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Applications of logic of correlated knowledge to quantum mechanics

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Abstract. Knowledge of agents associated to particles of quantum system has been modelled in the paper. Also analysis of formulas and satisfiability in the model have been done.

Keywords: logic of correlated knowledge, quantum system, agent.

Introduction

Logic of correlated knowledge (LCK) has been introduced by Alexandru Baltag and Sonja Smets in [1]. LCK is an epistemic logic enriched by observational capabilities of agents. Applications of the epistemic logic cover fields such as distributed systems, merging of knowledge bases, robotics or network security in computer science and artificial intelligence. By adding observational capabilities to agents, logic of correlated knowledge can be applied to reason about systems where knowledge correlate between spatially distributed parts of the system. This includes any social system, quantum system, distributed information system, traffic light system or any other system where knowledge is correlated.

Quantum system may consist of one or more elementary particles. Associating agent to each particle, we get multi-agent system, where agents can perform observations and get results. Allowing communication between agents, correlations such as quantum entanglement can be extracted. This can not be done by traditional epistemic logic or logic of distributed knowledge.

We start by defining some basics of logic of correlated knowledge in Section 1. Then we describe some principles of quantum mechanics in Section 2, such as superposition and quantum entanglement. And we finalize by modelling knowledge of quantum system using logic of correlated knowledge in Section 3.

1 Logic of correlated knowledge

Consider a set $N = \{a_1, a_2, \ldots, a_n\}$ of agents or locations. Each agent can perform its local observations. Given sets $O_{a_1}, \ldots, O_{a_n}$ of possible observations for each agent, a joint observation is a tuple of observations $o = (o_a)_{a \in N} \in O_{a_1} \times \cdots \times O_{a_n}$ or $o = (o_a)_{a \in I} \in O_I$, where $O_I := \times_{a \in I} O_a$ and $I \subseteq N$. Joint observations together with results $r \in R$ make new atomic formulas o^r.

Definition 1 [Syntax of logic of correlated knowledge]. The language of logic of correlated knowledge has the following syntax:

$$A := p \mid o^r \mid \neg A \mid A \vee B \mid A \wedge B \mid A \to B \mid K_I A$$

where p is any atomic proposition, $o = (o_a)_{a \in I} \in O_I, r \in R$, and $I \subseteq N$.

States (configurations) of the system are functions $s : O_{a_1} \times \cdots \times O_{a_n} \to R$ or $s_I : O_I \to R$, where $I \subseteq N$ and a set of results R is in the structure (R, Σ) together with an abstract operation $\Sigma : \mathcal{P}(R) \to R$ of composing results. For every joint observation $e \in O_I$, the local state s_I is defined as:

$$s_I\big((e_a)_{a \in I}\big) := \Sigma\big\{s(o) : o \in O_{a_1} \times \cdots \times O_{a_n} \text{ such that } o_a = e_a \text{ for all } a \in I\big\}.$$

If s and t are two possible states of the system and a group of agents I can make exactly the same observations in these two states ($s_I = t_I$), then these states are observationally equivalent to I, and it is written as $s \overset{I}{\sim} t$. A model of logic of correlated knowledge is a multi-modal Kripke model [4], where the relations between states mean observational equivalence.

Definition 2 [Model of logic of correlated knowledge]. For a set of states S, a family of binary relations $\{\overset{I}{\sim}\}_{I \subseteq N} \subseteq S \times S$ and a function of interpretations $V : S \to (P \to \{\mathit{true}, \mathit{false}\})$, where P is a set of atomic propositions, a model of logic of correlated knowledge is a multi-modal Kripke model $(S, \{\overset{I}{\sim}\}_{I \subseteq N}, V)$ with some epistemic conditions [1].

The satisfaction relation $\models$ for formula o^r is defined as: $M, s \models o^r$ iff $s_I(o) = r$.

2 Quantum mechanics

2.1 Superposition

Quantum superposition is a fundamental principle of quantum mechanics. It says that an elementary particle, such as an electron, exists partly in all its theoretically possible states simultaneously. But when measured or observed, it gives a result corresponding to only one of the possible configurations.An example of a directly observable effect of superposition is a spin state of an electron. Electron spin can be modelled as a unit three-dimensional vector associated with the particle, representing an axis of rotation.

We fix an electron's position in space. In order to prepare an electron in a particular direction, the electron is surrounded in a powerful magnetic field. The magnetic field forces the electron's spin to end up in the desired direction after a certain amount of time. Suppose that an electron has been prepared in some unknown direction and we want to be able to measure the electron's spin. We could again surround the electron with a known magnetic field. What actually happens is that one photon is emitted or no photon is emitted by the electron. If a photon is emitted, then its associated frequency corresponds to the amount of energy that would be radiated if the electron had been prepared in the North – down position. Note that the actual result – that is, one photon emitted or no photon emitted – doesn't depend on either the prepared angle or the detection angle. In fact, the outcomes of any experiment are probabilistic. This probability depends on the angle. Qualitively, the smaller the angle (between prepared and detection states) the less likely that a photon is emitted. So, information about the prepared angle can be statistically recovered from repeated experiments, but to re-iterate, only one of two outcomes can occur per detection.

The state of the electron can be represented as spin up $|\uparrow\rangle$ and spin down $|\downarrow\rangle$ or equivalently $|0\rangle$ and $|1\rangle$ in Dirac's ket notation. These two states are known as basis states in which a qubit may be measured [2]. The qubit is a linear combination of two basis states $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are probability amplitudes and $\alpha, \beta \in \mathbb{C}$ (set of complex numbers). When we measure this qubit, the probability of outcome $|0\rangle$ is $|\alpha|^2$ and the probability of outcome $|1\rangle$ is $|\beta|^2$. If we have two qubits, the state of the system is defined by linear combination of four basis states $|\psi\rangle = \alpha_1|00\rangle + \alpha_2|01\rangle + \alpha_3|10\rangle + \alpha_4|11\rangle$ (the probability of outcome $|00\rangle$ is $|\alpha_1|^2$, the probability of outcome $|01\rangle$ is $|\alpha_2|^2$, etc.). Because the absolute squares of the amplitudes equate to probabilities, it follows that sum of probabilities must be equal to 1.

2.2 Quantum entanglement

Quantum entanglement is a special connection between pairs or groups of quantum systems. Like the quantum states of individual particles, the state of an entangled system is defined as a sum, or superposition, of basis states. Consider two systems A and B, with respective basis states $\{|0\rangle_A, |1\rangle_A\}$ for A, and $\{|0\rangle_B, |1\rangle_B\}$ for B. The following is an entangled Bell state:

$$\frac{1}{\sqrt{2}}\bigl(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B\bigr).$$

Suppose Alice is an observer for system A, and Bob is an observer for system B. If in the entangled state given above Alice makes a measurement, there are two possible outcomes, occurring with equal probability [5]:

- Alice measures 0, and the state of the system collapses to $|0\rangle_A|1\rangle_B$. Any subsequent measurement performed by Bob, will always return 1.
- Alice measures 1, and the state of the system collapses to $|1\rangle_A|0\rangle_B$. Any subsequent measurement performed by Bob will always return 0.

Entanglement is broken when the entangled particles decohere through interaction with the environment, for example, when a measurement is made [6]. Thus, system B has been altered by Alice performing a local measurement on system A. This remains true even if the systems A and B are spatially separated.

3 Logic of correlated knowledge and quantum mechanics

Suppose our quantum system consists of two elementary particles (electrons), which are spatially separated. To each particle we associate one agent. Thus our set of agents is $N = \{A, B\}$. Possible observations for agent A are $O_A = \{0, 1\}$, and possible observations for agent B are $O_B = \{0, 1\}$. The result set R is interpreted as

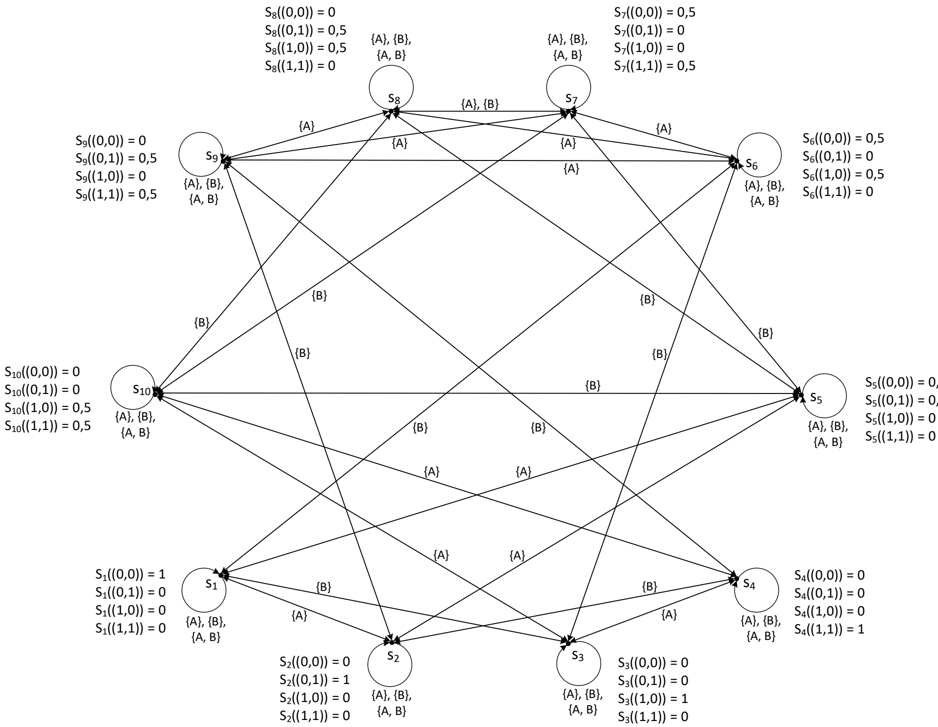

Fig. 1. Model of logic of correlated knowledge ($R = \{0, 0.5, 1\}$).

the probabilities of the outcomes for the local observations of the agents and for the joint observations of the group of agents. We model knowledge of agents, having result set $R = \{0, 0.5, 1\}$. For simplicity wider range of probabilities are not included as well as negative probabilities. We consider addition as operation Σ of result composition. As mentioned before in the section of quantum mechanics the sum of all probabilities of any state must be equal to 1.

Having result set $R = \{0, 0.5, 1\}$ our two-qubit system can be in ten possible states in the model of logic of correlated knowledge (Fig. 1), where states are defined by functions $s_i : O_A \times O_B \to R$ ($i \in \{1, \ldots, 10\}$).

Suppose real state of the world is s_1. In this situation agent A after local observation gets 0 with probability 1 (100%), and never gets 1. In the state s_2 he gets same observational results with same probabilities as in s_1. Despite the fact that global states s_1 and s_2 are different, agent A does not distinguish them and does'nt know that real state is s_1. Thus we have equivalence relation $\{A\}$ between states s_1 and s_2.

Considering knowledge of agent A and agent B, we can analyze some formulas and their satisfiability:

- $s_1 \models (0_A)^1$. Agent A after local observation gets 0 with probability 1. This is true at state s_1.
- $s_1 \models K_A(0_A)^1$. Agent A knows that after local observation he gets 0 with probability 1. This is true at state s_1.

- $s_1 \not\models K_A(0_B)^1$. Agent A knows that after local observation agent B gets 0 with probability 1. This is incorrect because actually agent A does'nt know what agent B gets after local observation. Also we can notice that agent B gets 0 with probability 1 at state s_1 but agent A does'nt know this.
- $s_1 \models \neg K_A(0_B)^1$. Agent A does'nt know what agent B gets after local observation. This is true at state s_1.
- $s_1 \models K_A((0_B)^0 \vee (0_B)^{0.5} \vee (0_B)^1)$. Agent A knows that after local observation agent B gets 0 with probability 0, 0.5 or 1. This is true at state s_1.
- $s_1 \models K_A K_B((1_B)^0 \vee (1_B)^{0.5} \vee (1_B)^1)$. Agent A knows that agent B knows that after local observation agent B gets 1 with probability 0, 0.5 or 1. This is true at state s_1.
- $s_8 \models K_A((0_A)^{0.5} \wedge (1_A)^{0.5}) \wedge K_B((0_B)^{0.5} \wedge (1_B)^{0.5})$. Agent A knows that after local observation he gets 0 with probability 0.5 and he gets 1 with probability 0.5. Also agent B knows that after local observation he gets 0 with probability 0.5 and he gets 1 with probability 0.5. This is true at state s_8. This state is also known as Bell state. Pooling together all information of both agents (and closing under logical inference) does not lead us to Bell state where two particles are entangled. Joint observation of both agents needs to be done, to extract the correlation of knowledge.
- $s_8 \models K_{\{A,B\}}((0_A, 1_B)^{0.5} \wedge (1_A, 0_B)^{0.5})$. Group of agents $\{A, B\}$ know that after joint observation they get $(0_A, 1_B)$ with probability 0.5 and they get $(1_A, 0_B)$ with probability 0.5. This is true at state s_8. The correlation between knowledge of agents associated to entangled particles has been extracted.

Logic of correlated knowledge allows us to model knowledge of agents, associated to quantum systems. Also using Gentzen type sequent calculus GS-LCK presented in [3], logical inference about such knowledge can be done. Having premises satisfiable at some set of models, conclusions can be checked if they also true in these models.

Conclusions

Associating agents to particles of quantum systems, allows us to model knowledge of agents and extract states of quantum entanglement, using logic of correlated knowledge. This can not be done by traditional epistemic logics. Pooling together all information of agents to one place and closing under logical inference does not lead us to correlation of knowledge. Observational capabilities of agents expands the range of applications of family of epistemic logics and possibly captures deeper knowledge of the group of agents.

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REZIUMĖ

Koreliatyvių žinių logikos taikymas kvantinėje mechanikoje

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Straipsnyje pateikiamas agentų, susietų su kvantinės sistemos dalelėmis, žinių modeliavimas. Taip pat buvo atlikta formulių ir jų įvykdomumo modelyje analizė.

Raktiniai žodžiai: koreliatyvių žinių logika, kvantinė sistema, agentas.