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A note on the value-distribution of the periodic zeta-function

Audrius KAČĖNAS* (VU), Antanas LAURINČIKAS (VU, ŠU)
e-mail: antanas.laurincikas@maf.vu.lt

Let $\mathbb{Z}$ denote the set of all integer numbers, and $\mathfrak{A} = \{a_m, m \in \mathbb{Z}\}$ be a sequence of complex numbers with period $k > 0$. In [1] the periodic zeta-function $\zeta(s;\mathfrak{A})$, $s = \sigma + it$, was introduced and studied. For $\sigma > 1$, the function $\zeta(s;\mathfrak{A})$ is defined by

$$\zeta(s;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

If $\mathfrak{A} = \{1\}$ and $k = 1$, then the function $\zeta(s;\mathfrak{A})$ reduces to the classical Riemann zeta-function $\zeta(s)$. The equality

$$\begin{aligned}\zeta(s;\mathfrak{A}) &= \sum_{m=1}^{\infty} \frac{a_m}{m^s} = \sum_{q=1}^{k} a_q \sum_{m=0}^{\infty} \frac{1}{(mk+q)^s} \\ &= \frac{1}{k^s} \sum_{q=1}^{k} a_q \zeta\left(s, \frac{q}{k}\right),\end{aligned}$$

which is valid for $\sigma > 1$, and $\zeta(s,\alpha)$ denotes the Hurwitz zeta-function, gives the analytic continuation of the function $\zeta(s;\mathfrak{A})$ into the entire s-plane where it is regular with the possible exception of a simple pole at $s = 1$ with the residue

$$a = \frac{1}{k} \sum_{m=0}^{k-1} a_m.$$

In [3] the mean square of $\zeta(s;\mathfrak{A})$ in the strip $\frac{1}{2} \leqslant \sigma \leqslant 1$ was studied and some limit theorems in the sense of the weak convergence of probability measures were proved.

Denote by $meas\{A\}$ the Lebesgue measure of the set A, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T} meas\{\tau \in [0,T], \ldots\},$$

where instead of dots a condition satisfied by τ is to be written. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. Denote by $H(G)$ the space of analytic on G functions

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equipped with the topology of uniform convergence on compacta. Moreover, let γ be the unit circle on the complex plane,

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for all primes p. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and let, for natural m,

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p).$$

Let $\mathbb{C}$ be the complex plane, and $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where m_H denotes the Haar measure on $(\Omega, \mathcal{B}(\Omega))$, define an $H(D)$-valued random element $\zeta(s, \omega; \mathfrak{A})$ by the formula

$$\zeta(s, \omega; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega, \quad s \in D.$$

Theorem A [3]. *The probability measure*

$$\nu_T(\zeta(s + i\tau; \mathfrak{A}) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the distribution of the random element $\zeta(s, \omega; \mathfrak{A})$ *as* $T \to \infty$.

The aim of this note is to obtain a similar limit theorem in the space of functions defined in the half-plane $D_1 = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$.

Theorem 1. *Suppose that* $a = 0$. *Then the probability measure*

$$\nu_T\big(\zeta(s + i\tau; \mathfrak{A}) \in A\big), \quad A \in \mathcal{B}\big(H(D_1)\big),$$

converges weakly to the distribution of the random element

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega, \quad s \in D_1,$$

as $T \to \infty$.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let $d(s_1, s_2)$ be a metric on $\mathbb{C}_\infty$ given by the formulae

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1 + |s_1|^2}\sqrt{1 + |s_2|^2}}, \quad d(s, \infty) = \frac{2}{\sqrt{1 + |s|^2}}, \quad d(\infty, \infty) = 0.$$

Here $s, s_1, s_2 \in \mathbb{C}$. This metric is compatible with the topology of $\mathbb{C}_\infty$. Denote by $M(D_1)$ the space of meromorphic on D_1 functions $f : D_1 \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $\{f_n, f_n \in M(D_1)\}$ converges to the function $f \in M(D_1)$ if

$$d\bigl(f_n(s), f(s)\bigr) \to 0$$

as $n \to \infty$ uniformly in s on compact subsets of D_1.

Theorem 2. *Suppose that $a \neq 0$. Then the probability measure*

$$\nu_T\bigl(\zeta(s+i\tau;\mathfrak{A}) \in A\bigr), \quad A \in \mathcal{B}\bigl(M(D_1)\bigr),$$

converges weakly to the distribution of the random element

$$\sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega, \quad s \in D_1.$$

Proof of Theorem 1 completely coincides with that of Theorem A.

Proof of Theorem 2. Let

$$\zeta_1(s;\mathfrak{A}) = (1 - 2^{1-s})\zeta(s;\mathfrak{A}).$$

Then the function $\zeta_1(s;\mathfrak{A})$ is analytic in D_1, and, for $\sigma > 1$, it is given by an absolutely convergent Dirichlet series. Let B be a factor bounded by a constant. Since, for $\sigma > \frac{1}{2}$, [3]

$$\int_1^T \bigl|\zeta(\sigma + it;\mathfrak{A})\bigr|^2 \,\mathrm{d}t = BT, \quad T \to \infty,$$

clearly,

$$\int_1^T \bigl|\zeta_1(\sigma + it;\mathfrak{A})\bigr|^2 \,\mathrm{d}t = BT, \quad T \to \infty.$$

Moreover, $\zeta_1(s;\mathfrak{A})$ is a function of finite order. Therefore, the probability measure

$$Q_T(A) = \nu_T\bigl(\zeta_1(s+i\tau;\mathfrak{A}) \in A\bigr), \quad A \in \mathcal{B}\bigl(H(D_1)\bigr)$$

converges weakly to the distribution of the random element

$$(1 - 2^{1-s}\omega(2)) \sum_{m=1}^{\infty} \frac{a_m \omega(m)}{m^s}, \quad \omega \in \Omega, \quad s \in D_1,$$

as $T \to \infty$. This can be obtained similarly to the proof of Theorem A and Theorem 1. The function

$$p(s) = 1 - 2^{1-s}$$

is a Dirichlet polynomial. Therefore, the probability measure

$$Q_{T,p}(A) = \nu_T\big(p(s+i\tau) \in A\big), \quad A \in \mathcal{B}\big(H(D_1)\big),$$

converges weakly to the distribution of the random element.

$$1 - 2^{1-s}\omega(2), \quad \omega \in \Omega, \quad s \in D_1,$$

as $T \to \infty$ [4]. The weak convergence of the probability measures Q_T and $Q_{T,p}$ implies the weak convergence of the probability measure

$$P_T(A) = \nu_T\Big(\big(\zeta_1(s+i\tau;\mathfrak{A}), 1 - 2^{1-s-i\tau}\big) \in A\Big), \quad A \in \mathcal{B}\big(H^2(D_1)\big),$$

to the distribution of the random element

$$\left(\left(1 - 2^{1-s}\omega(2)\right) \sum_{m=1}^{\infty} \frac{a_m\omega(m)}{m^s}, \quad 1 - 2^{1-s}\omega(2)\right), \quad \omega \in \Omega, \quad s \in D_1 \qquad (1)$$

as to $T \to \infty$. The details are the same as, for example, in [5].

Let the function $h : H^2(D_1) \to M(D_1)$ be given by the formula

$$h(f_1, f_2) = \frac{f_1}{f_2}, \quad f_1, f_2 \in H(D_1).$$

Since the metric d satisfies the equality

$$d(f_1, f_2) = d\left(\frac{1}{f_1}, \frac{1}{f_2}\right),$$

the function h is continuous. Hence, by Theorem 5.1 from [2], we obtain that the measure $P_T h^{-1}$ converges weakly to Ph^{-1}as $T \to \infty$, where P is the distribution of the random element (1). Thus, in view of the definition of the function h, Theorem 2 follows.

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Pastaba apie periodinės dzeta funkcijos reikšmių pasiskirstymą

A. Kačėnas, A. Laurinčikas

Straipsnyje įrodytos ribinės teoremos periodinei dzeta funkcijai analizinių ir meromorfinių funkcijų erdvėse.