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Some decidable classes of modal logic S5

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1. Introduction

It is know (see, for example [1]) that monadic quantifier modal logic S5 is undecidable. We describe some decidable by deducibility classes of quantifier modal logic S5 containing only monadic predicate variables. In this paper we consider only closed formulas, i.e., the formulas without free variables. Also every atomic formula occurs in the scope of modal connective $\Box$ (or $\Diamond$). We assume that there are no two bound variables in a formula with the same name. The formulas F contain only logical connectives $\vee, \wedge, \neg$ and no logical or modal symbol in F lies in the scope of a negation.

Definition 1. *We say the occurence of modal connective $\Box$ (or $\Diamond$) is the nearest modal connective for the occurence of atomic formula P if the following conditions holds:*

- *the occurence of P belongs to the scope of considerable $\Box$ ($\Diamond$),*
- *if in the scope of any other occurence of $\Box$ ($\Diamond$)is a considerable occurence of P, then the nearest modal connective belong also to the same scope of $\Box$ ($\Diamond$)*

Definition 2. *A sequence Q_i, Q_j ($Q_i, Q_j \in \forall, \exists, \Box, \Diamond$) is called F-prefix of the occurence of a atomic formula $G(x)$ in F if $Q_j x$ ($Q_j \in \forall, \exists$) belongs to the scope of the nearest modal connective Q_i ($Q_i \in \Box, \Diamond$) of considerable G(x). Contrary F-prefix is Q_j, Q_i.*

Definition 3. *A sequence Q_i, Q_j ($Q_i, Q_j \in \forall, \exists, \Box, \Diamond$), which is the F-prefix of same atomic formula in F, is called F-prefix of the formula F*

We will consider three decidable classes.

The formulas of the first class the following conditions hold:

1. all occurences of literal containing x (for all variables x) belong to the scope of the same nearest modal connective; the occurences of different variables belong to different scopes of nearest modal connectives as well,
2. every F-prefix of the formula F ends either with $\forall, \Box$ or F-prefix does not contain $\forall, \Box$.

For example, the following formula belongs to a considerable class:

$$\forall y \Diamond \forall x(P(x) \vee \Box((P(y) \wedge Q(y)) \vee \exists z \Diamond Q(z))$$

The next formula does not belong to the first class:

$$\forall x \exists y \Diamond (P(x) \wedge Q(y))$$

We will describe two more classes. The second class consists of all formulas which every F-prefix ends either with $\forall$ or $\Box$.

For example, the formulas of following forms belong to the second class:

$$Q_1 x_1 \ldots Q_m x_m \forall y_1 \ldots \forall y_s G \qquad Q \in \{\Box, \Diamond\}, i = 1, ..., m$$

$$Q_1 x_1 \ldots Q_m x_m \Box y_1 \ldots \Box y_s G \qquad Q \in \{\forall, \exists\}, i = 1, ..., m$$

where G is a formula which does not contain $\Box, \Diamond, \forall, \exists$.

Third class. Every formula contains only two occurences of $\Diamond, \exists$, or more precisely, any formula holds one of following conditions:

1. there are only two occurences of $\Diamond$,
2. there are only two occurences of $\exists$,
3. there is only one occurence of $\Diamond$ and one occurence of $\exists$.

Besides the considerable occurences do not occur in a scope of $\forall$ or $\Box$.

2. Indexing modal logic S5

In this section, we will consider modal logic formulas containing predicate variables with arity n ($n \geqslant 1$). In next section we will consider formulas containing only monadic predicate variables which become two-place predicate variables after the transformation into a formula of S5i.

We define the formulas of modal logic $S5i$ in the following way:

1. If $P(x_1, ..., x_n)$ is n-place predicate variable, then $P(0, x_1, ..., x_n)$ is a formula.
2. If F is a formula, then $\neg F$ is also a formula.
3. If F, G are formulas, then $(F \wedge G)$, $(F \vee G)$, $(F \rightarrow G)$ are also formulas.
4. If F is a formula, x is an individual variable, then $\forall x F$ and $\exists x F$ are also formulas.
5. If $P_1(0, x_1^1, ..., x_{n_1}^1), P_2(0, x_1^2, ..., x_{n_2}^2), \ldots, P_m(0, x_1^m, ..., x_{n_m}^m)$ are any occurences of atomic formulas (we do not require all of the possible occurences of such formulas) in F, then $\Box z F'$ and $\Diamond z F'$ are also formulas. F' is obtained from F by replacing $P_k(0, x_1^k, ..., x_{n_k}^k)$ by $P_k(z, x_1^k, ..., x_{n_k}^k)$ ($k = 1, ..., m$), and z is a new individual variable not occuring in F.

We present modal sequent system $S5i$. We will describe a part of inference rules. The following rules $(\vdash \wedge)$, $(\wedge \vdash)$, $(\vdash\rightarrow)$, $(\rightarrow\vdash)$, $(\vdash \exists)$, $(\exists \vdash)$, $(\vdash \Diamond)$, $(\Diamond \vdash)$ are described similarly.

Axiom scheme: $\Gamma_1, F, \Gamma_2 \vdash \Delta_1, F, \Delta_2$

Inference rules:

$$(\vdash \neg) \quad \frac{\Gamma_1, F, \Gamma_2 \vdash \Delta_1, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \neg F, \Delta_2} \qquad (\neg \vdash) \quad \frac{\Gamma_1, \Gamma_2 \vdash \Delta_1, F, \Delta_2}{\Gamma_1, \neg F, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

$$(\vdash \vee) \quad \frac{\Gamma, \vdash \Delta_1, F, G, \Delta_2}{\Gamma \vdash \Delta_1, F \vee G, \Delta_2} \qquad (\vee \vdash) \quad \frac{\Gamma_1, F, \Gamma_2 \vdash \Delta \quad \Gamma_1, G, \Gamma_2 \vdash \Delta}{\Gamma_1, F \vee G, \Gamma_2 \vdash \Delta}$$

$$(\vdash \forall) \quad \frac{\Gamma \vdash \Delta_1, F(y), \Delta_2}{\Gamma \vdash \Delta_1, \forall x F(x), \Delta_2} \qquad (\forall \vdash) \quad \frac{\Gamma_1, F(t), \forall x F(x), \Gamma_2 \vdash \Delta}{\Gamma_1, \forall x F(x), \Gamma_2 \vdash \Delta}$$

$$(\vdash \Box) \quad \frac{\Gamma \vdash \Delta_1, F(i), \Delta_2}{\Gamma \vdash \Delta_1, \Box x F(x), \Delta_2} \qquad (\Box \vdash) \quad \frac{\Gamma_1, F(k), \Box x F(x), \Gamma_2 \vdash \Delta}{\Gamma_1, \Box x F(x), \Gamma_2 \vdash \Delta}$$

$\Gamma, \Gamma_1, \Gamma_2, \Delta, \Delta_1, \Delta_2$ are finite sets of formulas. F, G are formulas. In the rule $(\vdash \forall)$ the variable y has no free occurences in conclusion of the rule. Term t is free for x in $F(x)$. i is a natural number not occuring in the conclusion of $(\vdash \Box)$. k is any natural number (including zero).

Theorem 1. *For any formula F of modal logic one can construct F' of S5i such that $\vdash F$ is derivable in indexing system S5 iff $\vdash F'$ is derivable in S5i.*

For any formula F, we construct F' in following way. Any occurence of a atomic formula $P(x_1, ..., x_n)$ not occuring in a scope of modal connective $\Box(\Diamond)$ is replaced by $P(0, x_1, ..., x_n)$. Suppose that $P_1(x_1^1, ..., x_{n_1}^1), ..., P_m(x_1^m, ..., x_{n_m}^m)$ is a complete sequence of atomic formulas which nearest modal connective is occurence of $\Box(\Diamond)$. We replace $P_i(x_1^i, ..., x_{n_i}^i)$ $(i = 1, ..., m)$ by $P_i(0, x_1^i, ..., x_n^i)$ and considerable occurence of $\Box(\Diamond)$ is replaced by $\Box z(\Diamond z)$ (where z is a new variable). If all such sequences are replaced, then we delete all occurences of $\Box(\Diamond)$ without variables.

For example,

$$F: \quad \forall x \Box\Box(P(x) \wedge \Diamond \exists y Q(y))$$

$$F': \quad \forall x \Box z(P(z, x) \wedge \Diamond u \exists y Q(u, y))$$

One shows that for any formula F of modal logic one can construct the deductive equivalent sequent which is derivable in K_1S5 ([2]) iff $\vdash F'$ is derivable in $S5i$.

Definition 4. *A equivalent formula for F of the form $Q_1x_1Q_2x_2...Q_nx_nG$ (where $Q_j \in \{\exists, \forall, \Box, \Diamond\}$ and G is quantifier-free formula not occuring $\Box, \Diamond$) is called generalized prenex normal form of F.*

Theorem 2. *For any formula of S5i there exists deductive equivalent generalized prenex normal form.*

3. Decidable classes

Theorem 3. *For any formula F of the first class there exists deductive equivalent formula F' of the generalized prenex normal form $F' = Q_1x_1...Q_mx_mG$ which belongs to first class as well and the following condition holds: for any literal in G there exists such j $(j = 1, 2, ..., m-1)$ that $x_j = y$ and $x_{j+1} = z$ (or $x_j = z, x_{j+1} = y$).*

Theorem 4. *Assume we have a derivation of sequent $\vdash F$ (F belongs to the first class) and in deduction tree the literal $P([i/x], [a/y])$ is meeted. Then we can construct deduction tree in* *which does be not the literals of the form $Q(i,b)$ (i.e., the literals with $b \neq a$) or $R(j,a)$ (i.e., the literals with $j \neq i$).*

Theorem 5. *The first class is decidable by deducibility.*

The second class. Class is decidable because the class of the formulas of the classical predicate logic without function symbols and equality containing only the formulas which all F-prefixes end with $\forall$ belongs to class **K** [3]. **K** is decidable [3].

The third class. For any formula of the third class one can construct a equivalent formula F' occuring in $S5i$ of the following generalized prenex normal form

$$Q_1xQ_2yQ_3z_1Q_4z_2...Q_{n+2}z_nGQ_1,Q_2 \in \{\exists,\Diamond\}, \quad Q_i \in \{\forall,\Box\} \ (i=3,4,...,n+2)$$

where G is a formula does not contain occurences of $\forall, \exists, \Box, \Diamond$.

Class is decidable because any F-prefix the following conditions holds:

- it ends with $\Box$ or $\forall$.
- it is one of the following forms $\exists\exists, \exists\Diamond, \Diamond\exists, \Diamond\Diamond$; also the considerable occurences of $\exists, \Diamond$ not occur in the scope of a occurence of $\forall$ or $\Box$.

Such formulas (after the replacing $\Box$ by $\forall$ and $\Diamond$ by $\exists$) occur in decidable class **K** as well.

References

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Kai kurios išsprendžiamos modalinės logikos S5 klasės

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Naudojantis Maslovo atvirkštiniu metodu įrodomas kai kurių modalinės logikos S5 klasių su vienviečiais predikatiniais kintamaisais išsprendžiamumas pagal išvedimą.