

# On integer parts of some sequences

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## 1. Introduction

Let  $a_n$  be some increasing sequence of positive real numbers. With this sequence we generate for each  $\alpha > 0$  a sequence of natural numbers  $A(n, \alpha)$ . Examples are:  $A(n, \alpha) = [a_n \alpha]$ ,  $[a_n^\alpha]$ ,  $[\alpha^{a_n}]$ .

With some sequence of subsets of natural numbers  $S_1, S_2, \dots$  let us define

$$\mathcal{A}(\alpha) = \{n : A(n, \alpha) \in S_n\}.$$

We are interested on the conditions which imply that for almost all  $\alpha > 0$  the sets  $\mathcal{A}(\alpha)$  are infinite. An interesting instance of this problem was investigated by G. Harman [1]. With  $S_1 = S_2 = \dots = \mathcal{P}$  being the set of all prime numbers he proved that  $\mathcal{A}(\alpha)$  are infinite for almost all  $\alpha > 0$  if and only if the series

$$\sum_{n=1}^{\infty} \frac{1}{\log a_n} \left( \sum_{\substack{m \leq n \\ |a_m - a_n| < 1}} 1 \right)^{-1}$$

diverges.

We shall consider the outlined problem with

$$S_n = \{m : m \equiv m_n \pmod{M_n}\},$$

where  $0 \leq m_n < M_n$  are given natural numbers. Note, that the case of bounded  $M_n$  was investigated in [3].

**Theorem.** Let  $a_n$  be a sequence of positive numbers,  $0 \leq m_n < M_n$ ,  $M_n/a_n \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $A(n, \alpha) = [a_n \alpha]$  and

$$\mathcal{A}(\alpha) = \{n : A(n, \alpha) \equiv m_n \pmod{M_n}\}.$$

If the series

$$\sum_{n=1}^{\infty} \frac{1}{M_n} \tag{1}$$

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converges, then  $\mathcal{A}(\alpha)$  is finite for almost all  $\alpha > 0$ . If the series (1) diverges and

$$M_n \ll \frac{a_n}{a_m}, \quad m \leq n - \Delta_n, \quad \Delta_n = \sum_{m \leq n} \frac{1}{M_n}, \quad (2)$$

then  $\mathcal{A}(\alpha)$  is infinite for almost all  $\alpha > 0$ .

For increasing sequence  $a_n$  the condition (2) may be replaced by

$$M_n \ll \frac{a_n}{a_{n-[\Delta_n]}}. \quad (3)$$

If  $a_n = q^n$ , then (3) is satisfied with  $M_n \ln q \cdot n$ .

## 2. Proof of the theorem

Our main tool is the following proposition. The Lebesgue measure on the real line is denoted by  $\lambda$ .

**Lemma** ([2], Lemma 6.1, p.171). *Let  $J$  be a subinterval of the real line and  $\mathcal{D}_n$  be a sequence of subsets of  $J$ . For each open interval  $I \subset J$  suppose that there is a sequence of sets  $\mathcal{B}_n \subset \mathcal{D}_n \cap I$  such that*

$$\sum_{n=1}^{\infty} \lambda(\mathcal{B}_n) = +\infty$$

and

$$\limsup_{N \rightarrow \infty} \left( \sum_{n \leq N} \lambda(\mathcal{B}_n) \right)^2 \left( \sum_{m, n \leq N} \lambda(\mathcal{B}_n \cap \mathcal{B}_m) \right)^{-1} \geq \delta \lambda(I), \quad (4)$$

where  $\delta$  is a positive constant independent on  $I$ . Then almost all  $\alpha \in J$  belong to infinitely many  $\mathcal{D}_n$ .

With the notation of our theorem we set

$$\mathcal{D}_n = \{ \alpha : A(n, \alpha) \equiv m_n \pmod{M_n} \}.$$

The condition  $\alpha \in \mathcal{D}_n$  is equivalent then to the existence of some natural number  $s$  satisfying

$$m_n + sM_n \leq a_n \alpha < m_n + 1 + sM_n, \quad \text{or} \\ \alpha \in \left[ \frac{m_n}{a_n} + s \frac{M_n}{a_n}, \frac{m_n + 1}{a_n} + s \frac{M_n}{a_n} \right).$$

Then

$$\mathcal{D}_n = \bigcup_{s \geq 0} J(n, s), \quad J(n, s) = \left[ \frac{m_n}{a_n} + s \frac{M_n}{a_n}, \frac{m_n + 1}{a_n} + s \frac{M_n}{a_n} \right).$$

We fix now an interval  $I = (a, a + b)$ ,  $a, b > 0$  and denote  $\mathcal{B}_n = \mathcal{D}_n \cap I$ . The set  $\mathcal{B}_n$  is a union of some intervals  $J(n, s)$ , two of them are shortened, if necessary, by taking the lower (upper) range equal to  $a$  ( $a + b$ ).

The number of intervals in the union is equal to the number of integers  $s$ , satisfying  $a \leq (m_n + sM_n)/a_n < a + b$  increased by 1. Hence

$$\begin{aligned} \lambda(\mathcal{B}_n) &= \frac{1}{a_n} \# \left\{ s : a \leq s \frac{M_n}{a_n} + \frac{m_n}{a_n} < a + b \right\} \\ &+ O\left(\frac{1}{a_n}\right) = \frac{b}{M_n} + O\left(\frac{1}{a_n}\right) = \frac{b}{M_n} (1 + o(1)). \end{aligned}$$

It follows now from the Borel–Cantelli lemma that the convergence of the series (1) imply that  $\lambda(\limsup \mathcal{B}_n) = 0$ , which means that for almost all  $\alpha \in I$  the sets  $\mathcal{A}(\alpha)$  are finite. This proves the first part of Theorem.

For the second part we need to prove (4) with  $\mathcal{B}_n$  defined as above. Let

$$L(N) = \sum_{1 \leq k, l \leq N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) = \sum_{k \leq N} \lambda(\mathcal{B}_k) + \sum_{1 \leq k < l \leq N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l). \quad (5)$$

For the first sum we have

$$\sum_{k \leq N} \lambda(\mathcal{B}_k) = \lambda(I)(1 + o(1)) \sum_{k \leq N} \frac{1}{M_k}, \quad \lambda(I) = b. \quad (6)$$

To prove (4), we need an upper bound for  $L(N)$ . It follows from (5) and (6) that it suffices to prove that with some  $C > 0$  independent of  $I$

$$\sum_{1 \leq k < l \leq N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) < C \lambda(I) \left( \sum_{m \leq N} \frac{1}{M_m} \right)^2 \quad (7)$$

holds. Consider now the summands on the left-hand side of (7). Let

$$\mathcal{B}_k = \cup_s^* J(k, s), \quad \mathcal{B}_l = \cup_t^* J(l, t),$$

where  $*$  indicates that the sums are taken over some appropriate subsets of  $s$  and  $t$  and two intervals in each union are, if necessary, shortened.

We fix some  $s$  and consider when

$$J(k, s) \cap J(l, t) \neq \emptyset. \quad (8)$$

A sufficient condition for this is

$$\frac{m_k}{a_k} + s \frac{M_k}{a_k} \leq \frac{m_l}{a_l} + t \frac{M_l}{a_l} < \frac{m_k + 1}{a_k} + s \frac{M_k}{a_k}. \quad (9)$$

Both inequalities are also necessary for all  $t$ , except, probably for one case. The number of  $t$  satisfying (9) is bounded by  $a_l/(a_k M_k) + 1$ , hence, the number of non-empty intersections in (8) is

$$\frac{a_l}{a_k M_l} + O(1).$$

As a consequence we obtain

$$\lambda(J(k, s) \cap \mathcal{B}_l) = \frac{1}{a_l} \left( \frac{a_l}{a_k M_l} + O(1) \right).$$

How large is the number of intervals  $\mathcal{B}_k$  consists of? For all these intervals except one the inequality

$$a < \frac{m_k}{a_k} + s \frac{M_k}{a_k} < a + b$$

must hold. The number of intervals is then bounded by  $ba_k/M_k + O(1)$ . We have now

$$\begin{aligned} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) &= \left( \frac{ba_k}{M_k} + O(1) \right) \left( \frac{1}{a_k M_l} + O\left(\frac{1}{a_l}\right) \right) \\ &= \frac{b}{M_k M_l} \left( 1 + O\left( \frac{M_k}{ba_k} + \frac{M_k M_l}{ba_l} + \frac{a_k M_l}{a_l} \right) \right). \end{aligned}$$

We use this result for  $k_0 \leq k \leq l - \Delta_l$ . Then  $M_l \ll a_l/a_k$ , and

$$\lambda(\mathcal{B}_k \cap \mathcal{B}_l) \ll \frac{b}{M_k M_l} = \frac{\lambda(I)}{M_k M_l} \quad (10)$$

with the constant in  $\ll$  independent of  $I$ . For  $k \leq l$  not in the range  $k_0 \leq k < l - \Delta_l$  we use the trivial bound

$$\lambda(\mathcal{B}_k \cap \mathcal{B}_l) \ll \lambda(\mathcal{B}_l) \ll \frac{b}{M_l}. \quad (11)$$

We are now ready to prove (7). Using the bounds (10) and (11) we have

$$\begin{aligned} \sum_{1 \leq k < l \leq N} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) &= \sum_{l=1}^n \sum_{k \in [k_0, l - \Delta_n]} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) + \sum_{l=1}^n \sum_{k \notin [k_0, l - \Delta_n]} \lambda(\mathcal{B}_k \cap \mathcal{B}_l) \\ &\ll b \sum_{1 \leq k \leq l \leq N} \frac{1}{M_k M_l} + b \sum_{l=1}^n (k_0 + \Delta_l) \frac{1}{M_l} \ll b \left( \sum_{n \leq N} \frac{1}{M_n} \right)^2. \end{aligned}$$

The theorem is now proved.

## References

- [1] G. Harman, Metrical theorems on prime values of the integer parts of real sequences, *Proc. London Math. Soc.* (3), **75**, 481–496 (1997).
- [2] G. Harman, *Metric Number Theory*, Clarendon Press, Oxford (1998).
- [3] V. Stakėnas, Tikimybinė skaičių teorija ir kontinuumas, *Lietuvos matematikų draugijos mokslo darbai*, **3**, 93–99 (1999).

## Sveikosios tam tikrų sekų dalys

V. Stakėnas

Darbe nagrinėjamos natūraliųjų skaičių sekos  $[a_n\alpha]$ , čia  $a_n$  – teigiamų realiųjų skaičių skaičių seka  $\alpha > 0$ . Įrodytoje teoremoje tvirtinama, kad esant tam tikroms sąlygoms beveik visiems  $\alpha > 0$  skaičiai  $[a_n\alpha]$  tenkina lyginius  $[a_n\alpha] \equiv m_n \pmod{M_n}$  be galo daug kartų.