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A limit theorem for the Lerch zeta-function

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Let $s = \sigma + it$ be a complex variable, and let $L(\lambda, \alpha, s)$ denote the Lerch zeta-function, defined for $\sigma > 1$ by the following Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}.$$

Here $\lambda \in \mathbb{R}$, $0 < \alpha \leqslant 1$ are fixed parameters. The function $L(\lambda, \alpha, s)$ is analytically continuable over the complex plane.

Let $D = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$, and let $H(D)$ stand for the space of analytic functions on D with the topology of uniform convergence on compacta. Let $N \in \mathbb{N} \cup \{0\}$, and let $h > 0$ be a fixed number such that $\exp\{2\pi k/h\}$ is rational for all $k \in \mathbb{Z}$. We suppose that $\lambda \notin \mathbb{Z}$, and α is a transcendental number. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and let

$$\mu_N(\ldots) = \frac{1}{N+1} \#\{0 \leqslant k \leqslant N, \ldots\},$$

where in the place of dots we write a condition satisfied by k.

In [1] a limit theorem for the probability measure

$$P_N(A) = \mu_N(L(\lambda, \alpha, s + ikh) \in A), \quad A \in \mathcal{B}(H(D))$$

was proved.

The purpose of this article is to obtain an explicit form of the limit measure P.

Define the probability measure

$$P_{N,p_n}(A) = \mu_N(p_n(s + ikh, \alpha) \in A), \quad A \in \mathcal{B}(H(D)),$$

where

$$p_n(s, \alpha) = \sum_{m=0}^{n} \frac{a(m)}{(m+\alpha)^s}, \quad a(m) \in \mathbb{C},$$

is an arbitrary Dirichlet polynomial.

Lemma 1. *There is a probability measure P_{p_n} on $(H(D), \mathcal{B}(H(D)))$ such that the measure P_{N,p_n} converges weakly to P_{p_n} as $N \to \infty$.*

Poof of the lemma is given in [1].

Let $g(m)$, $m = 0, 1, \ldots$, be an unimodular function, and define an arbitrary Dirichlet polynomial by

$$p_n(s, g, \alpha) = \sum_{m=0}^{\infty} \frac{a(m)g(m)}{(m+\alpha)^s}, \quad a(m) \in \mathbb{C}.$$

Define the probability measure

$$\widetilde{P}_{N,p_n}(A) = \mu_N(p_n(s+ikh, g, \alpha) \in A), \quad A \in \mathcal{B}(H(D)).$$

Lemma 2. *The probability measures P_{N,p_n} and $\widetilde{P}_{N,p_n}$ both converge weakly to the same probability measure P_{p_n} on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$.*

Proof. Let

$$\Omega_n = \prod_{m=0}^{n} \gamma_m,$$

where $\gamma_m = \gamma = \{s \in \mathbb{C} : |s| = 1\}$ for all $m = 0, 1, \ldots, n$, and let m_n stand for the Haar measure on $(\Omega_n, \mathcal{B}(\Omega_n))$. Define the function $h : \Omega_n \longrightarrow H(D)$ by the formula

$$h(x_0, \ldots, x_n) = \sum_{m=0}^{n} \frac{a(m)}{(m+\alpha)^s x_m}, \quad (x_0, \ldots, x_n) \in \Omega_n.$$

In the proof of Lemma 1 it was shown that $P_{p_n} = m_n h^{-1}$. Define the function $\widetilde{h}$ in a similar manner as h. By Lemma 1 the probability measure $\widetilde{P}_{N,p_n}$ converges weakly to $m_n h^{-1}$ as $N \to \infty$. Define the function $h_1 : \Omega_n \longrightarrow \Omega_n$ by the formula

$$h_1(x_0, \ldots, x_n) = \left(x_0 e^{-i\Theta_0}, \ldots, e^{-i\Theta_n}\right),$$

where Θ_m =arg$g(m)$, $m = 0, 1, \ldots, n$. Obviously,

$$\begin{aligned}\widetilde{h}(x_0, \ldots, x_n) &= \sum_{m=0}^{n} \frac{a(m)g(m)}{(m+\alpha)^s x_m} \\ &= \sum_{m=0}^{n} \frac{a(m)}{(m+\alpha)^s x_m} \exp\{i\Theta_m\} = h(h_1(x_0, \ldots, x_n)).\end{aligned}$$

Consequently,

$$m_n \widetilde{h}^{-1} = m_n(h(h_1))^{-1} = (m_n h_1^{-1})h^{-1}.$$

Since the Haar measure m_n is invariant with respect to the translation by points in Ω_n, it follows that $m_n\widetilde{h}^{-1} = m_n h^{-1}$. This proves the lemma.

Let

$$a_h = \{(m+\alpha)^{-ih}, m = 0, 1, \dots\},$$

and

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

$\gamma_m = \gamma = \{s \in \mathbb{C} : |s| = 1\}$. Define a transformation φ_h by

$$\varphi_h(\omega) = a_h\omega, \quad \omega \in \Omega.$$

A set $A \in \mathcal{B}(\Omega)$ is called an invariant set with respect to the transformation φ_h if the sets A and $A_h = \varphi_h(A)$ differ one from another by the set of zero m_H-measure. The transformation φ_h is called ergodic if its σ-field of invariant sets consists only of sets having m_H-measure equal to 0 or 1 (see [5]).

Lemma 3. *The transformation φ_h is ergodic.*

Proof. Let $A \in \mathcal{B}(\Omega)$ be such that $m_H(A \triangle A_h) = 0$. Let χ be a nontrivial character of Ω. It is known (see [2]) that there exists a positive rational number $r = \frac{l}{m}$ such that

$$\chi(\omega) = \omega(r) = \frac{\omega(l)}{\omega(k)}.$$

Consequently,

$$\chi(a_h) = (r+\alpha)^{-ih}.$$

If $\chi(a_h) = 1$ then

$$r + \alpha = \exp\left\{\frac{2\pi k_0}{h}\right\}$$

with some k_0. Since $\exp\{2\pi k/h\}$ is rational for all integers, hence we deduce that $k_0 = 0$ and therefore $r + \alpha = 1$. However then χ is the trivial character of Ω. Consequently, $\chi(a_h) \neq 1$.

Since the set A is invariant, the Fourier transform of its indicator is

$$\widehat{I}_A(\chi) = \int_\Omega \chi(\omega) I_A(\omega) m_H(d\omega) = \int_\Omega \chi(a_h\omega) I_A(a_h\omega) m_H(d\omega) = \chi(a_h)\widehat{I}_A(\chi).$$

Hence $\widehat{I}_A(\chi) = 0$ for all nontrivial characters of Ω. Let χ_0 be the trivial character of Ω. Suppose that $\widehat{I}_A(\chi_0) = u$. Therefore

$$\widehat{I}_A(\chi) = u \int\limits_{\Omega} \chi(\omega) m_H(d\omega) = \widehat{u}(\chi)$$

for each character χ of Ω. Since $I_A(\omega)$ is uniquely determined by its Fourier transform, we obtain that $m_H(A) = 0$ or $m_H(A) = 1$. The lemma is proved.

Let $\omega(m)$, $m = 0, 1, \ldots$, stand for the projection of $\omega \in \Omega$ to the coordinate space γ_m.
Set

$$L(\lambda, \alpha, s, \omega) = \sum_{m=1}^{\infty} L_m(\lambda, \alpha, s, \omega),$$

where

$$L_m(\lambda, \alpha, s, \omega) = \frac{\omega(m) e^{2\pi i \lambda m}}{(m+\alpha)^s}, \quad m = 1, 2, \ldots,$$

and

$$L_0(\lambda, \alpha, s, \omega) = \left|L(\lambda, \alpha, s, \omega)\right|^2.$$

It is known (see [4]) that $L(\lambda, \alpha, s, \omega)$ is a $H(D)$-valued random element on $(\Omega, \mathcal{B}(\Omega), m_H)$.

Lemma 4. *We have*

$$\sum_{k=0}^{N} \left|L(\lambda, \alpha, s + ikh, \omega)\right|^2 = BN, \quad N \to \infty,$$

for $\sigma > \frac{1}{2}$ *and for almost all* $\omega \in \Omega$.

Proof. Taking into account the pairwise orthogonality of the random variables $L_m(\lambda, \alpha, s, \omega)$, we find that

$$EL_0(\lambda, \alpha, s, \omega) = \sum_{m=1}^{\infty} E\left|L_m(\lambda, \alpha, s, \omega)\right|^2 = \sum_{m=1}^{\infty} \frac{1}{(m+\alpha)^{2\sigma}} < \infty.$$

It is obvious that

$$L_0(\lambda, \alpha, s, \varphi_h^k(\omega)) = \left|L(\lambda, \alpha, s, a_{kh}\omega)\right|^2 = \left|L(\lambda, \alpha, s + ikh, \omega)\right|^2.$$

In view of Lemma 3 and the well-known Birkhoff theorem (see [5])

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} L_0(\lambda, \alpha, s, \varphi_h^k(\omega)) = EL_0(\lambda, \alpha, s, \omega) < \infty$$

for almost all $\omega \in \Omega$. This proves the lemma.

The series

$$\sum_{m=0}^{\infty} \frac{\omega(m) e^{2\pi i \lambda m}}{(m+\alpha)^s}$$

converges uniformly on compact subsets of D for almost all $\omega \in \Omega$. This fact was obtained in [4]. Denote by Ω_1 the subset of Ω such that both the latter assertion and Lemma 4 are true for $\omega \in \Omega_1$. Then $m(\Omega_1) = 1$.

Lemma 5. *Let K be a compact subset of D. Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left| L(\lambda, \alpha, s+ikh, \omega) - L_n(\lambda, \alpha, s+ikh, \omega) \right| = 0$$

for $\omega \in \Omega_1$.

Poof is a slight modification of the proof in [1, Lemma 6].

In [1] it was proved that for $\sigma_1 > \frac{1}{2}$

$$L_n(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s} \exp\left\{ - \left(\frac{m+\alpha}{n+\alpha} \right)^{\sigma_1} \right\},$$

and the series being absolutely convergent for $\sigma > \frac{1}{2}$. Also the probability measure

$$P_{N,n}(A) = \mu_N(L_n(\lambda, \alpha, s+ikh) \in A), \quad A \in \mathcal{B}(H(D)),$$

was defined.

Lemma 6. *There exists a probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ such that the measure $P_{N,n}$ converges weakly to P_n as $N \to \infty$.*

Proof of the lemma is given in [1].

Define the probability measure

$$\widetilde{P}_{N,n}(A) = \mu_N(L_n(\lambda, \alpha, s+ikh, \omega) \in A), \quad A \in \mathcal{B}(H(D)),$$

for $\omega \in \Omega_1$.

Lemma 7. *There exists a probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ such that the measures $P_{N,n}$ and $\widetilde{P}_{N,n}$ both converge weakly to P_n as $N \to \infty$.*

The *proof* of the lemma is similar to that of Lemma 6.

Now we will deal with the probability measure

$$\widetilde{P}_N(A) = \mu_N(L(\lambda, \alpha, s+ikh, \omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

Reasoning similarly to the proof of the Theorem in [1] we obtain the following result.

Lemma 8. *There exists a probability measure P on $(H(D), \mathcal{B}(H(D)))$ such that the measures P_N and $\widetilde{P}_N$ both converge weakly to P as $N \to \infty$.*

Let P_L denote the distribution of $L(\lambda, \alpha, s, \omega)$, i.e.

$$P_L(A) = m_H(\omega \in \Omega : L(\lambda, \alpha, s, \omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

Theorem. *The measure P_N converges weakly to P_L on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$.*

Proof. Let $A \in \mathcal{B}(H(D))$ be a continuity set of P. The properties of weak convergence and the assertion of Lemma 8 imply that

$$\lim_{N\to\infty} \mu_N(L(\lambda, \alpha, s+ikh, \omega) \in A) = P(A)$$

for $\omega \in \Omega_1$. Let us fix the set A and define the random variable θ on $(\Omega, \mathcal{B}(\Omega), m_H)$ by

$$\theta(\omega) = \begin{cases} 1, & \text{if } L(\lambda, \alpha, s, \omega) \in A, \\ 0, & \text{if } L(\lambda, \alpha, s, \omega) \notin A. \end{cases}$$

Hence $E\theta = \int\limits_{\Omega} \theta dm_H = m(\omega : L(\lambda, \alpha, s, \omega) \in A) = P_L(A)$. By Lemma 3 and the Birkhoff theorem

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \theta(\varphi_h^k(\omega)) = E\theta$$

for almost all $\omega \in \Omega$. Thus

$$P(A) = P_L(A)$$

for any continuity set of P. Since the continuity sets constitute the determining class, we have

$$P(A) = P_L(A)$$

for all $A \in \mathcal{B}(H(D))$. The theorem is proved.

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Ribinė teorema Lercho dzeta funkcijai

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Straipsnyje įrodoma diskrečioji ribinė teorema Lercho dzeta funkcijai analizinių funkcijų erdvėje.