ASYMPTOTIC DISTRIBUTION OF BEURLING INTEGERS

Ramūnas Garunkštis and Laima Kaziulytė
Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University
Naugarduko 24, 03225 Vilnius, Lithuania
ramunas.garunkstis@mif.vu.lt
laima.kaziulyte@mif.vu.lt

We study generalised prime systems $\mathcal{P}$ and generalised integer systems $\mathcal{N}$ obtained from them. The asymptotic distribution of generalised integers is deduced assuming that the generalised prime counting function $\pi_{\mathcal{P}}(x)$ behaves as $\pi_{\mathcal{P}}(x) = b\,\mathrm{li}(x) + \mathrm{O}(x^{\alpha})$ for some $b > 0$ and $\alpha \in (1/2, 1)$.

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1. Introduction

As usual, let s denote a complex variable with σ and t its real and imaginary parts respectively.

A *generalised prime system* $\mathcal{P}$ is a sequence of positive reals $p_1, p_2, \dots$ satisfying

$$1 < p_1 \le p_2 \le \dots \le p_n \le \dots$$

and for which $p_n \to \infty$ as $n \to \infty$. The numbers of the form

$$p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

where $m \in \mathbb{N}$ and $k_1, \dots, k_m \in \mathbb{N}_0$ constitute the so called system $\mathcal{N}$ of *generalised integers* or *Beurling integers*. Here and henceforth $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. This system generalises the notion of prime numbers and the natural numbers obtained from them. Such systems (along with the attached zeta functions) were first introduced by Beurling [1] and have been studied by numerous authors since then (see, for instance, the papers by Diamond [4], Diamond, Montgomery and Vorhauer [5], Fainleib [6], Hall [7,8], Hilberdink and Lapidus [9], Landau [10], Stankus [15] and Zhang [20,21,22]).

First define the *counting functions* $\pi_{\mathcal{P}}(x)$ and $N_{\mathcal{P}}(x)$ by

$$\pi_{\mathcal{P}}(x) = \sum_{p \le x,\, p \in \mathcal{P}} 1, \tag{1.1}$$

$$N_{\mathcal{P}}(x) = \sum_{n \le x,\, n \in \mathcal{N}} 1. \tag{1.2}$$

Here, as elsewhere in the paper, we write $\sum_{p\in\mathcal{P}}$ to mean a sum over all the generalised primes, counting multiplicities. Similarly for $\sum_{n\in\mathcal{N}}$. Much of the research on this subject has been about connecting the asymptotic behaviour of the *generalised prime counting function* (1.1) and of the *generalised integer counting function* (1.2) as $x \to \infty$. In this paper we are interested in the following question. Given the asymptotic behaviour of $\pi_{\mathcal{P}}(x)$, what can be said about the behaviour of $N_{\mathcal{P}}(x)$?

Let $\mathrm{li}(x)$ be the logarithmic integral defined by

$$\mathrm{li}(x) = \lim_{\varepsilon\to 0^+}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x}\right)\frac{\mathrm{d}t}{\log t}.$$

In 1949 Nyman [14] showed that

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + \mathrm{O}\left(\frac{x}{(\log x)^A}\right) \quad (\forall A) \iff N_{\mathcal{P}}(x) = ax + \mathrm{O}\left(\frac{x}{(\log x)^A}\right) \quad (\forall A).$$

In 1960 Bredikhin [2] proved that if

$$\pi_{\mathcal{P}}(x) = \frac{bx}{\log x} + O\left(\frac{x}{(\log x)^{1+\varepsilon}}\right)$$

for some $b > 0$ and $\varepsilon > 0$, then

$$N_{\mathcal{P}}(x) = Cx(\log x)^{b-1} + O\left(\frac{x(\log x)^{b-1}}{(\log\log x)^{\varepsilon_1}}\right), \tag{1.3}$$

where $\varepsilon_1 = \min(1,\varepsilon)$ and $C > 0$ is a constant.

In 1961 Malliavin [11] showed that

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + \mathrm{O}\left(x\mathrm{e}^{-c_2(\log x)^{\beta}}\right) \tag{1.4}$$

for some $\beta \in (0,1)$ and $c_2 > 0$, implies

$$N_{\mathcal{P}}(x) = ax + \mathrm{O}\left(x\mathrm{e}^{-c_1(\log x)^{\alpha}}\right) \tag{1.5}$$

for $\alpha = \frac{\beta}{2+\beta}$ and some $a, c_1 > 0$. Diamond ([3], 1970) improved this to $\alpha = \frac{\beta}{1+\beta}$, and furthermore, Diamond's result contains $\log x \log\log x$ instead of $\log x$ in the exponent of (1.5).

In 2008 Hilberdink and Lapidus [9] showed that if

$$\pi_{\mathcal{P}}(x) = \mathrm{li}(x) + O\left(x^{\alpha}\right)$$

for some $\alpha \in (1/2, 1)$, then there exist positive constants C and δ such that

$$N_{\mathcal{P}}(x) = Cx + O\left(xe^{-\delta\sqrt{\log x\log\log x}}\right). \tag{1.6}$$

In this paper we consider the asymptotic distribution of generalised integers assuming

$$\pi_{\mathcal{P}}(x) = b\,\mathrm{li}(x) + \mathrm{O}(x^{\alpha})$$

for some $b > 0$ and $\alpha \in (1/2, 1)$.

In Section 2 we define the Beurling zeta-function, state our results (Theorem 1 and Corollary 2) and give sufficient background to understand them. Section 3 is devoted to auxiliary statements needed to prove the theorem and corollary in Section 4.

2. The Beurling Zeta-Function and the Main Result

In this section we formulate our theorem. To do this, several notations and results are needed.

With the generalised prime system $\mathcal{P}$ we associate a *zeta function*, which we refer to as a *Beurling zeta-function* and define formally by the *Euler product*

$$\zeta_{\mathcal{P}}(s) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}. \tag{2.1}$$

This infinite product may be formally multiplied out to give the *Dirichlet series*

$$\zeta_{\mathcal{P}}(s) = \sum_{n\in\mathcal{N}} \frac{1}{n^s}.$$

Note that when $\mathcal{P}$ is the set of rational primes, and hence $\mathcal{N}$ is the set of natural numbers, $\zeta_{\mathcal{P}}$ coincides with the classical Riemann zeta function. All the classical functions (when $\mathcal{N} = \mathbb{N}$) are written without any index: $\zeta(s), \Lambda(n)$.

We shall see below that the following function plays a special role. Let $b > 0$ (from now on b is fixed and denotes the same number unless otherwise stated). Set the function

$$Z(s) = s^{-1}((s-1)\zeta(s))^b,$$

where $Z(s)$ is defined on any simply connected domain which does not contain a zero of $\zeta(s)$ and does not contain the point $s = 0$. We shall always suppose that this domain includes the real half-line $[1, +\infty)$. We can then choose the principal value of the complex logarithm, so that $Z(1) = 1$.

By Tenenbaum [16, p. 182] we know, that the function $Z(s)$ is holomorphic in the disc $|s-1| < 1$, and can be represented there by the Taylor series

$$Z(s) = \sum_{j=0}^{\infty} \frac{1}{j!}\gamma_j(b)(s-1)^j \tag{2.2}$$

where the coefficients $\gamma_j(b)$ for all $\varepsilon > 0$ satisfy the upper bound

$$\frac{1}{j!}\gamma_j(b) \ll_{b,\varepsilon} (1+\varepsilon)^j.$$

It could be that $\zeta_{\mathcal{P}}(s)$ has no meromorphic continuation to a neighbourhood of $s = 1$. Because of that we will use an auxiliary function

$$G(s) = \zeta_{\mathcal{P}}(s)\zeta(s)^{-b} \qquad (\sigma > 1).$$

Assuming that

$$\pi_{\mathcal{P}}(x) = b\,\mathrm{li}(x) + O\left(x^{\alpha}\right) \tag{2.3}$$

for some $\alpha \in (1/2, 1)$, we can continue $G(s)$ analytically to the neighbourhoud of $s = 1$ (see Lemma 7 below). Hence, for $k \in \mathbb{N}_0$, we can set

$$\lambda_k = \frac{1}{\Gamma(b-k)} \sum_{\substack{h+j=k \\ h,j \in \mathbb{N}_0}} \frac{1}{h!j!} G^{(h)}(1)\gamma_j(b), \tag{2.4}$$

where the $\gamma_j(b)$ are the coefficients appearing in formula (2.2).

Our aim is to prove the following theorem.

Theorem 1. *Let N be a non-negative integer, $b > 0$, and $\alpha \in (1/2, 1)$. Then the formula (2.3) implies that*

$$N_{\mathcal{P}}(x) = x(\log x)^{b-1} \left\{ \sum_{k=0}^{N} \frac{\lambda_k}{(\log x)^k} + \mathrm{O}\left(R_N(x)\right) \right\} \tag{2.5}$$

with

$$R_N(x) = \mathrm{e}^{-c_6\sqrt{\log x \log\log x}} + \left(\frac{c_4 N + 1}{\log x}\right)^{N+1}.$$

The positive constants c_4, c_6 and the implicit constant in the Landau symbol depend at most on b and α. The coefficients λ_k are defined by formula (2.4).

We prove this theorem in Section 4. The following corollary easily follows from Theorem 1. Its proof is given at the end of Section 4.

Corollary 2. *Let $b \in \mathbb{N}$ and $\alpha \in (1/2, 1)$. If*

$$\pi_{\mathcal{P}}(x) = b\,\mathrm{li}(x) + O\left(x^{\alpha}\right),$$

then we obtain

$$N_{\mathcal{P}}(x) = x(\log x)^{b-1} \left\{ P\left(\frac{1}{\log x}\right) + \mathrm{O}\left(\mathrm{e}^{-c_6\sqrt{\log x \log\log x}}\right) \right\},$$

where P is a polynomial of degree at most $b-1$, $c_6 > 0$ is a constant which depends at most on b and α.

Theorem 1 can be compared to the results of Bredikhin (formula (1.3)), Hilberdink and Lapidus (formula (1.6)) and Tenenbaum [16, §5.3].

We note that the coefficients λ_k depend on the behaviour of the Beurling zeta-function and its derivatives near $s = 1$. Even though we assume quite a strong condition on the distribution of generalised primes, the problem of finding coefficients in asymptotic expansion in terms of $\log x$ of generalised integers is rather computational in nature and this appears to prohibit one from obtaining more explicit results.

However, a particular case when $\mathcal{P}$ is the subset of rational primes has been investigated considerably. For example, the classical result of asymptotic behaviour of the sums of multiplicative functions on primes and on integers was derived by Wirsing [19] in 1961.

For the interest of the reader we show some results towards the calculation of the constant λ_0 when $\mathcal{P}$ is a set of prime numbers from arithmetic progression.

Let $g_{d,a}(n) = 0$ if n has a prime divisor p satisfying $p \not\equiv a(mod\ d)$ and $g_{d,a}(n) = 1$ otherwise (note that $g_{d,a}(1) = 1$). We let

$$N(x; d, a) = \sum_{n \le x} g_{d,a}(n).$$

Note that by abuse of notation for this example we write $N(x; d, a)$ instead of $N_{\mathcal{P}}(x)$ to emphasize the parameters d and a from our particular set $\mathcal{P}$. From Moree and Cazaran [13, Theorem 6] together with the prime number theorem for arithmetic progressions, it follows that there exist constants $C_{d,a}, C_{d,a}(1), C_{d,a}(2), \ldots$ such that for each integer $m \ge 0$ we have

$$N(x; d, a) = \frac{C_{d,a} x}{(\log x)^{1-1/\varphi(d)}} \left(1 + \sum_{j=1}^{m} \frac{C_{d,a}(j)}{(\log x)^j} + \mathrm{O}\left(\frac{1}{(\log x)^{m+1}} \right) \right),$$

where the implied constant may depend on m, a, d and $C_{d,a} \ge 0$. Moree [12] obtained the following expression of the above constants.

$$C_{3,1} = \frac{\sqrt{2}}{3^{\frac{5}{4}}} \prod_{n=1}^{\infty} \left(\frac{L(2^n, \chi_{-3})}{(1 - 3^{-2^n})\zeta(2^n)} \right)^{\frac{1}{2^{n+1}}} = 0.301 \cdots, \quad C_{3,2} = \frac{2}{3\pi C_{3,1}} = 0.704 \cdots.$$

He used these evaluations to prove, for example, that $N(x; 3, 2) \ge N(x; 3, 1)$ for every x, not only sufficiently large one. This amazing result holds because of the phenomenon called Chebysev's bias, for which "more often" $\pi(x; 3, 2) > \pi(x; 3, 1)$ than the other way around, even though $\pi(x; 3, 2)$ and $\pi(x; 3, 1)$ are asymptotically equal due to the prime number theorem for arithmetic progressions.

3. Auxiliary Statements

Given a positive parameter r, we designate by *Hankel contour* the path in the complex plane continuing from $-\infty$ along the real line (arbitrary close, but below it) to $-r$, counterclockwise around a circle of radius r centered at 0, back to $-r$ on the real line, and back to $-\infty$ along the real line (arbitrary close, but above it).

Lemma 3. *For each $x > 1$, let $\mathcal{H}(x)$ denote the part of the Hankel contour situated in the half-plane $\sigma > -x$. Then we have uniformly for $z \in \mathbb{C}$*

$$\frac{1}{2\pi i} \int_{\mathcal{H}(x)} s^{-z} \mathrm{e}^{s} \, \mathrm{d}s = \frac{1}{\Gamma(z)} + \mathrm{O}\left(47^{|z|} \, \Gamma(1 + |z|) \, \mathrm{e}^{-\frac{1}{2}x} \right).$$

The proof can be found in [16, p. 184].

Lemma 4. ***(Stirling's formula)*** *Let $\delta > 0$. Then there is $c = c(\delta)$, such that*

$$\left|\log \Gamma(s) - \left((s-1/2)\log s - s + \log\sqrt{2\pi}\right)\right| < \frac{c}{|s|}$$

for $-\pi + \delta \le \arg s \le \pi - \delta$, $s \neq 0$. Here we take the principal part of the logarithm.

For a proof refer to [17, p. 151].

Lemma 5. ***(Perron's formula)*** *Let*

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

be a Dirichlet series with abscissa of convergence σ_c and

$$A(x) = \sum_{n \le x} a_n \qquad (x \ge 0)$$

be the summatory function of its coefficients. Then, for $\kappa > \max(0, \sigma_c)$ and $x \ge 1$, we have

$$\int_0^x A(t)\ \mathrm{d}t = \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} F(s) x^{s+1} \frac{\mathrm{d}s}{s(s+1)}.$$

For the proof see [16, p. 134].

Lemma 6. *For some positive constant A the region*

$$\sigma \ge 1 - A(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}} \qquad (t \ge 3)$$

is free of zeros of the function $\zeta(s)$ and in this region

$$\frac{1}{\zeta(s)} \ll (\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}} \qquad (t \to \infty).$$

The proof can be found in [18, p. 135].

Throughout this paper, we shall use the weighted counting function

$$\psi_{\mathcal{P}}(x) = \sum_{p^k \le x, k \in \mathbb{N}} \log p = \sum_{n \le x, n \in \mathcal{N}} \Lambda_{\mathcal{P}}(n).$$

Here $\Lambda_{\mathcal{P}}$ denotes the (generalised) von Mangoldt function, defined for n in the multiset $\mathcal{N}$ by $\Lambda_{\mathcal{P}}(n) = \log p$ if $n = p^m$ for some $p \in \mathcal{P}$ and $m \in \mathbb{N}$, and $\Lambda_{\mathcal{P}}(n) = 0$ otherwise. The formal Euler product (2.1) gives

$$-\frac{\zeta'_{\mathcal{P}}(s)}{\zeta_{\mathcal{P}}(s)} = \sum_{n \in \mathcal{N}} \frac{\Lambda_{\mathcal{P}}(n)}{n^s} =: \phi_{\mathcal{P}}(s). \tag{3.1}$$

The counting functions $N_{\mathcal{P}}(x)$ and $\psi_{\mathcal{P}}(x)$ are related to $\zeta_{\mathcal{P}}(s)$ and $\phi_{\mathcal{P}}(s)$ via

$$\zeta_{\mathcal{P}}(s) = s\int_1^{\infty} \frac{N_{\mathcal{P}}(x)}{x^{s+1}}\ \mathrm{d}x \quad \text{and} \quad \phi_{\mathcal{P}}(s) = s\int_1^{\infty} \frac{\psi_{\mathcal{P}}(x)}{x^{s+1}}\ \mathrm{d}x.$$

As a result, it is often more convenient to work with $\psi_{\mathcal{P}}(x)$, rather than $\pi_{\mathcal{P}}(x)$. Note that for $\alpha \in [\frac{1}{2}, 1)$, $b > 0$, the statements

$$\pi_{\mathcal{P}}(x) = b\,\mathrm{li}(x) + \mathrm{O}\left(x^{\alpha+\varepsilon}\right) \quad (\forall \varepsilon > 0) \quad \text{and} \quad \psi_{\mathcal{P}}(x) = bx + \mathrm{O}\left(x^{\alpha+\varepsilon}\right) \quad (\forall \varepsilon > 0),$$

are equivalent. Thus, let us assume that $\psi_{\mathcal{P}}(x) = bx + \mathrm{O}\left(x^{\alpha}\right)$ for $\alpha \in (0,1)$ and prove the theorem with this condition. In fact, one can use this assumption instead of more classical (2.3) with $\alpha \in (1/2, 1)$ in the formulation of Theorem 1. Denote

$$\begin{aligned} R(\alpha) = &\left\{ s \in \mathbb{C} : \sigma \geq \max\left(1 - \frac{A}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}, \alpha\right), \quad |t| \geq 3 \right\} \\ &\bigcup \left\{ s \in \mathbb{C} : \sigma \geq \max\left(1 - \frac{A}{(\log 3)^{\frac{2}{3}}(\log\log 3)^{\frac{1}{3}}}, \alpha\right), \quad |t| \leq 3 \right\}, \end{aligned} \tag{3.2}$$

where $\alpha \in (0,1)$, A is a positive constant from Lemma 6. We can assume that $A < 1$.

Lemma 7. *Suppose that for some $\alpha \in (0,1)$ and $b > 0$, we have*

$$\psi_{\mathcal{P}}(x) = bx + \mathrm{O}(x^{\alpha}). \tag{3.3}$$

Then $G(s) = \zeta_{\mathcal{P}}(s)\zeta(s)^{-b}$ has an analytic continuation to the region $R(\alpha)$, which is defined above.

Proof. By the lemma's assumption (3.3) and by the Dirichlet series representation (3.1) we see that the function $\frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s)$ is analytic in the half plane $\sigma > 1$. Thus, for $\sigma > 1$, we have

$$G(s) = \frac{\zeta_{\mathcal{P}}(s)}{\zeta(s)^b} = \exp\left(\int_2^s f(u)\,\mathrm{d}u + \log\frac{\zeta_{\mathcal{P}}(2)}{\zeta(2)^b}\right),$$

where

$$f(s) = \frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) - b\,\frac{\zeta'}{\zeta}(s).$$

To prove the lemma it is enough to show that $f(s)$ is analytic in the region $R(\alpha)$.

We write $f(s)$ as

$$f(s) = \frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) + b\,\zeta(s) - \left(b\,\frac{\zeta'}{\zeta}(s) + b\,\zeta(s)\right). \tag{3.4}$$

The sum in the parentheses of (3.4) is analytic in the region $R(\alpha)$. It follows from the facts that the function $\zeta(s)$ is analytic in the whole plane, except for a simple pole at $s = 1$, with residue 1 and $\zeta(s)$ does not have any zeros in the region $R(\alpha)$ (see Lemma 6 above and [18], p. 389).

Further,

$$\frac{\zeta'_{\mathcal{P}}}{\zeta_{\mathcal{P}}}(s) + b\,\zeta(s) = -\phi_{\mathcal{P}}(s) + b\,\zeta(s).$$

Now we show that $\phi_{\mathcal{P}}(s)$ has the analytic continuation to $\{s \in \mathbb{C} : \sigma > \alpha\}$ except for a simple pole at $s = 1$ with residue b. By our assumption $\psi_{\mathcal{P}}(x) = bx + r(x)$, where $r(x) = \mathrm{O}(x^{\alpha})$. Thus

$$\phi_{\mathcal{P}}(s) = s \int_1^{\infty} \frac{bx + r(x)}{x^{s+1}} \, \mathrm{d}x = \frac{bs}{s-1} + \int_1^{\infty} \frac{r(x)}{x^{s+1}} \, \mathrm{d}x.$$

The latter integral converges for $\sigma > \alpha$ and represents an analytic function in this half-plane. It follows that $-\phi_{\mathcal{P}}(s) + b\,\zeta(s)$ is holomorphic for $\sigma > \alpha$. Hence, the function $f(s)$ is analytic in the region $R(\alpha)$. This proves Lemma 7. □

Put $\varepsilon(t) = A(1-\alpha)\frac{\log\log|t|}{\log|t|}$ for $|t| \geq 3$, where A is a constant from Lemma 6.

Lemma 8. *For sufficiently large $|t|$ we have*

$$|\zeta_{\mathcal{P}}(\sigma + it)| \leq |t|^{\frac{2b}{\log\log|t|}}, \tag{3.5}$$

where $\sigma \in [1 - \varepsilon(t), 3/2)$.

Proof. The proof is essentially identical to that leading up to (2.6) of Theorem 2.2 of Hilberdink and Lapidus paper [9]. The minor difference is that in assuming (3.3) we generalize the case $b = 1$ discussed in [9]. Thus our $\phi_{\mathcal{P}}(s)$ is basically $b\phi(s)$, where $\phi(s)$ is the equivalent function corresponding to the system for which $b = 1$. The estimate (3.5) is explicitly written only for $\sigma = 1 - \varepsilon(t)$ in [9], but, following their argument, holds for $\sigma \in [1 - \varepsilon(t), 1)$. If $\sigma \in [1, 3/2)$ we have to slightly modify the end of the proof in [9], leading to a better upper bound of the logarithm

$$\begin{aligned}
\log|\zeta_{\mathcal{P}}(\sigma + it)| &\leq b \int_{\sigma}^{2} \frac{|t|^{\frac{1-y}{1-\alpha}} - 1}{1-y} \, \mathrm{d}y + \mathrm{O}(1) \\
&= b \int_{-1}^{1-\sigma} \frac{|t|^{\frac{u}{1-\alpha}} - 1}{u} \, \mathrm{d}u + \mathrm{O}(1) \\
&= b \int_{\sigma-1}^{1} \frac{1 - |t|^{-\frac{v}{1-\alpha}}}{v} \, \mathrm{d}v + \mathrm{O}(1) \\
&\leq b \int_{0}^{1} \frac{1 - |t|^{-\frac{v}{1-\alpha}}}{v} \, \mathrm{d}v + \mathrm{O}(1) \\
&= b \int_{0}^{\frac{\log|t|}{1-\alpha}} \frac{1 - \mathrm{e}^{-x}}{x} \, \mathrm{d}x + \mathrm{O}(1) \\
&= \mathrm{O}(\log\log|t|).
\end{aligned}$$

Thus,

$$|\zeta_{\mathcal{P}}(\sigma + it)| \leq \exp\left(c_7 \log\log|t|\right) \leq \exp\left(\frac{2b\log|t|}{\log\log|t|}\right),$$

for $\sigma \in [1, 3/2)$ and sufficiently large $|t|$, which is our claim.

□

4. Proofs of the Theorem 1 and Corollary 2

For the proof we use the Selberg-Delange method, see Chapter II.5 in Tenenbaum [16].

For a beginning of the proof of Theorem 1 define the domain D by deleting the real segment $(\alpha, 1]$ from the region $R(\alpha)$, where $R(\alpha)$ is defined by (3.2).

Let $\kappa = 1 + \frac{1}{\log x}$. Let $T > \exp(\mathrm{e}^{6b})$ (where b is from the equality (2.3)) be a parameter whose value will be determined later. Then Perron formula (Lemma 5) allows us to write

$$\int_1^x N_{\mathcal{P}}(t)\,\mathrm{d}t = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \zeta_{\mathcal{P}}(s)x^{s+1}\frac{\mathrm{d}s}{s(s+1)} \tag{4.1}$$
$$=: \frac{1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \zeta_{\mathcal{P}}(s)x^{s+1}\frac{\mathrm{d}s}{s(s+1)} + E(T) =: I_0 + E(T).$$

Next we change the path of the integration in the integral I_0. Put $\varepsilon(t) = A(1-\alpha)\frac{\log\log|t|}{\log|t|}$ for $|t| \geq 3$, where A is a constant from Lemma 6. It is worth noting that $\alpha < 1 - \varepsilon(t) < 1$. The residue theorem allows us to deform the segment of integration $[\kappa - iT, \kappa + iT]$ into some path joining the end-points $\kappa - iT, \kappa + iT$ and contained entirely in D. We choose the path symmetrically with respect to the real axis (see Fig. 1 below). Its upper part is made up of: the truncated Hankel contour Γ, surrounding the point $s = 1$, with radius $r = \frac{1}{2\log x}$, and linear part joining $1 - r$ to $1 - \varepsilon(3)$; the vertical segment $[1 - \varepsilon(3), 1 - \varepsilon(3) + 3i]$; the curve

$$\Upsilon : \left\{ s : \sigma = 1 - \varepsilon(t) = 1 - A(1-\alpha)\frac{\log\log|t|}{\log|t|}, \quad t \in [3, T] \right\};$$

and the horizontal segment $[1 - \varepsilon(T) + iT, \kappa + iT]$.

The contour is entirely contained in D since

$$A(1-\alpha)\frac{\log\log|t|}{\log|t|} < \frac{A}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}$$

for $|t| \geq 3$, which implies that $1 - \varepsilon(t)$ is in the region $R(\alpha)$ for all $|t| \leq T$.

We shall see that the main contribution arises from the integral over the truncated Hankel contour Γ. We denote this integral by I_1:

$$I_1 = \frac{1}{2\pi i}\int_\Gamma \frac{\zeta_{\mathcal{P}}(s)x^{s+1}}{s(s+1)}\,\mathrm{d}s.$$

Let the notation $\int_{[s_1,s_2]}$ mean an integral over the interval starting at s_1 and ending at s_2. Then the other parts of the path are denoted as follows.

$$I_2 = \frac{1}{2\pi i}\int_{[1-\varepsilon(3),1-\varepsilon(3)+3i]} \frac{\zeta_{\mathcal{P}}(s)x^{s+1}}{s(s+1)}\,\mathrm{d}s + \frac{1}{2\pi i}\int_{[1-\varepsilon(3)-3i,1-\varepsilon(3)]} \frac{\zeta_{\mathcal{P}}(s)x^{s+1}}{s(s+1)}\,\mathrm{d}s.$$

$$I_3 = \frac{1}{2\pi i}\int_{\Upsilon\cup\bar{\Upsilon}} \frac{\zeta_{\mathcal{P}}(s)x^{s+1}}{s(s+1)}\,\mathrm{d}s, \tag{4.2}$$

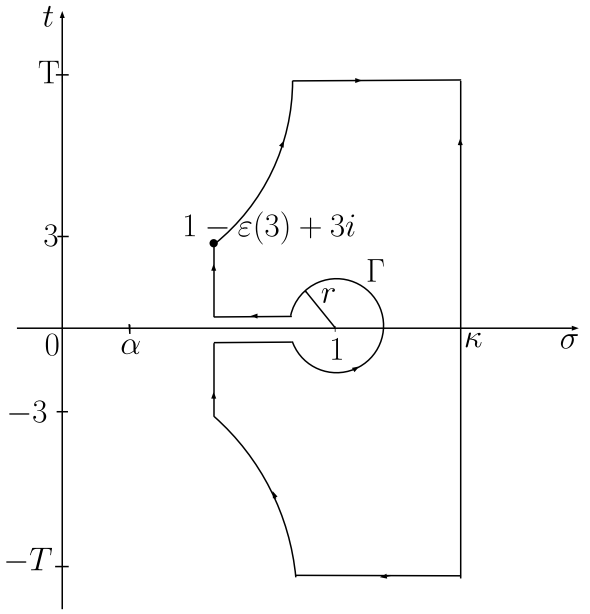

Fig. 1. Deformation of the segment $[\kappa - iT, \kappa + iT]$

where $\bar{\Upsilon}$ denotes the part of our integration contour (see Fig. 1) which is symmetric to the curve Υ with respect to the real axis.

$$I_4 = \frac{1}{2\pi i}\int_{[1-\varepsilon(T)+iT,\kappa+iT]} \frac{\zeta_{\mathcal{P}}(s)x^{s+1}}{s(s+1)}\,\mathrm{d}s + \frac{1}{2\pi i}\int_{[\kappa-iT,1-\varepsilon(T)-iT]} \frac{\zeta_{\mathcal{P}}(s)x^{s+1}}{s(s+1)}\,\mathrm{d}s.$$

Using these notations (see Fig. 1) we have $I_0 = I_1 + I_2 + I_3 + I_4$. Then in view of the formula (4.1) we get

$$\int_1^x N_{\mathcal{P}}(t)\,\mathrm{d}t = I_1 + I_2 + I_3 + I_4 + E(T).$$

Next we will show that it is possible to choose $T = T(x)$ such that $I_2 + I_3 + I_4 + E(T) \ll x^2 \mathrm{e}^{-c_1\sqrt{\log x \log\log x}}$. Appealing to Lemma 8, we see immediately that

$$E(T) \ll \int_T^\infty \frac{t^{\frac{2b}{\log\log t}} x^2}{t^2}\,\mathrm{d}t \ll x^2 \int_T^\infty t^{\frac{2b}{\log\log T}-2}\,\mathrm{d}t \ll x^2 T^{\frac{2b}{\log\log T}-1}.$$

This upper bound is equally valid for the integral I_4 since

$$I_4 \ll \int_{1-\varepsilon(T)}^{1+\frac{1}{\log x}} \frac{T^{\frac{2b}{\log\log T}} x^2}{T^2}\,\mathrm{d}\sigma \ll x^2 T^{\frac{2b}{\log\log T}-1}.$$

The integral I_2 is

$$I_2 \ll x^{2-\varepsilon(3)}.$$

Finally, to get the upper bound for the integral I_3 we choose a number $k_0 > \exp(\mathrm{e}^{5b})$ such that Lemma 8 is valid for all $|t| \geq k_0$. Splitting the integral (4.2), we have

$$I_3(x) := \mathrm{O}\left(I_{31}(x)\right) + \mathrm{O}\left(I_{32}(x)\right), \tag{4.3}$$

where

$$\begin{aligned}
I_{31}(x) :&= \int_3^{k_0} \frac{|\zeta_{\mathcal{P}}(1-\varepsilon(t)+it)|x^{2-\varepsilon(k_0)}}{t^2}\,\mathrm{d}t = \mathrm{O}\left(x^{2-\varepsilon(k_0)}\right), \\
I_{32}(x) :&= \int_{k_0}^{T} \frac{|\zeta_{\mathcal{P}}(1-\varepsilon(t)+it)|x^{2-\varepsilon(t)}}{t^2}\,\mathrm{d}t \\
&= \mathrm{O}\left(x^2 \int_{k_0}^{T} \exp\left\{\frac{2b\log t}{\log\log t} - 2\log t - A(1-\alpha)\frac{\log\log t\log x}{\log t}\right\}\,\mathrm{d}t\right) \\
&= \mathrm{O}\left(x^2 \int_{\log k_0}^{\log T} \exp\left\{-\frac{1}{2}u - A(1-\alpha)\frac{\log u\log x}{u}\right\}\,\mathrm{d}u\right).
\end{aligned}$$

Let $\eta > 0$ and split up the latter integral into two parts with ranges $\left[\log k_0, \eta\sqrt{\log x\log\log x}\right]$ and $\left[\eta\sqrt{\log x\log\log x}, \log T\right]$. For $u \leq \eta\sqrt{\log x\log\log x}$, we have

$$\frac{\log u\log x}{u} \geq \frac{1}{2\eta}\sqrt{\log x\log\log x},$$

since $(\log u)/u$ decreases with u for $u \geq \mathrm{e}$. Hence

$$\begin{aligned}
&\int_{\log k_0}^{\log T} \exp\left\{-\frac{1}{2}u - A(1-\alpha)\frac{\log u\log x}{u}\right\}\,\mathrm{d}u \\
&\quad\leq \mathrm{e}^{-\frac{A(1-\alpha)}{2\eta}\sqrt{\log x\log\log x}} \int_{\log k_0}^{\eta\sqrt{\log x\log\log x}} \mathrm{e}^{-\frac{u}{2}}\,\mathrm{d}u + \int_{\eta\sqrt{\log x\log\log x}}^{\log T} \mathrm{e}^{-\frac{u}{2}}\,\mathrm{d}u \\
&\quad= \mathrm{O}\left(\mathrm{e}^{-\frac{A(1-\alpha)}{2\eta}\sqrt{\log x\log\log x}}\right) + \mathrm{O}\left(\mathrm{e}^{-\frac{\eta}{2}\sqrt{\log x\log\log x}}\right) \\
&\quad= \mathrm{O}\left(\mathrm{e}^{-\eta'\sqrt{\log x\log\log x}}\right),
\end{aligned}$$

for some $\eta' > 0$. In fact, the optimal choice is obtained by taking η such that $\eta = \frac{A(1-\alpha)}{\eta}$; i.e., $\eta = \sqrt{A(1-\alpha)}$, which gives $\eta' = \frac{1}{2}\sqrt{A(1-\alpha)}$. Hence (4.3) becomes

$$I_3(x) = \mathrm{O}\left(x^2\mathrm{e}^{-\frac{1}{2}\sqrt{A(1-\alpha)\log x\log\log x}}\right).$$

Selecting $T = \exp\left(\frac{1}{2}\log x\log\log x\right)$ for $x > \exp(\mathrm{e}^{12b})$, leads us to the main formula

$$\int_1^x N_{\mathcal{P}}(t)\,\mathrm{d}t = I_1(x) + \mathrm{O}\left(x^2\mathrm{e}^{-c_1\sqrt{\log x\log\log x}}\right), \tag{4.4}$$

with

$$I_1(x) = \frac{1}{2\pi i}\int_\Gamma \zeta_{\mathcal{P}}(s)\frac{x^{s+1}}{s(s+1)}\,\mathrm{d}s,$$

where Γ is the truncated Hankel contour.

Here and for the rest of the proof we make the convention that all constants, explicit $(c_1, c_2, c_3, c_4, \ldots)$ or implicit, depend at most on b and α.

It remains to study the main term $I_1(x)$ of (4.4). Clearly, $I_1(x)$ is an infinitely differentiable function of x on $\mathbb{R}^+$, and in particular we have

$$I_1'(x) = \frac{1}{2\pi i}\int_\Gamma \zeta_{\mathcal{P}}(s)x^s\frac{\mathrm{d}s}{s}, \qquad I_1''(x) = \frac{1}{2\pi i}\int_\Gamma \zeta_{\mathcal{P}}(s)x^{s-1}\,\mathrm{d}s.$$

Recall that $Z(s) = s^{-1}((s-1)\zeta(s))^b$. For $s \in D$ (where D is defined in the begining of Section 4) we then can write

$$\zeta_{\mathcal{P}}(s) = s\,G(s)Z(s)(s-1)^{-b}.$$

From this and the result of Lemma 7, for $s \in \Gamma$,

$$\zeta_{\mathcal{P}}(s) \ll |s-1|^{-b}.$$

Since $r = 1/(2\log x)$, it follows that

$$I_1''(x) \ll \int_\Gamma \left(\frac{1}{2\log x}\right)^{-b} x^{s-1}\,\mathrm{d}s \ll (\log x)^b. \tag{4.5}$$

As both $G(s)$ and $Z(s)$ are holomorphic in the open set containing the disk $|s-1| < 1-\varepsilon(3)$, so is their product, which can be represented there by the Taylor series

$$G(s)Z(s) = \sum_{k=0}^{\infty} g_k(b)(s-1)^k$$

with

$$g_k(b) = \frac{1}{k!}\sum_{\substack{h+j=k\\ h,j\in\mathbb{N}_0}} \binom{k}{j} G^{(h)}(1)\gamma_j(b) = \Gamma(b-k)\lambda_k.$$

In addition, since $G(s)Z(s)$ is $\mathrm{O}(1)$ in the disk $|s-1| < 1-\varepsilon(3)$, Cauchy's formula implies that

$$g_k(b) \ll (1-\varepsilon(3))^{-k} \qquad (x \to +\infty).$$

Observing that Γ is contained in the disk $|s-1| \le 1-\varepsilon(3)$, we can write for $s \in \Gamma$ and $N \ge 0$,

$$G(s)Z(s) = \sum_{k=0}^{N} g_k(b)(s-1)^k + \mathrm{O}\left(\left(\frac{|s-1|}{1-\varepsilon(3)}\right)^{N+1}\right).$$

Therefore

$$I_1'(x) = \sum_{k=0}^{N} g_k(b)\frac{1}{2\pi i}\int_\Gamma x^s(s-1)^{k-b}\,\mathrm{d}s + \mathrm{O}((1-\varepsilon(3))^{-N-1}B(x)) \tag{4.6}$$

with

$$
\begin{aligned}
B(x) &= \int_\Gamma |x^s(s-1)^{N+1-b}|\,|\mathrm{d}s| \\
&\ll \int_{1-\varepsilon(3)}^{1-r} (1-\sigma)^{N+1-b} x^\sigma \ \mathrm{d}\sigma + x^{r+1} r^{N+2-b}.
\end{aligned}
$$

Using the change of variable $t = (1-\sigma)\log x$, we obtain

$$
\begin{aligned}
B(x) &\ll x(\log x)^{b-2-N} \left(\int_{\frac{1}{2}}^{\infty} t^{N+1-b} e^{-t} \ \mathrm{d}t + 2^{-N} \right) \\
&\ll x(\log x)^{b-2-N} \left(\int_{\frac{1}{2}}^{1} \left(\frac{1}{2}\right)^{1-b} e^{-1/2} \ \mathrm{d}t + \int_1^\infty t^{N+1+b} \mathrm{e}^{-t} \ \mathrm{d}t + 2^{-N} \right) \\
&\ll x(\log x)^{b-2-N} \Gamma(N+b+2) \ll x(\log x)^{b-1} \left(\frac{N+1}{\log x} \right)^{N+1}. \qquad (4.7)
\end{aligned}
$$

To estimate the integral which appears in (4.6), we change the variable s to w by the relation $w = (s-1)\log x$. Then, with the notation of Lemma 3 and the use of Stirling's formula (Lemma 4), we get

$$
\begin{aligned}
\frac{1}{2\pi i} \int_\Gamma x^s (s-1)^{k-b} \ \mathrm{d}s &= \frac{x}{2\pi i} (\log x)^{b-1-k} \int_{\mathcal{H}(\varepsilon(3)\log x)} w^{k-b} e^w \ \mathrm{d}w \\
&= x(\log x)^{b-1-k} \left\{ \frac{1}{\Gamma(b-k)} + \mathrm{O}\left(47^{b-k} \Gamma(|b-k|+1)\, \mathrm{e}^{-\frac{\varepsilon(3)}{2}\log x} \right) \right\} \\
&= x(\log x)^{b-1-k} \left\{ \frac{1}{\Gamma(b-k)} + \mathrm{O}\left((c_2 k + 1)^k x^{-\varepsilon(3)/2} \right) \right\}.
\end{aligned}
$$

Thus for the main term of (4.6) we have

$$
\sum_{k=0}^{N} g_k(b) \frac{1}{2\pi i} \int_\Gamma x^s (s-1)^{k-b} \ \mathrm{d}s = x(\log x)^{b-1} \left\{ \sum_{k=0}^{N} \frac{\lambda_k}{(\log x)^k} + \mathrm{O}(E_N) \right\} \qquad (4.8)
$$

with

$$
\begin{aligned}
E_N &= x^{-\frac{\varepsilon(3)}{2}} \sum_{k=0}^{N} |g_k(b)| \left(\frac{c_2 k + 1}{\log x} \right)^k \\
&\ll x^{-\frac{\varepsilon(3)}{2}} \sum_{k=0}^{N} (1-\varepsilon(3))^{-k} \left(\frac{c_2 k + 1}{\log x} \right)^k \\
&\ll x^{-\frac{\varepsilon(3)}{2}} \left(\frac{c_3 N + 1}{\log x} \right)^N \ll \left(\frac{c_3 N + 1}{\log x} \right)^{N+1}.
\end{aligned}
$$

Upon substituting (4.7), (4.8) and the latter E_N expression in (4.6), it follows that

$$I_1'(x) = x(\log x)^{b-1}\left\{\sum_{k=0}^{N}\frac{\lambda_k}{(\log x)^k} + \mathrm{O}\left(\frac{c_4N+1}{\log x}\right)^{N+1}\right\}. \tag{4.9}$$

We next show that $I_1'(x)$ is a suitable approximation for $N_{\mathcal{P}}(x)$. The proof of Theorem 1 will be finished after the following lemma.

Lemma 9. $N_{\mathcal{P}}(x) = I_1'(x) + \mathrm{O}\left(x\mathrm{e}^{-c_5\sqrt{\log x\log\log x}}\right)$.

Proof. We follow the argument in [9] and use their notation $N_1(x) = \int_1^x N_{\mathcal{P}}(t)\,\mathrm{d}t$. Since $N_{\mathcal{P}}(x)$ is increasing, we have for every $0 < h < x$,

$$\frac{N_1(x) - N_1(x-h)}{h} \le N_{\mathcal{P}}(x) \le \frac{N_1(x+h) - N_1(x)}{h}.$$

Remember that by (4.4)

$$N_1(x) = I_1(x) + \mathrm{O}\left(x^2\mathrm{e}^{-c_1\sqrt{\log x\log\log x}}\right).$$

Insert this into the above inequality and use the estimate

$$\begin{aligned} I_1(x+h) - I_1(x) &= hI_1'(x) + h^2\int_0^1 (1-t)I_1''(x+th)\,\mathrm{d}t \\ &= hI_1'(x) + \mathrm{O}\left(h^2(\log x)^b\right), \end{aligned}$$

implied by (4.5). Choosing

$$h = x\mathrm{e}^{-\frac{1}{2}c_1\sqrt{\log x\log\log x}},$$

we obtain

$$N_{\mathcal{P}}(x) = I_1'(x) + \mathrm{O}\left(x\mathrm{e}^{-c_5\sqrt{\log x\log\log x}}\right), \tag{4.10}$$

as required. □

Theorem 1 follows by combining (4.9) and (4.10).

The proof of Corollary 2 is as follows. By formula (2.4) we have $\lambda_k = 0$ whenever $k \ge b$. We can hence choose N so as to minimise the error term in (2.5). By choosing $N = [\sqrt{\log x}/c_4]$ in Theorem 1, we get the desired result.

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