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On the value-distribution of Matsumoto zeta-function on the complex plane

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Let $\mathbb{N}$ and $\mathbb{C}$ denote the sets of natural and complex numbers, respectively. For any integer m, we define a positive integer $g(m)$. Let $a_m^{(j)}$ be complex numbers, and $f(j,m)$, $1 \leqslant j \leqslant g(m), m \in \mathbb{N}$, be natural numbers. We define the polynomials

$$A_m(X) = \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)} X^{f(j,m)}\right)$$

of degree $f(1,m) + \ldots + f(g(m),m)$. In [7] K. Matsumoto introduced the zeta-function

$$\varphi(s) = \prod_{m=1}^{\infty} A_m^{-1}(p_m^{-s}),$$

where $s = \sigma + it$ is a complex variable, and p_m denotes the mth prime number. Under some hypotheses on $g(m)$, $a_m^{(j)}$ and $\varphi(s)$ he proved the limit theorems for $\log \varphi(s)$ in the complex plane.

Let B denote a number (not always the same) bounded by a constant. Suppose that

$$g(m) = Bp_m^{\alpha}, \quad \left|a_m^{(j)}\right| \leqslant p_m^{\beta} \tag{1}$$

with non-negative constants α and β. Then $\varphi(s)$ is a holomorphic function in the half-plane $\sigma > \alpha + \beta + 1$ with no zeros. Let, for $\sigma > \beta$,

$$\log \varphi(s) = -\sum_{m=1}^{\infty} \sum_{j=1}^{g(m)} \mathrm{Log}\left(1 - a_m^{(j)} p_m^{-f(j,m)s}\right),$$

and let R denote a closed rectangle on the complex plane with the edges parallel to the axes. The first theorem of [7] asserts that the limit

$$\lim_{T\to\infty} \frac{1}{T} meas\{t \in [0,T],\ \log \varphi(\sigma_0 + it) \in R\}$$

exists. Here $meas\{A\}$ stands for the Lebesgue measure of the set A, and $T > 0$.

Let ρ_0 be a constant with $\alpha+\beta+\frac{1}{2}\leqslant\rho_0\leqslant\alpha+\beta+1$, and we assume that $\varphi(s)$ can be meromorphically continued to the region $\sigma\geqslant\rho_0$. All poles of $\varphi(s)$ belong to a compact set, for $\sigma\geqslant\rho_0$,

$$|\varphi(\sigma+it)|=B|t|^{\delta}$$

with some positive δ, and

$$\int_0^T|\varphi(\rho_0+it)|^2\,\mathrm{d}t=BT.$$

We put

$$G=\{s\in\mathbb{C},\ \sigma\geqslant\rho_0\}\setminus\bigcup_{s'=\sigma'+it'}\{s=\sigma+it',\ \rho_0\leqslant\sigma\leqslant\sigma'\},$$

where $s'=\sigma'+it'$ runs all possible zeros and poles of $\varphi(s)$ in the strip $\rho_0\leqslant\sigma\leqslant\alpha+\beta+1$. Define $\varphi(\sigma_0+it_0)$ for $\sigma_0+it_0\in G$ by analytic continuation along the path $s=\sigma+it_0$, $\sigma\geqslant\sigma_0$. In the second theorem of [7] is proved the existence of the limit

$$\lim_{T\to\infty}\frac{1}{T}meas\{t\in[0,T],\ \sigma_0+it\in G,\ \log\varphi(\sigma_0+it)\in R\}\tag{2}$$

for $\sigma_0\geqslant\rho_0$.

The lower and upper bounds for (2) were obtained in [1], [8], [9].

Limit theorems for the function $\varphi(s)$ in the spaces of analytic and meromorphic functions were proved in [3]. The explicit form of a limit measure in these theorems was given in [4] and [5]. In [6] the universality property for the function $\varphi(s)$ was obtained.

Let h be a fixed number, and let, for $N\in\mathbb{N}$,

$$\mu_N(\ldots)=\frac{1}{N+1}\#\{0\leqslant k\leqslant N,\ldots\},$$

where instead of dots a condition satisfied by k is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and define a probability measure

$$P_N(A)=\mu_N\big(\varphi(\sigma+ikh)\in A\big),\quad A\in\mathcal{B}(\mathbb{C}).$$

Denote by γ the unit circle on $\mathbb{C}$, i.e., $\gamma=\{s\in\mathbb{C}:\ |s|=1\}$, and let

$$\Omega=\prod_p\gamma_p,$$

where $\gamma_p=\gamma$ for all primes p. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological group. Then there exists a probability Haar measure m_H on $(\Omega,\mathcal{B}(\Omega))$. This yields a probability space $(\Omega,\mathcal{B}(\Omega),m_H)$.

Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ onto the coordinate space γ_p. Setting

$$\omega(k) = \prod_{p^r \| k} \omega^r(p),$$

where $p^r \| k$ means that $p^r | k$ but $p^{r+1} \not| \, k$, we obtain an extention of the function $\omega(p)$ to the set $\mathbb{N}$ as a completely multiplicative unimodular function.

For $\sigma > \alpha + \beta + \frac{1}{2}$, define on $(\Omega, \mathcal{B}(\Omega), m_H)$ the complex-valued random element $\varphi(\sigma + it, \omega)$ by

$$\varphi(\sigma + it, \omega) = \sum_{k=1}^{\infty} \frac{b(k)\omega(k)}{k^{\sigma+it}}.$$

Denote by P_φ a distribution of the random element $\varphi(\sigma + it, \omega)$, i.e.,

$$P_\varphi(A) = m_H\big(\varphi(\sigma + it, \omega) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem. *Suppose that* $\exp\left\{\frac{2\pi k}{h}\right\}$ *is irrational for all integers* $k \neq 0$*. Then, for* $\sigma > \alpha + \beta + \frac{1}{2}$*, the probability measure* P_N *converges weakly to* P_φ *as* $N \to \infty$*.*

We will give the sketch of the proof only.

We begin the proof of the Theorem with a discrete limit theorems for a trigonometrical polynomial

$$p_n(t) = \sum_{k=1}^{n} a_k k^{-it}, \quad a_m \in \mathbb{C}.$$

Define a probability mesure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$

$$P_{N,p_n}(A) = \mu_N\big(p_n(mh) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Than we prove that there exists a probability measure P_{p_n} on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that the measure P_{N,p_n} converges weakly to P_{p_n} as $N \to \infty$.

After this we define

$$p_n(t, g) = \sum_{k=1}^{n} a_k g(k) k^{-it},$$

and

$$\widetilde{P}_{N,p_n} = \mu_N\big(p_n(mh, g) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

where $g(k)$, $k \in \mathbb{N}$, is a completely multiplikative function, and show that the probability measures P_{N,p_n} and $\widetilde{P}_{N,p_n}$ both converge weakly to the same measure as $N \to \infty$.

Now we prove assertion for absolutely convergent Dirichlet series. Let $\sigma_1 > \frac{1}{2}$, and

$$\varphi_n(s) = \sum_{m=1}^{\infty} \frac{b(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

Define the function

$$l_n(s) = \frac{s}{\sigma_1}\Gamma\left(\frac{s}{\sigma_1}\right) n^s,$$

and

$$a_n(m) = \frac{1}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{l_n(s)\mathrm{d}s}{sm^s},$$

where $\Gamma(s)$ is the Euler gamma-function. Let

$$\varphi_n(s,\omega) = \sum_{m=1}^{\infty} \frac{b(m)\omega(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}, \quad \omega \in \Omega.$$

Define two probability measures

$$P_{N,n}(A) = \mu_N\big(\varphi_n(\sigma + imh) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$\widetilde{P}_{N,n}(A) = \mu_N\big(\varphi_n(\sigma + imh, \omega) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

After this we show that there exists a probability measure P_n on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that both the measures $P_{N,n}$ and $\widetilde{P}_{N,n}$ converge weakly to P_n as $N \to \infty$.

We approximate the function $\varphi(s)$ in the mean by $\varphi_n(s)$, i.e., we prove that in the half-plane $\sigma > \alpha + \beta + \frac{1}{2}$

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \big|\varphi(\sigma + ikh) - \varphi_n(\sigma + ikh)\big| = 0. \tag{3}$$

Let $a_h = \{p^{-ih}, p \text{ is prime}\}$. We define a transformation φ_h on Ω taking the value $\varphi_h(\omega) = a_h\omega$ for $\omega \in \Omega$. Then φ_h is a measurable measure-preserving transformation on $(\Omega, \mathcal{B}(\Omega), m_H)$. Applying elements of ergodic theory [10] we proof that φ_h is ergodic transformation.

Now let T be a measurable measure-preserving ergodic transformation on the space $(\widetilde{\Omega}, F, m)$. Then in [10] is proved that for every $f \in L^1(\Omega, F, m)$, for almost all $\omega \in \widetilde{\Omega}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega) = E(f).$$

Denote by Ω_1 a subset of Ω such that for $\omega \in \Omega_1$ the series

$$\sum_{k=1}^{\infty} \frac{b(k)\omega(k)}{k^{\sigma+it}}$$

converges and, for $\sigma > \alpha + \beta + \frac{1}{2}$,

$$\sum_{k=0}^{N} \left|\varphi(\sigma + ikh, \omega)\right|^2 \mathrm{d}t = BN.$$

We have that $m_H(\Omega_1) = 1$.

We can proof that in the half-plane $\sigma > \alpha + \beta + \frac{1}{2}$, for $\omega \in \Omega_1$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \left|\varphi(\sigma + ikh, \omega) - \varphi_n(\sigma + ikh, \omega)\right| = 0.$$

Now let, for $\omega \in \Omega_1$,

$$\widetilde{P}_N(A) = \mu_N\left(\varphi(\sigma + ikh, \omega) \in A\right), \quad A \in \mathcal{B}(\mathbb{C}).$$

It is proved that both the measures $P_{N,n}$ and $\widetilde{P}_{N,n}$ converge weakly to the same measure P_n as $N \to \infty$. From this it follows that the family of the probability measures P_n is relatively compact. We obtain by the Prochorov theorem that it is also tight.

Let $A \in \mathcal{B}(\mathbb{C})$ be a continuity set of P. For $\omega \in \Omega_1$ we have

$$\lim_{N\to\infty} \mu_N(\varphi(s + ikh, \omega) \in A) = P(A). \tag{4}$$

Now we fix the set A and define the random variable η on $(\Omega, \mathcal{B}(\Omega), m_H)$ by the formula

$$\eta(\omega) = \begin{cases} 1 & \text{if } \varphi(\sigma, \omega) \in A, \\ 0 & \text{if } \varphi(\sigma, \omega) \notin A. \end{cases}$$

Then, clearly,

$$E(\eta) = \int_{\Omega} \eta dm_H = m_H(\omega : \varphi(\sigma, \omega) \in A) = P_\varphi(A). \tag{5}$$

We find that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \eta(\varphi_h^k(\omega)) = E\eta \tag{6}$$

for almost all $\omega \in \Omega$. However, the definitions of η and of φ_h give

$$\frac{1}{N+1}\sum_{k=0}^{N}\eta\big(\varphi_h^k(\omega)\big) = \mu_N\big(\varphi(\sigma+ikh,\omega)\in A\big).$$

From this, (5) and (6) we find that

$$\lim_{N\to\infty}\mu_N\big(\varphi(\sigma+ikh,\omega)\in A\big) = P_\varphi(A)$$

for almost all ω. Therefore, by (4)

$$P(A) = P_\varphi(A)$$

for any continuity set A of P. Since the continuity sets constitute the determining class, we obtain that

$$P(A) = P_\varphi(A)$$

for all $A \in \mathcal{B}(\mathbb{C})$. The theorem is proved.

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Apie Matsumoto dzeta funkcijos reikšmių pasiskirstymą kompleksinėje plokštumoje

R. Kačinskaitė

Straipsnyje įrodoma diskrečioji ribinė teorema Matsumoto dzeta funkcijai tikimybinių matų silpno konvergavimo prasme kompleksinėje plokštumoje.