Liet. matem. rink., T. 40, spec. nr., 270–274

A resolution calculus for modal logic S4

Stasys NORGĖLA (VU)

e-mail: stasys.norgela@maf.vu.lt

1. Introduction

In this paper, we present a resolution calculus for the first-order modal logic S4. The formulas are given not necessary in a clausal form. This method can be used for automatizable proof procedure of a quantified modal logic. We will consider formulas for which the following conditions hold:

1. the formulas F contain only logical connectives $\neg$, &, $\vee$, and no logical or modal symbol in F lies in the scope of a negation,
2. the formulas are closed, i.e., we consider the formulas without free variables,
3. the formulas are transformed into Skolem normal form (see [1],[2]),
4. the formulas are of the form $G_1 \vee G_2 \vee \ldots \vee G_s$, where G_i is a literal or a formula beginning with $\Box$, $\Diamond$.

The order of formulas is not fixed in a disjunction or in a conjunction. In what follows, P, P_1, P_2 denote the atomic formulas. Formulas are denoted by F, G, K, H and M. Moreover, H and M can be the empty formulas as well. The symbol $\bot$ denotes an empty formula.

2. The resolution rules

2.1. *Classical rules*

$$(c1) \quad \frac{[P_1 \vee H, \neg P_2 \vee M]\theta}{[H \vee M]\theta}$$

θ is a most general unifier of $\{P_1, P_2\}$. We assume that the formulas written over the line have no common individual variables (this if necessary can be obtained by renaming variables). Substitution θ is a finite set of the form $t_1/x_1, \ldots, t_n/x_n$, where every x_i is a variable, every t_i is a term, different from x_i, and for all i, j such that $i \neq j$, x_i differs from x_j. Moreover, if the level (see [1]) of x is n and if the term t contains some symbol whose level is greater than n, then the substitution of t for x is forbidden.

$$(c2)\ \frac{(F\&G)\vee H}{F\vee H} \qquad (c3)\ \frac{res(P,\neg P)}{\bot}$$

$$(c4)\ \frac{res(F\vee K,G)}{res(F,G)\vee K} \qquad (c5)\ \frac{res(F\&K,G)}{K\&res(F,G)}$$

$$(c6)\ \frac{res(F\vee G)}{G\vee resF} \qquad (c7)\ \frac{res(F\&G)}{res(F,G)}$$

$$(c8)\ \frac{res(F\&G)}{G\&resF}$$

2.2. *Modal rules*

$$(m1)\ \frac{[H\vee\Box F, M\vee\Box G]\theta}{[H\vee M\vee\Box res(F,G)]\theta} \qquad (m2)\ \frac{[H\vee\Box F, M\vee\Diamond G]\theta}{[H\vee M\vee\Diamond res(F,G)]\theta}$$

$$(m3)\ \frac{[H\vee\Box F]\theta}{[H\vee\Box resF]\theta} \qquad (m4)\ \frac{[H\vee\Diamond F]\theta}{[H\vee\Diamond resF]\theta}$$

$$(m5)\ \frac{res(\Box F,\Box H)}{\Box res(F,H)} \qquad (m6)\ \frac{res(\Box H,\Diamond F)}{\Diamond res(H,F)}$$

$$(m7)\ \frac{res(\Box F,H)}{res(F^-,H)} \qquad (m8)\ \frac{res(\Box F,H)}{res(\Box\Box F^+,H)}$$

$$(m9)\ \frac{[H\vee\Box F,K]\theta}{[H\vee res(F^-,K)]\theta} \qquad (m10)\ \frac{[H\vee\Box F,K]\theta}{[H\vee res(\Box\Box F^+,K)]\theta}$$

F^- is obtained from F (see [1]) by subtracting one from the level of those symbols that have a level greater than the modal degree of $\Box F$.

F^+ is obtained from F by adding one to the level of those symbols whose level is greater than the modal degree of $\Box F$.

2.3. *Simplification rules*

$$(s1)\ \frac{F\vee\bot}{F} \qquad (s2)\ \frac{F\&\bot}{\bot} \qquad (s3)\ \frac{\Box\bot}{\bot}$$

$$(s4)\ \frac{\Diamond\bot}{\bot} \qquad (s5)\ \frac{res(\bot,H)}{\bot} \qquad (s6)\ \frac{res(\bot\vee F,H)}{res(F,H)}$$

$$(s7)\ \frac{res(\bot\&F,H)}{\bot} \qquad (s8)\ \frac{res(\Box\bot,H)}{\bot} \qquad (s9)\ \frac{res(\Diamond\bot,H)}{\bot}$$

2.4. *Duplication rule*

$$(d1)\quad \frac{F(x^n)}{F(x^n)\&F(y^n)}.$$

Here y is a new variable, x^n occurs only in $F(x^n)$, $F(x^n)$ is not in the scope of more than n modal, and $F(x^n)$ is not in the scope of a negation.

2.5. *Factorization rule*

$$(f1)\quad \frac{F \vee F \vee H}{F \vee H}.$$

The main results

We define the *generalized formulas* as follows:

1. If F is a formula, then $resF$ is a generalized formula.
2. If F and G are formulas, then $res(F, G)$ is a generalized formula.
3. If F is a generalized formula, then $\neg F$ is also a generalized formula.
4. If F is a formula and G is a generalized formula, then $(F \vee G), (F\&G), (F \to G), (G \to F), \Box G, \Diamond G$ are generalized formulas.

Note that we consider only Skolemized formulas.The formulas F, G, K, H and M met in the resolution rules do not contain res.

A derivation of the formula (generalized formula) F from a set of formulas S is a finite sequence $G_1, G_2, \ldots, G_s$ such that

1. $G_s = F$.
2. G_i is a formula or a generalized formula.
3. For every $i \leqslant s$ at least one of the following conditions holds:
   (a) $G_i \in S$.
   (b) For some $j, k < i$ F_i follows from G_j, G_k by one of the rules $(c1)$, $(c2)$, $(m1)$–$(m4)$, $(m9)$, $(m10)$ or $(s1)$– $(s4)$.
   (c) For some $j (j < i)$ $G_j = G(resK)$, i.e., $resK$ is a generalized subformula of G, $G_i = G(resH)$ (or $G_i = G(H)$) and $resH$ (or H) follows from $resK$ by one of the rules $(c3)$–$(c8)$, $(m5)$–$(m8)$ or $(s5)$–$(s9)$.
   (d) For some j $G_j = G(F(x^n))$ and $G_i = G(F(x^n)\&F(y^n))$. Here y is a new variable satisfying the conditions of the rule $(d1)$.
   (e) For some $j < i$ $G_j = G(K)$ is a formula, $G_j = G(M)$ and M follows from K by one of the rules $(s1)$–$(s4)$ or $(f1)$.

Theorem 1. *$S \vdash \perp$ if and only if S is refutable.*

Proof. Soundness and completness of a resolution modal system $S4$ is proved in [1]. We will show that every application of a rule of resolution modal system in [1] is simulated by a finite sequence of applications of . onsidered calculus.

Assume that a formula which does not satisfy the Condition 4 described in the introduction is obtained. In this case, we can obtain the required form by applying a finite number of rule $(c2)$.

Each application of rules $(m1)$–$(m4)$, $(m9)$ and $(m10)$ introduces generalized formulas containing res. The rules $(c3)$–$(c8)$, $(m5)$–$(m8)$, $(s5)$–$(s9)$ and $(d1)$ present recursive transformation of generalized formulas,i.e., of the formulas containing res. We simulate the applications of the rules $(c1)$, $(c2)$, $(m1)$–$(m4)$, $(m9)$ and $(m10)$ for the subformulas which are in the scope of res using the above-introduced resolution rules. As a result a simplified formula not containing res can be obtained by applying the rules $(s5)$–$(s9)$.

The rule $(c2)$ from [1] of the form *if C is a θ-resolvent of $S' \cup \{A\}$, then $C \vee B\theta$ is a θ-resolvent of $S' \cup \{A \cup B\}$* is simulated by rules $(c1)$, $(c4)$, $(c6)$ of the calculus in question.

Rule $(c3)$ from [1] of the form *if C is a θ-resolvent of $S' \cup \{A\}$, then $C\&B\theta$ is a θ-resolvent of $S' \cup \{A\&B\}$* is simulated by rules $(c5)$ and $(c8)$ of a considered calculus.

Rule $(c4)$ from [1] of the form *if C is a θ-resolvent of $\{A, B\}$, then C is a θ-resolvent of $\{A\&B\}$* is simulated by rule $(c7)$ of a considered calculus.

Rules $(m1)$–$(m4)$ from [1] are simulated by the corresponding rules $(m2)$, $(m3)$, $(m1)$ and $(m4)$ of a considered calculus.

The simplifications rules from [1] are simulated by rules $(s1)$–$(s9)$ of a respective calculus. Moreover, each formula of a considered calculus is a particular case of some rule from [1]. The theorem is proved.

Consider now the formulas of propositional modal logic for which the following conditions hold:

- the formulas F contain only logical connectives $\neg$ and $\vee$,
- no logical or modal symbol lies in the scope of a negation.

Now, we shall present our calculus in this particular case (p denotes a propositional variable).

Calculus MS4

$$(c1)\ \frac{p \vee H, \neg p \vee M}{H \vee M} \qquad (c2)\ \frac{res(p \vee H, \neg p \vee M)}{H \vee M}$$

$$(m1)\ \frac{H \vee \Box p, \neg p \vee M}{H \vee M} \qquad (m2)\ \frac{H \vee \Box p, \Diamond\neg p \vee M}{H \vee M}$$

$$(m3)\ \frac{H \vee \Box F, M \vee \Box G}{H \vee M \vee \Box res(F, G)} \qquad (m4)\ \frac{H \vee \Box F, M \vee \Diamond G}{H \vee M \vee \Diamond res(F, G)}$$

$$(m5)\ \frac{res(H \vee \Box p, \neg p \vee M)}{H \vee M} \qquad (m6)\ \frac{res(H \vee \Box p, \Diamond\neg p \vee M)}{H \vee M}$$

$$(m7)\ \frac{res(H \vee \Box F, M \vee \Box G)}{H \vee M \vee \Box res(F, G)} \qquad (m8)\ \frac{res(H \vee \Box F, M \vee \Diamond G)}{H \vee M \vee \Diamond res(F, G)}$$

$$(s1)\ \frac{\Box F}{F} \qquad (s2)\ \frac{\Box F}{\Box\Box F} \qquad (s3)\ \frac{\Box \bot}{\bot}$$

$$(s4)\ \frac{\Diamond F}{\bot} \qquad (s5)\ \frac{F \lor \bot}{F} \qquad (f1)\ \frac{F \lor F \lor H}{F \lor H}$$

DEFINITION 1. A derivation of a formula F from the set of formulas S is a finite sequence $G_1, G_2, \ldots, G_s$ such that

1. $G_i (i = 1, 2, \ldots, s)$ is a formula or a generalized formula.
2. $G_s = F$.
3. For every $i \leqslant s$ at least one of the following conditions holds:
   (a) $G_i \in S$,
   (b) For some j and $k < i$ F follows from G_j and G_k by one of the rules $(c1)$, $(m1)$–$(m4)$.
   (c) For some $j < iG_j = G(resK)$, i.e., $resK$ is a generalized subformula of G, $G_i = G(H)$ and H follows from $resK$ by one of rules $(c2)$, $(m5)$–$(m8)$.
   (d) For some $j < i$ $G_j = G(K)$ (K does not contain res), $G_i = G(H)$ and H follows from K by one of the rules $(s1)$–$(s5)$, $(f1)$.

Disjunctions of modal literals are called *modal clauses*. *Modal literals* are expressions of the form $q, \Box q$ or $\Diamond q$, where q is a propositional variable or its negation. *Initial modal clauses* are expressions of the form $\Box C$, where C is a modal clause. The following proposition is improved in [3]: *for any formula F one can construct (by introduction of new variables) the list X_p of initial clauses and a propositional variable g such that* $\vdash_{S4} F$ if and only if $\vdash_{S4} \& X_F \to g$.

It means that, in the general case, we can consider the set S of input formulas containing only modal and initial clauses. Note that the rules of MS4 allow us to derive from S formulas which are not initial (or modal) clauses.

For example, $\Box\neg p \lor \Box q, \Box(r \lor \neg q \lor \neg s) \vdash_{MS4} \Box\neg p \lor \Box(r \lor \neg s)$.

References

[1] M. Cialdea, Resolution for some first-order modal systems, *Theoretical Computer Science*, **85**, 213–229 (1991).

[2] M. Cialdea, Herbrand style proof procedures for modal logic, *Journal of Applied Non-Classical Logics*, 3(2), 205–223 (1993).

[3] G. Mints, Gentzen-type systems and resolution rule. Part I. Propositional logic, In: P. Martin-Löf, G. Mints (Eds.), COLOG-88, *Lecture Notes in Computer Sci.*, **417**, Springer-Verlag, pp. 198–231 (1990).

Rezoliucijų skaičiavimas modalumų logikai S4

S. Norgėla

Darbe nagrinėjamos bendro pavidalo modalumų logikos formulės. Aprašomas rezoliucijų skaičiavimas modalumų logikai S4 bei įrodomas jo pilnumas ir korektiškumas.