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Fractional Brownian Vector Field in the Framework of Euclidean Geometry

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Abstract

A new fractional Brownian vector field (FBVF) is created for modeling multidimensional and multivariate fractal data. It is shown that the FBVF is a multidimensional and multivariate generalization of the classical Kolmogorov–Wiener process, allowing the distribution of field increments to be defined solely through fractal Euclidean distances between observation points. Conditions are established under which the family of field distributions satisfies the Kolmogorov consistency theorem. Maximum likelihood and variogram-based methods are developed to analytically estimate the mean and covariance of the FBVF, while the Hurst parameter is computed using an one-variable optimization algorithm. A kriging method is constructed for solving prediction problems using observations of fractal data. For computer simulation of field realizations, recursive and kriging-based algorithms are applied. A computational Monte Carlo experiment confirms the reliability of the proposed methods, particularly in accurately estimating the Hurst parameter. Applications to heavy metal concentrations in soil and climate data analysis demonstrate the effectiveness of the model in representing and analyzing multifractal, multidimensional processes.

Keywords: random fields; Brownian fields; Hurst parameter; maximum likelihood; variogram

MSC: 60G22; 60G20; 60G60; 62M30; 62F10; 65C05

1. Introduction

Many applied fields—such as economics, finance, epidemiology, geology, and geoinformatics—rely on stochastic processes that display fractal behavior [1–4]. Reviews of fractal process theory, along with detailed analyses of its core properties, are presented in both the foundational and contemporary literature [5,6].

Fractional Brownian motion (FBM) is widely used models for characterizing fractal behavior in stochastic processes. The mathematical framework of FBM was formulated in the work of A. Kolmogorov [7] and later developed in the study by B. Mandelbrot and J. Van Ness [8]. It describes one-dimensional Gaussian stochastic processes with stationary increments.

Although early applications of FBM were restricted to one-dimensional temporal contexts, many contemporary problems in applied sciences involve multidimensional variables. Consequently, there is an increasing demand for methodologies capable of analyzing intrinsically multi-output data, necessitating more sophisticated and flexible modeling frameworks [9–11]. In this regard, fractional Brownian fields (FBFs) constitute a natural extension of the one-dimensional FBM to higher-dimensional spaces. A prominent example is the



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Lévy FBF, which generalizes FBM to $\mathbb{R}^p$ while preserving fundamental properties, including stationary increments and isotropy. Subsequent advancements, notably by Molchanov [12], introduced constructions based on non-Euclidean norms, thereby facilitating the formulation of random fields in Minkowski spaces and other non-standard geometries.

In applied studies, FBFs are often represented as two-dimensional Gaussian rough surfaces [13,14]. Within geostatistics, FBF models have been studied in the framework of intrinsic kriging, serving as examples of intrinsically stationary processes for which predictions are typically based on the variogram rather than the covariance function [9,15]. In particular, ref. [9,16] constructed FBF models for computer simulation response surface metamodeling.

The covariance function for FBF, based on works [9,17], is expressed as:

$$k(x, x') = \frac{1}{2} \left(\|x\|^{2H} + \|x'\|^{2H} - \|x - x'\|^{2H} \right), \quad (1)$$

where $x, x' \in \mathbb{R}^p$, and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^p$, $H \in [0, 1)$.

This work introduces a fractional Brownian vector field (FBVF) for modeling multidimensional and multi-output data. Notably, most existing studies focus on stochastic scalar fields defined as mappings $\mathbb{R}^p \rightarrow \mathbb{R}$. In contrast, the present study addresses stochastic vector fields of the form $\mathbb{R}^p \rightarrow \mathbb{R}^m$, which motivates the designation FBVF, adopted following [10,18]. It is also worth noting that a related model was previously investigated in [10] for the case $p \geq k$. The mathematical theory is also developed, supported by rigorous proofs of the theorems.

FBVF can be defined through two primary approaches: constructing positive covariance functions or implementing the Kolmogorov model [19]. The application of the Kolmogorov model to random fields was suggested in [9,16,17]. The Kolmogorov model, in particular, involves computing fractal distances between measurement points and a reference point, conventionally set at the origin of the coordinate system. This aspect becomes very inconvenient in modeling. Our work builds on the framework introduced in [20,21], which constructs random fields using Euclidean fractional distance matrices. This approach removes the necessity of explicitly specifying positive-definite correlation functions—a fundamental component in traditional formulations (e.g., [19])—and eliminates the requirement for defining an origin of coordinates.

We develop a method and algorithm for generating FBVF realizations using a recursive construction based on kriging model. We also propose two techniques for estimating the Hurst parameter: a maximum likelihood approach and a variogram-based method. Both generation and estimation procedures are designed to handle multivariate and multi-output fractal data. Their performance is assessed through a Monte Carlo simulation study, which shows that the maximum likelihood estimator consistently achieves higher accuracy. Consequently, we focus on this estimator for real-world applications. The methodology is applied to a dataset of heavy metal concentrations from the Meuse River and to climate data. The results demonstrate the proposed approach's effectiveness in revealing the multifractal characteristics of the data.

The paper is organized as follows. Section 2 introduces the notations and preliminary results, including a proof of the compatibility of the proposed model with the Kolmogorov Existence Theorem. Section 3 presents the derivation of the kriging model based on FBVF. Section 4 describes numerical experiments involving the simulation of FBVF. Section 5 illustrates the applications of the developed models to real datasets. Finally, Section 6 provides concluding remarks and gives directions for future research.

2. Notation and Preliminary Results

A Gaussian field is defined on a probability space $(\Omega, \mathcal{E}, P)$, where Ω denotes the set of possible outcomes (realizations), $\mathcal{E}$ is a σ -algebra on Ω , and P is a Gaussian measure. Within this framework, a Gaussian vector field can be represented as a mapping $z : \Omega \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ such that for any finite collection of input points, the corresponding outputs follow a joint Gaussian distribution, in accordance with the Kolmogorov Existence Theorem [19], and are fully determined by their mean vector and covariance matrix.

A fractional Brownian vector field (FBVF) extends the concept of Gaussian fields by incorporating fractional-order smoothness. Let $X = (x_1, x_2, \dots, x_k)$, with $x_i \in \chi \subseteq \mathbb{R}^p$, denote a sequence of distinct input points, and let $Z = (z_1, z_2, \dots, z_k)$, with $z_i \in \zeta \subseteq \mathbb{R}^m$, represent their corresponding outputs. The FBVF satisfies the following consistency conditions:

- The joint distribution of Z is compatible with a family of Gaussian distributions $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_k)$.
- The joint distribution Φ is invariant under permutations of the observations.

To fully specify the multivariate Gaussian distribution Φ , it is necessary to define:

1. The mean vector $\mu \in \mathbb{R}^m$.
2. The covariance matrix $\Sigma \in \mathbb{R}^{m \times m}$, which encodes the dependencies among the output variables.

Assume that, at any point $x \in \chi$ and for any scenario $\omega \in \Omega$, the random field $z(x, \omega)$ follows a normal distribution with mean vector μ and positive definite covariance matrix Σ . For simplicity, we assume that μ and Σ are constant across the domain χ . This assumption can be expressed as

$$z(x, \omega) \sim \mathcal{N}(\mu, \Sigma), \quad (2)$$

where

$$\mu = \mathbb{E}[z(x, \omega)],$$

and

$$\Sigma = \text{cov}[z(x, \omega)] = \mathbb{E}[(z(x, \omega) - \mu)(z(x, \omega) - \mu)^\top]. \quad (3)$$

2.1. Fractional Euclidean Distances Matrix Geometry

In this work, a key component in modeling FBVFs for multivariate, multi-output data is the matrix of fractional Euclidean distances between observation points, defined as

$$A = [(x_i - x_j)^\top (x_i - x_j)^H]_{i,j=1}^k, \quad (4)$$

where $x_i, x_j \in \chi \subseteq \mathbb{R}^p$, $x_i \neq x_j$ for all $1 \leq i, j \leq k$. The parameter $H \in [0, 1)$ characterizes the degree of self-similarity and smoothness of the FBVFs. This construction is motivated by classical results from Euclidean geometry and its extensions to fractional spaces [20,22].

The covariance structure of FBVFs, following the approaches in [20,23], is defined via a kernel matrix that generalizes Kolmogorov's model in Equation (1). This kernel matrix is expressed as

$$F = (1 - s^\top \mathbf{1}) \mathbf{1} a^\top (I - s \mathbf{1}^\top) + (1 - s^\top \mathbf{1}) (I - \mathbf{1} s^\top) a \mathbf{1}^\top - (I - \mathbf{1} s^\top) A (I - s \mathbf{1}^\top), \quad (5)$$

where

- I is the identity matrix of size $k \times k$.
- $\mathbf{1}$ is a k -dimensional vector of ones.
- $s \in \mathbb{R}^k$ is a vector of weights.

- $c = ((x_1 - x_0)^\top (x_1 - x_0)^H, (x_2 - x_0)^\top (x_2 - x_0)^H, \dots, (x_k - x_0)^\top (x_k - x_0)^H)$ is the vector of fractional Euclidean distances from each observation point to a reference point $x_0 \in \mathbb{R}^p$, with $x_0 \neq x_i$ for all $1 \leq i \leq k$.

Example 1. The FBVF with covariance function Equation (1) can be expressed with covariance kernel matrix Equation (5) if $s = 0$:

$$F = \mathbf{1}c^\top + c\mathbf{1}^\top - A. \quad (6)$$

As noted in [10], the model defined by Equation (5) is valid for $p \geq k$.

Assuming that all points are mutually distinct, the following propositions summarize the principal properties of the kernel matrix F and the fractional distance matrix A .

Proposition 1. The matrix F is positive definite if $s^\top \mathbf{1} \neq 1$ for the proof; see [Theorem 1] [21]. In [Theorem 1] [21], the definition of the kernel matrix differs.

Proposition 2. The determinant of Equation (4) satisfies $|A| \neq 0$ [20,21].

Remark 1. Propositions (1) and (2) imply that the matrix A is invertible; consequently, $\mathbf{1}^\top A^{-1} \mathbf{1} \neq 0$. Note $\mathbf{1}^\top A^{-1} \mathbf{1} > 0$ [22].

Proposition 3. If measurement points are mutually distinct, $s^\top \mathbf{1} \neq 1$, then the inverse of the kernel matrix F is given:

$$F^{-1} = \frac{A^{-1} \mathbf{1} \mathbf{1}^\top A^{-1}}{\mathbf{1}^\top A^{-1} \mathbf{1}} - A^{-1} + \frac{Q Q^\top}{D}, \quad (7)$$

where

$$D = (1 - s^\top \mathbf{1})^2 \left(c^\top A^{-1} c - \frac{(1 - \mathbf{1}^\top A^{-1} c)^2}{\mathbf{1}^\top A^{-1} \mathbf{1}} \right),$$

$$Q = s + (1 - s^\top \mathbf{1}) A^{-1} \left(c + \mathbf{1} \frac{1 - \mathbf{1}^\top A^{-1} c}{\mathbf{1}^\top A^{-1} \mathbf{1}} \right). \quad (8)$$

The determinant of F can be expressed in relation to the determinant of A as follows:

$$|F| = \text{abs}(|A|) D \mathbf{1}^\top A^{-1} \mathbf{1}. \quad (9)$$

For the proof, see [Theorem 1] [20].

Note,

$$\mathbf{1}^\top Q = 1. \quad (10)$$

Corollary 1. After straightforward manipulations, (see [Corollary 5] [20]), Equation (7) yields:

$$F^{-1} - \frac{F^{-1} \mathbf{1} \mathbf{1}^\top F^{-1}}{\mathbf{1}^\top F^{-1} \mathbf{1}} = -A^{-1} + \frac{A^{-1} \mathbf{1} \mathbf{1}^\top A^{-1}}{\mathbf{1}^\top A^{-1} \mathbf{1}}.$$

2.2. Probabilistic Model of FBVF

The construction of the fractional distance matrix A and the kernel matrix F enables the development of flexible covariance structures that simultaneously incorporate fractal geometry and multivariate dependencies. These mathematical constructs are crucial for capturing the inherent complexity of multi-output systems while preserving computational feasibility. The interaction between the Hurst parameter H and the fractal distance matrix

A directly governs the smoothness and scale of the modeled fields, providing a robust framework for representing multivariate relationships.

The established propositions highlight two important implications:

- The kernel matrix F is positive definite, allowing it to serve as a covariance matrix for modeling dependencies.
- The invertibility of both F and A ensures that computations involving their inverses are feasible, which is critical for statistical inference and prediction.

Given these properties, the vector-valued observations $Z = (z_1, z_2, \dots, z_k)$, with $z_i = z(x_i, \omega)$ for $i = 1, 2, \dots, k$, at the multivariate points $X = (x_1, x_2, \dots, x_k)$, can be modeled as jointly following a multivariate Gaussian distribution:

$$\text{vec}(Z(X)) \sim \mathcal{N}(\text{vec}(\mu \mathbf{1}^\top), F \otimes \Sigma), \quad (11)$$

where

- $\mu \mathbf{1}^\top$ denotes the mean matrix replicated across all observation points;
- $F \otimes \Sigma$ is the covariance matrix, expressed as the Kronecker product of the kernel matrix F (representing column-wise covariances) and the row covariance matrix Σ (as defined previously).

Here, $\text{vec}(\cdot)$ denotes the vectorization operator, which stacks the columns of a matrix into a single column vector, and $\otimes$ denotes the Kronecker product. For convenience, the main properties of the vectorization and Kronecker product operators are provided in Appendix A.

Note that the mean vector μ is assumed to be constant across the domain, as the results can be readily extended to the case of a spatially varying mean vector.

The expected value of the vectorized observations is given by:

$$\mathbb{E}[\text{vec}(Z(X))] = \text{vec}(\mu \mathbf{1}^\top),$$

and the covariance matrix of the vectorized deviations from the mean is:

$$\mathbb{E}[\text{vec}(Z(X) - \mu \mathbf{1}^\top) \text{vec}(Z(X) - \mu \mathbf{1}^\top)^\top] = F \otimes \Sigma. \quad (12)$$

Here, the kernel matrix F , derived in Equation (5), characterizes the covariance structure among points in the input space, while the row covariance matrix Σ , defined in Equation (3), describes the dependencies among the components of the output space.

2.3. Consistency of FBVF Distribution Family

In this subsection we define a family of distribution characterizing a given FBVF and examine their consistency. According to Kolmogorov's Existence Theorem [19], the assumptions of compatibility and symmetry are required to hold. In the remainder of this subsection, we explicitly indicate the dimension k of the vectors and matrices, and subsequently omit it thereafter whenever no ambiguity arises.

Theorem 1. Assume that the matrix of observations Z at a set of mutually distinct points X is given, distributed with respect to Equation (11), where $\Sigma > 0$. Then, the probability density function of the matrix Z is given by:

$$\rho(Z|\mu, \Sigma) = \frac{e^{-\frac{1}{2} \text{tr}((Z - \mu \mathbf{1}^\top) \Sigma^{-1} (Z - \mu \mathbf{1}^\top)^\top F^{-1})}}{(2\pi)^{\frac{mk}{2}} |\Sigma|^{\frac{k}{2}} |F|^{\frac{m}{2}}} \quad (13)$$

$$= \frac{e^{-\frac{1}{2} \text{tr} \left(Z \Sigma^{-1} Z^{\top} \left(\frac{A^{-1} \mathbf{1} \mathbf{1}^{\top} A^{-1}}{\mathbf{1}^{\top} A^{-1} \mathbf{1}} - A^{-1} \right) \right) - \frac{(ZQ - \mu)^{\top} \Sigma^{-1} (ZQ - \mu)}{2D}}}{(2\pi)^{\frac{mk}{2}} |\Sigma|^{\frac{k}{2}} \text{abs}(|A|)^{\frac{m}{2}} (\mathbf{1}^{\top} A^{-1} \mathbf{1} D)^{\frac{m}{2}}}. \quad (14)$$

Proof. Based on the model defined in Equation (11), the corresponding density function Equation (13) is derived from

$$\rho(Z|\mu, \Sigma) = \frac{e^{-\frac{1}{2} \text{vec}(Z - \mu \mathbf{1}^{\top})^{\top} (\Sigma \otimes F)^{-1} \text{vec}(Z - \mu \mathbf{1}^{\top})}}{2\pi^{\frac{mk}{2}} |\Sigma \otimes F|^{\frac{1}{2}}} = \frac{e^{-\frac{1}{2} \text{vec}(Z - \mu \mathbf{1}^{\top})^{\top} \text{vec}(F^{-1} (Z - \mu \mathbf{1}^{\top})^{\top} \Sigma^{-1})}}{2\pi^{\frac{mk}{2}} |\Sigma|^{\frac{k}{2}} |F|^{\frac{m}{2}}}.$$

using Formulas (A1)–(A7) presented in Appendix A.

Equation (14) is consequence of Equation (13) and Proposition (3) Equations (7), (9) and (10). $\square$

The compatibility condition of Kolmogorov Existence Theorem for Gaussian fields requires that the density of the sample Z_k from k observations and added new measurement $Z_{k+1} = (Z_k, z_{k+1})$, where z_{k+1} is new observation, is equal to product of density of this sample and conditional density of new observation.

Define

$$B_k = \frac{A_k^{-1} \mathbf{1}_k \mathbf{1}_k^{\top} A_k^{-1}}{\mathbf{1}_k^{\top} A_k^{-1} \mathbf{1}_k} - A_k^{-1}.$$

Lemma 1. Consider the following block representation:

$$B_{k+1} = \begin{bmatrix} P_{k+1} & b_{k+1} \\ b_{k+1}^{\top} & g_{k+1} \end{bmatrix}, \quad (15)$$

where

$$g_{k+1} = \frac{1}{(a_{k+1}^{\top} A_k^{-1} a_{k+1} - \frac{(1 - \mathbf{1}_k^{\top} A_k^{-1} a_{k+1})^2}{\mathbf{1}_k^{\top} A_k^{-1} \mathbf{1}_k})}. \quad (16)$$

$$b_{k+1} = -g_{k+1} A_k^{-1} \left(a_k + \mathbf{1}_k \frac{1 - \mathbf{1}_k^{\top} A_k^{-1} a_k}{\mathbf{1}_k^{\top} A_k^{-1} \mathbf{1}_k} \right) \quad (17)$$

$$P_{k+1} = B_k + \frac{b_{k+1} b_{k+1}^{\top}}{b_{k+1}} \quad (18)$$

Here, a_{k+1} denotes the vector of fractional distances between the points in X_k and the point x_{k+1} .

Proof. The proof is provided in Appendix B. $\square$

The next theorem considers the distribution properties of increments of the output.

Theorem 2. The family of densities

$$\begin{aligned} \rho(Z_{k+1} | \Sigma) &= \frac{e^{-\frac{1}{2} \text{tr} \left(Z_{k+1} \Sigma^{-1} Z_{k+1}^{\top} \left(\frac{A_{k+1}^{-1} \mathbf{1}_{k+1} \mathbf{1}_{k+1}^{\top} A_{k+1}^{-1}}{\mathbf{1}_{k+1}^{\top} A_{k+1}^{-1} \mathbf{1}_{k+1}} - A_{k+1}^{-1} \right) \right)}}{(2\pi)^{\frac{mk}{2}} |\Sigma|^{\frac{k+1}{2}} \left(\text{abs}(|A_{k+1}|) \mathbf{1}_{k+1}^{\top} A_{k+1}^{-1} \mathbf{1}_{k+1} \right)^{\frac{m}{2}}} \\ &= \rho(Z_k | \Sigma) \frac{e^{-\frac{1}{2} (z_{k+1}^{\top} - A_k^{-1} \left(a_k + \mathbf{1}_k \frac{1 - \mathbf{1}_k^{\top} A_k^{-1} a_k}{\mathbf{1}_k^{\top} A_k^{-1} \mathbf{1}_k} \right) Z_i) \Sigma^{-1} (z_{k+1} - Z_k^{\top} A_k^{-1} \left(a_k + \mathbf{1}_k \frac{1 - \mathbf{1}_k^{\top} A_k^{-1} a_k}{\mathbf{1}_k^{\top} A_k^{-1} \mathbf{1}_k} \right))}}{(2\pi)^{\frac{m}{2}} |\Sigma|^{\frac{1}{2}} (a_{k+1}^{\top} A_k^{-1} a_{k+1} - \frac{(1 - \mathbf{1}_k^{\top} A_k^{-1} a_{k+1})^2}{\mathbf{1}_k^{\top} A_k^{-1} \mathbf{1}_k})} \end{aligned} \quad (19)$$

is consistent with the Kolmogorov Existence Theorem.

Proof. By using Formula (A14) as well as Formulas (A9) and (A10) from Appendix A, one can verify that

$$\begin{aligned} & \text{tr}(Z_{k+1}\Sigma^{-1}Z_{k+1}^\top B_{k+1}) \\ &= \text{tr}\left(Z_k\Sigma^{-1}Z_k^\top\left(P_{k+1}-\frac{b_{k+1}b_{k+1}^\top}{g_{k+1}}\right)\right) + \text{tr}\left((z_{k+1}+Z_k^\top\frac{b_{k+1}}{g_{k+1}})(z_{k+1}^\top+\frac{b_{k+1}^\top}{g_{k+1}}Z_k)\Sigma^{-1}g_{k+1}\right) \\ &= \text{tr}(Z_k\Sigma^{-1}Z_k^\top B_k) + (z_{k+1}^\top+\frac{b_{k+1}^\top}{g_{k+1}}Z_k)\Sigma^{-1}(z_{k+1}+Z_k^\top\frac{b_{k+1}}{g_{k+1}})g_{k+1} \\ &= \sum_{i=1}^k(z_{i+1}^\top+\frac{b_{i+1}^\top}{g_{i+1}}Z_i)\Sigma^{-1}(z_{i+1}+Z_i^\top\frac{b_{i+1}}{g_{i+1}})g_{i+1}. \end{aligned} \quad (20)$$

Then Equation (19) follows from Equation (20) and Lemma (1). $\square$

Notably, the family defined in Equation (19) of Theorem (2) represents a multi-output, multivariate generalization of the classical Wiener process, as it is formulated in terms of increments between observations and the distances between the corresponding observation points.

Example 2. If $p = 1$, Equation (19) represents the multi-output Kolmogorov process [7]. When $H = 0.5$, Equation (19) reduces to the multi-output Wiener process. In this case, $\rho(Z) = \sum_{i=1}^k \frac{(z_{i+1}-z_i)^\top \Sigma^{-1} (z_{i+1}-z_i)}{x_{i+1}-x_i}$, where $x_1 < x_2 < \dots < x_{k+1}$, where observation points are ordered.

The following theorem establishes the consistency of the multivariate Kolmogorov model given in Equation (6).

Theorem 3. Assume $s_k = 0, c_k = ((x_1 - x_0)^\top (x_1 - x_0)^H, (x_2 - x_0)^\top (x_2 - x_0)^H, \dots, (x_k - x_0)^\top (x_k - x_0)^H), x_0, x_k \in \mathbb{R}^p, x_0 \neq x_k, k = 2, 3, \dots$. Then the corresponding family of densities Equation (6) is consistent and represents the certain FBVF.

Proof. Let us consider the sequences (x_0, X) and (μ, Z) , and correspondingly extend the matrix A by adding the fractal distance vector c to its top and left. Then, the considered family of distributions can be consistently expressed using Theorem (2). $\square$

Remark 2. Note, the classical Kolmogorov-Wiener process is the case of FBVF if $p = 1, m = 1$.

The next theorem provides general consistency conditions of FBVF distribution family.

Theorem 4. Considering two sequences $s_k \in \mathbb{R}^k, c_k \in \mathbb{R}_+^k, k = 2, 3, \dots$, the family of densities:

$$\rho(Z_k | \mu, \Sigma) = \frac{e^{-\frac{1}{2} \text{tr}(Z_k \Sigma_k^{-1} Z_k^\top B_k) - \frac{(Z_k Q_k - \mu)^\top \Sigma_k^{-1} (Z_k Q_k - \mu)}{2D_k}}}{(2\pi)^{\frac{mk}{2}} |\Sigma_k|^{\frac{k}{2}} \text{abs}(|A_k|)^{\frac{m}{2}} \left(\mathbf{1}_k^\top A_k^{-1} \mathbf{1}_k D_k\right)^{\frac{m}{2}}}$$

where $\Sigma > 0, s_k^\top \mathbf{1}_k \neq 1$,

$$D_k = (1 - s_k^\top \mathbf{1}_k)^2 \left(c_k^\top A_k^{-1} c_k - \frac{(1 - \mathbf{1}_k^\top A_k^{-1} c_k)^2}{\mathbf{1}_k^\top A_k^{-1} \mathbf{1}_k} \right), D_k > 0,$$

$$Q_k = s_k + (1 - s_k^\top \mathbf{1}_k) A_k^{-1} \left(c_k + \mathbf{1}_k \frac{1 - \mathbf{1}_k^\top A_k^{-1} c_k}{\mathbf{1}_k^\top A_k^{-1} \mathbf{1}_k} \right), \text{ is consistent, iff}$$

$$Q_k = q_{k+1} - \frac{b_{k+1}(Q_{k+1})_{k+1}}{g_{k+1}}. \quad (21)$$

$$D_k = D_{k+1} + \frac{(Q_{k+1})_{k+1}^2}{g_{k+1}} \quad (22)$$

Proof. Proof is given in Appendix B. $\square$

Notably, the sequence c_k in Theorem (4) has a geometric interpretation, representing distances from a specific reference point. Further investigation is required to determine an appropriate choice of the distance sequences c_k and their corresponding weights s_k .

Corollary 2. It follows from Equation (5) that

$$\mathbb{E}(z_1 - z_2)(z_1 - z_2)^\top = 2\Sigma \|x_1 - x_2\|^{2H}. \quad (23)$$

Proof. Assume that $k = 2$. It can be readily verified that $F_2 = \begin{pmatrix} \|x_1 - x_2\|^{2H} & 0 & 0 \\ 0 & \|x_1 - x_2\|^{2H} & 0 \\ 0 & 0 & \|x_1 - x_2\|^{2H} \end{pmatrix}$. Substituting this expression into Equation (17) consequently yields Equation (23). $\square$

2.4. Likelihood Estimation of Hyperparameters

The maximum likelihood estimates (MLE) of the FBVF hyperparameters are obtained by minimizing the negative log-likelihood of the observed data. According to Theorem (1), the negative twice log-likelihood function is expressed as:

$$\begin{aligned} \mathcal{L}_F(\mu, \Sigma, H \mid Z) &= -2 \ln(\rho(Z \mid \mu, \Sigma, H)) = k \ln(|\Sigma|) + m \ln(|F|) \\ &+ \text{tr}\left((Z - \mu \mathbf{1}^\top) \Sigma^{-1} (Z - \mu \mathbf{1}^\top)^\top F^{-1}\right) + k m \ln(2\pi). \end{aligned} \quad (24)$$

The likelihood function, constructed from the observed data, depends on the covariance matrix Σ , the mean vector μ , and the Hurst parameter H .

Theorem 5. The MLEs of μ and Σ can be obtained by minimizing the likelihood function in Equation (24):

$$\hat{\mu} = \frac{ZF^{-1}\mathbf{1}}{\mathbf{1}^\top F^{-1}\mathbf{1}} \quad (25)$$

$$\hat{\Sigma} = \frac{1}{k} \left(Z - \hat{\mu} \mathbf{1}^\top \right) F^{-1} \left(Z - \hat{\mu} \mathbf{1}^\top \right)^\top. \quad (26)$$

Proof. The estimates $\hat{\mu}$ and $\hat{\Sigma}$ are obtained by taking the partial derivatives of the negative log-likelihood function.

Using Equation (A12) from Appendix A, the derivative of the log-likelihood function with respect to μ is given by

$$\frac{\partial \mathcal{L}_F}{\partial \mu}(\mu, \Sigma, H \mid Z) = -2, \Sigma^{-1} \left(Z - \mu \mathbf{1}^\top \right) F^{-1} \mathbf{1}.$$

Similarly, Equations (A13) and (A14) from Appendix A imply the derivatives of the log-likelihood function with respect to Σ

$$\frac{\partial \mathcal{L}_F}{\partial \Sigma}(\mu, \Sigma, H \mid Z) = k \Sigma^{-1} - \Sigma^{-1} \left(Z - \mu \mathbf{1}^\top \right) F^{-1} \left(Z - \mu \mathbf{1}^\top \right)^\top \Sigma^{-1}.$$

Setting these derivatives to zero and solving the resulting equations yields Equations (25) and (26). $\square$

Corollary 3. Using Equations (25) and (8), the MLE of the mean can be expressed as:

$$\hat{\mu} = ZQ.$$

Corollary 4. The MLE, $\hat{\Sigma}$, can be obtained by substituting the mean estimate $\hat{\mu}$ from Equation (25) into the expression in Equation (26):

$$\begin{aligned}\hat{\Sigma} &= \frac{1}{k} \left(Z - \frac{ZF^{-1}\mathbf{1}}{\mathbf{1}^\top F^{-1}\mathbf{1}} \mathbf{1}^\top \right) F^{-1} \left(Z - \frac{ZF^{-1}\mathbf{1}}{\mathbf{1}^\top F^{-1}\mathbf{1}} \mathbf{1}^\top \right)^\top \\ &= \frac{1}{k} \left(ZF^{-1}Z^\top - \frac{ZF^{-1}\mathbf{1}\mathbf{1}^\top F^{-1}Z^\top}{\mathbf{1}^\top F^{-1}\mathbf{1}} \right).\end{aligned}\quad (27)$$

Corollary 5. Using Corollary (1), the MLE of Σ can be expressed in terms of the matrix A .

$$\hat{\Sigma} = \frac{1}{k} \left(\frac{ZA^{-1}\mathbf{1}\mathbf{1}^\top A^{-1}Z^\top}{\mathbf{1}^\top A^{-1}\mathbf{1}} - ZA^{-1}Z^\top \right). \quad (28)$$

Note that the latter estimate does not depend on the auxiliary hyperparameters s and a .

2.5. Estimation of Hurst Parameter

Substituting Equations (25) and (27) into Equation (24), dividing by mk for normalization and omitting the constant term, the corollary follows.

Corollary 6. The MLE $\hat{H}$ is obtained as the solution to the following optimization problem:

$$\hat{H} = \underset{0 < \hat{H} < 1}{\operatorname{argmin}} \mathcal{L}_F(H | Z) \quad (29)$$

where

$$\begin{aligned}\mathcal{L}_F(H | Z) &= \frac{1}{m} \ln(|\hat{\Sigma}|) + \frac{1}{k} \ln(|F|) \\ &= \frac{1}{m} \ln\left(\left|\frac{1}{k} \left(ZF^{-1}Z^\top - \frac{ZF^{-1}\mathbf{1}\mathbf{1}^\top F^{-1}Z^\top}{\mathbf{1}^\top F^{-1}\mathbf{1}} \right)\right|\right) + \frac{1}{k} \ln(|F|).\end{aligned}\quad (30)$$

Corollary 7. For the distribution family defined in Theorem (2), the estimate is derived as follows:

$$\hat{H} = \underset{0 < \hat{H} < 1}{\operatorname{argmin}} \mathcal{L}_A(H | Z) \quad (31)$$

where

$$\begin{aligned}\mathcal{L}_A(H | Z) &= \frac{1}{m} \ln(|\hat{\Sigma}|) + \frac{1}{k} \ln(\operatorname{abs}(|A|)\mathbf{1}^\top A^{-1}\mathbf{1}) = \\ &= \frac{1}{m} \ln\left(\left|\frac{1}{k} \left(\frac{ZA^{-1}\mathbf{1}\mathbf{1}^\top A^{-1}Z^\top}{\mathbf{1}^\top A^{-1}\mathbf{1}} - ZA^{-1}Z^\top \right)\right|\right) + \frac{1}{k} \ln(\operatorname{abs}(|A|)\mathbf{1}^\top A^{-1}\mathbf{1}).\end{aligned}\quad (32)$$

Thus, the optimal $\hat{H}$ is obtained by minimizing the normalized negative log-likelihood function Equation (32).

2.6. Variogram Estimation

The parametric power variogram is governed by Corollary (1).

The variogram function for FBVF is expressed as:

$$V(H) = \underset{\Sigma}{\operatorname{argmin}} \left[\operatorname{tr} \left(\sum_{i=1}^{k-1} \sum_{j=i}^k ((z_i - z_j)(z_i - z_j)^\top) - 2\Sigma((x_i - x_j)^\top(x_i - x_j))^H \right) \right. \\ \left. ((z_i - z_j)(z_i - z_j)^\top) - 2\Sigma((x_i - x_j)^\top(x_i - x_j))^H \right)^\top \right] \quad (33)$$

Theorem 6. Let $X = (x_1, x_2, \dots, x_k)$, with $x_i \in \mathbb{R}^p$, $p \geq 1$, for $1 \leq i \leq k$ denote the sampling locations, and $Z = (z_1, z_2, \dots, z_k)$, $z_i \in \zeta \subseteq \mathbb{R}^m$, $m \geq 1$ be a random vector set. The variogram estimate of $\hat{H}$ is then obtained as follows.

$$\hat{H} = \underset{\hat{H}}{\operatorname{argmax}} \left[W(H) \right]. \quad (34)$$

$$\text{where } W(H) = \operatorname{tr} \left(\frac{2(\sum_{i=1}^{k-1} \sum_{j=i}^k (z_i - z_j)^\top (z_i - z_j) ((x_i - x_j)^\top (x_i - x_j))^H)^2}{k(k-1) \sum_{i=1}^{k-1} \sum_{j=i}^k ((x_i - x_j)^\top (x_i - x_j))^{2H}} \right)$$

Proof. By differentiating Equation (33) and setting the derivative to zero, the variogram estimate is obtained as:

$$\hat{\Sigma} = \frac{\sum_{i=1}^{k-1} \sum_{j=i}^k (z_i - z_j)^\top (z_i - z_j) ((x_i - x_j)^\top (x_i - x_j))^H}{\sum_{i=1}^{k-1} \sum_{j=i}^k ((x_i - x_j)^\top (x_i - x_j))^{2H}}.$$

Subsequently, the estimate $\hat{\Sigma}$ is substituted into the variogram function in Equation (33), obtaining $V(H) = C - W(H)$ where $C = \frac{2 \sum_{i=1}^{k-1} \sum_{j=i}^k ((z_i - z_j)^\top (z_i - z_j))^2}{k(k-1)}$ does not depend on H , which in turn is then minimized with respect to H . $\square$

The computational complexity of this method is $\mathcal{O}(k^2)$. Some practitioners select models visually; however, such selections are often unreliable and prone to fluctuations.

Once the Hurst parameter is estimated using Equations (29), (31) and (34), the fractal dimension can be computed as $D = n_E + 1 - H$, where n_E denotes the Euclidean measure [24].

3. Kriging

To model multidimensional spatial data, the kriging model is constructed by combining the covariance matrix of the FBVF and fractional distances matrix.

Theorem 7. Suppose the set of FBVF values $Z = (z_1, z_2, \dots, z_k)$ is observed at mutually distinct points $X = (x_1, x_2, \dots, x_k)$, where $z_i \in \mathbb{R}^m$, $x_i \in \mathbb{R}^p$, and $1 \leq i \leq k$. Furthermore, assume that an additional value $z \in \mathbb{R}^m$ is observed at the point $x \in \mathbb{R}^p$.

The kernel matrix F' corresponding to the augmented vector (Z, z) can then be represented as a block matrix, where F denotes the kernel matrix associated with Z .

$$F' = \begin{bmatrix} F & f \\ f^\top & v \end{bmatrix},$$

F' , F are consistent. The kriging predictor and the associated covariances at the new observation point are given by:

$$y(x) = \mu + (Z - \mathbf{1}\mu^\top)F^{-1}f, \quad (35)$$

$$\Sigma(x) = \frac{\Sigma}{k}(v - f^{\top} F^{-1} f). \quad (36)$$

Proof. The proof follows from the formulas for the conditional mean and conditional covariance of the distribution given in Equation (17). $\square$

One can use likelihood-based estimates of μ and Σ to obtain kriging predictors [25].

Corollary 8. *The kriging predictor and its associated covariances*

$$y(x) = Z^{\top} F^{-1} (f + \mathbf{1} \frac{1 - \mathbf{1}^{\top} F^{-1} f}{\mathbf{1}^{\top} F^{-1} \mathbf{1}}), \quad (37)$$

$$\Sigma(x) = \frac{\Sigma}{k} (v - f F^{-1} f + \frac{(1 - \mathbf{1}^{\top} F^{-1} f)^2}{\mathbf{1}^{\top} F^{-1} \mathbf{1}}), \quad (38)$$

Corollary 9. *The kriging predictor and its associated covariances can be expressed in terms of the distance matrix A as follows:*

$$y(x) = Z^{\top} A^{-1} (a + \mathbf{1} \frac{1 - \mathbf{1}^{\top} A^{-1} a}{\mathbf{1}^{\top} A^{-1} \mathbf{1}}), \quad (39)$$

$$\Sigma(x) = \frac{\Sigma}{k} (a A^{-1} a - \frac{(1 - \mathbf{1}^{\top} A^{-1} a)^2}{\mathbf{1}^{\top} A^{-1} \mathbf{1}}), \quad (40)$$

where a denotes the vector of fractal distances from the sample points X to the prediction point x . The result follows from Proposition (3) and Corollary (1).

An MLE or variogram-based estimate of Σ can be used in Equation (40).

Simulation Algorithms

In this section, we present a method for generating FBVF data using a recursive algorithm (see Appendix C) for the compatible family of FBVF distributions in Equation (19). While computationally efficient, its complexity is $\mathcal{O}(k^3)$ and depends on factors described below.

Recursive generation of FBVF values is convenient for practical applications. Accordingly, Corollary (1) can be used to simulate the first two observations at x_1 and x_2 , which define the baseline FBVF values, as follows:

$$z_1 = L\zeta_1 \quad (41)$$

$$z_2 = z_1 + \sqrt{2}L\zeta_2 \left((x_1 - x_2)^{\top} (x_1 - x_2) \right)^{\frac{H}{2}}. \quad (42)$$

The generalized expression of the recursive algorithm, based on Theorem (7) and the properties of the kernel matrix, is given for $i = 2, 3, \dots$:

$$z_{i+1} = y(x_i) + L\zeta_i \sqrt{\Sigma(x_i)}. \quad (43)$$

4. Numerical Analysis

In this section, we examine the MLE of the parameters of the generalized Kolmogorov model and compare them with the previously constructed generalized variogram.

Monte Carlo Simulation Study

We present the procedural steps for conducting a Monte Carlo experiment using a FBVF generation algorithm (see Appendix C), which is essential for comparing maximum likelihood and variogram methods.

- (i) Primarily, we selected values of the parameter H , specifically $H = 0.1, 0.2, \dots, 0.9$.
- (ii) Two-, five-, and ten-dimensional observations $p = 2$ and $m = 2, 5, 10$, with different sample sizes $k = 20, 50, 100$ were generated using a recursive algorithm.
- (iii) The MLE and variogram estimates of H are computed for each data set using the methods specified in Equations (31) and (34).
- (iv) To evaluate the impact of the number of variates, the Monte Carlo simulation was repeated 100 times for $m = 2, 5$, and 10.

Figures 1 and 2 present boxplots of the estimated Hurst parameters using two approaches: the variogram-based estimator and the MLE. Each panel shows the distribution of estimated values across different true parameter values H (ranging from 0.1 to 0.9).

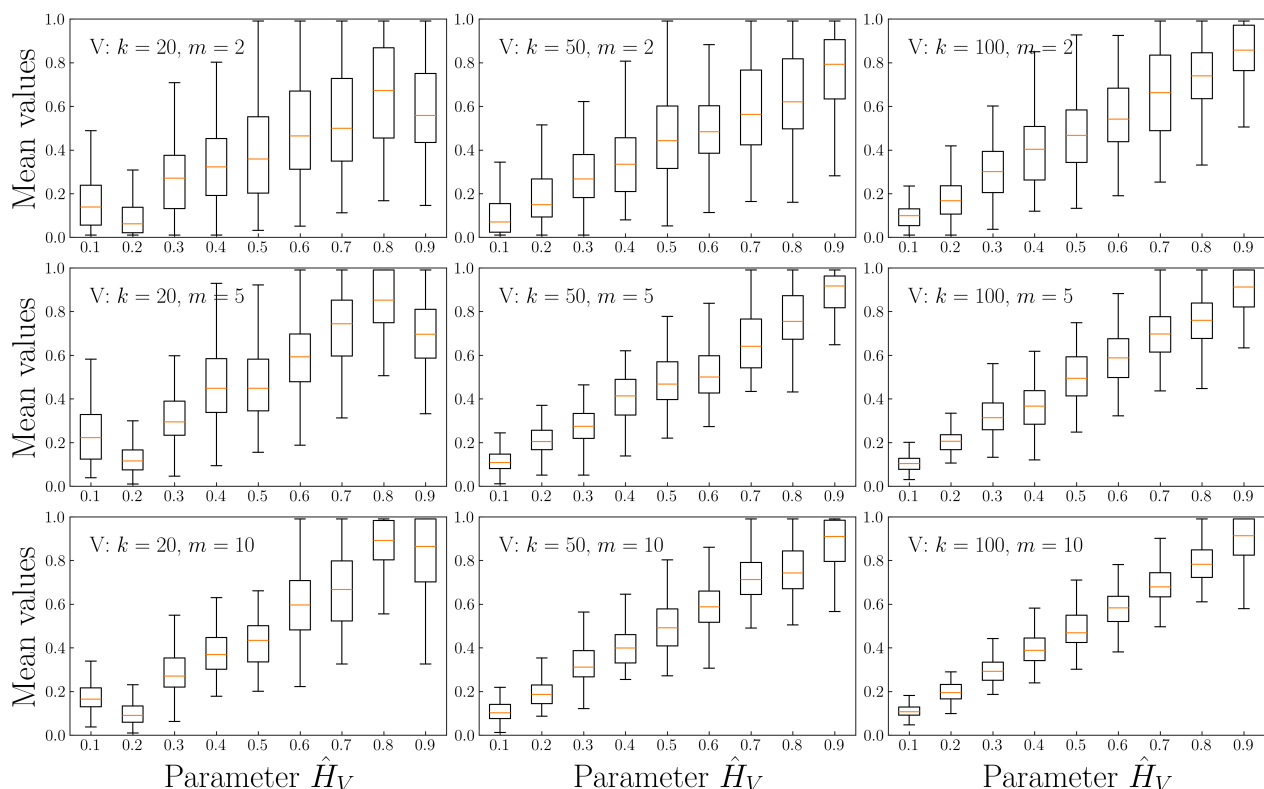


Figure 1. Variogram estimates of Hurst parameter.

For the variogram-based results, the boxplots are widely scattered, exhibiting high variability and frequent deviations from the true parameter values. The spread remains substantial even as the sample size ($k = 20, 50, 100$) increases, indicating limited estimation accuracy and stability.

In contrast, the MLE method demonstrates improved performance. The estimated Hurst parameters closely align with the true values, and their variability decreases systematically as both k (sample size) and m (dimension) increase. The narrower boxplots indicate higher precision and greater robustness of the MLE compared to the variogram-based approach.

The comparison demonstrates that the variogram-based approach performs poorly, producing unstable and widely scattered estimates, whereas the MLE method provides reliable and consistent estimates of the Hurst parameter. These results justify focusing subsequent modeling efforts exclusively on MLE for higher-dimensional data.

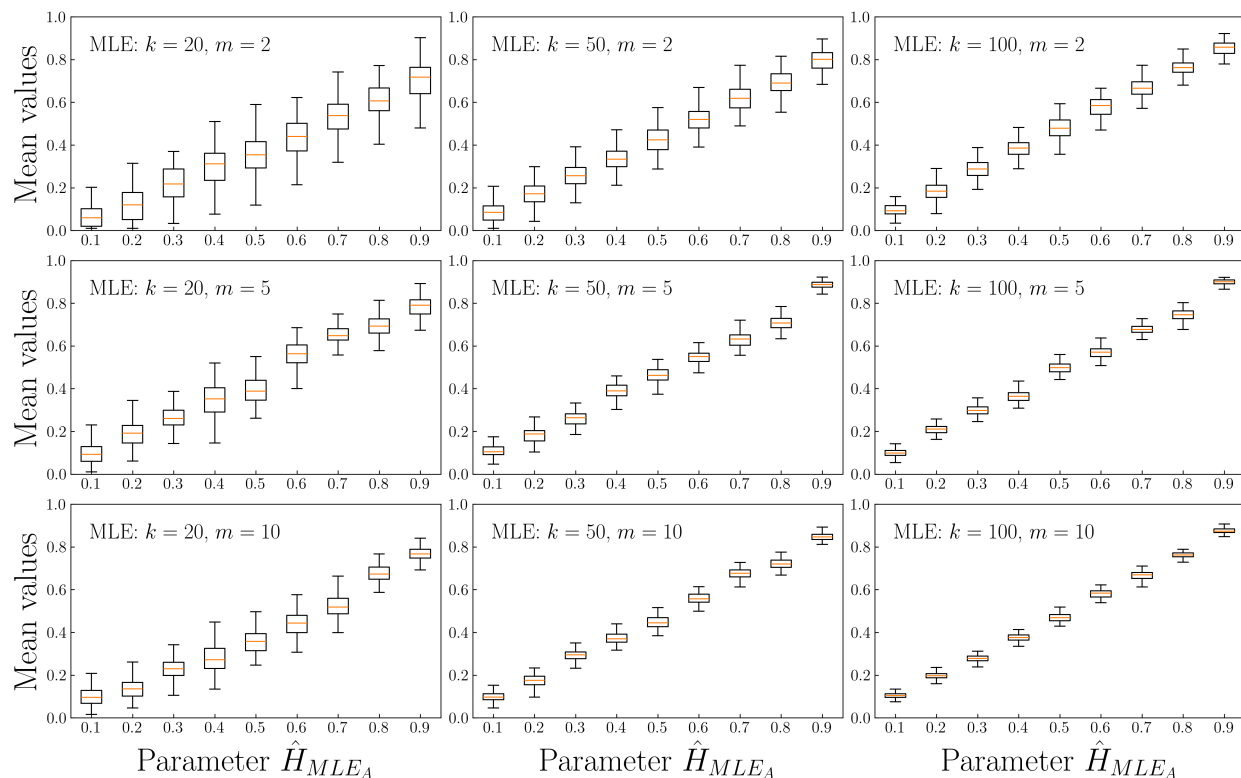


Figure 2. Maximum likelihood estimates of Hurst parameter.

Figure 3 shows examples of the likelihood and variogram function curves obtained for a predefined Hurst parameter of $H = 0.8$, with $k = 50$ and $m = 2$.

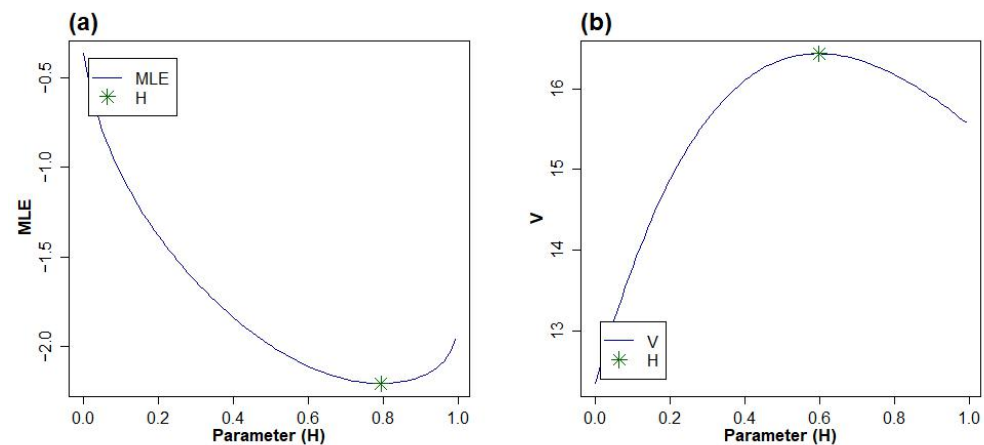


Figure 3. Estimates obtained via (a) maximum likelihood and (b) variogram.

Figure 3 shows that the variogram estimate $\hat{H}_V = 0.5970$ is less accurate than the likelihood-based estimate $\hat{H}_{MLE} = 0.7927$.

The computer simulation results indicate that the likelihood function exhibits a convex shape.

5. Real Data Applications for Multi-Fractal Analysis

Practical parameter estimation methods were applied to both the Meuse and climate datasets to ensure completeness.

5.1. Heavy Metal Pollution

We use the Meuse dataset [26,27], which contains $k = 155$ topsoil samples collected from the Meuse River floodplain near Stein, the Netherlands. The dataset reports concentrations (ppm) of $n = 4$ heavy metals: cadmium (Cd), copper (Cu), lead (Pb), and zinc (Zn).

This study evaluates the likelihood of different fractal structures (F.S.). Each structure partitions the metals into non-overlapping subsets. The likelihood of a given structure is computed as follows:

$$\mathcal{L}(Z) = \frac{1}{m} \sum_{j=1}^n \mathcal{L}_A(H_j | Z_j) m_j, \quad (44)$$

where n is the number of non-overlapping subset ($1 \leq n \leq 4$), m_j is the number of metals in the j -th subset ($\sum_{j=1}^n m_j = m$), and Z_j represents the data for that subset. The parameter $H_j = \arg \min_{0 < H < 1} \mathcal{L}_A(H | Z_j)$ is the estimated Hurst exponent for the j -th structure using Equation (32). In total, 15 structures are considered, each comprising subsets of n metals.

The likelihood function (LF) corresponding to different fractal structures (F.S.) of heavy metals is presented in Table 1, with distinct fractals within each structure separated by commas. The analysis was performed by estimating the Hurst parameter for all metals collectively, as well as for each metal individually.

Table 1. MLE analysis of fractal structures.

LF	F.S.	LF	F.S.	LF	F.S.
5.609	1234	6.22625	124,3	6.648724	14,23
5.864048	24,1,3	6.42325	123,4	6.791926	23,1,4
6.0295	134,2	6.4309	14,2,3	6.829765	13,2,4
6.102975	234,1	6.459813	13,2,4	7.00975	1,2,3,4
6.104712	12,34	6.6336	12,3,4	7.256612	34,1,2

The combination of metals into distinct fractal structures was evaluated using the likelihood function. The most probable structure, (1234), corresponds to the joint analysis of all metals rather than separate subsets. For this structure, the maximum likelihood estimates of the means μ_0 , standard deviations σ_0 , and correlations Q_0 are computed as follows:

From the correlation matrix in Table 2, the metals exhibit strong mutual correlations. This supports the conclusion that all metals belong to a single fractal structure with a low Hurst exponent $\hat{H} = 0.176$. The combination of high correlations and a low Hurst exponent suggests that the joint distribution of the four metals is driven by a common extreme process [28]. Overall, these results indicate that a multidimensional fractal representation is more plausible than modeling the metals individually.

Table 2. Summary statistics and correlation matrix for heavy metal concentrations.

	μ_0	σ_0	Q_0			
			Cd	Cu	Pb	Zn
Cd	5.149	5.267	1	0.880	0.717	0.875
Cu	54.251	33.897	0.880	1	0.756	0.871
Pb	203.858	159.906	0.717	0.756	1	0.935
Zn	660.048	504.876	0.875	0.871	0.935	1

5.2. Climate Data

Climate processes often exhibit FBVFs, reflecting complex spatiotemporal dependencies. In this study, we evaluate the applicability of the proposed model to multivariate climate data. Precipitation and temperature records for Lithuania (2017–2020) were obtained from the Global Climate Monitor System [29], developed by the Climate Research Group at the University of Seville.

The dataset was prepared to estimate the Hurst parameter (H), where locations are defined by geographic coordinates ($p = 2$) and observations consist of temperature and precipitation ($m = 2$). We first examined the fractality of the joint temperature–precipitation process (T–P). In this context, H denotes the Hurst parameter $\hat{H}$, obtained via joint estimation based on the log-likelihood function $\mathcal{L}$:

$$\mathcal{L}(H | Z) = \frac{1}{T} \sum_{t=1}^{66} \mathcal{L}(H_t | z_t), \quad (45)$$

where $Z = z_{t=1}^T$ represents the observed data for all $T = 66$ months, z_t denotes the observations for month t , and H_t is the corresponding monthly estimate.

Table 3 presents the values of $H_t = \arg \min_{0 < H < 1} \mathcal{L}(H | z_t)$ for each month from 2017 to 2020.

Table 3. Hurst index of temperature–precipitation (T–P) fractal.

Months	Years			
	2017	2018	2019	2020
01	0.845	0.812	0.836	0.832
02	0.818	0.824	0.801	0.818
03	0.782	0.729	0.642	0.706
04	0.672	0.723	0.709	0.747
05	0.759	0.697	0.650	0.743
06	0.721	0.768	0.726	0.658
07	0.703	0.761	0.651	0.711
08	0.634	0.665	0.708	0.718
09	0.607	0.666	0.589	0.571
10	0.669	0.647	0.637	0.639
11	0.734	0.760	0.707	0.747
12	0.728	0.831	0.787	0.779

The results indicate that winter months (January and February) consistently exhibit $\hat{H} > 0.8$, reflecting strong positive correlations between monthly precipitation and temperature. The weakest correlations occur in autumn (September and November), with $0.571 \leq \hat{H} \leq 0.734$. The minimum, $\hat{H} = 0.571$, was observed in September 2020, and the maximum, $\hat{H} = 0.845$, occurred in January 2017. In all cases, $\hat{H} > 0.5$, confirming persistent spatiotemporal dependencies.

Table 4 presents the aggregated statistical measures from the multifractal analysis of the climate data. The first row (LF) reports the mean values of the likelihood function, representing the overall average behavior of each parameter. The second row (SD) shows the standard deviations, reflecting the temporal variability of the estimates. The third row (RMSE) provides the root mean square errors, capturing both bias and variability in a single metric.

Notably, the joint temperature–precipitation model (T–P) exhibits relatively high LF values. In contrast, the independent estimation of the Hurst indices for temperature and precipitation, calculated as $H_t = \frac{1}{m} (\arg \min_{0 < H < 1} \mathcal{L}(H | z_{p,t}) + \arg \min_{0 < H < 1} \mathcal{L}(H | z_{\tau,t}))$, where $z_{p,t}$ and $z_{\tau,t}$ are the monthly precipitation and temperature data, shows substantially

lower LF values. The correlation between the monthly Hurst indices of temperature and precipitation is negligible, with $cor = -0.02 \pm 0.02$.

Table 4. Likelihood functions (LFs) are shown for the bivariate fractal (T–P) model, the multifractal (T and P) model with a constant Hurst exponent, and the seasonal trend.

	T–P	T and P	Const	Trend
LF	0.343	0.303	0.506	0.308
SD	0.381	0.357	0.254	0.265
RMSE	0.074	0.061	0.072	0.074

$$H_p(t) = 0.884 + 0.069 \sin\left(\frac{\pi t}{6} + \alpha\right) \quad (46)$$

$$H_\tau(t) = 0.537 + 0.044 \cos\left(\frac{\pi t}{6} - \alpha\right), \quad (47)$$

$\alpha = -0.036$, t -month serial number.

We analyzed seasonal variations in the Hurst index separately for temperature and precipitation. Using the maximum likelihood method, the seasonal trend parameters were estimated by minimizing the likelihood function $\frac{1}{m} \sum_{t=1}^{48} (\mathcal{L}(H_p(t)|z_{p,t}) + \mathcal{L}(H_\tau(t)|z_{p,t})) \rightarrow \min$, and the resulting model was compared with one that excluded the seasonal component. Table 4 summarizes the results: the “Const” column shows the likelihood function values for the model without a seasonal trend ($H_t = \text{Const}$), while the “Trend” column reports the values for the model including the seasonal trend. The estimated seasonal trend parameters are provided in Equations (46) and (47). Figure 4 illustrates the identified monthly trends and their estimates, highlighting the seasonal bias in the dynamics of the Hurst parameter in the climate data.

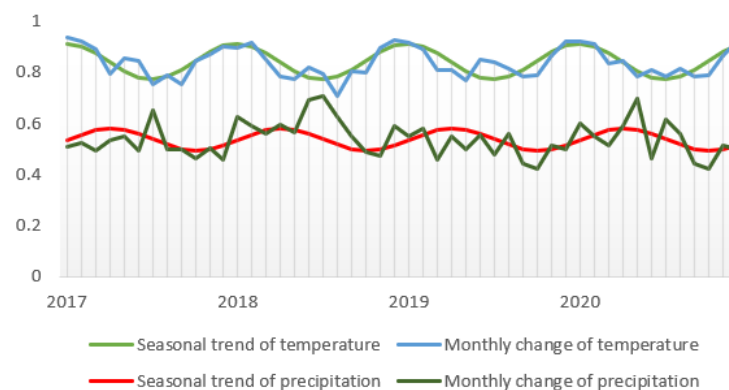


Figure 4. Seasonal variation in the Hurst index for the precipitation and temperature multifractal.

All computations presented in Sections 4 and 5 were performed using the R Studio software package (R 4.2.2) on a computer equipped with an Intel® Core™ i7-2600 CPU @ 3.40 GHz and 8 GB RAM, ensuring computational accuracy. Part of the implemented programs has been shared in the FrackrigingR package CRAN (<https://cran.r-project.org/web/packages/FracKrigingR/index.html>) (accessed on 8 November 2021).

6. Conclusions

In this study, an FBVF model is formulated and implemented based on a kernel matrix and expressed solely through the Euclidean matrix of fractal distances. The compatibility of the FBVF distribution family with the Kolmogorov extension theorem is established. It is also demonstrated that, for the analysis of fractal data, it is sufficient to employ the fractal distance matrix, without introducing a positive-definite correlation function, as is commonly done in other studies [19].

The developed model represents a multivariate and multi-output generalization of the one-dimensional Wiener process, enabling the modeling of FBVFs using only increments of the field values between different observations and the fractal Euclidean distances between observation points.

Maximum likelihood and variogram-based methods for parameter estimation are constructed. An efficient algorithm for estimating the Hurst parameter is developed by solving a one-dimensional optimization problem. FBVF kriging algorithms are derived. Efficient algorithms for generating FBVFs with a given Hurst parameter are also developed, utilizing kriging for this purpose.

A Monte Carlo experiment demonstrates that the proposed maximum likelihood estimator of the Hurst parameter provides more reliable estimates than the variogram-based method. The application of the developed methodology to the analysis of heavy metal distributions in soil data and to climate datasets confirms its suitability as a tool for investigating multifractal, multi-output, and multidimensional processes.

This study provides a foundation for future research. The proposed methodology can be extended to more complex fractal models, including those defined using the Minkowski metric and other distance metrics. Furthermore, the proposed framework has the potential to support the development of multifractal spectrum analysis and control methodologies for complex systems, thereby broadening its applicability across a range of scientific and engineering disciplines.

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Abbreviations

The following abbreviations are used in this manuscript:

FBVF	Fractional Brownian Vector Field
FBM	Fractional Brownian motion
FBF	Fractional Brownian fields
LF	Likelihood function

Appendix A. Additional Formulas

1. $\text{vec}(ACB) = (B^T \otimes A) \text{vec}(C)$ [30] [p. 401]
2. $|A \otimes B| = |A|^k |B|^m$, A is $m \times m$, B is $k \times k$ [30] [p. 402]
3. $\text{vec}(C)^T \text{vec}(ACB) = \text{tr}(C^T ACB)$
4. Trace cyclic property $\text{tr}(ABCD) = \text{tr}(BCDA) = \text{tr}(CDAB) = \text{tr}(DABC)$
5. $(A \otimes B)^T = A^T \otimes B^T$ [30] [p. 400]
6. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ [30] [p. 401]
7. $\text{tr}(A \otimes B) = \text{tr}(A) \text{tr}(B)$ [30] [p. 402]

8. Block matrix inverse:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}.$$

A, B, C, D – matrices, their sizes $n \times n, n \times m, m \times n$ or $m \times m$ [30] [p. 108].

9. $\operatorname{tr} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \operatorname{tr}(A) + \operatorname{tr}(D)$

10. Woodbury matrix identity: $(A + DCB)^{-1} = A^{-1} - A^{-1}D(C^{-1} + BA^{-1}D)^{-1}BA^{-1}$ [30] [p. 157, Remark after Fact 2.16.21].

11. Matrix Determinant Lemma: $\left| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right| = |A||D - CA^{-1}B|$, if A is invertible,
 $|A + BDC^T| = |D^{-1} + CA^{-1}B^T||D||A|$, if A, D are invertible [30] [p. 147, Fact 2.14.13].

12. $d \operatorname{tr}(XBX^TC) = \operatorname{tr}((dX)BX^TC) + \operatorname{tr}(XB(dX^T)C)$

13. $\frac{d(\operatorname{tr}(C^T A^{-1}CB^{-1}))}{dA} = -A^{-1}CB^{-1}CA^{-1}$ [30] [p. 692]

14. $\frac{d(\ln(|A|))}{dA} = A^{-1}, |A| > 0$ [30] [p. 692]

Sizes of vectors and matrices are compatible in applied operations.

Appendix B. Additional Proofs

Proof of Lemma (1).

Proof. According to the Matrix Determinant Lemma (see Appendix A), Equation (14) implies,

$$g_{k+1} = \frac{|A_k|}{|A_{k+1}|} \frac{\mathbf{1}_k^T A_k^{-1} \mathbf{1}_k}{\mathbf{1}_{k+1}^T A_{k+1}^{-1} \mathbf{1}_{k+1}}. \quad (\text{A1})$$

$$\frac{|A_{k+1}|}{|A_k|} = a_k^T A_k^{-1} a_k \quad (\text{A2})$$

Using the block matrix inversion formula (see Appendix A) one can split the vector:

$$A_{k+1}^{-1} \mathbf{1}_{k+1} = (r_{k+1}, (A_{k+1}^{-1} \mathbf{1}_{k+1})_{k+1}) \quad (\text{A3})$$

where $r_{k+1} = A_k^{-1} \left(\mathbf{1}_k + a_{k+1} \frac{1 - \mathbf{1}_k^T A_k^{-1} a_{k+1}}{a_{k+1}^T A_k^{-1} a_{k+1}} \right), (A_{k+1}^{-1} \mathbf{1}_{k+1})_{k+1} = -\frac{1 - \mathbf{1}_k^T A_k^{-1} a_{k+1}}{a_{k+1}^T A_k^{-1} a_{k+1}}$

Equations (A1)–(A3) imply Equation (16).

Note that Equations (17) and (A3), and

$$B^{k+1} = \frac{A_{k+1}^{-1} \mathbf{1}_{k+1} (\mathbf{1}_{k+1}^T A_{k+1}^{-1})^{k+1}}{\mathbf{1}_{k+1}^T A_{k+1}^{-1} \mathbf{1}_{k+1}} - (A_{k+1}^{-1})^{k+1} = \frac{(A_{k+1}^{-1})^{k+1}}{\mathbf{1}_{k+1}^T A_{k+1}^{-1} \mathbf{1}_{k+1}} (A_{k+1}^{-1} \mathbf{1}_{k+1} \mathbf{1}_{k+1}^T - \mathbf{1}_{k+1}^T A_{k+1}^{-1} \mathbf{1}_{k+1} I_{k+1}) \quad (\text{A4})$$

imply the simplification

$$b_{k+1} = \frac{(r_{k+1} \mathbf{1}_k^\top - \mathbf{1}_{k+1}^\top A_{k+1}^{-1} \mathbf{1}_{k+1} I_k) A_k^{-1} a_{k+1} - r_{k+1}}{\mathbf{1}_{k+1}^\top A_{k+1}^{-1} \mathbf{1}_{k+1} (a_{k+1}^\top A_k^{-1} a_{k+1})} = \quad (\text{A5})$$

$$= \frac{A_k^{-1} \mathbf{1}_k \frac{\mathbf{1}_k A_k^{-1} a_{k+1} - 1}{a_{k+1}^\top A_k^{-1} a_{k+1}} - (\mathbf{1}_k^\top A_k^{-1} \mathbf{1}_k) \frac{A_k^{-1} a_{k+1}}{a_{k+1}^\top A_k^{-1} a_{k+1}}}{\mathbf{1}_{k+1}^\top A_{k+1}^{-1} \mathbf{1}_{k+1}}, \quad (\text{A6})$$

which in turn implies Equation (17). Proof of Equation (18) is similar to a proofs of Equations (16) and (17). $\square$

Proof of Theorem (4).

Proof. Following the proof of Theorem (1), the analysis of model compatibility reveals that

$$W = K + L = \text{tr} \left((z_{k+1} + Z_k^\top \frac{b_{k+1}}{g_{k+1}}) (z_{k+1}^\top + \frac{b_{k+1}^\top}{g_{k+1}} Z_k) \Sigma^{-1} g_{k+1} \right) + \\ + \text{tr} \left((Z_{k+1}^\top - \mu \mathbf{1}_{k+1}^\top) (Z_{k+1}^\top - \mu \mathbf{1}_{k+1}^\top)^\top \Sigma^{-1} \frac{Q_{k+1} Q_{k+1}^\top}{D_{k+1}} \right).$$

Now following the proof of Theorem (2) decompose W as $W = U + V$, where U denotes the conditional mean of z_{k+1} , and the remainder, after further simplifications, is

$$V = \text{tr} \left((Z_k^\top - \mu \mathbf{1}_k^\top) (Z_k^\top - \mu \mathbf{1}_k^\top)^\top \Sigma^{-1} \frac{b_{k+1} b_{k+1}^\top}{g_{k+1}} \right) + \text{tr} \left((Z_k^\top - \mu \mathbf{1}_k^\top) (Z_k^\top - \mu \mathbf{1}_k^\top)^\top \Sigma^{-1} \frac{q_{k+1} q_{k+1}^\top}{D_{k+1}} \right) - \\ - \text{tr} \left((Z_k^\top - \mu \mathbf{1}_k^\top) (Z_k^\top - \mu \mathbf{1}_k^\top)^\top \Sigma^{-1} \frac{(\frac{(Q_{k+1})_{k+1} q_{k+1}}{D_{k+1}} + b_{k+1}) (\frac{(Q_{k+1})_{k+1} q_{k+1}}{D_{k+1}} + b_{k+1})^\top}{g_{k+1} + \frac{(Q_{k+1})_{k+1}^2}{D_{k+1}}} \right) = \\ = \text{tr} \left((Z_k^\top - \mu \mathbf{1}_k^\top) (Z_k^\top - \mu \mathbf{1}_k^\top)^\top \Sigma^{-1} \frac{\left(q_{k+1} - \frac{b_{k+1} (Q_{k+1})_{k+1}}{g_{k+1}} \right) \left(q_{k+1} - \frac{b_{k+1} (Q_{k+1})_{k+1}}{g_{k+1}} \right)^\top}{D_{k+1} + \frac{(Q_{k+1})_{k+1}^2}{g_{k+1}}} \right).$$

Since V is given by $\text{tr} \left((Z_k^\top - \mu \mathbf{1}_k^\top) (Z_k^\top - \mu \mathbf{1}_k^\top)^\top \Sigma^{-1} \frac{Q_k Q_k^\top}{D_k} \right)$, it follows that

$$\frac{Q_k Q_k^\top}{D_k} = \frac{\left(q_{k+1} - \frac{b_{k+1} (Q_{k+1})_{k+1}}{g_{k+1}} \right) \left(q_{k+1} - \frac{b_{k+1} (Q_{k+1})_{k+1}}{g_{k+1}} \right)^\top}{D_{k+1} + \frac{(Q_{k+1})_{k+1}^2}{g_{k+1}}}. \quad (\text{A7})$$

Since $\mathbf{1}_k Q_k = 1$ and $\mathbf{1}_k q_{k+1} = 1 - (Q_{k+1})_{k+1}$, multiplying both sides of Equation (A7) by $\mathbf{1}_k^\top$ and $\mathbf{1}_k$ yields Equations (21) and (22). $\square$

Appendix C. Algorithm

Algorithm A1: Fractional Brownian Vector Field Recursive Generation Algorithm.

```

1 Procedure RecMatrix( $A, F, \alpha, k$ ):
   Input : The size of realizations  $k$ ,  $A$ -matrix of fractional distances,  $F$ -auxiliary
           vector,  $\alpha$ -auxiliary value.
   Output:  $a$ 
2   for  $i \leftarrow 1$  to  $k - 1$  do
3     for  $j \leftarrow 1$  to  $k - 1$  do
4        $a_{i,j} = A_{i,j} - \frac{F_i F_j}{\alpha}$ 
5        $a_{i,k} = \frac{F_i}{\alpha}$ 
6        $a_{k,i} = a_{i,k}$ 
7    $a_{k,k} = \frac{-1}{\alpha}$ 
8 Procedure RecField( $k, m, H, X, \Sigma$ ):
   Input : Site locations  $X \in \mathbb{R}^{k \times m}$ , the size of realizations  $k$ , the number of
           columns  $m$ , the value of Hurst parameter  $H \in [0, 1)$ ,  $\Sigma$ -positive
           definite matrix ( $m \times m$ ).
   Output: The matrix of simulated values of FBVF  $Z \in \mathbb{R}^{k \times m}$ .
9   Compute Cholesky decomposition  $L = \text{Chol}(\Sigma)$ 
10  Generate vector with standard normal distribution  $\zeta \sim \mathcal{N}(0, 1)$ ,  $\zeta \in \mathbb{R}^m$ ;
11   $u = ((X_1 - X_2)^\top (X_1 - X_2))^{H/2}$ 
12  Set  $A_{1,1} = 0$ ;  $A_{2,2} = 0$ ;  $A_{1,2} = \frac{1}{u}$ ;  $A_{2,1} = \frac{1}{u}$ ;
13  Compute first and second values of FBVF:
14   $Z_1 = L\zeta$  Equation (41)
15   $Z_2 = Z_1 + \sqrt{2}L\zeta u$  Equation (42)
16   $\mathbf{1} = (1, 1)$ 
17  for  $i \leftarrow 2$  to  $k - 1$  do
18    for  $j \leftarrow 1$  to  $i$  do
19      for  $l \leftarrow 1$  to  $p$  do
20         $g_l = (X_{j,l} - X_{i+1,l})^2$ 
21       $f_j = (\sum_{l=1}^p f_l)^H$ 
22       $F = Af$ 
23       $\alpha = f^\top F$ 
24       $E = A\mathbf{1}$ 
25       $vd = z(F + E \frac{\mathbf{1} - \mathbf{1}^\top F}{\mathbf{1}^\top E})$ 
26       $dy = \alpha - \frac{(\mathbf{1} - \mathbf{1}^\top F)^2}{\mathbf{1}^\top E}$ 
27       $A = \text{RecMatrix}(A, F, \alpha, i + 1)$ 
28       $\mathbf{1}_{i+1} = \mathbf{1}$ 
29       $Z_{i+1} = vd + L\zeta \sqrt{dy}$  Equation (43)

```

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