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On Sprindjuk's works in "Lietuvos matematikos rinkinys"

Antanas LAURINČIKAS* (VU, ŠU)
e-mail: antanas.laurincikas@maf.vu.lt

A famous Belorussian number-theorist V.G. Sprindjuk (1936–1987) began his scientific activity in 1959 at Vilnius university under supervision of Professor J. Kubilius. At that moment J. Kubilius had already obtained some important results on the classification of transcendental numbers [4], [5], in particular, concerning the Mahler conjecture, and proposed to V.G. Sprindjuk to continue the investigations in the field.

In 1962 V.G. Sprindjuk published five papers [9]–[13] in "Lietuvos matematikos rinkinys" on the classification of thanscendental numbers and on the approxsimation of a number by algebraic numbers, also he obtained some results related to the Mahler conjecture.

The first paper [9] of V.G. Sprindjuk was related to the classification of transcendental numbers and to the approximation by algebraic numbers.

Let $P(x) = a_0 + a_1x + \ldots + a_nx^n$ be a polynomial with integer coefficients, and let $h = \max\limits_i |a_i|$ be the height of $P(x)$. A transcendental number ω is called the S-number if exists a constant λ such that the inequality

$$\left|P(\omega)\right| > ch^{-n\lambda}$$

is satisfied for every $n = 1, 2, \ldots$ with $c = c(n, \lambda, \omega)$. Denote by $\theta_n(\omega)$ and $\tilde{\theta}_n(\omega)$ the infimum of such λ for real and complex ω, respectively. Then [8] the inequalities

$$\theta_n(\omega) \geqslant 1, \quad \tilde{\theta}_n(\omega) \geqslant \frac{1}{2} - \frac{1}{2n}, \quad n = 1, 2, \ldots$$

hold. K. Mahler [7] obtained that almost all numbers in the sense of the Lebesgue measure are S-numbers, and also he proved that almost sure

$$\theta(\omega) \leqslant 4, \quad \tilde{\theta}(\omega) \leqslant 3.5.$$

Here

$$\theta(\omega) = \sup_n \theta_n(\omega), \quad \tilde{\theta}(\omega) = \sup_n \tilde{\theta}_n(\omega).$$

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Moreover, K. Mahler conjectured that for almost all ω

$$\theta(\omega) = 1 \quad \text{and} \quad \tilde{\theta}(\omega) = \frac{1}{2}.$$

Some years later J.F. Koksma proved [3] that

$$\theta(\omega) \leqslant 3, \quad \tilde{\theta}(\omega) \leqslant 2.5,$$

and W.J. Le Veque gave [6] more precise inequalities

$$\theta(\omega) \leqslant 2, \quad \tilde{\theta}(\omega) \leqslant 1.5.$$

F. Kash [2] and B. Volkmann [16] showed that $\tilde{\theta}(\omega) \leqslant 1$ for almost all ω.

We note that the Mahler conjecture is a consequence of inequalities

$$\theta_n(\omega) \leqslant 1 \quad \text{and} \quad \tilde{\theta}_n(\omega) \leqslant \frac{1}{2},$$

if they are satisfied for all n. The case $n = 1$ is not difficult. For $n = 2$ J. Kubilius obtained [4], [5] $\theta_2(\omega) = 1$. F. Kash proved [1] that $\tilde{\theta}(\omega) = \frac{1}{4}$, and B. Volkmann [15] found that $\tilde{\theta}_3(\omega) = \frac{1}{3}$ for almost all ω. Finally, B. Volkmann [17] proved that $\theta_3(\omega) = 1$, and [18]

$$\theta_n(\omega) \leqslant \frac{3}{2}, \quad \tilde{\theta}_n(\omega) \leqslant \frac{3}{4} - \frac{1}{2n}, \quad n = 4, 5, \ldots. \tag{1}$$

V.G. Sprindjuk proved in [9] that there exist numbers θ_n and $\tilde{\theta}_n$,

$$1 \leqslant \theta_n \leqslant 2 - \frac{2}{n}, \quad \frac{1}{2} - \frac{1}{2n} \leqslant \tilde{\theta}_n \leqslant 1 - \frac{3}{2n}, \quad n \geqslant 2,$$

such that for almost all ω

$$\theta_n(\omega) = \theta_n, \quad \tilde{\theta}_n(\omega) = \tilde{\theta}_n.$$

Also, in the same paper V.G. Sprindjuk considered the S^*-numbers introduced by J.F. Koksma. A transcendental number ω is called the S^*-number if there exists a constant λ such that for all algebraic numbers α of degree $\leqslant n$ and of height $\leqslant h$ the inequality

$$|\omega - \alpha| > ch^{-1-n\lambda}$$

is satisfied with $c = c(n, \lambda, \omega)$.

In similar notation as above V.G. Sprindjuk obtained that there exist numbers θ_n^* and $\tilde{\theta}_n^*$,

$$\frac{2\theta_n - 1}{3} \leqslant \theta_n^* \leqslant 1, \quad n \geqslant 3,$$

$$\frac{4\tilde{\theta}_n - 1}{6} - \frac{1}{6n} \leqslant \tilde{\theta}_n^* \leqslant \frac{1}{2} - \frac{1}{2n}, \quad n \geqslant 4,$$

such that for almost all ω

$$\theta_n^*(\omega) = \theta_n^*, \quad \tilde{\theta}_n^*(\omega) = \tilde{\theta}_n^*.$$

It follows from Volkmann's inequalities (1) that

$$1 \leqslant \theta(\omega) \leqslant \frac{3}{2} \quad \text{and} \quad \frac{1}{2} \leqslant \tilde{\theta}(\omega) \leqslant \frac{3}{4}. \tag{2}$$

V.G. Sprindjuk returned to the Mahler problem in [11]. He improved the inequalities (2) showing that

$$1 \leqslant \theta(\omega) \leqslant \frac{4}{3}, \quad \frac{1}{2} \leqslant \tilde{\theta}(\omega) \leqslant \frac{2}{3}.$$

In [12] V.G. Sprindjuk proposed a new classification of transcendental numbers. Denote by $\mathcal{P}_n(h)$ the set of all polynomials with integer coefficients of degree $\leqslant n$ and of height $\leqslant h$. Let

$$\nu_n(\omega, h) = \min |P(\omega)|,$$

where the minimum is taken over all $P \in \mathcal{P}_n(h)$, $P(\omega) \neq 0$. Moreover, we put

$$\begin{aligned} \nu(\omega, h) &= \limsup_{n\to\infty} \frac{\log\log(1/\nu_n(\omega, h))}{\log n}, \\ \nu(\omega) &= \sup_{h\in\mathbf{N}} \nu(\omega, h). \end{aligned}$$

Then $\nu(\omega)$ is called the order of the number ω. In the case $\nu < \infty$ we set

$$\begin{aligned} t(\omega, h) &= \limsup_{n\to\infty} \frac{\log(1/\nu_n(\omega, h))}{n^\nu}, \\ t(\omega) &= \limsup_{h\to\infty} \frac{t(\omega, h)}{\log h}. \end{aligned}$$

Then $t(\omega)$ is called the type of the number ω. If $\nu = \infty$, but there exists h such that $\nu(\omega, h) = \infty$, then denote by h_0 the smallest of such h.

After this notation and definitions the classes of $\tilde{A}$, $\tilde{S}$, $\tilde{T}$ and $\tilde{U}$ – numbers are introduced as follows.

$$\begin{aligned} &\tilde{A}: \ 0 \leqslant \nu(\omega) \leqslant 1; \quad \text{if} \quad \nu(\omega) = 1, \quad \text{then} \quad t(\omega) = 0; \\ &\tilde{S}: \ 1 \leqslant \nu(\omega) < \infty; \quad \text{if} \quad \nu(\omega) = 1, \quad \text{then} \quad t(\omega) > 0; \\ &\tilde{T}: \ \nu(\omega) = \infty; h_0(\omega) = \infty; \\ &\tilde{U}: \ \nu(\omega) = \infty; h_0(\omega) < \infty. \end{aligned}$$

This Sprindjuk's classification differs from Mahler's classification: in Mahler's classification first the limit $h \to \infty$ is taken, and then $n \to \infty$.

V.G. Sprindjuk obtained in [12] that all algebraic numbers are of order $\leqslant 1$, all transcendental numbers are of order $\leqslant 1$. Moreover, almost all numbers are of order $\leqslant 2$, and thus they are $\tilde{S}$-numbers.

One paper of Sprindjuk [10] is devoted to Diophantine approximation. It is well known that the classification of transcendental numbers is closely related to Diophantine approximation of dependent variables. Denote by $\|x\|$ the distance of x from the nearest integer, and let $\varphi(q)$ be a positive decreasing function. Then the well-known Khinchine theorem asserts that for almost all real vectors $(\omega_1, \ldots, \omega_m)$ the system of Diophantine inequalities

$$\max\left(\|\omega_1 q\|, \ldots, \|\omega_m q\|\right) < \varphi(q)$$

has a finite or infinite number of solutions provided the series

$$\sum_{q=1}^{\infty} \varphi^m(q)$$

is convergent or divergent, respectively. There exists a general problem on the number of solutions of similar system when instead of ω_j a polynomial $P_j(\omega_1, \ldots, \omega_k)$ with integer coefficients is taken. In particular, K. Mahler conjecture is equivalent to the assumption that the system of Diophantine inequalities

$$\max\left(\|\omega q\|, \|\omega^2 q\|, \ldots, \|\omega^n q\|\right) < q^{-1/n-\delta}$$

for almost all real numbers q has only a finite number of solutions provided $\delta > 0$. The case $n = 2$ with some improvements was completely solved by J. Kubilius in [4] and [5], and this gave the Mahler conjecture for $n = 2$. V.G. Sprindjuk in [10] obtained the following result. Suppose that the series

$$\sum_{q=1}^{\infty} \varphi^n(q) q^{n/m-1}, \quad m = \frac{n(n+1)}{2} + n$$

is convergent. Then the system of Diophantine inequalities

$$\max_{1 \leqslant i,j \leqslant n} \left(\|\omega_i q\|, \|\omega_i \omega_j q\|\right) < \varphi(q)$$

for almost all $(\omega_1, \ldots, \omega_n)$ has only a finite number of solutions. Note that the Mahler conjecture by using of generalization of Kubilius's ideas was completely solved by V.G. Sprindjuk in 1964, see [14].

The last paper of Sprindjuk in "Lietuvos matematikos rinkinys" generalizes the results of [9] for polynomials with coefficients from the field of powers series. Let K be a finite field of characteristics p, and let $K\langle x\rangle$ be a field of the formal Laurent series $\omega(x)$ over K. The field $K\langle x\rangle$ can be normalized, and also the Lebesgue measure can be introduced. Let $K[x]$ denote the ring of $K\langle x\rangle$, and

$$P(z) = a_0(x) + a_1(x)z + \ldots + a_n(x)z^n, \quad a_j \in K[x].$$

Denote by $|*|$ the norm in $K\langle x\rangle$, and let

$$h = \max\left(|a_0(x)|, |a_1(x)|, \ldots, |a_n(x)|\right)$$

be the height of $P(z)$. We put

$$w_n(\omega, h) = \min\left|P(\omega)\right),$$

where the minimum is taken over all polynomials $P(z)$ of degree $\leqslant n$ and of height $\leqslant h$, $P(\omega) \neq 0$. Let

$$w_n(\omega) = -\liminf_{h\to\infty} \frac{\log w_n(\omega, h)}{\log h}.$$

Then V.G. Sprindjuk proved that for almost all elements $\omega \in K\langle x\rangle$ the inequalities

$$\frac{1}{n} w_n(\omega) \leqslant \max\left(\frac{5}{4} - \frac{3}{8n}, \frac{4}{3} - \frac{1}{4n}\right), \quad n = 2, 3, \ldots,$$

hold. Moreover, there exist numbers w_n such that

$$w_n(\omega) = w_n, \quad n = 1, 2, \ldots$$

for almost all ω.

We note that the publications of V.G. Sprindjuk in "Lietuvos matematikos rinkinys" were the basic for his future researches.

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Sprindžuko darbai „Lietuvos matematikos rinkinyje“

A. Laurinčikas

Straipsnyje pateikta žymaus baltarusių matematiko V.G. Sprindžuko darbų „Lietuvos matematikos rinkinyje“ apžvalga.