq 0$$

by the properties of the support. Hence,

$$P_{f_2}(G)=P_Lu^{-1}(G)=P_L(u^{-1}G)>0.$$

Since g and G are arbitrary, this shows that the support of P_{f_2} is the set $H(D_V)$. Thus, it suffices to prove (1.32).

It is well known that the approximation in the space $H(D_V)$ reduces to that on compact sets with connected complements. Therefore, Lemma 1.16 can be applied. By this lemma, fore every $\varepsilon>0$ and compact subset K with connected complement, there exists a polynomial $p=p(s)$ such that

$$\sup_{s\in K}|g(s)-p(s)|<\varepsilon,$$

since $g\in H(D_V)$, thus, it is continuous on K and analytic in the interior of K. If ε is small enough, then the polynomial p lies in the set G. Thus, the set $u^{-1}\{p\}$ lies in the set $u^{-1}G$, and to prove (1.32) it suffices to show that

$$\big(u^{-1}\{p\}\big)\cap S_V^r\neq\emptyset.\tag{1.33}$$

It is well known that the non-vanishing of a polynomial in a bounded region, for example, in D_V, can be controlled by its constant term. Therefore, there exist $g_1, ..., g_r \in S_V$ such that

$$u(g_1, ..., g_r) = p.$$

Actually, we have seen that at least two coefficients b_j are distinct from zero. Without loss of generality, we may suppose that $b_1 \neq 0$ and $b_2 \neq 0$. Then there exists $C \in \mathbb{C}$ with sufficiently large $|C|$ such that, for $s \in D_V$,

$$g_1(s) = \frac{p(s) + C}{b_1} \neq 0$$

and

$$g_2(s) = -\frac{C + b_2 + ... + b_r}{b_2} \neq 0.$$

Thus, taking $g_3(s) = ... = g_r(s) = 1$, we have that $g_1, ..., g_r \in S_V$ and

$$\sum_{j=1}^{r} b_j g_j(s) = p(s).$$

This shows that (1.32) is true, and we have that the support of P_{f_2} is the whole of $H(D_V)$.

By the construction, $\{\omega_1(p)\}$ is a sequence of independent random variables. Moreover, in each $L(s, \omega_1, \chi_j)$, the characters χ_j modulo bq occur. Therefore, in

$$L(s, \omega_1, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)\omega_1(m)}{m^s},$$

the terms with $m, (m, b) > 1$, are equal to zero. From this, it follows that $\omega_1(b)$ and $L(s, \omega_1, \chi)$ are independent random elements for $k = 1, ..., r$. These remarks show that the random elements $f_1(s, \omega_1)$ and $f_2(s, \omega_1)$ are independent. Since the random element $f_1(s, \omega_1)$ is not degenerated at zero, and the support of the rando element $f_2(s, \omega_1)$ is the whole $H(D_V)$, this shows that the support of the product $f_1(s, \omega_1)f_2(s, \omega_1)$ is the whole of $H(D_V)$. In other words, the support of the measure $P_{1\zeta}$ is the whole of $H(D_V)$. The proposition is proved. □

Proof of Theorem 1.2. The theorem, similarly as Theorem 1.1, follows from Proposition 1.4 and Lemma 1.16.

Let $V > 0$ be such that $K \subset D_V$. Define the set

$$G_\varepsilon = \{g \in H(D_V) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\},$$

where, by Lemma 1.16, $p(s)$ is a polynomial such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{1.33}$$

By the second part of Proposition 1.4, the polynomial $p(s)$ is an element of the support of the measure $P_{1\zeta}$, therefore, the set G_ε is an open neighbourhood of an element of the support $P_{1\zeta}$. Hence,

$$P_{1\zeta}(G_\varepsilon) > 0. \tag{1.34}$$

By the first part of Proposition 1.4 and Lemma 1.17, we have that

$$\liminf_{T\to\infty} P_{1T}(G_\varepsilon) \geq P_{1\zeta}(G_\varepsilon).$$

Therefore, the definitions of P_{1T} and G_ε together with (1.34) give the inequality

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathrm{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, \frac{a}{b}; \mathfrak{a}) - p(s)| < \frac{\varepsilon}{2}\Big\} > 0. \tag{1.35}$$

Let τ satisfy the inequality

$$\sup_{s\in K} |\zeta(s+i\tau, \frac{a}{b}; \mathfrak{a}) - p(s)| < \frac{\varepsilon}{2}.$$

Then, in view of (1.33), for such τ

$$\sup_{s\in K} |\zeta(s+i\tau, \frac{a}{b}; \mathfrak{a}) - f(s)| < \varepsilon.$$

This, the monotonicity of the Lebesgue measure and (1.35) give the inequality

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathrm{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau, \frac{a}{b}; \mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

The theorem is proved. □

1.3 A discrete universality theorem involving the set $L(\alpha, h, \pi)$

In this section, we approximate analytic functions from the class $H(D)$, $K \in \mathcal{K}$, by discrete shifts $\zeta(s+ikh, \alpha; \mathfrak{a})$, $k \in \mathbb{N}_0$, where $h > 0$ is a fixed number.

Define the set

$$L(\alpha, h, \pi) = \big\{\big(\log(m+\alpha) : m \in \mathbb{N}_0\big), \frac{2\pi}{h}\big\}.$$

The main result of this section is the following theorem.

Theorem 1.3. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

As it was noted in Introduction, the algebraic independence of the numbers α and $\exp\left\{\frac{2\pi}{h}\right\}$ implies the linear independence over $\mathbb{Q}$ of the set $L(\alpha, h, \pi)$. For example, we can take $\alpha = \frac{1}{\pi}$ and rational h in Theorem 1.3.

Our proof of Theorem 1.3 is based on a limit theorem for

$$P_N(A) \overset{def}{=} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \zeta(s+ikh, \alpha; \mathfrak{a}) \in A\Big\}, A \in \big(H(D), \mathcal{B}(H(D)\big),$$

as $N \to \infty$. We use the same probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$ as in Section 1.1 and the notation of the random element $\zeta(s, \alpha, \omega; \mathfrak{a})$.

Proposition 1.5. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then P_N converges weakly to P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the whole of $H(D)$.*

We begin the proof of Proposition 1.5 with a discrete theorem on the torus Ω. Let, for $A \in \mathcal{B}(\Omega)$,

$$Q_N(A) = \frac{1}{N+1}\#\Big\{0 \le k \le N : \big((m+\alpha)^{-ikh} : m \in \mathbb{N}\big) \in A\Big\}.$$

Lemma 1.18. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. As in the proof of Lemma 1.3, we will apply the Fourier transform method. Let $\underline{k} = \{k_m : k_m \in \mathbb{Z}, m \in \mathbb{N}_0\}$. Then

$$g_N(\underline{k}) = \int_\Omega \prod_{m=0}^{\infty} \omega^{k_m}(m) dQ_N,$$

where only a finite number of integers k_m are distinct from zero, is the Fourier transform of Q_N. Thus, by the definition of Q_N,

$$g_N(\underline{k}) = \frac{1}{N+1}\sum_{k=0}^{N}\prod_{m=0}^{\infty}(m+\alpha)^{ikk_mh} =$$

$$\frac{1}{N+1}\sum_{k=0}^{N}\exp\Big\{-ikh\sum_{m=0}^{\infty}k_m\log(m+\alpha)\Big\}. \tag{1.36}$$

The linear independence of the set $L(\alpha, h, \pi)$ shows that

$$\sum_{m=0}^{\infty}k_m\log(m+\alpha) = 0 \tag{1.37}$$

if and only if $\underline{k} = \underline{0}$ (we have in mind that in (1.37) there is only a finite sum). Moreover, we observe that, for $\underline{k} \neq \underline{0}$,

$$\exp\Big\{-ih\sum_{m=0}^{\infty}k_m\log(m+\alpha)\Big\} \neq 1. \tag{1.38}$$

Indeed, if the latter inequality is not true, then, for some $l \in \mathbb{R}$,

$$\sum_{m=0}^{\infty}k_m\log(m+\alpha) = \frac{2\pi l}{h},$$

and this contradicts the linear independence of the set $L(\alpha, h, \pi)$. Now, using the formula for the sum of a geometric progression, we deduce from (1.36) and (1.38) that

$$g_N(\underline{k}) = \begin{cases} 1 \quad \text{if} \quad \underline{k} = \underline{0}, \\ \dfrac{1-\exp\Big\{-i(N+1)\sum_{m=0}^{\infty}k_m\log(m+\alpha)\Big\}}{(N+1)\Big(1-\exp\Big\{-ih\sum_{m=0}^{\infty}k_m\log(m+\alpha)\Big\}\Big)} & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Obviously, this implies

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

Since the function

$$g(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

is the Fourier transform of the Haar measure, the lemma is proved. □

In the sequel, we use the same function $\zeta_n(s, \alpha; \mathfrak{a})$ as in Lemma 1.6. For $A \in \big(\mathcal{B}H(D)\big)$, define

$$P_{N,n}(A) = \frac{1}{N+1} \#\Big\{0 \le k \le N : \zeta_n(s + ikh, \alpha; \mathfrak{a}) \in A\Big\}.$$

Lemma 1.19. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,n}$ converges weakly to a certain probability measure $\hat{P}_n$ on $\big(H(D), \mathcal{B}(H(D))\big)$ as $N \to \infty$.*

Proof. We repeat the proof of Lemma 1.6 with using Lemma 1.18 in place of Lemma 1.3.

Moreover, if $u_n : \Omega \to H(D)$ is defined by the formula

$$u_n(\omega) = \zeta(s, \alpha, \omega; \mathfrak{a}), \omega \in \Omega,$$

then we have that $\hat{P}_n = m_H u_n^{-1}$.

For the approximation in the mean of the function $\zeta(s+ikh, \alpha; \mathfrak{a})$ by $\zeta_n(s+ikh, \alpha; \mathfrak{a})$, the following Gallangher lemma is useful.

Lemma 1.20. *Suppose that $T_0, T \ge \delta > 0$ are real numbers, and $\mathcal{T}$ is a finite non-empty set in the interval $\big[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}\big]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t - x| < \delta}} 1.$$

Let S(x) be a complex-valued continuous function on $[T_0, T + T_0]$ having a continuous derivative on $(T_0, T + T_0)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t) |S(t)|^2 \le \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 dx + \Big(\int_{T_0}^{T_0+T} |S(x)|^2 dx \int_{T_0}^{T_0+T} |S'(x)|^2 dx \Big)^{\frac{1}{2}}.$$

Proof of the lemma is given in [43, Lemma 1.4]. □

Let ρ be the metric on $H(D)$ defined in Section 1.1.

Lemma 1.21. *The equality*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho\big(\zeta(s + ikh, \alpha; \mathfrak{a}), \zeta_n(s + ikh, \alpha; \mathfrak{a})\big) = 0$$

holds for all $0 < \alpha \le 1$ and $\mathfrak{a}$.

Proof. We apply similar arguments to those used in the proof of Lemma 1.7, however, with an estimate for the discrete mean square of the function $\zeta(s, \alpha; \mathfrak{a})$. For this, we apply Lemma 1.20.

For $\frac{1}{2} < \sigma < 1$, the estimates

$$\int_0^T |\zeta(\sigma + it, \alpha; \mathfrak{a})|^2 dt \ll T \tag{1.39}$$

and

$$\int_0^T |\zeta'(\sigma + it, \alpha; \mathfrak{a})|^2 dt \ll T \tag{1.40}$$

are valid. The first of them was already used in Section 1.1 and was obtained in [13], while the second is implied by the first and Cauchy integral formula. By Lemma 1.20, we find that

$$\sum_{k=0}^{N} |\zeta(\sigma + ikh + it, \alpha; \mathfrak{a})|^2 \ll$$

$$\frac{1}{h}\int_0^{Nh} |\zeta(\sigma + i\tau + it, \alpha; \mathfrak{a})|^2 d\tau + \Big(\int_0^{Nh} |\zeta(\sigma + i\tau + it, \alpha; \mathfrak{a})|^2 d\tau \int_0^{Nk} |\zeta'(\sigma + i\tau + it, \alpha; \mathfrak{a})|^2 d\tau\Big)^{\frac{1}{2}}.$$

This inequality together with (1.39) and (1.40) shows that, for $\frac{1}{2} < \sigma < 1$,

$$\sum_{k=0}^{N} |\zeta(\sigma + ikh + it, \alpha; \mathfrak{a})|^2 \ll N(1 + |t|). \tag{1.41}$$

The further part of the proof runs in the same way than that of Lemma 1.7. Let K be an arbitrary compact set of the strip D. Then, repeating the proof of Lemma 1.7 with obvious changes, we find in the notation of Lemma 1.7 that

$$\frac{1}{N+1}\sum_{k=0}^{N} \sup_{s\in K} \big|\zeta(s + ikh, \alpha; \mathfrak{a}) - \zeta_n(s + ikh, \alpha; \mathfrak{a})\big| \ll$$

$$\sup_{s\in K} \int_{-\infty}^{+\infty} \big|l_n(\sigma_1 - \sigma + iu, \alpha)\big| \Big(\frac{1}{N+1}\sum_{k=0}^{N} |\zeta(\sigma_1 + ikh + it + iu, \alpha; \mathfrak{a})|^2\Big)^{\frac{1}{2}} du + o(1)$$

as $N \to \infty$, where t is bounded by a constant depending on K. This and (1.41) give the estimate

$$\frac{1}{N+1}\sum_{k=0}^{N} \sup_{s\in K} \big|\zeta(s + ikh, \alpha; \mathfrak{a}) - \zeta_n(s + ikh, \alpha; \mathfrak{a})\big| \ll$$

$$\int_{-\infty}^{+\infty} \big|l_n(\sigma_2 + it, \alpha)\big| \big(1 + |t|\big)^{\frac{1}{2}} dt + o(1)$$

as $N \to \infty$ with $\sigma_2 < 0$. Hence,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1}\sum_{k=0}^{N} \sup_{s\in K} \big|\zeta(s + ikh, \alpha; \mathfrak{a}) - \zeta_n(s + ikh, \alpha; \mathfrak{a}) = 0,$$

and lemma follows from the definition of the metric ρ. □

Proof of Proposition 1.5. We begin with similar arguments as in proof of Lemma 1.14. On a certain probability space with probability measure μ, define a random variable ξ_N with the distribution

$$\mu(\xi_N = hk) = \frac{1}{N+1}, k = 0, 1, ..., N,$$

and put

$$X_{N,n} = X_{N,n}(s) = \zeta(s + i\xi_N, \alpha; \mathfrak{a}).$$

Let X_n be the $H(D)$-valued random element with the distribution $\hat{P}_n$, where $\hat{P}_n$ is the limit measure in Lemma 1.19. Then Lemma 1.19 implies the relation

$$X_{N,n} \xrightarrow{\mathcal{D}} X_n. \tag{1.42}$$

Let K_l be a compact set from the definition of the metric ρ. Then, for $M_l > 0$, we find that

$$\mu\big(\sup_{s\in K_l} |X_{N,n}(s)| > M_l\big) \leq \frac{1}{(N+1)M_l} \sum_{k=0}^{N} \sup_{s\in K_k} |\zeta_n(s + ikh, \alpha; \mathfrak{a})|. \tag{1.43}$$

By Lemma 1.20, for $\sigma > \frac{1}{2}$,

$$\sum_{k=0}^{N} |\zeta_n(\sigma + ikh, \alpha; \mathfrak{a})|^2 \ll$$

$$\int_0^{Nh} |\zeta_n(\sigma + it, \alpha; \mathfrak{a})|^2 dt + \Bigg(\int_0^{Nh} |\zeta_n(\sigma + it, \alpha; \mathfrak{a})|^2 dt \int_0^{Nh} |\zeta_n'(\sigma + it, \alpha; \mathfrak{a})|^2 dt\Bigg)^{\frac{1}{2}}. \tag{1.44}$$

Since the series for $\zeta(s, \alpha; \mathfrak{a})$ and $\zeta_n'(s, \alpha; \mathfrak{a})$ are absolutely convergent for $\sigma > \frac{1}{2}$,

$$\limsup_{N\to\infty} \frac{1}{N} \int_0^N |\zeta_n(\sigma + it, \alpha; \mathfrak{a})|^2 dt = \sum_{m=0}^{\infty} \frac{|a_m|^2 v_n^2(m, \alpha)}{(m+\alpha)^{2\sigma}} \leq \sum_{m=0}^{\infty} \frac{|a_m|^2}{(m+\alpha)^{2\sigma}} \leq C_{1,\alpha} < \infty$$

and

$$\limsup_{N\to\infty} \frac{1}{N} \int_0^N |\zeta_n'(\sigma+it, \alpha; \mathfrak{a})|^2 dt = \sum_{m=0}^{\infty} \frac{|a_m|^2 \log^2(m+\alpha) v_n^2(m, \alpha)}{(m+\alpha)^{2\sigma}} \leq \sum_{m=0}^{\infty} \frac{|a_m|^2 \log^2(m+\alpha)}{(m+\alpha)^{2\sigma}} \leq C_{2,\alpha} < \infty.$$

These estimates and (1.44) show that, for $\sigma > \frac{1}{2}$,

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} |\zeta(\sigma + ikh, \alpha; \mathfrak{a})|^2 \ll \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \Big(\frac{1}{N+1} \sum_{k=0}^{N} |\zeta(\sigma + ikh, \alpha; \mathfrak{a})|^2\Big)^{\frac{1}{2}} \leq C_\alpha < \infty.$$

Now, an application of the Cauchy integral formula implies

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \sup_{s\in K_l} |\zeta_n(s + ikh, \alpha; \mathfrak{a})| \leq R_{l,\alpha} < \infty.$$

We fix $\varepsilon > 0$ and put $M_{l,\alpha} = M_{l,\alpha}(\varepsilon) = 2^l R_{l,\alpha} \varepsilon^{-1}$. Then the above inequality and (1.42) show that

$$\limsup_{N\to\infty} \mu\big(\sup_{s\in K_l} |X_{N,n}(s)| > M_{l,\alpha}\big) \leq \frac{\varepsilon}{2^l} \tag{1.45}$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$. Define the set

$$H_\varepsilon = \big\{g \in H(D) : \sup_{s\in K_l} |g(s)| \leq M_{l,\alpha}, l \in \mathbb{N}\big\}.$$

Then H_ε is a compact set in the space $H(D)$, and, in view of (1.45),

$$\mu\big(X_N(s) \in H_\varepsilon\big) \geq 1 - \varepsilon.$$

Consequently, by the definition of X_n,

$$\hat{P}_n(H_\varepsilon) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus, we proved that the family of probability measures $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight. Therefore, by Lemma 1.12, this family is relatively compact. Therefore, every sequence of that family contains weakly convergent subsequence $\hat{P}_{n_r}$ to a certain probability measure P on $\big(H(D), \mathcal{B}(H(D)\big)$ as $r \to \infty$. Thus,

$$X_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{1.46}$$

Let the $H(D)$-valued random element $Y_N = Y_N(s)$ be given by the formula

$$Y_N(s) = \zeta(s + i\xi_N, \alpha; \mathfrak{a}).$$

Then, in view of Lemma 1.21, we obtain that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu\big(\rho(X_{N,n}, Y_N)\big) =$$

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \rho\big(\zeta(s + ikh, \alpha; \mathfrak{a}), \zeta_n(s + ikh, \alpha; \mathfrak{a})\big) \geq \epsilon\Big\} \leq$$

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \rho\big(\zeta(s + ikh, \alpha; \mathfrak{a}), \zeta_n(s + ikh, \alpha; \mathfrak{a})\big) = 0.$$

This and (1.42), (1.46) show that the hypothesis of Lemma 1.13 are satisfied. Therefore, we have that

$$Y_N \xrightarrow[N\to\infty]{\mathcal{D}} P, \tag{1.47}$$

and the definition of Y_N shows that P_N converges weakly to P as $N \to \infty$. Moreover, the relation (1.47) implies that the limit measure P is independent of the choice of the measure $\{P_{n_r}\}$. From this and the relative compactness of $\{\hat{P}_n\}$, it follows that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

This means that $\{\hat{P}_n\}$ converges weakly to P as $n \to \infty$.

For the identification of the measure P, we apply Propositions 1.2 and 1.3. The linear independence over $\mathbb{Q}$ of the set $L(\alpha, h, \pi)$ implies that of the set $L(\alpha)$. Therefore, under the hypothesis of the proposition,

$$\frac{1}{T} \operatorname{meas}\big\{\tau \in [0,T] : \zeta(s + i\tau, \alpha; \mathfrak{a}) \in A\big\}, A \in \mathcal{B}\big(H(D)\big),$$

by the proof of Proposition 1.2, converges weakly as $T \to \infty$ to the limit measure P of $\hat{P}_n$ as $n \to \infty$, and that P coincides with the measure P_ζ. Moreover, the support of P_ζ is the whole $H(D)$. Since P_N also converges weakly to the same measure P as $N \to \infty$, from this the proposition follows. □

Proof of Theorem 1.3. We argue similarly to the proof of Theorem 1.1. Let $p(s)$ be a polynomial satisfying (1.22), and

$$G_\varepsilon = \Big\{g \in H(D) : \sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}\Big\}.$$

Then, by the second part of Proposition 1.5, the set G_ε is an open neighborhood of an element $p(s)$ of the support of the measure P_ζ. Therefore, by the properties of the support,

$$P_\zeta(G_\varepsilon) > 0.$$

Hence, by the first part of Proposition 1.5 and Lemma 1.17, we have

$$\liminf_{N\to\infty} P_N(G_\varepsilon) \geq P_\zeta(G_\varepsilon) > 0,$$

or, in view of the definitions of P_N and G_ε,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - p(s)| < \frac{\varepsilon}{2}\Big\} > 0. \tag{1.48}$$

Suppose that $k \in \mathbb{N}_0$ satisfies the inequality

$$\sup_{s\in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - p(s)| < \frac{\varepsilon}{2}.$$

Then, for such k, the inequality (1.22) implies

$$\sup_{s\in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - f(s)| < \varepsilon.$$

Therefore,

$$\Big\{0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - p(s)| < \frac{\varepsilon}{2}\Big\} \subset \Big\{0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - f(s)| < \varepsilon\Big\}.$$

From this and (1.48), we obtain the inequality

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh, \alpha; \mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

The theorem is proved.

□

Theorem 1.3 is published in [42].

1.4 Discrete universality of the function $\zeta(s, \alpha; \mathfrak{a})$ with rational parameter α

This section is devoted to a discrete version of Theorem 1.2.

Theorem 1.4. *Suppose that* $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a, b) = 1, b \neq 2$ *and that* $\mathrm{rad}(q)$ *divides* b. *Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} |\zeta(s+ikh, \frac{a}{b}; \mathfrak{a}) - f(s)| < \varepsilon\Big\} > 0.$$

As in the case of Theorem 1.2, we will apply some elements of the joint value distribution of Dirichlet L-functions. Also, we will use the notation of Section 1.2.

Let $\mathbb{P}$ be the set of all prime numbers. We divide the set of all positive numbers h into two parts. We say that h is type 1 if numbers $\exp\left\{\frac{2\pi m}{h}\right\}$ are irrational for all $m \in \mathbb{Z}\setminus\{0\}$, and h is of type 2 if it is not of a type 1.

Let Ω_{1h} be a closed subgroup of the group Ω_1 generated by the element $(p^{-ih} : p \in \mathbb{P})$. If h is of type 2, then there exists a minimal $m_0 \in \mathbb{N}$ such that the number $\exp\left\{\frac{2\pi m_0}{h}\right\}$ is rational. Suppose that

$$\exp\left\{\frac{2\pi m_0}{h}\right\} = \frac{m_1}{m_2}, m_1, m_2 \in \mathbb{N}, (m_1, m_2) = 1.$$

Extend the function $\omega_1(p), p \in \mathbb{P}$, to the set $\mathbb{N}$ by the formula

$$\omega_1(m) = \prod_{\substack{p^l | m\\ p^{l+1}\nmid m}} \omega_1^l(p), m \in \mathbb{N}.$$

Then it is known [1], see also [34], that

$$\Omega_{1h} = \begin{cases} \Omega_1 & \text{if } h \text{ is of type 1,}\\ \{\omega_1 \in \Omega_1 : \omega_1(m_1) = \omega_2(m_1)\} & \text{if } h \text{ is of type 2.}\end{cases}$$

On $\big(\Omega_{1h}, \mathcal{B}(\Omega_{1h})\big)$, as on $\big(\Omega_1, \mathcal{B}(\Omega_1)\big)$, also the probability Haar measure m_{1Hh} exists. Denote by $\omega_{1h}(p)$ the component of $\omega_{1h} \in \Omega_{1h}$. Now, on the probability space $\big(\Omega_{1h}, \mathcal{B}(\Omega_{1h})\big)$, define the $H^r(D)$-valued random element $\underline{L}(s, \omega_{1h}, \underline{\chi}), \underline{\chi} = (\chi_1, ..., \chi_r)$, by

$$L(s_1, \omega_{1h}, \underline{\chi}) = \big(L(s_1, \omega_{1h}, \chi_1), ..., L(s_1, \omega_{1h}, \chi_r)\big),$$

where

$$L(s, \omega_{1h}, \chi_j) = \prod_p \Big(1 - \frac{\omega_{1h}(p)\chi_j(p)}{p^s}\Big)^{-1}, j = 1, ..., r.$$

We remind that $\chi_1, ..., \chi_r$ are distinct Dirichlet characters modulo bq. Moreover, for $A \in \mathcal{B}\big(H^r(D)\big)$ and $h > 0$, we set

$$P_{N,h}(A) = \frac{1}{N+1}\#\Big\{0 \le k \le N : \big(L(s+ikh, \chi_1), ..., L(s+ikh, \chi_r)\big) \in A\Big\}$$

and

$$P_{\underline{L},h}(A) = m_{1Hh}\big\{\omega_{1h} \in \Omega_{1h} : \underline{L}(s, \omega_{1h}, \underline{\chi}) \in A\big\},$$

i.e., $P_{\underline{L},h}$ is the distribution of the random element $P_{\underline{L},h}$. Let

$$S = \big\{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\big\}.$$

Lemma 1.21. *For every $h > 0$, $P_{N,h}$ converges weakly to $P_{\underline{L},h}$ as $N \to \infty$. Moreover, the support of the measure $P_{\underline{L},h}$ is the set S^r.*

Proof. The lemma was obtained in [1], the proof of Theorem 5.3.1. In the case of h of type 1, the lemma for a more general case of non-equivalent Dirichlet characters was given in [20]. The case of h of type 2 is considered similarly, see, for example, the paper [34]. □

Lemma 1.21 is not convenient for the investigation of the function $\zeta(s,\alpha;\mathfrak{a})$. As in Section 1.2, in place of $H(D)$, we will consider the space $H(D_V)$ of analytic functions defined on a bounded region $D_V = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1, |t| < V\}$. We recall that

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Let $P_{N,V,h}(A)$ and $P_{\underline{L},V,h}(A)$ be the corresponding analogues of $P_{N,h}(A)$ and $P_{\underline{L},h}(A)$ defined for $A \in \mathcal{B}(H^r(D_V))$.

By same method as Lemma 1.21, the following statement is obtained.

Lemma 1.22. *For every $h > 0$, $P_{N,V,h}$ converges weakly to $P_{\underline{L},V,h}$ as $N \to \infty$. Moreover, the support of the measure $P_{\underline{L},V,h}$ is the set S_V^r.*

We note that the weak convergence for $P_{N,V,h}$ follows from Lemma 1.21 by using Lemma 1.5 and a certain continuous mapping. However, since $H^r(D_V) \supset H^r(D)$, Lemma 1.21 does not imply the explicit form of the support of the measure$P_{\underline{L},V,h}$, and the proof must be repeated.

Now we are in position to prove a discrete limit theorem for the function $\zeta(s,\frac{a}{b};\mathfrak{a})$. For $A \in \mathcal{B}(H(D_V))$, define

$$Q_{N,V,h}(A) = \frac{1}{N+1}\#\Big\{0 \le k \le N : \Big(\zeta\big(s+ikh,\frac{a}{b};\mathfrak{a}\big)\Big) \in A\Big\}.$$

Moreover, on the probability space $(\Omega_{1h}, \mathcal{B}(\Omega_{1h}), m_{1Hh})$, define two $H(D_V)$-valued random elements

$$\zeta_1(s,\omega_{1h}) = \frac{b^s \overline{\omega}_{1h}(b)}{r}$$

and

$$\zeta_2(s,\omega_{1h}) = \sum_{j=1}^{r} b_j L(s,\omega_{1h},\chi_j),$$

and set

$$\zeta\Big(s,\frac{a}{b},\omega_{1h},\omega;\mathfrak{a}\Big) = \zeta_1(s,\omega_{1h})\zeta_2(s,\omega_{1h}).$$

Here the coefficients b_j are the same as in Section 1.2. Denote by $P_{\zeta,V,h}$ the distribution of the random element $\zeta\Big(s,\frac{a}{b},\omega_{1h},\omega;\mathfrak{a}\Big)$, i.e.,

$$P_{\zeta,V,h}(A) = m_{1Hh}\Big\{\omega_{1h} \in \Omega_{1h} : \zeta\Big(s,\frac{a}{b},\omega_{1h},\omega;\mathfrak{a}\Big) \in A\Big\}, A \in \mathcal{B}(H(D_V)).$$

Proposition 1.6. *Suppose that a, b and q are as in Theorem 1.4. Then $Q_{N,V,h}$ converges weakly to the measure $P_{\zeta,V,h}$ as $N \to \infty$. Moreover, the support of the measure $P_{\zeta,V,h}$ is the space $H(D_V)$.*

Proof. The function

$$\zeta_1(s) = \frac{b^s}{r}$$

is a Dirichlet polynomial. Therefore, we find by a standard method (the case of h of type 2 is discussed in[34]) that

$$\frac{1}{N+1}\#\{0 \le k \le N : \zeta_1(s+ikh) \in A\}, A \in \mathcal{B}(H(D_V)) \tag{1.49}$$

converges weakly to the distribution of the random element $\zeta_1(s, \omega_{1h})$ as $n \to \infty$.

For the proof of a discrete limit theorem for the function

$$\zeta_2(s) \overset{def}{=} \sum_{j=1}^{r} b_j L(s, \chi_j),$$

we will apply Lemma 1.22. Let the function $u : H^r(D_V) \to H(D_V)$ be given by the formula

$$u(g_1, ..., g_r) = \sum_{j=1}^{r} b_j g_j, \ g_1, ..., g_r \in H(D_V).$$

Then the function u is continuous. Moreover, for $A \in \mathcal{B}(H(D_V))$,

$$Q_{2,N,V,h}(A) \overset{def}{=} \frac{1}{N+1}\#\{0 \le k \le N : \zeta_2(s+ikh) \in A\} =$$

$$\frac{1}{N+1}\#\Big\{0 \le k \le N : u\big(L(s+ikh, \chi_1), ..., L(s+ikh, \chi_r)\big) \in A\Big\} =$$

$$\frac{1}{N+1}\#\Big\{0 \le k \le N : \big(L(s+ikh, \chi_1), ..., L(s+ikh, \chi_r)\big) \in u^{-1}A\Big\} =$$

$$P_{N,V,h}(u^{-1}A) = P_{N,V,h}u^{-1}(A).$$

Therefore, the continuity of the function u, Lemma 1.5 and Lemma 1.22 show that $Q_{2,N,V,h}$ converges weakly to the measure $P_{\underline{L},V,h}u^{-1}$ as $N \to \infty$. We observe that the measure $P_{\underline{L},V,h}u^{-1}$ is the distribution of the random element $\zeta_2(s, \omega_{1h})$. Actually, for $A \in \mathcal{B}(H(D_V))$,

$$P_{\underline{L},V,h}u^{-1}(A) = P_{\underline{L},V,h}(u^{-1}A) = m_{1Hh}\{\omega_{1h} \in \Omega_{1h} : \underline{L}(s, \omega_1, \underline{\chi}) \in u^{-1}A\} =$$

$$m_{1Hh}\{\omega_{1h} \in \Omega_{1h} : u(\underline{L}(s, \omega_1, \underline{\chi})) \in A\} =$$

$$m_{1Hh}\{\omega_{1h} \in \Omega_{1h} : \zeta_2(s, \omega_{1h}) \in A\}.$$

Now, the weak convergence of the measures (1.49) and $Q_{2,N,V,h}$ together with a modified Cramér-Wold method implies the weak convergence for

$$\frac{1}{N+1}\#\Big\{0 \le k \le N : \big(\zeta_1(s+ikh), \zeta_2(s+ikh)\big) \in A\Big\}, A \in \mathcal{B}(H^2(D_V)),$$

to the distribution of the random element $\big(\zeta_1(s, \omega_{1h}), \zeta_2(s, \omega_{1h})\big)$ as $N \to \infty$. From this, using the function $u_1 : H^2(D) \to H(D_V)$ given by $u_1(g_1, g_2) = g_1 g_2$, $g_1, g_2 \in H(D_V)$, we find that, in view of the equality

$$\zeta\Big(s, \frac{a}{b}; \mathfrak{a}\Big) = \zeta_1(s)\zeta_2(s),$$

$Q_{N,V,h}$ converges weakly to the distribution of the random element $\zeta_1(s, \omega_{1h})\zeta_2(s, \omega_{1h}) = \zeta\left(s, \frac{a}{b}, \omega_h; \mathfrak{a}\right)$ as $N \to \infty$.

It remains to find the support of the measure $P_{\zeta,V,h}$. Let g be an arbitrary element of $H(D_V)$, and G be its any open neighbourhood. Since the function u is continuous, we have that the set $u^{-1}G$ is open as well. If $K \subset D_V$ is a compact subset with connected complement, then, by Lemma 1.16, for every $\varepsilon > 0$, there exists a polynomial $p = p(s)$ such that,

$$\sup_{s \in K} = |g(s) - p(s)| < \varepsilon.$$

It is well known, see, for example, [21], that the approximation in the space $H(D_V)$ can be restricted to that on compact set with connected complements. Therefore, we can choose the polynomial $p(s)$ to lie in the set G. The region D_V is bounded, thus, the non-vanishing of the polynomial $p(s)$ can be controlled by its constant form. Hence, there exists $g_1, ..., g_r \in S_V$ such that

$$u(g_1, ..., g_r) = p.$$

Really, since $\alpha \neq \frac{1}{2}$, we have that $b \geq 3$, hence, $r \geq 2$. Therefore, at least two coefficients b_j in (1.25) are distinct from zero. Thus, without loss of generality, we may suppose that $b_1 \neq 0$ and $b_2 \neq 0$. Then we can find $C \in \mathbb{C}$ with sufficiently large $|C|$ such that, for $s \in D_V$,

$$g_1(s) = \frac{p(s) + C}{b_1} \neq 0$$

and

$$g_2(s) = -\frac{C + b_3 + ...b_r}{b_2} \neq 0.$$

If $g_3(s) = ... = g_r(s) = 1$, then $g_1, ..., g_r \in S_V$, and

$$\sum_{j=1}^{r} b_j g_j(s) = p(s).$$

This shows that

$$\left(u^{-1}\{p\}\right) \cap S_V^r \neq \emptyset.$$

Since $p(s)$ lies in G, hence,

$$(u^{-1}G) \cap S_V^r \neq \emptyset.$$

Therefore, there exists $g_1 \in S_V^r$ such that $g_1 \in u^{-1}G$, i.e., $u^{-1}G$ is an open neighbourhood of an element of the set S_V^r. By Lemma 1.22, the set S_V^r is the support of the measure $P_{\underline{L},V,h}$. Hence, $P_{\underline{L},V,h}(u^{-1}G) > 0$. Therefore,

$$P_{\underline{L},V,h}u^{-1}(G) = P_{\underline{L},V,h}(u^{-1}G) > 0.$$

Since g and G are arbitrary, this shows that the support of the measure $P_{\underline{L},V,h}u^{-1}$ is the whole $H(D_V)$. Hence, the support of the random element $\zeta_2(s, \omega_{1h})$ is also the whole $H(D_V)$.

By the construction, $\{\omega_{1h}(p) : p \in \mathbb{P}\}$ is a sequence of independent random variables. If $p|b$, then $p|qb$, thus, $\chi_j(p) = 0$. Hence,

$$L(s, \omega_{1h}, \chi_j) = \prod_{p \nmid b} \Big(1 - \frac{\omega_{1h}(p)\chi_j(p)}{p^s}\Big)^{-1}, j = 1, ..., r.$$

From this, it follows that the random elements $\zeta_1(s, \omega_{1h})$ and $\zeta_2(s, \omega_{1h})$ are independent. Since the random element $\zeta_1(s, \omega_{1h})$ is not degenerated at zero, and the support of the random element $\zeta_2(s, \omega_{1h})$ is the whole $H(D_V)$, we obtain that the support of the random element

$$\zeta\Big(s, \frac{a}{b}, \omega_{1h}; \mathfrak{a}\Big) = \zeta_1(s, \omega_{1h})\zeta_2(s, \omega_{1h})$$

is $H(D_V)$. The proposition is proved. □

Proof of Theorem 1.4. By Lemma 1.16, there exits a polynomial $p(s)$ such that

$$\sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2}. \tag{1.50}$$

We take $V > 0$ such that $K \subset D_V$, and define the set

$$G = \Big\{g \in H(D_V) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\Big\}.$$

In view of Proposition 1.6, the set G is an open neighbourhood of the element $p(s)$ of the support of the measure $P_{\zeta,V,h}$. Therefore,

$$P_{\zeta,V,h}(G) > 0. \tag{1.51}$$

By the first part of Proposition 1.6 and Lemma 1.17,

$$\liminf_{N \to \infty} Q_{N,V,h}(G) \geq P_{\zeta,V,h}(G),$$

thus, by the definitions of $Q_{N,V,h}$ and G, and (1.51)

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s \in K} \Big|\zeta(s + ikh, \frac{a}{b}; \mathfrak{a}) - p(s)\Big| < \frac{\varepsilon}{2}\Big\} > 0. \tag{1.52}$$

However, if $k \in \mathbb{N}_0$ satisfies the equality

$$\sup_{s \in K} \Big|\zeta(s + ikh, \frac{a}{b}; \mathfrak{a}) - p(s)\Big| < \frac{\varepsilon}{2},$$

then, in view of (1.50), the number k also satisfies the inequality

$$\sup_{s \in K} \Big|\zeta(s + ikh, \frac{a}{b}; \mathfrak{a}) - f(s)\Big| < \varepsilon$$

Therefore, inequality (1.52) implies

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s \in K} \Big|\zeta(s + ikh, \frac{a}{b}; \mathfrak{a}) - f(s)\Big| < \varepsilon\Big\} > 0.$$

The theorem is proved. □

Chapter 2

Application of uniform distribution modulo 1

In this chapter, we extend, in a certain sense, Theorem 1.3. In Theorem 1.3, the functions of the class $H(K)$, $K \in \mathcal{K}$, are approximated by discrete shifts $\zeta(s+ikh, \alpha; \mathfrak{a})$, $s = \sigma + it$. Thus, for discrete shifts the arithmetical progression $\{kh : k \in \mathbb{N}\}$ with a fixed $h > 0$ is used. In this chapter, in place of the latter set, we will use the set $\{hk^{\beta_1} \log^{\beta_2} k : k \geq 2\}$, where $h > 0$, $0 < \beta_1 < 1$ and $\beta_2 > 0$ are fixed numbers. For the proof of universality, we apply good properties of sequences that are uniformly distributed modulo 1.

We remind that a sequence $\{x_m : m \in \mathbb{N}\}$ is uniformly distributed modulo 1 if, for every interval $I = [a, b) \subset [0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I\big(\{x_k\}\big) = b - a,$$

where $\{u\}$ is the fractional part of $u \in \mathbb{R}$, and χ_I is the indicator function of I.

In this chapter, we prove the following theorem.

Theorem 2.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$ and that β_1, $0 < \beta_1 < 1$, and $\beta_2 > 0$ are fixed numbers. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N-1} \# \Big\{ 2 \leq k \leq N : \sup_{s\in K} \big|\zeta(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a}) - f(s)\big| < \varepsilon \} > 0.$$

2.1 A limit theorem

For the proof of Theorem 2.1, as for the proofs of universality theorems of Chapter 1, we apply a limit theorem for weakly convergent probability measures in the space of analytic functions $H(D)$. The proof of Theorem 2.1 is based on the weak convergence of

$$P_N(A) = \frac{1}{N-1} \# \Big\{ 2 \leq k \leq N : \zeta(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a}) \in A \Big\}, A \in \mathcal{B}\big(H(D)\big).$$

For other objects, we preserve the notation of Section 1.1. Thus,

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$, and, on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$ with the measure m_H, the $H(D)$-valued random element $\zeta(s, \alpha, \omega; \mathfrak{a})$ is defined by

$$\zeta(s, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m+\alpha)^s}.$$

The probability measure P_ζ is the distribution of the random element $\zeta(s, \alpha, \omega; \mathfrak{a})$.

The main result of this section is the following proposition.

Proposition 2.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and β_1, β_2 and h are as in Theorem 2.1. Then P_N converges weakly to P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the whole of $H(D)$.*

We start with two lemmas on uniform distribution modulo 1.

Lemma 2.1. *A sequence $\{x_k\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if and only if, for every $m \in \mathbb{Z} \setminus \{0\}$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i m x_k} = 0.$$

The lemma is called the Weyl criterion. Its proof is given, for example, in [16], Theorem 2.1.

Lemma 2.2. *The sequence $\big\{a k^{\beta_1} \log^{\beta_2} k : k \geq 2\big\}$ with $a \in \mathbb{R} \setminus \{0\}$ and β_1, $0 < \beta_1 < 1$, and $\beta_2 > 0$ is uniformly distributed modulo 1.*

The lemma is a part of a more general assertion for the sequence of the same type with $\beta_1 > 0, \beta \notin \mathbb{Z}$, and arbitrary $\beta_2 \in \mathbb{R}$ [16], p.31. For its proof the so-called difference theorems are applied.

Lemmas 2.1 and 2.2 are applied for the proof of the weak convergence for

$$Q_N(A) = \frac{1}{N-1} \# \Big\{2 \leq k \leq N : \Big((m+\alpha)^{-ihk^{\beta_1} \log^{\beta_2} k} : m \in \mathbb{N}_0\Big) \in A\Big\}, A \in \mathcal{B}(\Omega).$$

Lemma 2.3. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that β_1, β_2 and h are as in Theorem 2.1. Then Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. As usual, we apply the Fourier transform method. Let $g_N(\underline{k})$, $\underline{k} = \{k_m : k_m \in \mathbb{Z}, m \in \mathbb{N}_0\}$ be the Fourier transform of Q_N. Then we have that

$$g_N(\underline{k}) = \int_{\Omega} \prod_{m=0}^{\infty} \omega^{k_m}(m) dQ_N,$$

where only a finite number of integers k_m are distinct from zero. Thus, by the definition of Q_N,

$$g_N(\underline{k}) = \frac{1}{N-1} \sum_{k=2}^{N} \prod_{m=0}^{\infty} (m+\alpha)^{-ihk_m k^{\beta_1} \log^{\beta_2} k} =$$

$$\frac{1}{N-1}\sum_{k=2}^{N}\exp\Big\{-ihk^{\beta_1}\log^{\beta_2}k\sum_{m=0}^{\infty}{}'k_m\log(m+\alpha)\Big\}, \tag{2.1}$$

where $\sum'$ means that in

$$\sum_{m=0}^{\infty}k_m\log(m+\alpha)$$

only a finite number of integers k_m are distinct from zero. Clearly,

$$g_N(\underline{0})=1. \tag{2.2}$$

Consider the case $\underline{k}\neq\underline{0}$. The linear independence over $\mathbb{Q}$ of the set $L(\alpha)$ ensure that, in this case,

$$\sum_{m=0}^{\infty}{}'k_m\log(m+\alpha)\neq 0.$$

Since $h>0$, this and Lemma 2.2 shows that the sequence

$$\Big\{-\frac{hk^{\beta_1}\log^{\beta_2}k}{2\pi}\sum_{m=0}^{\infty}{}'k_m\log(m+\alpha):k\geq 2\Big\}$$

is uniformly distributed modulo 1. Therefore, in view of Lemma 2.1 and (2.1),

$$\lim_{N\to\infty}g_N(\underline{k})=0$$

for $\underline{k}\neq\underline{0}$. Thus, by (2.2),

$$\lim_{N\to\infty}g_N(\underline{k})=\begin{cases}1 & \text{if } \underline{k}=\underline{0},\\ 0 & \text{if } \underline{k}\neq\underline{0}.\end{cases}$$

Since the right-hand side of the later equality is the Fourier transform of the Haar measure, this proves the lemma. □

Furthermore, we will deal with a limit theorem for absolutely convergent Dirichlet series

$$\zeta_n(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m v_n(m,\alpha)}{(m+\alpha)^s}$$

and

$$\zeta_n(s,\alpha,\omega;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m\omega(m)v_n(m,\alpha)}{(m+\alpha)^s}$$

that are the same as in Section 1.1. From the absolute convergence of the series for $\zeta_n(s,\alpha,\omega;\mathfrak{a})$, it follows that the function $u_n:\Omega\to H(D)$ given by the formula

$$u_n(\omega)=\zeta_n(s,\alpha,\omega;\mathfrak{a}),\omega\in\Omega,$$

is continuous one. For $A\in\mathcal{B}(H(D))$, let

$$P_{N,n}(A)=\frac{1}{N-1}\#\Big\{2\leq k\leq N:\zeta_n(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})\Big)\in A\Big\}.$$

Moreover, we put $\hat{P}_n=m_H u_n^{-1}$, where the measure $m_H u_n^{-1}$ is defined by

$$m_H u_n^{-1}(A)=m_H(u_n^{-1}A),A\in\mathcal{B}(H(D)).$$

Lemma 2.4. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that β_1, β_2 and h are as in Theorem 2.1. Then $P_{N,n}$ converges weakly to $\hat{P}_n$ as $n \to \infty$.*

Proof. By the definition of the function u_n, we have

$$u_n\big((m+\alpha)^{-ihk^{\beta_1}\log^{\beta_2}k} : m \in \mathbb{N}_0\big) = \zeta_n(s + ihk^{\beta_1}\log^{\beta_2}k, \alpha; \mathfrak{a}).$$

Therefore, for $A \in \mathcal{B}\big(H(D)\big)$,

$$P_{N,n}(A) = \frac{1}{N-1}\#\Big\{2 \le k \le N : u_n\Big((m+\alpha)^{-ihk^{\beta_1}\log^{\beta_2}k} : m \in \mathbb{N}_0\Big) \in A\Big\} =$$
$$\frac{1}{N-1}\#\Big\{2 \le k \le N : \Big((m+\alpha)^{-ihk^{\beta_1}\log^{\beta_2}k} : m \in \mathbb{N}_0\Big) \in u_n^{-1}A\Big\} =$$
$$Q_N(u_n^{-1}A) = Q_N u_n^{-1}A.$$

This, the continuity of the function u_n and Lemma 1.5 show that $P_{N,n}$ converges weakly to $\hat{P}$ as $N \to \infty$. □

Now we will approximate the function $\zeta(s + ihk^{\beta_1}\log^{\beta_2}k, \alpha; \mathfrak{a})$ by $\zeta_n(s + ihk^{\beta_1}\log^{\beta_2}k, \alpha; \mathfrak{a})$ in the mean. Let ρ be the metric in $H(D)$ defined in Section 1.1.

Lemma 2.5. *For all α and $\mathfrak{a}$, the equality*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N-1}\sum_{k=1}^{N}\rho\Big(\zeta(s + ihk^{\beta_1}\log^{\beta_2}k, \alpha; \mathfrak{a}), \zeta_n(s + ihk^{\beta_1}\log^{\beta_2}k, \alpha; \mathfrak{a})\Big) = 0.$$

holds.

Proof. For $2 \le k \le N$ with sufficiently large N, we have

$$(k+1)^{\beta_1}\log^{\beta_2}(k+1) - k^{\beta_1}\log^{\beta_2}k =$$
$$k^{\beta_1}\Big(1 + \frac{1}{k}\Big)^{\beta_1}\Big(\log k + \log\Big(1 + \frac{1}{k}\Big)\Big)^{\beta_2} - k^{\beta_1}\log^{\beta_2}k =$$
$$k^{\beta_1}\Big(1 + \frac{\beta_1}{k} + \frac{\beta_1(\beta_1-1)}{2k^2} + ...\Big)\Big(\log k + \frac{1}{k} - \frac{1}{2k^2} + ...\Big)^{\beta_2} - k^{\beta_1}\log^{\beta_2}k =$$
$$\Big(k^{\beta_1} + \frac{\beta_1}{k^{1-\beta_1}} + \frac{\beta_1(\beta_1-1)}{2k^{2-\beta_1}} + ...\Big)\log^{\beta_2}k\Big(1 + \frac{1}{k\log k} - \frac{1}{2k^2\log k} + ...\Big)^{\beta_2} - k^{\beta_1}\log^{\beta_2}k \ge \frac{c\log^{\beta_2}N}{N^{1-\beta_1}}$$

with suitable constant $c > 0$ not depending on N. Moreover, the estimates (1.39) and (1.40) imply, for $\frac{1}{2} < \sigma < 1$, the estimates

$$\int_0^T \big|\zeta(\sigma + it + i\tau, \alpha; \mathfrak{a})\big|^2 dt \ll T(1 + |\tau|)$$

and

$$\int_0^T \big|\zeta'(\sigma + it + i\tau, \alpha; \mathfrak{a})\big|^2 dt \ll T(1 + |\tau|)$$

for $\tau \in \mathbb{R}$. Therefore, taking

$$\delta = \frac{ch\log^{\beta_2 N}}{N^{1-\beta_1}}$$

in Lemma 1.20, we find that

$$\sum_{k=2}^{N} \left|\zeta(\sigma + ihk^{\beta_1}\log^{\beta_2} k + it, \alpha; \mathfrak{a})\right|^2 \ll$$

$$N^{1-\beta_1}\log^{-\beta_2} N \int\limits_{1}^{hN^{\beta_1}\log^{\beta_2} N} |\zeta(\sigma + i\tau + it, \alpha; \mathfrak{a})|^2 d\tau +$$

$$\left(\int\limits_{1}^{hN^{\beta_1}\log^{\beta_2} N} |\zeta(\sigma + i\tau + it, \alpha; \mathfrak{a})|^2 d\tau \int\limits_{1}^{hN^{\beta_1}\log^{\beta_2} N} |\zeta'(\sigma + i\tau + it, \alpha; \mathfrak{a})|^2 d\tau\right)^{\frac{1}{2}} \ll N(1 + |t|) \tag{2.3}$$

for $\frac{1}{2} < \sigma < 1$. Let K be a compact subset of the strip D. Then, repeating the proof of Lemmas 1.7 and 1.21, we obtain that

$$\frac{1}{N-1}\sum_{k=2}^{N} \sup_{s\in K} \left|\zeta(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}) - \zeta_n(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\right| \ll$$

$$\int\limits_{-\infty}^{+\infty} \left|l_n(\sigma_1 + iu, \alpha)\right| \left(\frac{1}{N-1}\sum_{k=2}^{N} \left|\zeta(\sigma_1 + ihk^{\beta_1}\log^{\beta_2} k + it + iu, \alpha; \mathfrak{a})\right|^2\right)^{\frac{1}{2}} du + o(1)$$

as $N \to \infty$, where $\frac{1}{2} < \sigma < 1$, $\sigma_1 < 0$, and t is bounded by a constant depending on K. Therefore, by (2.3),

$$\frac{1}{N-1}\sum_{k=2}^{N} \sup_{s\in K} \left|\zeta(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}) - \zeta_n(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\right| \ll$$

$$\int_{-\infty}^{+\infty} \left|l_n(\sigma_1 + it, \alpha)\right| \left(1 + |t|\right)^{\frac{1}{2}} dt + o(1)$$

as $N \to \infty$ with $\sigma_1 < 0$. Since, by the definition of $l_n(s, \alpha)$,

$$\lim_{n\to\infty} l_n(\sigma_1 + it, \alpha) = 0,$$

from this it follows that

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N-1}\sum_{k=2}^{N} \sup_{s\in K} \left|\zeta(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}) - \zeta_n(s + ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\right| = 0,$$

and the definition of the metric ρ proves the lemma. □

Proof of Proposition 2.1. On a certain probability space $(\hat{\Omega}, A, \mathbb{P})$, define a random variable θ_N by the formula

$$\mathbb{P}(\theta_N = hk^{\beta_1}\log^{\beta_2} k) = \frac{1}{N-1}, k = 2, ..., N.$$

Let $H(D)$-valued random element $X_{N,n}$ be given by

$$X_{N,n} = X_{N,n}(s) = \zeta_n(s + i\theta_N, \alpha; \mathfrak{a}).$$

Moreover, let $\hat{P}_n$ be the limit measure in Lemma 2.4, and $\hat{X}_n$ be a $H(D)$-valued random element having the distribution $\hat{P}_n$. Then the assertion of Lemma 2.4 can be rewritten in the form

$$X_{N,n} \xrightarrow[n\to\infty]{\mathcal{D}} \hat{X}_n. \tag{2.4}$$

At first, we will prove that the family of probability measures $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight, i.e., that, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset H(D)$ such that

$$\hat{P}_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Since the series for $\zeta_n(s, \alpha; \mathfrak{a})$ is absolutely convergent for $\sigma > \frac{1}{2}$, we have that

$$\sup_{s\in K} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(\sigma + it, \alpha; \mathfrak{a})|^2 dt = \sup_{s\in K} \sum_{m=0}^{\infty} \frac{|a_m|^2 v_n^2(m, \alpha)}{(m+\alpha)^{2\sigma}} \leq \sum_{m=0}^{\infty} \frac{|a_m|^2}{(m+\alpha)^{2\sigma}} \leq C < \infty.$$

Thus, for $\frac{1}{2} < \sigma < 1$,

$$\int_0^T |\zeta_n(\sigma + it, \alpha; \mathfrak{a})|^2 dt \ll T,$$

and, by the Cauchy integral formula,

$$\int_0^T |\zeta_n'(\sigma + it, \alpha; \mathfrak{a})|^2 dt \ll T.$$

Now, an application of Lemma 1.20, as in the proof of Lemma 2.5, gives for $\frac{1}{2} < \sigma < 1$ and $\tau \in \mathbb{R}$,

$$\sum_{k=2}^{N} \left|\zeta_n(\sigma + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})\right|^2 \ll$$

$$N^{1-\beta_1} \log^{-\beta_2} N \int\limits_{1}^{hN^{\beta_1} \log^{\beta_2} N} |\zeta(\sigma + it, \alpha; \mathfrak{a})|^2 dt +$$

$$\Bigg(\int\limits_{0}^{hN^{\beta_1} \log^{\beta_2} N} |\zeta(\sigma + it, \alpha; \mathfrak{a})|^2 dt \int\limits_{1}^{hN^{\beta_1} \log^{\beta_2} N} |\zeta'(\sigma + it, \alpha; \mathfrak{a})|^2 dt \Bigg)^{\frac{1}{2}} \ll N.$$

Hence,

$$\limsup_{N\to\infty} \frac{1}{N-1} \sum_{k=2}^{N} \left|\zeta_n(\sigma + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})\right|^2 \leq C_1 < \infty$$

for all $n \in \mathbb{N}$. Therefore, by the Cauchy inequality,

$$\limsup_{N\to\infty} \frac{1}{N-1} \sum_{k=2}^{N} \left|\zeta_n(\sigma + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})\right| \leq C_2 < \infty \tag{2.5}$$

for all $n \in \mathbb{N}$. Let $K_l, l \in \mathbb{N}$, be compact sets in the definition of the metric ρ. Then (2.5) together with the Cauchy integral formula shows that

$$\limsup_{N\to\infty} \frac{1}{N-1} \sum_{k=2}^{N} \sup_{s\in K_l} \left|\zeta_n(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})\right| \leq R_l < \infty \tag{2.6}$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be an arbitrary number, and $M_l = M_l(\varepsilon) = 2^{\varepsilon} R_l \varepsilon^{-1}$. Then, taking into account (2.6), we find, for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$\limsup_{N\to\infty} \mathbb{P}\big(\sup_{s\in K_l} |X_{N,n}(s)| > M_l \big)$$

$$= \limsup_{N\to\infty} \frac{1}{N-1} \#\Big\{2 \leq k \leq N : \sup_{s\in K_l} \left|\zeta_n(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})\right| > M_l \Big\}.$$

$$\leq \limsup_{N\to\infty} \frac{1}{(N-1)M_l} \sum_{k=2}^{N} \sup_{s\in K_l} \left|\zeta_n(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\right| \leq \frac{\varepsilon}{2^l}.$$

Hence, by relation (2.4), for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$\mathbb{P}\Big(\sup_{s\in K_l} |\hat{X}_n(s)| > M\Big) \leq \frac{\varepsilon}{2^l}. \tag{2.7}$$

Putting

$$K = K(\varepsilon) = \Big\{g \in \big(H(D)\big) : \sup_{S\in K_l} |g(s)| \leq M_l, l \in \mathbb{N}\Big\},$$

we have that K is a compact subset of $H(D)$ because it is uniformly bounded on compact subsets of the strip D. Moreover, by (2.7), for all $n \in \mathbb{N}$,

$$\mathbb{P}\big(\hat{X}_n(s) \in K\big) \geq 1 - \varepsilon,$$

or, equivalently, for all $n \in \mathbb{N}$,

$$\hat{P}_n(K) \geq 1 - \varepsilon,$$

and the tightness of the family $\{\hat{P}_n : n \in \mathbb{N}\}$ is proved.

Since the family $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight, by Lemma 1.12, it is relatively compact. Therefore, there exists a sequence $\{\hat{P}_{n_k}\} \subset \{\hat{P}_n\}$ such that $\hat{P}_{n_k}$ converges weakly to a certain probability measure P on $\Big(H(D), \mathcal{B}\big(H(D)\big)\Big)$ as $k \to \infty$. Thus,

$$\hat{X}_{n_k} \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{2.8}$$

On the probability space $(\hat{\Omega}, \mathcal{A}, \mathbb{P})$, define one more $H(D)$-valued random element $X_N = X_N(s)$ by the formula

$$X_N(s) = \zeta(s + i\theta_N, \alpha; \mathfrak{a}).$$

Then, by Lemma 2.5, we find that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}\big(\rho(X_N, X_{N,n}) \geq \varepsilon\big)$$

$$= \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N-1} \#\Big\{2 \leq k \leq N : \rho\big(\zeta(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}), \zeta_n(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\big) \geq \varepsilon\Big\}$$

$$\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N-1)\varepsilon} \sum_{k=2}^{N} \rho\big(\zeta(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}), \zeta_n(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\big) = 0.$$

This equality, (2.4) and (2.8) show that the hypotheses of Lemma 1.13 are satisfied. Therefore, the relation

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} P \tag{2.9}$$

is true, thus, P_N converges weakly to P as $n \to \infty$. Moreover, (2.9) shows that the limit measure P is independent of the choice of the sequence $\{\hat{P}_{n_k}\}$. Since the family $\{\hat{P}_n\}$ is relatively compact, this implies the relation

$$\hat{X}_n \xrightarrow[N\to\infty]{\mathcal{D}} P.$$

Thus, we have that $\{\hat{P}_n\}$ converges weakly to P as $n \to \infty$.

For the identification of the measure P, we apply the weak convergence of the measure

$$\frac{1}{T}\,\text{meas}\big\{\tau \in [0,T] : \zeta(s+i\tau,\alpha;\mathfrak{a})\big\}, A \in \mathcal{B}\big(H(D)\big).$$

Under hypothesis that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, Proposition 1.1 asserts that this measure converges weakly to P_ζ as $n \to \infty$. In the proof of this fact, it was obtained that P_ζ is also the limit measure of $\hat{P}_n$ as $n \to \infty$. Thus, we have shown that $P = P_\zeta$.

Moreover, by Proposition 1.2, the support of the measure P_ζ is the whole of $H(D)$. The proposition is proved. □

2.2 Proof of universality

Proof of Theorem 2.1. As the proofs of universality theorems of Chapter 1, the proof of Theorem 2.1 uses a limit theorem in the space of analytic functions for the function $\zeta(s,\alpha;\mathfrak{a})$, and the Mergelyan theorem on the approximation of analytic functions by polynomials.

Using Lemma 1.16, we find a polynomial $p(s)$ such that

$$\sup_{s\in K}|f(s)-p(s)| < \frac{\varepsilon}{2}, \tag{2.10}$$

and define the set

$$G = \Big\{g \in H(D) : \sup_{s\in K}|g(s)-p(s)| < \frac{\varepsilon}{2}\Big\}.$$

Then, G is an open set in the space $H(D)$, therefore, by the first part of Proposition 2.1 and Lemma 1.17, we have the inequality

$$\liminf_{N\to\infty} P_N(G) \geq P_\zeta(G). \tag{2.11}$$

Moreover, the set G is an open neighbourhood of the polynomial $p(s)$ that, by the second part of Proposition 2.1, is an element of the support of the measure P_ζ. Thus, by a property of the support,

$$P_\zeta(G) > 0.$$

This, (2.11) and the definitions of P_N and G give the inequality

$$\liminf_{N\to\infty}\frac{1}{N-1}\#\Big\{2 \leq k \leq N : \sup_{s\in K}\Big|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-p(s)\Big| < \frac{\varepsilon}{2}\Big\} > 0. \tag{2.12}$$

Suppose that $k \in \mathbb{N}$ satisfies the inequality

$$\sup_{s\in K}\Big|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-p(s)\Big| < \frac{\varepsilon}{2}.$$

Then, for such k in view of (2.10),

$$\sup_{s\in K}\Big|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-f(s)\Big| \leq$$

$$\sup_{s\in K}\left|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-p(s)\right|+\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

This shows that

$$\left\{2\le k\le N:\sup_{s\in K}\left|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-p(s)\right|<\frac{\varepsilon}{2}\right\}\subset$$
$$\left\{2\le k\le N:\sup_{s\in K}\left|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-f(s)\right|<\varepsilon\right\}.$$

Therefore,

$$\liminf_{N\to\infty}\frac{1}{N-1}\#\left\{2\le k\le N:\sup_{s\in K}\left|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-f(s)\right|<\varepsilon\right\}\ge$$
$$\liminf_{N\to\infty}\frac{1}{N-1}\#\left\{2\le k\le N:\sup_{s\in K}\left|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-p(s)\right|<\frac{\varepsilon}{2}\right\}>0.$$

This together with (2.12) gives the inequality

$$\liminf_{N\to\infty}\frac{1}{N-1}\#\left\{2\le k\le N:\sup_{s\in K}\left|\zeta(s+ihk^{\beta_1}\log^{\beta_2}k,\alpha;\mathfrak{a})-f(s)\right|<\varepsilon\right\}>0.$$

The theorem is proved.

Theorem 2.1 is published in [36]. □

Chapter 3

Universality of composite functions of the periodic Hurwitz zeta-function

As it was noted in Introduction, it is important to extend the class of universal functions in the Voronin sense. This chapter is devoted to the universality for the functions $F\big(\zeta(s,\alpha;\mathfrak{a})\big)$, where $F: H(D)\to H(D)$ is a certain operator. We recall that $H(D)$, $D=\{s=\sigma+it\in\mathbb{C}: \frac{1}{2}<\sigma<1\}$, is the space of analytic functions on D endowed with topology of uniform convergence on compacta.

3.1 Generalization of continuous universality theorems

In this section, we generalize Theorems 1.1 and 1.2 for composite functions.

Theorem 3.1. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $F: H(D)\to H(D)$ is a continuous operator such that, for every open set $G\subset H(D)$, the set $F^{-1}G$ is not empty. Let $K\in\mathcal{K}$ and $f\in H(K)$. Then, for every $\varepsilon>0$*

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]: \sup_{s\in K}\big|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big)-f(s)\big|<\varepsilon\Big\}>0.$$

Proof. We will apply elements of the proof of Theorem 1.1 and properties of the operator F.

By Proposition 1.2,

$$P_T(A)=\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]: \zeta(s+i\tau,\alpha;\mathfrak{a})\in A\Big\}, A\in\mathcal{B}\big(H(D)\big),$$

converges weakly to P_ζ as $T\to\infty$, where P_ζ is the distribution of the random element

$$\zeta(s,\alpha,\omega;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m\omega(m)}{(m+\alpha)^s}.$$

For $A\in\mathcal{B}\big(H(D)\big)$, define

$$P_{T,F}(A)=\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]: F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big)\in A\Big\}.$$

Then we have that, for $A \in \mathcal{B}(H(D))$,

$$P_{T,F}(A) = \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau,\alpha;\mathfrak{a}) \in F^{-1}A\right\}.$$

Hence, $P_{T,F} = P_T F^{-1}$, where

$$P_T F^{-1}(A) = P_T(F^{-1}A), A \in \mathcal{B}(H(D)).$$

Since P_T converges weakly to P_ζ as $T \to \infty$, and the operator F is continuous, the latter equality and Lemma 1.5 show that $P_{T,F}$ converges weakly to $P_\zeta F^{-1}$ as $T \to \infty$.

It remains to find the support of the measure $P_\zeta F^{-1}$. Let g be an arbitrary element of the space $H(D)$, and G be any open neighbourhood of g. By the hypothesis of the theorem, the set $F^{-1}G$ is not empty, and because of the continuity of the operator F is open. Hence, $F^{-1}G$ is an open neighbourhood of a certain element $g_1 \in H(D)$. Since, by Proposition 1.2, the support of the measure P_ζ is the whole $H(D)$, hence we obtain by properties of the support that

$$P_\zeta(F^{-1}G) > 0.$$

Therefore,

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0.$$

Since the objects g and G are arbitrary, this shows that the support of the measure $P_\zeta F^{-1}$ is the whole $H(D)$.

The remaining part of the proof is standard. By Lemma 1.16, there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.1}$$

Define

$$G = \left\{g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\right\}.$$

Then G is an open neighbourhood of $p(s)$ which, in view of the above remark, is an element of the support of the measure $P_\zeta F^{-1}$. Therefore, $P_\zeta F^{-1}(G) > 0$. Using the weak convergence of $P_{T,F}$ to $P_\zeta F^{-1}$ as $T \to \infty$, and applying Lemma 1.17, we obtain that

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big) \in G\right\} \geq P_\zeta F^{-1}(G) > 0.$$

Hence, by the definition of G,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big) - p(s)\right| < \frac{\varepsilon}{2}\right\} > 0.$$

This together with inequality (3.1) proves the theorem. □

The hypothesis of Theorem 3.1 that $F^{-1}G \neq \emptyset$ for every open set $G \subset H(D)$ can be replaced by a stronger but simpler one. Thus, we have the following theorem. Denote by $F^{-1}\{p\}$ the preimage of a polynomial p.

Theorem 3.2. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 3.1 is true.*

Proof. We observe that the compact sets K_l in the definition of the metric ρ can be chosen to be with connected complements. For example, we can take closed rectangles. Moreover, the quantity $\rho(g_1, g_2)$, $g_1, g_2 \in H(D)$, is small if

$$\sup_{s\in K_l} |g_1(s) - g_2(s)|$$

is small enough for sufficiently large $l \in \mathbb{N}$. Thus, the approximation in the space $H(D)$ reduces to that on compact subsets of the strip D with connected complements.

We will prove that, for every non-empty open set $G \subset H(D)$, the set $F^{-1}G$ is non-empty. Let $G \subset H(D) \neq \emptyset$ be an arbitrary open set, and $g \in G$. Suppose that $K \in \mathcal{K}$. Then, by Lemma 1.16, for every $\varepsilon > 0$, there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon.$$

Therefore, if ε is small enough, we may assume that $p \in G$, too. By the hypothesis of the theorem, this shows that the set $F^{-1}G$ is non-empty. Thus, we obtained the hypothesis of Theorem 3.1, and the assertion of the theorem follows from Theorem 3.1. □

In the next theorem, we replace the hypothesis on the continuity of the operator F by a certain analogue of the Lipschitz type condition in the space of analytic functions.

Theorem 3.3. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that the operator $F : H(D) \to H(D)$ is a such that, for every polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is not empty, and for each $K \in \mathcal{K}$, there exist positive constants c and β, and $K_1 \in \mathcal{K}$ such that*

$$\sup_{s\in K} \left|F\big(g_1(s)\big) - F\big(g_2(s)\big)\right| \le c \sup_{s\in K_1} \left|g_1(s) - g_2(s)\right|^{\beta}$$

for all $g_1, g_2 \in H(D)$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 3.1 is true.

Proof. By Lemma 1.16, there exists a polynomial $p = p(s)$ such that inequality (3.1) is valid. Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{s\in K_1} \left|\zeta(s + i\tau, \alpha; \mathfrak{a}) - g(s)\right| \le c^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}, \tag{3.2}$$

where $g \in F^{-1}\{p\}$, and $K_1 \in \mathcal{K}$ corresponds the set K in hypothesis of the theorem. Then, for the same τ, by the inequality of the theorem, we have that

$$\sup_{s\in K} \left|F\big(\zeta(s + i\tau, \alpha; \mathfrak{a})\big) - p(s)\right| \le c \sup_{s\in K_1} \left|\zeta(s + i\tau, \alpha; \mathfrak{a}) - g(s)\right|^{\beta} \le c\left(c^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}\right)^{\beta} = \frac{\varepsilon}{2}. \tag{3.3}$$

By Theorem 1.1, the set of reals τ satisfying inequality (3.2), has a positive lower density, i.e.,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{s\in K_1} \left|\zeta(s + i\tau, \alpha; \mathfrak{a}) - g(s)\right| \le c^{-\frac{1}{\beta}} \left(\frac{\varepsilon}{2}\right)^{\frac{1}{\beta}}\right\} > 0.$$

This shows that

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big) - p(s)\right| < \frac{\varepsilon}{2}\right\} > 0.$$

Combining this with (3.3) proves the theorem. □

We will prove that the operator $F : H(D) \to H(D)$ given by

$$F(g) = cg^{(n)},\ c \neq 0,$$

where $g^{(n)}$ denotes the n-th derivative of g, satisfies the hypothesis of Theorem 3.1. For this, we recall the Cauchy integral formula, see, for example, [49].

Lemma 4.1. *Suppose that G is a domain in the complex plane, $g(s)$ is an analytic function in G, and L is a simple contour with its interior int L in G. Then, for $s_0 \in$ int L,*

$$g^{(x)}(s_0) = \frac{n!}{2\pi i} \int\limits_L \frac{g(s)}{(s-s_0)^{n+1}} ds.$$

Obviously, the set $F^{-1}\{p\}$ is not empty for each polynomial $p = p(s)$ because the equation

$$cg^{(n)}(s) = p(s) = a_k s^k + a_{k-1}s^{k-1} + ... + a_0$$

has the solution

$$g(s) = \frac{1}{c}\left(\frac{a_k s^{k+n}}{(k+1)...(k+n)} + ... + \frac{a_0 s^n}{1...n}\right) \in H(D).$$

Now, let $K, K_1 \in \mathcal{K}$ and G is an open set such that $K \subset G \subset K_1$, and let L be a simple closed contour lying in $K_1 \setminus G$, and containing inside the set K. Then, in view of Lemma 4.1, for $g_1, g_2 \in H(D)$ and $s \in K$,

$$F\big(g_1(s)\big) - F\big(g_2(s)\big) = c\frac{n!}{2\pi i} \int\limits_L \frac{g(z)}{(z-s)^{n+1}} dz.$$

Therefore,

$$\sup_{s\in K} \left|F\big(g_1(s)\big) - F\big(g_2(s)\big)\right| = \frac{cn!}{2\pi} \left|\int\limits_L \frac{g_1(z)-g_2(z)}{(z-s)^{n+1}} dz\right| \leq$$

$$\frac{cn!}{2\pi} \int\limits_L \frac{|g_1(z)-g_2(z)|}{|z-s|^{n+1}} |dz| \leq$$

$$\frac{cn!}{2\pi} \sup_{s\in L} |g_1(s)-g_2(s)| \frac{|dz|}{(z-s)^{n+1}} \leq C \sup_{s\in L} |g_1(s)-g_2(s)| \leq C \sup_{s\in K_1} |g_1(s)-g_2(s)|$$

with a certain positive constant C. Thus, in this case $\beta = 1$.

Now, let $a_1, ..., a_r$ be distinct complex numbers, and

$$H_{a_1,...,a_r}(D) = \big\{g \in H(D) : g(s) \neq a_j, j = 1, ..., r\big\}.$$

Thus, $H_{a_1,...,a_r}(D)$ is a subset of the space of analytic functions not taking any value $a_1, ..., a_r$. In the next theorem, we approximate analytic functions from the set $H_{a_1,...,a_r}(D)$.

Theorem 3.4. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator such that $F\big(H(D)\big) \supset H_{a_1,\dots,a_r}(D)$. For $r = 1$, let $K \in \mathcal{K}$ and $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. For $r \geq 2$, let $K \subset D$ be an arbitrary compact set, and $f(s) \in H_{a_1,\dots,a_r}(D)$. Then the same assertion as in Theorem 3.1 is true.*

Proof. As in the proof of Theorem 3.1, we have that

$$P_{T,F}(A) \stackrel{def}{=} \frac{1}{T} \text{meas}\Big\{\tau \in [0,T] : F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big) \in A\Big\}, A \in \mathcal{B}\big(H(D)\big),$$

converges weakly to the measure $P_\zeta F^{-1}$ as $T \to \infty$. Thus, it remains to give explicitly the support of $P_\zeta F^{-1}$. Let g be an arbitrary element of the set $F\big(H(D)\big)$, and G be any open neighbourhood of g. Then there exists $g_1 \in H(D)$ such that $F(g_1) = g$. Hence, by the continuity of F, the set $F^{-1}G$ is an open neighbourhood of the element g_1. Since the support of the measure P_ζ, by Proposition 1.2, is the whole of $H(D)$, this shows that $P_\zeta(F^{-1}G) > 0$. Thus,

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0. \tag{3.4}$$

Moreover,

$$P_\zeta F^{-1}\Big(F\big(H(D)\big)\Big) = P_\zeta\Big(F^{-1}F\big(H(D)\big)\Big) = P_\zeta\big(H(D)\big) = 1.$$

Since the support is a closed set, this together with (3.4) proves that the support of the measure $P_\zeta F^{-1}$ is the closure of the set $F\big(H(D)\big)$.

The case r=1. By the Lemma 1.16, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4}. \tag{3.5}$$

Since $f(s) \neq a_1$ on K, we have that $p(s) \neq a_1$ on K as well if ε is small enough. Therefore, we can define on K a continuous branch of the logarithm $\log\big(p(s) - a_1\big)$ which will be analytic in the interior of K. Applying Lemma 1.16 once more, we find a polynomial $p_1(s)$ such that

$$\sup_{s\in K} \big|\big(p(s) - a_1\big) - e^{p_1(s)}\big| < \frac{\varepsilon}{4}. \tag{3.6}$$

We put $g_1(s) = e^{p_1(s)} + a_1$. Then $g_1(s) \in H(D)$ and $g_1(s) \neq a_1$, in other words, $g_1(s) \in H_{a_1}(D)$. Since by the hypothesis of the theorem, $H_1(D) \subset F\big(H(D)\big)$, and the support of $P_\zeta F^{-1}$ is the closure of $F\big(H(D)\big)$, we have that g_1 is an element of the support of the measure $P_\zeta F^{-1}$. Define

$$G_1 = \Big\{g \in H(D) : \sup_{s\in K} |f(s) - g_1(s)| < \frac{\varepsilon}{2}\Big\}.$$

Then G_1 is an open neighbourhood of the element g_1, therefore, we have that $P_\zeta F^{-1}(G_1) > 0$. Using the weak convergence of $P_{T,F}$ to $P_\zeta F^{-1}$ as $T \to \infty$, hence we obtain by Lemma 1.17 that

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} \big|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big) - p(s)\big| < \frac{\varepsilon}{2}\Big\}$$
$$\geq P_\zeta F^{-1}(G_1) > 0. \tag{3.7}$$

In view of inequalities (3.5) and (3.6),

$$\sup_{s\in K}|f(s)-g_1(s)| \le \sup_{s\in K}|f(s)-p(s)| + \sup_{s\in K}|p(s)-g_1(s)| < \frac{\varepsilon}{2}. \tag{3.8}$$

Suppose that $\tau \in \mathbb{R}$ satisfies the inequality

$$\sup_{s\in K}\left|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big)-g_1(s)\right| < \frac{\varepsilon}{2}.$$

Then, by (3.8),

$$\sup_{s\in K}\left|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big)-f(s)\right| \le \sup_{s\in K}\left|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big)-g_1(s)\right| + \sup_{s\in K}|f(s)-g_1(s)| < \varepsilon.$$

Therefore,

$$\left\{\tau\in[0,T]:\sup_{s\in K}\left|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big)-g_1(s)\right|<\frac{\varepsilon}{2}\right\}\subset\left\{\tau\in[0,T]:\sup_{s\in K}\left|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big)-f(s)\right|<\varepsilon\right\}.$$

This together with (3.7) proves the theorem in case r=1.

The case $r \ge 2$. Since $f(s) \in H_{a_1,\dots,a_r}(D)$ and $H_{a_1,\dots,a_r}(D) \subset F\big(H(D)\big)$, and the support of $P_\zeta F^{-1}$ is the closure of $F\big(H(D)\big)$, we have that f is an element of the support of the measure $P_\zeta F^{-1}$. Define the set

$$G_2 = \left\{g \in H(D) : \sup_{s\in K}|g(s)-f(s)| < \varepsilon\right\}.$$

Then $P_\zeta F^{-1}(G_2) > 0$, and the definition of G_2 and Lemma 1.17 give the inequality

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]:\sup_{s\in K}\left|F\big(\zeta(s+i\tau,\alpha;\mathfrak{a})\big)-f(s)\right|<\varepsilon\right\} \ge P_\zeta F^{-1}(G_2) > 0.$$

The theorem is proved. □

We give some examples of operators satisfying the hypothesis of Theorem 3.4.

1. Let $F(g) = \sin g$. Then $F\big(H(D)\big) \supset H_{-1,1}(D)$. Actually, it is well known that

$$\sin s = \frac{e^{is}-e^{-is}}{2i}.$$

We take an arbitrary function $g \in H_{-1,1}(D)$ and solve the equation

$$\frac{e^{if}-e^{-if}}{2i} = g.$$

Hence,

$$e^{if}-e^{-if}-2ig = 0.$$

Therefore, taking $x = e^{if}$, we obtain the equation

$$x^2 - 2ixg - 1 = 0.$$

Hence,

$$x = ig \pm \sqrt{-g^2+1}.$$

Since $g \in H_{-1,1}(D)$, $g \neq -1, 1$ on D. Therefore, $ig \pm \sqrt{-g^2+1} \neq 0$ and lies in $H(D)$, and we find that

$$f = \frac{1}{i} \log\left(ig + \sqrt{-g^2+1}\right)$$

is an element of $H(D)$. This shows that $F\big(H(D)\big) \supset H_{-1,1}(D)$, and, by Lemma 3.4, the functions from the set $H_{-1,1}(D)$ can be approximated by shifts $\sin\big(\zeta(s+i\tau, \alpha; \mathfrak{a})\big)$ with, for example, transcendental α.

2. Let $F(g) = \cos hg$. We will prove that $F\big(H(D)\big) \supset H_{-1,1}(D)$. Similarly as above, we take arbitrary $g \in H_{-1,1}(D)$ and consider the equation

$$\frac{e^f - e^{-f}}{2} = g.$$

From this, putting $x = e^f$, we find the equation

$$x^2 - 2xg + 1 = 0.$$

Thus,

$$x = g \pm \sqrt{g^2 - 1}.$$

Since $g \in H_{-1,1}(D)$, $g \neq -1, 1$. Therefore,

$$f = \log\left(g + \sqrt{g^2-1}\right) \in H(D),$$

and $F\big(H(D)\big) \supset H_{-1,1}(D)$. Thus, by Theorem 3.4, the functions from the set $H_{-1,1}(D)$ can be approximated by shifts $\cos h\big(\zeta(s+i\tau, \alpha; \mathfrak{a})\big)$ with transcendental α.

Theorem 3.5. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator. Let $K \in D$ be an arbitrary compact subset, and $f(s) \in F\big(H(D)\big)$. Then the same assertion as in Theorem 3.1 is true.*

Proof. In the beginning of the proof of Theorem 3.4, it was obtained that the support of the measure $P_\zeta F^{-1}$ is the closure of $F\big(H(D)\big)$. Thus, in Theorem 3.5, we approximate analytic functions from the support of the measure $P_\zeta F^{-1}$. Therefore, the proof of Theorem 3.5 coincides with that of Theorem 3.4 with $r \geq 2$. We note that in the proof we do not use the Mergelyan theorem (Lemma 1.16). □

Theorem 1.2 also can be generalized for composite functions.

Theorem 3.6. *Suppose that $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a,b) = 1, b \neq 2$ and that $\operatorname{rad}(q)$ divides b, and that, for $V > 0$, $F : H(D_V) \to H(D_V)$ is a continuous operator such that, for every open set $G \subset H(D_V)$, the set $F^{-1}G$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \sup_{s\in K} \left|F\Big(\zeta(s+i\tau, \frac{a}{b}; \mathfrak{a})\Big) - f(s)\right| < \varepsilon\right\} > 0.$$

Proof. By Proposition 1.4,

$$P_{1T}(A)=\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:\zeta\Big(s+i\tau,\frac{a}{b};\mathfrak{a}\Big)\in A\Big\}, A\in\mathcal{B}\big(H(D)\big),$$

converges weakly to the measure $P_{1\zeta}$ as $T\to\infty$, where $P_{1\zeta}$ is the distribution of the random element

$$\zeta\Big(s,\frac{a}{b},\omega_1;\mathfrak{a}\Big)=\frac{b^s\overline{\omega_1(p)}}{r}\sum_{j=1}^{r}b_jL(s,\omega_1,\chi_j).$$

For $A\in\mathcal{B}\big(H(D_V)\big)$, define

$$P_{1T,F}(A)=\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:F\Big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\Big)\in A\Big\}.$$

From the definitions P_{1T} and $P_{1T,F}$, it follows that, for $A\in\mathcal{B}\big(H(D_V)\big)$,

$$P_{1T,F}(A)=\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:\zeta\Big(s+i\tau,\frac{a}{b};\mathfrak{a}\Big)\in F^{-1}A\Big\}.$$

Therefore, $P_{1T,F}=P_{1T}F^{-1}$, thus, the continuity of F, weak convergence of P_{1T} to $P_{1\zeta}$ as $T\to\infty$, and Lemma 1.5 imply the weak convergence of $P_{1T,F}$ to $P_{1\zeta}F^{-1}$ as $T\to\infty$.

We will prove that the support of the measure $P_{1\zeta}$ is the whole of $H(D_V)$. We take an arbitrary element $g\in H(D_V)$ and its any open neighbourhood G. Since the set $F^{-1}G$ is not empty and the operator F is continuous, we have that $F^{-1}G$ is an open neighbourhood of a certain element $g_1\in H(D_V)$. By the second part of Proposition 1.4, the support of he measure $P_{1\zeta}$ is the whole of $H(D_V)$. Thus,

$$P_{1\zeta}(F^{-1}G)>0.$$

Hence,

$$P_{1\zeta}F^{-1}(G)=P_{1\zeta}(F^{-1}G)>0.$$

Since g and G are arbitrary, this shows that the support of the measure $P_{1\zeta}F^{-1}$ is the whole of $H(D_V)$.

Now, let $V>0$ be such that $K\subset H(D_V)$. Define the set

$$G=\Big\{g\in H(D_V):\sup_{s\in K}|g(s)-f(s)|<\frac{\varepsilon}{2}\Big\},$$

where $p(s)$ is a polynomial satisfying (3.1). Then G is an open neighbourhood of $p(s)$, i.e., is an open neighbourhood of an element of the support of the measure $P_{1\zeta}F^{-1}$. Hence, $P_{1\zeta}F^{-1}(G)>0$. Therefore, in view of weak convergence of $P_{1\zeta,F}$ to $P_{1\zeta}F^{-1}$ and Lemma 1.17, we find that

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:F\Big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\Big)\in G\Big\}\geq P_{1\zeta}F^{-1}(G)>0,$$

or, by the definition of G,

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:\sup_{s\in K}\Big|F\Big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\Big)-p(s)\Big|<\frac{\varepsilon}{2}\Big\}>0.$$

Combining this with (3.1) proves the theorem.

Next theorem is an analogue of Theorem 3.2.

□

Theorem 3.7. *Suppose that* $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a,b) = 1, b \neq 2$ *and that* $\mathrm{rad}(q)$ *divides* b, *and that* $F : H(D_V) \to H(D_V)$ *is a continuous operator such that, for every polynomial* $p = p(s)$, *the set* $F^{-1}\{p\}$ *is not empty. Let* $K \in \mathcal{K}$ *and* $f(s) \in H(K)$. *Then the same assertion as in Theorem 3.6 is true.*

Proof. We will prove that the hypotheses of the theorem satisfy those of Theorem 3.6, i.e, that for every non-empty set $G \subset H(D_V)$, its preimage $F^{-1}G$ is also non-empty. Thus, let $G \subset H(D_V)$, $G \neq \emptyset$, and let $g \in G$. We take $K \in \mathcal{K}$, $K \in D_V$. Then, using Lemma 1.16, for every $\varepsilon > 0$, we find a polynomial $p = p(s)$ such that

$$\sup_{s \in K} |g(s) - p(s)| < \varepsilon.$$

If $\varepsilon > 0$ is small enough, this shows that the polynomial $p(s)$ lies in G. Therefore, by the hypothesis of the theorem, the set $F^{-1}G$ is non-empty. Thus, by Theorem 3.6, the assertion of the theorem follows. □

Let

$$H_{a_1,\ldots,a_r}(D) = \{g \in H(D_V) : g(s) = a_j, j = 1, \ldots, r\}.$$

Theorem 3.8. *Suppose that* $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a,b) = 1, b \neq 2$ *and that* $\mathrm{rad}(q)$ *divides* b, *and that* $F : H(D_V) \to H(D_V)$ *is a continuous operator such that,* $F\big(H(D_V)\big) \subset H_{a_1,\ldots,a_r}(D_V)$. *For* $r = 1$, *let* $K \in \mathcal{K}$, $f(s) \in H(K)$ *and* $f(s) \neq a_1$ *on* K. *For* $r \geq 2$, *let* $K \subset D$ *be an arbitrary compact set, and* $f(s) \in H_{a_1,\ldots,a_r}(D_V)$. *Then the assertion of Theorem 3.6 is true.*

Proof. We argue similarly to the proof of Theorem 3.4. We consider the support of the measure $P_{1\zeta}F^{-1}$. Let g be an arbitrary element of the set $H_{a_1,\ldots,a_r}(D_V)$, and G be any open neighbourhood of g. Then there exists an element $g_1 \in H(D)$ such that $F(g_1) = g$. Hence, by the continuity of the operator F, the set $F^{-1}G$ is an open neighbourhood of the element g_1. By Proposition 1.4, the support of the measure $P_{1\zeta}$ is the whole of $H(D_V)$, thus, $P_{1\zeta}(F^{-1}G) > 0$. Hence,

$$P_{1\zeta}F^{-1}(G) = P_{1\zeta}(F^{-1}G) > 0.$$

This shows that the support of $P_{1\zeta}F^{-1}$ contains the set $H_{a_1,\ldots,a_r}(D_V)$. Moreover, the support is closed set, thus, the support contains the closure of $H_{a_1,\ldots,a_r}(D_V)$.

The case r=1. Let $V > 0$ be such that $K \subset D_V$. By Lemma 1.16, there exists a polynomial $p(s)$ satisfying inequality (3.5), and there exists a polynomial $p_1(s)$ satisfying inequality (3.6). Let $g_1(s) = e^{p_1(s)} + a_1$. Then $g_1(s) \in H(D_V)$ and $g_1(s) \neq a_1$, hence, $g_1(s) \in H_{a_1}(D_V)$. Since the support of the measure $P_{1\zeta}F^{-1}$ contains the closure of $H_{a_1}(D_V)$, we have that g_1 is an element of the support of $P_{1\zeta}F^{-1}$. Define the set

$$G_1 = \Big\{g \in H(D_V) : \sup_{s \in K} |f(s) - g_1(s)| < \frac{\varepsilon}{2}\Big\},$$

The set G_1 is open, thus, it is an open neighbourhood of the element g_1, hence, the inequality $P_{1\zeta}F^{-1}(G_1) > 0$ is true. Since $P_{1T,F}$ converges weakly to $P_{1\zeta}F^{-1}$, we obtain that

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]:\sup_{s\in K}\Big|F\Big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\Big)-g_1(s)\Big|<\frac{\varepsilon}{2}\Big\}\geq P_{1\zeta}F^{-1}(G_1)>0. \tag{3.9}$$

In view of (3.5) and (3.6),

$$\sup_{s\in K}|f(s)-g_1(s)|<\frac{\varepsilon}{2}.$$

Hence, using the inequality

$$\sup_{s\in K}\big|F\big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\big)-g_1(s)\big|<\frac{\varepsilon}{2},$$

we find that

$$\sup_{s\in K}\big|F\big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\big)-f(s)\big|<\varepsilon.$$

Therefore, it follows from (3.9) that

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]:\sup_{s\in K}\Big|F\Big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\Big)-f(s)\Big|<\varepsilon\Big\}>0.$$

The case $r\geq 2$. Since the support of the measure $P_{1\zeta}F^{-1}$ contains the closure of the set $H_{a_1,\dots,a_r}(D_V)$ (as in the case $r=1$, we take $V>0$ such that $K\subset D_V$), we have that $f(s)$ is an element of the support of $P_{1\zeta}F^{-1}$. Let

$$G_2=\Big\{g\in H(D_V):\sup_{s\in K}|g(s)-f(s)|<\varepsilon\Big\}.$$

Then G_2 is an open set of the element $f(s)$, thus, $P_{1\zeta}F^{-1}(G_2)>0$. Therefore, the weak convergence of $P_{1T,F}$ to $P_{1\zeta}F^{-1}$ as $T\to\infty$ implies the inequality

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]:\sup_{s\in K}\Big|F\Big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\Big)-f(s)\Big|<\varepsilon\Big\}>0.$$

The theorem is proved. □

We give one example. Let

$$F\big(g(s)\big)=g^2(s)+4g(s)+2.$$

We consider the equation

$$g^2(s)+4g(s)+2=f(s),$$

where $f(s)\in H_{-2}(D_V)$. We find that

$$g(s)=-2\pm\sqrt{2^2-(2-f(s))}=-2\pm\sqrt{2+f(s)}.$$

Since $f(s)\neq -2$ on D_V, we have that $g(s)=-2+\sqrt{2+f(s)}$ belongs to $F\big(H(D_V)\big)$. Therefore, by Theorem 3.8, the functions of the set $H_{-2}(D_V)$ can be approximated by shifts $F\Big(\zeta(s+i\tau,\frac{a}{b};\mathfrak{a})\Big)$.

3.2 Generalization of discrete universality theorems

This section is devoted to generalizations of Theorem 1.3, 1.4 and 2.1 for composite functions.

We start with an analogue of Theorem 3.1.

Theorem 3.9. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{N+1}\#\Big\{0 \le k \le N : \sup_{s\in K}\Big|F\Big(\zeta(s+ikh,\alpha;\mathfrak{a})\Big) - f(s)\Big| < \varepsilon\Big\} > 0.$$

Proof. We use the probabilistic way as in all theorems of the thesis. Proposition 1.5 asserts that

$$P_N(A) = \frac{1}{N+1}\#\Big\{0 \le k \le N : \zeta(s+ikh,\alpha;\mathfrak{a}) \in A\Big\}, A \in \mathcal{B}\big(H(D)\big),$$

converges weakly to the measure P_ζ as $N \to \infty$, where P_ζ is the distribution of $H(D)$-valued random element

$$\zeta(s,\alpha,\omega;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m\omega(m)}{(m+\alpha)^s}.$$

We will deal with weak convergence of

$$P_{N,F}(A) \overset{def}{=} \frac{1}{N+1}\#\Big\{0 \le k \le N : F\Big(\zeta(s+ikh,\alpha;\mathfrak{a})\Big) \in A\Big\}, A \in \mathcal{B}\big(H(D)\big).$$

The definitions of P_N and $P_{N,F}$ show that, for $A \in \mathcal{B}\big(H(D)\big)$,

$$P_{N,F}(A) = \frac{1}{N+1}\#\Big\{0 \le k \le N : \zeta(s+ikh,\alpha;\mathfrak{a}) \in F^{-1}A\Big\},$$

thus, the equality $P_{N,F} = P_N F^{-1}$ holds. Since, by Proposition 1.5, P_N converges weakly to P_ζ as $N \to \infty$, and the operator F is continuous, the above equality together with Lemma 1.5 imply the weak convergence of $P_{N,F}$ to $P_\zeta F^{-1}$ as $N \to \infty$.

Now, we discuss the support of the measure $P_\zeta F^{-1}$. Since, by Proposition 1.5, the support of P_ζ is the whole of $H(D)$, we find by repeating the arguments of the proof of Theorem 3.1 that the support of $P_\zeta F^{-1}$ is the whole of $H(D)$.

Let $p(s)$ and G be the same as in the proof of Theorem 3.1. Then Lemma 1.17 and the weak convergence of $P_{N,F}$ give the inequality

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \le k \le N : F\Big(\zeta(s+ikh,\alpha;\mathfrak{a})\Big) \in G\Big\} =$$

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \le k \le N : \sup_{s\in K}\Big|F\Big(\zeta(s+ikh,\alpha;\mathfrak{a})\Big) - p(s)\Big| < \frac{\varepsilon}{2}\Big\} = P_\zeta F^{-1}(G) > 0.$$

This inequality and (3.1) give the assertion of the theorem. □

The next theorem is a discrete analogue of Theorem 3.2.

Theorem 3.10. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then the same assertion as in Theorem 3.9 is true.*

Proof. We will prove that the hypothesis of the theorem that the set $F^{-1}\{p\}$ is non-empty for every polynomial p implies that of Theorem 3.9 that the set $F^{-1}G$ is non-empty for every open set $G \subset H(D)$. This follows from Lemma 1.16. Actually, let $g \in G$. Then, by Lemma 1.16, there exists a polynomial such that

$$\sup_{s\in K} |g(s) - p(s)| < \varepsilon$$

for every $K \in \mathcal{K}$. Hence, if $\varepsilon > 0$ is small enough, we have that $p(s) \in G$. Therefore, since $F^{-1}\{p\} \neq \emptyset$, we obtain that $F^{-1}G \neq \emptyset$ as well. This remark and Theorem 3.9 prove the theorem. □

Let the set $H_{a_1,\dots,a_r}(D)$ be the same as in Theorem 3.4.

Theorem 3.11. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$, and that $F : H(D) \to H(D)$ is a continuous operator such that $F\big(H(D)\big) \supset H_{a_1,\dots,a_r}(D)$. For $r = 1$, let $K \in \mathcal{K}$, $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. For $r \geq 2$, let $K \subset D$ be an arbitrary compact set, and $f(s) \in H_{a_1,\dots,a_r}(D)$. Then the same assertion as in Theorem 3.9 is true.*

Proof. The proof is analogical to that of Theorem 3.4. As it was noted in the proof of Theorem 3.9, $P_{N,F}$ converges weakly to the measure $P_\zeta F^{-1}$ as $N \to \infty$. Moreover, it was obtained that the support of the measure $P_\zeta F^{-1}$ is the closure of the set $F\big(H(D)\big)$.

The case r=1. Using the notation of the proof of Theorem 3.4 and the weak convergence of the measure $P_{T,F}$, we obtain that

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} \Big|F\Big(\zeta(s + ikh, \alpha; \mathfrak{a})\Big) - g_1(s)\Big| < \frac{\varepsilon}{2}\Big\} \geq P_\zeta F^{-1}(G_1) > 0. \tag{3.10}$$

Suppose that $k \in \mathbb{N}_0$ satisfies the inequality

$$\sup_{s\in K} \Big|F\Big(\zeta(s + ikh, \alpha; \mathfrak{a})\Big) - g_1(s)\Big| < \frac{\varepsilon}{2}.$$

Then, in view of (3.8),

$$\sup_{s\in K} \Big|F\Big(\zeta(s + ikh, \alpha; \mathfrak{a})\Big) - f(s)\Big| \leq \sup_{s\in K} \Big|F\Big(\zeta(s + ikh, \alpha; \mathfrak{a})\Big) - g_1(s)\Big| + \sup_{s\in K} |f(s) - g_1(s)| < \varepsilon.$$

Therefore,

$$\Big\{0 \leq k \leq N : \sup_{s\in K} \Big|F\Big(\zeta(s+ikh, \alpha; \mathfrak{a})\Big) - g_1(s)\Big| < \frac{\varepsilon}{2}\Big\} \subset \Big\{0 \leq k \leq N : \sup_{s\in K} \Big|F\Big(\zeta(s+ikh, \alpha; \mathfrak{a})\Big) - f(s)\Big| < \varepsilon\Big\}.$$

This and (3.10) give the inequality of the theorem for $r = 1$.

The case r $\geq$ 2. Let the set G_2 be from the proof of Theorem 3.4. Then $P_\zeta F^{-1}(G_2) > 0$, and Lemma 1.17 together with the definition of G_2 gives the inequality

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} \Big|F\Big(\zeta(s + ikh, \alpha; \mathfrak{a})\Big) - f(s)\Big| < \frac{\varepsilon}{2}\Big\} \geq P_\zeta F^{-1}(G_2) > 0.$$

The theorem is proved. □

Also, generalizations of the discrete universality for $\zeta(s, \alpha; \mathfrak{a})$ with rational parameter α for composite functions is possible. In virtue of similarity to the above theorems, we present only one theorem.

Theorem 3.12. *Suppose that $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a, b) = 1, \alpha \neq \frac{1}{2}$ and $(bl + a, bq) = 1$ for all $l = 0, ..., q-1$, and that, for $V > 0$, $F : H(D_V) \to H(D_V)$ is a continuous operator such that, for every open set $G \subset H(D_V)$, the set $F^{-1}G$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \sup_{s\in K} \Big| F\Big(\zeta(s + ikh, \frac{a}{b}; \mathfrak{a})\Big) - f(s)\Big| < \varepsilon\Big\} > 0.$$

Proof. By Proposition 1.6, we have that, for $V > 0$ and $h > 0$,

$$Q_{N,V,h}(A) = \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \zeta\Big(s + ikh, \frac{a}{b}; \mathfrak{a}\Big) \in A\Big\}, A \in \mathcal{B}\big(H(D_V)\big),$$

converges weakly to the measure $P_{\zeta,V,h}$ as $N \to \infty$, where $P_{\zeta,V,h}$ is the distribution of the random element

$$\zeta\Big(s, \frac{a}{b}, \omega_{1h}; \mathfrak{a}\Big) = \frac{b^s \overline{\omega_{1h}(b)}}{r} \sum_{j=1}^{r} b_j L(s, \omega_{1h}\chi_j)$$

in the notation of Section 1.4. For $A \in \mathcal{B}\big(H(D_V)\big)$, define

$$Q_{N,V,F,h}(A) = \frac{1}{N+1} \Big\{0 \leq k \leq N : F\Big(\zeta(s + ikh, \frac{a}{b}; \mathfrak{a})\Big) \in A\Big\}.$$

This and the definition of $Q_{N,V,h}$ show that, for $A \in \mathcal{B}\big(H(D_V)\big)$,

$$Q_{N,V,F,h}(A) = \frac{1}{N+1} \Big\{0 \leq k \leq N : \zeta\Big(s + ikh, \frac{a}{b}; \mathfrak{a}\Big) \in F^{-1}A\Big\}.$$

Therefore, the equality $Q_{N,V,F,h} = Q_{N,V,h}F^{-1}$ is true. Since $Q_{N,V,h}$ converges weakly to $P_{\zeta,V,h}$ as $N \to \infty$, and the operator F is continuous, this equality together with Lemma 1.5 implies the weak convergence for $Q_{N,V,F,h}$ to $P_{\zeta,V,h}F^{-1}$ as $N \to \infty$.

Now, consider the support of the measure $P_{\zeta,V,h}F^{-1}$. We take an arbitrary element $g \in H(D_V)$, and an arbitrary open neighbourhood G of g. Since the operator F is continuous, the set $F^{-1}G$ is open, too, and, by the hypothesis of the theorem, is not empty. This means that $F^{-1}G$ is an open neighbourhood of a certain element $g_1 \in H(D_V)$. Therefore, by second part of Proposition 1.6, we have that $P_{\zeta,V,h}(F^{-1}G) > 0$. Hence,

$$P_{\zeta,V,h}F^{-1}(G) = P_{\zeta,V,h}(F^{-1}G) > 0.$$

Since g and G are arbitrary, we obtain that the support of the measure $P_{\zeta,V,h}F^{-1}$ is the whole of $H(D_V)$.

Define the set

$$G = \Big\{g \in H(D_V) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2}\Big\},$$

where $V > 0$ is such that $K \subset D_V$, and $p(s)$ is a polynomial satisfying the inequality

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.11}$$

The polynomial $p(s)$ is an element of the support of the measure $P_{\zeta,V,h}F^{-1}$, therefore, $P_{\zeta,V,h}F^{-1}(G) > 0$. Hence, in view of the weak convergence of $Q_{N,V,F,h}$ and Lemma 1.7, we obtain that

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \le k \le N : F\Big(\zeta(s+ikh, \frac{a}{b}; \mathfrak{a})\Big) \in G\Big\} \ge P_{\zeta,V,h}F^{-1}(G) > 0,$$

or, by the definition of G,

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\Big\{0 \le k \le N : \sup_{s\in K}\Big|F\Big(\zeta(s+ikh, \frac{a}{b}; \mathfrak{a})\Big) - p(s)\Big| < \frac{\varepsilon}{2}\Big\} > 0.$$

This and (3.11) prove the theorem. □

Similarly, other versions of the generalization of Theorem 1.4 for composite functions can be obtained.

Now, we give one generalization of Theorem 2.1 for composite functions.

Theorem 3.13. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, $F : H(D) \to H(D)$ is a continuous operator such that, for every polynomial $p = p(s)$, the set $F^{-1}\{p\}$ is non-empty, and β_1, $0 < \beta_1 < 1$, and $\beta_2 > 0$ are fixed numbers. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N-1}\#\Big\{2 \le k \le N : \sup_{s\in K}\Big|F\Big(\zeta(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\Big) - f(s)\Big| < \varepsilon\Big\} > 0.$$

Proof. By the Proposition 2.1,

$$P_N(A) = \frac{1}{N-1}\#\Big\{2 \le k \le N : \zeta(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a}) \in A\Big\}, A \in \mathcal{B}\big(H(D)\big),$$

converges weakly to the distribution P_ζ of the $H(D)$-valued random element

$$\zeta(s, \alpha, \omega; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m\omega(m)}{(m+\alpha)^s}$$

as $N \to \infty$. Moreover, the support of P_ζ is the whole of $H(D)$. For $A \in \mathcal{B}\big(H(D)\big)$, define

$$P_{N,F}(A) = \frac{1}{N-1}\#\Big\{2 \le k \le N : F\Big(\zeta(s+ihk^{\beta_1}\log^{\beta_2} k, \alpha; \mathfrak{a})\Big) \in A\Big\}.$$

Then we have that $P_{N,F} = P_N F^{-1}$. This, the continuity of the operator F, the weak convergence of P_N to P_ζ as $N \to \infty$ and Lemma 1.5 show that $P_{N,F}$ converges weakly to $P_\zeta F^{-1}$.

First, we observe that, for every open set $G \subset H(D)$, the set $F^{-1}G$ is non-empty. Actually, let $G \neq \emptyset$ be an open set, and g be its arbitrary element. Then, in view of Lemma 1.16, for every $\delta > 0$, there exists a polynomial $p = p(s)$ such that

$$\sup_{s\in K} |g(s) - p(s)| < \delta.$$

for $K \in \mathcal{K}$. Hence, with sufficiently small δ, we obtain that the polynomial $p(s)$ also lies in G. Therefore, by the hypothesis of the theorem, the set $F^{-1}G$ is non-empty.

Since the operator F is continuous, the set $F^{-1}G$ is open. Thus, there exists an element $g_1 \in H(D)$ such that the set $F^{-1}G$ is its open neighbourhood. Hence, it follows that

$$P_\zeta(F^{-1}G) > 0,$$

and this shows that $P_\zeta F^{-1}(G) > 0$ for every open neighbourhood G of an arbitrary element $g \in H(D)$. This shows that the support of the measure $P_\zeta F^{-1}$ is the whole of $H(D)$.

Now, let

$$G = \Big\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \Big\},$$

where $p(s)$ is arbitrary polynomial. Then, by the above remark on the support of $P_\zeta F^{-1}$, we obtain the inequality

$$\liminf_{N \to \infty} \frac{1}{N-1} \# \Big\{ 2 \le k \le N : F\Big(\zeta(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})\Big) \in G \Big\} \ge P_\zeta F^{-1}(G) > 0. \tag{3.12}$$

Hence,

$$\liminf_{N \to \infty} \frac{1}{N-1} \# \Big\{ 2 \le k \le N : \sup_{s \in K} \Big| F\Big(\zeta(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})\Big) - p(s) \Big| < \frac{\varepsilon}{2} \Big\} > 0.$$

However, by Lemma 1.16, we may choose the polynomial $p(s)$ to satisfy the inequality

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}.$$

This and (3.12) give the assertion of the theorem. □

Chapter 4

Estimations for the number of zeros of the periodic Hurwitz zeta-functions

We recall that, for a certain function $f(s)$, the assertion $A(\sigma_1, \sigma_2; c, T)$ is valid if, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c > 0$ such that, for sufficiently large T, the function $f(s)$ has more than cT zeros lying in the rectangle

$$\{s = \sigma + it \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}.$$

In this chapter, we prove that, for periodic Hurwitz zeta-functions and some their generalizations, the assertion $A(\sigma_1, \sigma_2; c, T)$ and its discrete analogue are true.

4.1 Continuous case

We start with the classical Rouché theorem on the number of zeros of a certain pair of analytic functions.

Lemma 4.1. *Let G be a domain in the complex plane $\mathbb{C}$, K a compact subset of G, and $f(s)$ and $g(s)$ analytic functions in G such that*

$$|f(s) - g(s)| < |f(s)|$$

for every point s in the boundary of K. Then the functions $f(s)$ and $g(s)$ have the same number of zeros in the interior of K, taking into account multiplicities.

Proof of the lemma can be found, for example, in [49], Section X.12.

Theorem 4.1. *Suppose, that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then, for the function $\zeta(s, \alpha; \mathfrak{a})$, the assertion $A(\sigma_1, \sigma_2; c, T)$ is valid.*

Proof. We use the notation

$$\sigma_0 = \frac{\sigma_1 + \sigma_2}{2} \quad \text{and} \quad \rho_0 = \frac{\sigma_2 - \sigma_1}{2},$$

and apply Theorem 1.1 with $K = \{s \in \mathbb{C} : |s - \sigma_0| \le \rho_0\}$ and $f(s) = s - \sigma_0$. Then the inequality of Theorem 1.1

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} \left| \zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s) \right| < \varepsilon \right\} > 0$$

with $\varepsilon > 0$ means that the Lebegue measure of the set $\tau \in [0, T]$ such that

$$\sup_{s\in K} |\zeta(s, \alpha; \mathfrak{a}) - f(s)| < \varepsilon, \tag{4.1}$$

for sufficiently large T, is greater than cT, where $c = c(\sigma_1, \sigma_2, \alpha; \mathfrak{a})$ is a certain positive constant. We take ε to satisfy the inequalities

$$0 < \varepsilon < \frac{1}{2} \inf_{|s-\sigma_0|=\rho_0} |f(s)| = \frac{\rho_0}{2}.$$

Then the functions $f(s)$ and $\zeta(s+i\tau, \alpha; \mathfrak{a})$ on the disc K satisfy the hypotheses of Lemma 4.1. Actually, they are analytic in K, and, on the boundary of K,

$$\sup_{|s-\sigma_0|=\rho_0} |\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon < \sup_{|s-\sigma_0|=\rho_0} |f(s)|.$$

Therefore, since the function $f(s) = s - \sigma_0$ has precisely one zero in the interior of K, the function $\zeta(s+i\tau, \alpha; \mathfrak{a})$ also has one zero in the interior of that disc. However, the number of $\tau \in [0, T]$ satisfying (4.1) is greater than cT. Hence, for the function $\zeta(s, \alpha; \mathfrak{a})$ the assertion $A(\sigma_1, \sigma_2; c, T)$ is valid. □

We recall that q is the minimal period of the sequence $\mathfrak{a}$.

Theorem 4.2. *Suppose that* $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a, b) = 1, b \ne 2$ *and that* $\operatorname{rad}(q)$ *divides* b. *Then, for the function* $\zeta\left(s, \frac{a}{b}; \mathfrak{a}\right)$, *the assertion* $A(\sigma_1, \sigma_2; c, T)$ *is valid.*

Proof. We use the notation of the proof of Theorem 4.1. By Theorem 1.2, for every $\varepsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} \left| \zeta\left(s + i\tau, \frac{a}{b}; \mathfrak{a}\right) - f(s) \right| < \varepsilon \right\} > 0.$$

This means that the Lebesgue measure of the set $\tau \in [0, T]$ such that

$$\sup_{s\in K} \left| \zeta\left(s, \frac{a}{b}; \mathfrak{a}\right) - f(s) \right| < \varepsilon, \tag{4.2}$$

for sufficiently large T, is greater than cT, where $c = c(\sigma_1, \sigma_2; a, b; \mathfrak{a})$ is a certain positive constant. The further proof uses Lemma 4.1 and (4.2), and completely coincides with that of Theorem 4.1. □

4.2 Discrete case

We say that, for a certain function $f(s)$, the assertion $B(\sigma_1, \sigma_2; c; \varphi, k_0, N)$ is valid if, for every σ_1, σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, there exists a constant $c > 0$ such that, for sufficiently large N, the function $f\big(s + i\varphi(k)\big)$ has a zero in the disc

$$\Big|s - \frac{\sigma_1 + \sigma_2}{2}\Big| \le \frac{\sigma_2 - \sigma_1}{2}$$

for more than cN integers $k, k_0 \le k \le N$.

Theorem 4.3. *Suppose that the set $L(\alpha, h, \pi)$ is linearly independent over $\mathbb{Q}$. Then, for the function $\zeta(s, \alpha; \mathfrak{a})$, the assertion $B(\sigma_1, \sigma_2; c; kh, 0, N)$ is valid.*

Proof. Let σ_0, ρ_0, K and $f(s)$ be the same as in the proof of Theorem 4.1. Then, by Theorem 1.3, for every $\varepsilon > 0$, the set of integers $k \ge 0$ satisfying the inequality

$$\sup_{s \in K} \big|\zeta(s + ikh, \alpha; \mathfrak{a}) - f(s)\big| < \varepsilon \tag{4.3}$$

has a positive lower density. We take

$$0 < \varepsilon < \frac{1}{2} \inf_{|s-\sigma_0|=\rho_0} |s - \sigma_0| = \frac{\rho_0}{2}. \tag{4.4}$$

Then

$$\sup_{|s-\sigma_0|=\rho_0} \big|\zeta(s + ikh, \alpha; \mathfrak{a}) - f(s)\big| < \inf_{|s-\sigma_0|=\rho_0} |f(s)|.$$

Thus, the functions $\zeta(s + ikh, \alpha; \mathfrak{a})$ and $f(s)$ on the disc K satisfy the hypothesis of Lemma 4.1. Since the function $s - \sigma_0$ has one zero in that disc, the function $\zeta(s + ikh, \alpha; \mathfrak{a})$ also has precisely one zero in that disc. However, there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, \mathfrak{a}, h) > 0$ such that the number of $k, 0 \le k \le N$, for which inequality (4.3) holds, for sufficiently large N, is greater than cN. ☐

Theorem 4.4. *Suppose that $\alpha = \frac{a}{b}, a, b \in \mathbb{N}, a < b, (a, b) = 1, b \ne 2$ and that $\mathrm{rad}(q)$ divides b. Then, for the function $\zeta\big(s, \frac{a}{b}; \mathfrak{a}\big)$, the assertion $B(\sigma_1, \sigma_2; c, kh, 0, N)$ is valid.*

Proof. By Theorem 1.4, we have that, for every $\varepsilon > 0$, the set of integers $k \ge 0$ satisfying the inequality

$$\sup_{s \in K} \Big|\zeta\Big(s + ikh, \frac{a}{b}; \mathfrak{a}\Big) - f(s)\Big| < \varepsilon \tag{4.5}$$

has a positive lower density. Here we use the same notation as above. Suppose that the number ε satisfies inequalities (4.4). Then

$$\sup_{|s-\sigma_0|=\rho_0} \Big|\zeta\Big(s + ikh, \frac{a}{b}; \mathfrak{a}\Big) - f(s)\Big| < \inf_{|s-\sigma_0|=\rho_0} |f(s)|.$$

This shows that the functions $\zeta\big(s + ikh, \frac{a}{b}; \mathfrak{a}\big)$ and $f(s) = s - \sigma_0$ on the disc K satisfies the hypotheses of Lemma 4.1. From this and (4.5), the theorem follows with certain $c = c(\sigma_1, \sigma_2, a, b, \mathfrak{a}, h) > 0$. ☐

Theorem 4.5. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, and that $\beta_1, 0 < \beta_1 < 1$, and $\beta_2 > 0$ are fixed numbers. Then, for the function $\zeta(s, \alpha; \mathfrak{a})$, the assertion $B(\sigma_1, \sigma_2; c, hk^{\beta_1} \log^{\beta_2} k, 2, N)$ is valid.*

Proof. We recall that the same notation as above for K and $f(s)$ is used. By Theorem 2.1, we see that, for every $\varepsilon > 0$, the set of integers $k \geq 2$ satisfying the inequality

$$\sup_{s \in K} \left| \zeta(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a}) - f(s) \right| < \varepsilon \tag{4.5}$$

has a positive lower density. Therefore, choosing the number ε satisfying (4.4), and applying the properties of inequality (4.6), we obtain analogically as above that there exists a constant $c = c(\sigma_1, \sigma_2, \alpha, \mathfrak{a}, h, \beta_1, \beta_2) > 0$ such that the assertion $B(\sigma_1, \sigma_2, c, hk^{\beta_1} \log^{\beta_2} k, 2, N)$ is valid. □

Similarly, it is possible to obtain lower estimates for the number of zeros of composite functions of $\zeta(s, \alpha; \mathfrak{a})$.

Conclusions

1. The periodic Hurwitz zeta-function with parameter α such that the set $\{\log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$ has a continuous universality property.
2. The periodic Hurwitz zeta-function with parameter α such that the set $\{(\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h}\}$, $h > 0$, is linearly independent over $\mathbb{Q}$ has a discrete universality property.
3. The periodic Hurwitz zeta-function has a discrete universality property on the approximation of analytic functions by shifts $\zeta(s + ihk^{\beta_1} \log^{\beta_2} k, \alpha; \mathfrak{a})$, $0 < \beta_1 < 1$, $\beta_2 > 0$.
4. Composite functions $F(\zeta(s, \alpha; \mathfrak{a}))$ for some classes of the operator F in the space of analytic functions have continuous and discrete universality properties.
5. Universality theorems for the periodic Hurwitz zeta-function imply lower estimates for their number of zeros.

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Notation

p	prime number
j, k, l, m, n, r, q	non-negative integers
h	fixed positive number
$i = \sqrt{-1}$	imaginary unity
$\mathbb{P}$	set of all prime numbers
$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	set of all non-negative integers
$\mathbb{R}$	set of all real numbers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{C}$	set of all complex numbers
$s = \sigma + it$, $\sigma, t \in \mathbb{R}$	complex variable
$H(G)$	space of analytic functions on G
$\mathbb{B}(X)$	Borel σ-field of the space X
χ	Dirichlet character
$L(s, \chi)$	Dirichlet L-function defined, for $\sigma > 1$, by $L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}$, and by analytic continuation elsewhere
$\zeta(s, \alpha)$	Hurwitz zeta-function defined, for $\sigma > 1$, by $\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}$, and by analytic continuation elsewhere
$\zeta(s, \alpha; \mathfrak{a})$	periodic Hurwitz zeta-function defined, for $\sigma > 1$, by $\zeta(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}$, $\mathfrak{a} = \{a_m\}$, and by analytic continuation elsewhere
$\operatorname{meas} A$	Lebesgue measure of $A \subset \mathbb{R}$
$\#A$	cardinality of A
$F^{-1}G$	preimage of a set G
$F^{-1}\{p\}$	preimage of a polynomial p