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Three Algebraic Conjugates with Zero Sum: A Graph-Theoretic Approach

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Abstract

We give an alternative graph-theoretic proof that 20 is the least degree not divisible by 3 of an algebraic number having three distinct conjugates with zero sum. The proof excludes degree 16 through local constraints on vertex-transitive graphs of valency 6. The underlying graph construction can be studied in arbitrary degree, without the special degree restrictions of our group-theoretic arguments.

Keywords: algebraic numbers; linear relations in algebraic conjugates; vertex-transitive graphs

MSC: 11R04; 11R32; 05C25; 20B25

1. Introduction

Let α be an algebraic number of degree d over $\mathbb{Q}$, and suppose that three distinct algebraic conjugates $\alpha_1, \alpha_2, \alpha_3$ of α satisfy $\alpha_1 + \alpha_2 + \alpha_3 = 0$. In 2004, Dubickas and Smyth [1] asked whether d must be divisible by 3. Stong constructed a counterexample of degree 20: the irreducible polynomial

$$x^{20} + 4 \cdot 5^9 x^{10} + 16 \cdot 5^{15}$$

has three distinct roots whose sum is zero (see [1]). This raises the natural question of determining the least degree d not divisible by 3 for which such a relation can occur. Recently, we established the following result:

Theorem 1 ([2] (to appear in Proc. Am. Math. Soc.)). *Let $d > 1$ be the smallest positive integer not a multiple of 3 for which there exists an algebraic number of degree d whose three algebraic conjugates sum to zero. Then $d = 20$.*

The relation among three conjugates is a special case of

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_d\alpha_d = 0, \tag{1}$$

where $\alpha_1, \ldots, \alpha_d$ are the conjugates of an algebraic number of degree d and $a_1, \ldots, a_d \in \mathbb{Z}$ are not all zero. Such a relation is called trivial when $a_1 = \cdots = a_d$. For prime d, Kurbatov proved that every such relation is trivial [3]; related foundational results appear in [4,5], and Girstmair developed a representation-theoretic framework for the general problem [6]. Sparse additive relations have subsequently been studied in [7–17], with a multiplicative analogue considered in [18].

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This paper develops the graph-theoretic proof introduced in [19], giving an alternative proof of Theorem 1. In the remaining case $d = 16$, zero-sum triples yield a Galois-invariant linear 3-uniform hypergraph and a vertex-transitive graph of valency 6. Local constraints on its triangles reduce the arithmetic problem to explicit exclusions within the census of 40 such graphs [20,21].

Our group-theoretic arguments in [2] apply to the special degree families $d = p^m$ and $d = 2p^m$: they exclude zero-sum triples when $p \neq 3$ in the first family and $p \geq 5$ in the second, where p is prime and $m \geq 1$. These arguments use the corresponding orbit structure of Sylow subgroups. The graph construction has no such restriction on the degree and can, in principle, be studied for arbitrary d. Proposition 1 supplies the general incidence constraints, while the degree-16 analysis illustrates how the local restrictions in Lemma 8 can be combined with finite census computations. This formulation allows the structural arguments and the explicit graph witnesses to be checked separately.

Section 2 collects the auxiliary results, and Section 3 gives the proof of Theorem 1. Appendix A specifies the complete executable verification script, census data, and recorded output supplied with this manuscript.

2. Auxiliary Results

The first general result related to linear relations among algebraic conjugates of a given algebraic number was obtained by Kurbatov [3]:

Lemma 1. *The equality*

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d = 0$$

with conjugates $\alpha_1, \alpha_2, \ldots, \alpha_d$ of an algebraic number α of prime degree d over $\mathbb{Q}$ and $k_1, k_2, \ldots, k_d \in \mathbb{Z}$ can only hold if $k_1 = k_2 = \cdots = k_d$.

The result of Smyth [22] will be used several times in the proof of Theorem 1 to eliminate impossible relations among algebraic conjugates:

Lemma 2. *If $\alpha_1, \alpha_2, \alpha_3$ are three conjugates of an algebraic number satisfying $\alpha_1 \neq \alpha_2$, then $2\alpha_1 \neq \alpha_2 + \alpha_3$.*

Dubickas [23] (see also [5], Theorem 3′) generalized Lemma 2 as follows:

Lemma 3. *If $\alpha_1, \alpha_2, \ldots, \alpha_d$, where $d \geqslant 3$, are distinct algebraic numbers conjugate over a field K of characteristic zero, and $k_1, k_2, \ldots, k_d$ are non-zero rational numbers satisfying $|k_1| \geqslant |k_2| + \cdots + |k_d|$, then*

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d \notin K.$$

Also, we will use the result proved by Dubickas and Jankauskas [14].

Lemma 4. *The equality*

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d = 0$$

with conjugates $\alpha_1, \alpha_2, \ldots, \alpha_d$ of an algebraic number α of degree d over $\mathbb{Q}$ and $k_1, k_2, \ldots, k_d \in \mathbb{Z}$ satisfying $\sum_{i=1}^{d} k_i \neq 0$ can only hold if $tr(\alpha) := \alpha_1 + \alpha_2 + \cdots + \alpha_d = 0$.

3. Proof of Theorem 1

Proof. Let d be a positive integer not divisible by 3. Let α be an algebraic number over $\mathbb{Q}$ of degree d whose three algebraic conjugates sum to zero. Suppose that d is the smallest possible such positive integer.

By the example of Stong [1], $d \leqslant 20$. Dubickas and Jankauskas proved that $d \geqslant 10$ (see Theorem 1.2 in [14]). By Lemma 1, d cannot be a prime number. Thus $d \in \{10, 14, 16, 20\}$. Recently, Virbalas eliminated the cases $d = 10$ and $d = 14$ by proving that $d \neq 2p$, where $p \geqslant 5$ is a prime number (see Theorem 1.1 in [24]). So $d = 16$ or $d = 20$. Our goal is to show that $d \neq 16$. This will establish that $d = 20$.

Assume, to the contrary, that there exists an algebraic number α of degree $d = 16$ whose algebraic conjugates $\alpha_1, \alpha_2, \alpha_3$ satisfy the relation

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{2}$$

Clearly, these three conjugates cannot all be equal. Moreover, if some two conjugates are equal, say $\alpha_1 = \alpha_2$, then $2\alpha_1 + \alpha_3 = 0$, which is impossible (we can choose an automorphism π in the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ which maps α_1 to α' having the maximal absolute value m; then $2\alpha_1 + \alpha_3 = 0$ is mapped to $2\alpha' + \pi(\alpha_3) = 0$ and $2m = |2\alpha'| = |-\pi(\alpha_3)| \leqslant m$, which is impossible). Hence, all three conjugates in (2) are distinct.

Let $\alpha_1, \alpha_2, \ldots, \alpha_{16}$ be the algebraic conjugates of α. Lemma 4 implies that $tr(\alpha) = \alpha_1 + \alpha_2 + \cdots + \alpha_{16} = 0$. Let G be the Galois group of the normal closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Note that this normal closure is also the splitting field of the minimal polynomial of α over $\mathbb{Q}$, and therefore G is the Galois group of this polynomial. The group G corresponds to some transitive subgroup of the full symmetric group S_{16}.

Two relations $\alpha_i + \alpha_j + \alpha_k = 0$ and $\alpha_{i'} + \alpha_{j'} + \alpha_{k'} = 0$, where all the summands are algebraic conjugates of α, are called distinct if

$$\{\alpha_i, \alpha_j, \alpha_k\} \neq \{\alpha_{i'}, \alpha_{j'}, \alpha_{k'}\}.$$

Lemma 5. *For any two distinct relations $\alpha_i + \alpha_j + \alpha_k = 0$ and $\alpha_{i'} + \alpha_{j'} + \alpha_{k'} = 0$, we have $|\{i, j, k\} \cap \{i', j', k'\}| \leqslant 1$.*

Proof. Since the relations are distinct, we have that $|\{i, j, k\} \cap \{i', j', k'\}| \leqslant 2$. On the other hand, if, say, $\{j, k\} = \{j', k'\}$, then $\alpha_i = -\alpha_j - \alpha_k = -\alpha_{j'} - \alpha_{k'} = \alpha_{i'}$. Since the conjugates are distinct, $i = i'$, so the two relations coincide, a contradiction. Hence, the claim follows. □

Lemma 6. *For any fixed $i_0 \in \{1, 2, \ldots, 16\}$, the number of distinct relations of the form $\alpha_{i_0} + \alpha_j + \alpha_k = 0$ is less than 6.*

Proof. Assume, to the contrary, that for a fixed i_0 there exist six distinct relations of the form $\alpha_{i_0} + \alpha_j + \alpha_k = 0$. Lemma 5 implies that for any two distinct relations $\alpha_{i_0} + \alpha_{j_1} + \alpha_{k_1} = 0$ and $\alpha_{i_0} + \alpha_{j_2} + \alpha_{k_2} = 0$ the sets $\{j_1, k_1\}$ and $\{j_2, k_2\}$ are disjoint. Without loss of generality, we consider the following six relations (after relabeling the conjugates of α, if necessary):

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 &= 0,\\
\alpha_1 + \alpha_4 + \alpha_5 &= 0,\\
\alpha_1 + \alpha_6 + \alpha_7 &= 0,\\
\alpha_1 + \alpha_8 + \alpha_9 &= 0,\\
\alpha_1 + \alpha_{10} + \alpha_{11} &= 0,\\
\alpha_1 + \alpha_{12} + \alpha_{13} &= 0.
\end{aligned}$$

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By adding all of them and using $tr(\alpha) = 0$, we obtain

$$6\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{13} = 5\alpha_1 - \alpha_{14} - \alpha_{15} - \alpha_{16} = 0,$$

which is impossible by Lemma 3. □

We say that two relations $\alpha_i + \alpha_j + \alpha_k = 0$ and $\alpha_{i'} + \alpha_{j'} + \alpha_{k'} = 0$ are *conjugate* if there exists an automorphism $\pi \in G$ such that

$$\{\alpha_i, \alpha_j, \alpha_k\} = \{\pi(\alpha_{i'}), \pi(\alpha_{j'}), \pi(\alpha_{k'})\}.$$

One can easily see that this conjugacy relation is an equivalence relation on the set of all possible relations of the form $\alpha_i + \alpha_j + \alpha_k = 0$.

Lemma 7. *The following statements are true.*

(i) Any two relations of the form $\alpha_i + \alpha_j + \alpha_k = 0$ are conjugate.

(ii) There are exactly 16 distinct such relations.

(iii) Each algebraic conjugate of α appears in exactly 3 distinct such relations.

Proof. Consider a relation $\alpha_i + \alpha_j + \alpha_k = 0$ and its equivalence class $\mathcal{C}(\alpha_i, \alpha_j, \alpha_k)$, consisting of all the distinct relations that are conjugate to $\alpha_i + \alpha_j + \alpha_k = 0$. Let $N = |\mathcal{C}(\alpha_i, \alpha_j, \alpha_k)|$. Since the Galois group G acts transitively on the set $\{\alpha_1, \alpha_2, \ldots, \alpha_{16}\}$, each conjugate α_i appears an equal number of times, say l, in the relations in $\mathcal{C}(\alpha_i, \alpha_j, \alpha_k)$. We have N distinct relations in $\mathcal{C}(\alpha_i, \alpha_j, \alpha_k)$ and each such relation involves three distinct conjugates of α. Thus, there are $3N$ appearances of conjugates of α in $\mathcal{C}(\alpha_i, \alpha_j, \alpha_k)$. On the other hand, this number can be counted in a different way — each conjugate of α appears exactly l times. Hence, $3N = 16l$. Since the class is nonempty and G acts transitively, $l > 0$. As $\gcd(3, 16) = 1$, we have $3 \mid l$. Lemma 6 gives $l < 6$, so $l = 3$ and $N = 16$.

Assume that there are two distinct equivalence classes. Each contains three distinct relations involving α_1. Thus, in total, we have six distinct relations involving α_1. This contradicts Lemma 6. Hence, the lemma follows. □

By Lemma 7, we have exactly 16 distinct relations of the form $\alpha_i + \alpha_j + \alpha_k = 0$ and each α_i appears in exactly 3 distinct such relations. Without loss of generality, we assume the following system of distinct relations:

$$\begin{array}{ccccc}
\alpha_1 & + & \alpha_2 & + & \alpha_3 = 0, \\
\alpha_1 & + & \alpha_4 & + & \alpha_5 = 0, \\
\alpha_1 & + & \alpha_6 & + & \alpha_7 = 0, \\
\alpha_{i_{41}} & + & \alpha_{i_{42}} & + & \alpha_{i_{43}} = 0, \\
\alpha_{i_{51}} & + & \alpha_{i_{52}} & + & \alpha_{i_{53}} = 0, \\
\vdots & & \vdots & & \vdots \\
\alpha_{i_{161}} & + & \alpha_{i_{162}} & + & \alpha_{i_{163}} = 0.
\end{array} \tag{3}$$

This system of relations is equivalent to the following one:

$$\begin{array}{rcccccccl}
-\alpha_1 & = & \alpha_2 & + & \alpha_3 & = & \alpha_4 & + & \alpha_5 \quad = \quad \alpha_6 \quad + \quad \alpha_7, \\
-\alpha_2 & = & \alpha_1 & + & \alpha_3 & = & \alpha_{j_{23}} & + & \alpha_{j_{24}} \quad = \quad \alpha_{j_{25}} \quad + \quad \alpha_{j_{26}}, \\
-\alpha_3 & = & \alpha_1 & + & \alpha_2 & = & \alpha_{j_{33}} & + & \alpha_{j_{34}} \quad = \quad \alpha_{j_{35}} \quad + \quad \alpha_{j_{36}}, \\
-\alpha_4 & = & \alpha_1 & + & \alpha_5 & = & \alpha_{j_{43}} & + & \alpha_{j_{44}} \quad = \quad \alpha_{j_{45}} \quad + \quad \alpha_{j_{46}}, \\
-\alpha_5 & = & \alpha_1 & + & \alpha_4 & = & \alpha_{j_{53}} & + & \alpha_{j_{54}} \quad = \quad \alpha_{j_{55}} \quad + \quad \alpha_{j_{56}}, \\
-\alpha_6 & = & \alpha_1 & + & \alpha_7 & = & \alpha_{j_{63}} & + & \alpha_{j_{64}} \quad = \quad \alpha_{j_{65}} \quad + \quad \alpha_{j_{66}}, \\
-\alpha_7 & = & \alpha_1 & + & \alpha_6 & = & \alpha_{j_{73}} & + & \alpha_{j_{74}} \quad = \quad \alpha_{j_{75}} \quad + \quad \alpha_{j_{76}}, \\
-\alpha_8 & = & \alpha_{j_{81}} & + & \alpha_{j_{82}} & = & \alpha_{j_{83}} & + & \alpha_{j_{84}} \quad = \quad \alpha_{j_{85}} \quad + \quad \alpha_{j_{86}}, \\
\vdots & & & & \vdots & & & & \vdots \\
-\alpha_{16} & = & \alpha_{j_{161}} & + & \alpha_{j_{162}} & = & \alpha_{j_{163}} & + & \alpha_{j_{164}} \quad = \quad \alpha_{j_{165}} \quad + \quad \alpha_{j_{166}}.
\end{array} \tag{4}$$

Note that $-\alpha_1, -\alpha_2, \ldots, -\alpha_{16}$ are the algebraic conjugates of $-\alpha_1$. Since $\alpha_1 + \alpha_2 = -\alpha_3$ and $\mathbb{Q}(-\alpha_3) = \mathbb{Q}(\alpha_3)$, we have $\deg(\alpha_1 + \alpha_2) = 16$. Each sum $\alpha_i + \alpha_j$ appearing in (4) is therefore an algebraic conjugate of $\alpha_1 + \alpha_2$.

Note that each algebraic conjugate of $\alpha_1 + \alpha_2$ has exactly three distinct representations as a sum $\alpha_i + \alpha_j$ (see (iii) in Lemma 7).

Now, we will introduce the graph-theoretic approach to this problem. Let $\mathcal{G} = (V, E)$ be the finite simple undirected graph with $V := \{\alpha_1, \ldots, \alpha_{16}\}$. For $i \neq j$, the vertices α_i and α_j are joined by an edge, denoted (α_i, α_j) or (α_j, α_i), if $\alpha_i + \alpha_j$ is a conjugate of $\alpha_1 + \alpha_2$, equivalently, if this sum appears in (4).

Note that each vertex of $\mathcal{G}$ must have the same number of edges. In other words, the graph $\mathcal{G}$ is regular. This follows directly from the fact that each α_i appears exactly six times in (4) (or, equivalently, each α_i appears in three distinct relations $\alpha_i + \alpha_j + \alpha_k = 0$ and each such relation implies two edges (α_i, α_j) and (α_i, α_k), connecting the vertex α_i), i.e., $\deg(v) = 6$ for each $v \in V$. In such a setting, $\mathcal{G}$ must have $(16 \cdot 6)/2 = 48$ edges. Furthermore, since the Galois group G is transitive on the set of conjugates $\{\alpha_1, \alpha_2, \ldots, \alpha_{16}\}$ and, by Lemma 7, any two relations of the form $\alpha_i + \alpha_j + \alpha_k = 0$ are conjugate, the graph $\mathcal{G}$ must be vertex-transitive. It is known that there are exactly 40 vertex-transitive graphs on 16 vertices of degree 6. This was established by McKay and Royle [20] in their complete enumeration of transitive graphs on at most 26 vertices. The explicit list is maintained in [21] (https://doi.org/10.5281/zenodo.4010122) (see also the House of Graphs [25]). All 40 of these graphs, named $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{40}$, are given in Table 1. Note that these graphs are given in graph6 format (this format was invented by Brendan McKay [26,27] and is recognized by SageMath [28]).

The goal is to prove that for every $i \in \{1, 2, \ldots 40\}$, the graphs $\mathcal{G}$ and $\mathcal{G}_i$ are not isomorphic.

Note that a relation $\alpha_i + \alpha_j + \alpha_k = 0$ in (3) implies the three equalities $\alpha_i + \alpha_j = -\alpha_k, \alpha_i + \alpha_k = -\alpha_j$, and $\alpha_j + \alpha_k = -\alpha_i$. Thus, the three sums on the left are algebraic conjugates of $\alpha_1 + \alpha_2 = -\alpha_3$. In other words, edges $(\alpha_i, \alpha_j), (\alpha_i, \alpha_k)$, and (α_j, α_k) form a triangle in our graph $\mathcal{G}$. On the other hand, if, for three distinct conjugates $\alpha_a, \alpha_b, \alpha_c$, all three edges $(\alpha_a, \alpha_b), (\alpha_b, \alpha_c)$ and (α_a, α_c) belong to $\mathcal{G}$, then the relation $\alpha_a + \alpha_b + \alpha_c = 0$ need not hold. If it does, then we say that the vertices $\alpha_a, \alpha_b, \alpha_c$ form a *zero-sum triangle* $\triangle(\alpha_a, \alpha_b, \alpha_c)$.

Write $\deg_{\mathcal{G}}(v)$ for the degree of a vertex v in a graph $\mathcal{G}$, $N(v)$ for its open neighborhood, and $N[v] = N(v) \cup \{v\}$ for its closed neighborhood. The induced graph on a vertex set

retains exactly the edges with both endpoints in that set. We denote the graph induced on $N[v]$ by $\mathcal{G}(v)$; degrees in $\mathcal{G}(v)$ are taken within this induced graph.

Table 1. All 40 vertex-transitive graphs on 16 vertices of degree 6 (see [21]).

Graph	Code in **graph6** Format
$\mathcal{G}_1$	`OsaC???FzrMkYwXwJw@|?`
$\mathcal{G}_2$	`OsaC???RxnNKYw\WJs@}?`
$\mathcal{G}_3$	`OsaC???]ZrK{XwFwB{B{?`
$\mathcal{G}_4$	`OsaCB@_EWrKrXeFwB{B{?`
$\mathcal{G}_5$	`OsaKYCQGbRIjXKLWJEHqc`
$\mathcal{G}_6$	`OsaKYCcQXRBKSbLDmTBi_`
$\mathcal{G}_7$	`OsaKYDDGgdLBUELQeibr?`
$\mathcal{G}_8$	`OsaKYPDHOiDFEM[wMPRcg`
$\mathcal{G}_9$	`OsaKg?dQGid\[S[qLSQy_`
$\mathcal{G}_{10}$	`OsaKg?hSZEkkTIIdJWPyA`
$\mathcal{G}_{11}$	`OsaKiCaC‘RbMYKTYLabiC`
$\mathcal{G}_{12}$	`OsaKiCdPPDaUBR]WNAboS`
$\mathcal{G}_{13}$	`OsaSWSTOhDKiXQUEfBbr?`
$\mathcal{G}_{14}$	`OsaSXCdOha‘T[DRRNEAyC`
$\mathcal{G}_{15}$	`OsaSXDCSXRbKTBIdmSBY@`
$\mathcal{G}_{16}$	`OsaSYHBGpH‘XDL]SNDBoK`
$\mathcal{G}_{17}$	`Osedw?DOYFHBSZW\Fg@w‘`
$\mathcal{G}_{18}$	`Osedw?HOYFGbSZW\Fg@w‘`
$\mathcal{G}_{19}$	`Osedw@??pJhMP\UWEjBpA`
$\mathcal{G}_{20}$	`OsfDw?@G‘bgmW\RWFFAyA`
$\mathcal{G}_{21}$	`OsfDw@@GXPCZP]TSFDayA`
$\mathcal{G}_{22}$	`OsfLg?@WYHcZQYKXJcPuA`
$\mathcal{G}_{23}$	`Osqsw@@AXCclW]USEXaxA`
$\mathcal{G}_{24}$	`Osqsy@@GWRcdGtUEESqyA`
$\mathcal{G}_{25}$	`OtaCXOTHOphhRKSsKTJcK`
$\mathcal{G}_{26}$	`OtaLw?@OaRgn[S[Kdk?z@`
$\mathcal{G}_{27}$	`OtaLw?@ObBiNQ[P[fg@x@`
$\mathcal{G}_{28}$	`OtaLw?BOBBhNP[S[fa@y@`
$\mathcal{G}_{29}$	`OtaLy@@AWJg[WFSFfg@x@`
$\mathcal{G}_{30}$	`OtaLyD‘SYQGhKBKB‘eOXb`
$\mathcal{G}_{31}$	`Ota\W@@GYChJS]PZFc@u@`
$\mathcal{G}_{32}$	`Otakw?@QYbKLPLOufc@u@`
$\mathcal{G}_{33}$	`Otaky@@GWbhBPROlfc@u@`
$\mathcal{G}_{34}$	`OtrTOGBW@‘hIP]G|Bg_tP`
$\mathcal{G}_{35}$	`Ouj\w?@?YBcMSUILIhPTI`
$\mathcal{G}_{36}$	`O{fL_@HKQJCZCsBW_pzp_`
$\mathcal{G}_{37}$	`O{fL_CCQP‘GmCzB[C\Joo`
$\mathcal{G}_{38}$	`O}akqPPWOV@iHIDHcROcj`
$\mathcal{G}_{39}$	`O~aKYPDOxQBHHIGeacocj`
$\mathcal{G}_{40}$	`O~z\w?@?WB_M?Z?V_Fz{?`

Lemma 8. *The following statements are true for the graph $\mathcal{G}$.*

(i) Two distinct zero-sum triangles share at most one vertex.

(ii) Every edge (α, α'), $\alpha \neq \alpha'$, belongs to exactly one zero-sum triangle $\triangle(\alpha, \alpha', \alpha'')$ and the vertex α'' belongs to the open neighborhood $N(\alpha)$ of α.

(iii) If $\alpha, \alpha', \alpha''$ are three distinct vertices such that $\alpha', \alpha'' \in N(\alpha)$, $\deg_{\mathcal{G}(\alpha)}(\alpha') = 2$ and (α', α'') is an edge in $\mathcal{G}$, then $\triangle(\alpha, \alpha', \alpha'')$ is a zero-sum triangle.

(iv) If $\triangle(\alpha, \alpha_a, \alpha_b)$ and $\triangle(\alpha, \alpha_c, \alpha_d)$ are two distinct zero-sum triangles, then either (α_a, α_d) or (α_b, α_c) is not an edge in $\mathcal{G}$.

Proof. Part (i) follows from the definition of the graph $\mathcal{G}$ and Lemma 5.

(ii). Let (α, α'), $\alpha \neq \alpha'$, be an edge in $\mathcal{G}$. Then, according to the definition of $\mathcal{G}$, the sum $\alpha + \alpha'$ is an algebraic conjugate of $\alpha_1 + \alpha_2 = -\alpha_3$. Hence, $\alpha + \alpha' = -\alpha''$ for a conjugate α''. If α'' equaled either endpoint, we would obtain $2\beta + \gamma = 0$ for conjugates β, γ, which

the maximal-absolute-value argument following (2) excludes. Thus, the three vertices are distinct, and $\triangle(\alpha, \alpha', \alpha'')$ is a zero-sum triangle. Part (i) makes it the unique zero-sum triangle containing (α, α'). Moreover, (α, α'') is also an edge in $\mathcal{G}$, since $\alpha + \alpha'' = -\alpha'$ is an algebraic conjugate of $\alpha_1 + \alpha_2 = -\alpha_3$. So α'' is adjacent to α, $\alpha'' \neq \alpha$, and therefore $\alpha'' \in N(\alpha)$.

(iii). Since $\alpha' \in N(\alpha)$, we have that (α, α') is an edge in $\mathcal{G}$. Applying part (ii), we obtain that there exists a vertex $\tilde{\alpha} \in N(\alpha)$ such that the triangle $\triangle(\alpha, \alpha', \tilde{\alpha})$ is a zero-sum triangle. This, together with $\deg_{\mathcal{G}(\alpha)}(\alpha') = 2$, implies that in the induced graph $\mathcal{G}(\alpha)$ the vertex α' is adjacent only to α and $\tilde{\alpha}$. On the other hand, the edge (α', α'') is in $\mathcal{G}$ and $\alpha', \alpha'' \in N(\alpha)$. Thus, the edge (α', α'') belongs to the induced graph $\mathcal{G}(\alpha)$, and therefore α' is adjacent to α'' in $\mathcal{G}(\alpha)$. Hence, $\tilde{\alpha} = \alpha''$ and the claim follows.

(iv). Assume, to the contrary, that $\triangle(\alpha, \alpha_a, \alpha_b)$ and $\triangle(\alpha, \alpha_c, \alpha_d)$ are two distinct zero-sum triangles and both (α_a, α_d) and (α_b, α_c) are edges in $\mathcal{G}$. Part (ii) implies the existence of zero-sum triangles $\triangle(\alpha_a, \alpha_d, \alpha')$ and $\triangle(\alpha_b, \alpha_c, \alpha'')$. Note that $\alpha' \neq \alpha$, since, in view of part (i), zero-sum triangles $\triangle(\alpha, \alpha_a, \alpha_b)$ and $\triangle(\alpha_a, \alpha_d, \alpha')$ share exactly one vertex α_a. Similarly, $\alpha'' \neq \alpha$. Now, the four obtained zero-sum triangles imply the relations

$$\begin{aligned} \alpha + \alpha_a + \alpha_b &= 0, \\ \alpha + \alpha_c + \alpha_d &= 0, \\ \alpha_a + \alpha_d + \alpha' &= 0, \\ \alpha_b + \alpha_c + \alpha'' &= 0. \end{aligned} \tag{5}$$

By adding the first two relations in (5) and using the expressions for α' and α'' from the last two relations in (5), we obtain

$$2\alpha + (\alpha_a + \alpha_d) + (\alpha_b + \alpha_c) = 2\alpha - \alpha' - \alpha'' = 0.$$

This contradicts Lemma 2, since $\alpha' \neq \alpha$. □

Proposition 1. *Let V be the full set of n conjugates of an algebraic number over $\mathbb{Q}$, and suppose that its set $\mathcal{B}$ of zero-sum triples of distinct elements is nonempty. Then $\mathcal{B}$ is a Galois-invariant linear 3-uniform hypergraph: distinct blocks meet in at most one vertex, and each vertex belongs to the same positive number r of blocks. Its pair graph Γ, whose edges join distinct vertices in a common block, is vertex-transitive and $2r$-regular. Its nr edges are partitioned into the $nr/3$ block triangles. Moreover,*

$$rv + \sum_{w \in N_\Gamma(v)} w = 0 \qquad (v \in V),$$

so distinct vertices have distinct open neighborhoods. In the degree-16 setting above, $\Gamma = \mathcal{G}$, $r = 3$, and there are 16 blocks and 48 edges.

Proof. Two zero-sum triples containing the same pair have the same third vertex. Galois transitivity makes the block degree $r > 0$ constant and makes Γ vertex-transitive. Linearity gives $2r$ distinct neighbors per vertex; counting incidences gives the asserted totals. Summing the r block relations through v gives the displayed identity. Equal open neighborhoods would imply $r(v - v') = 0$, hence $v = v'$. Finally, Lemma 7 gives $r = 3$ in the degree-16 setting. □

First, consider the graph shown in Figure 1.

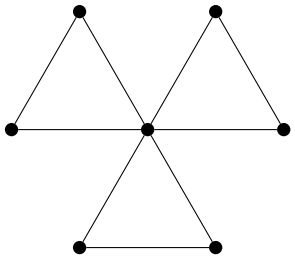

Figure 1. Graph `FyTAG` in graph6 format.

Notice that the first three equations in (3) imply that this graph must be a subgraph of $\mathcal{G}$. We check with SageMath [28] that none of the following 29 graphs

$$\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{19}; \quad \mathcal{G}_{25}, \mathcal{G}_{26}, \ldots, \mathcal{G}_{33}; \quad \mathcal{G}_{39}$$

has a subgraph isomorphic to the graph `FyTAG`. Hence, $\mathcal{G}$ is isomorphic to one of the graphs in the following set

$$\{\mathcal{G}_{20}, \mathcal{G}_{21}, \mathcal{G}_{22}, \mathcal{G}_{23}, \mathcal{G}_{24}, \mathcal{G}_{34}, \mathcal{G}_{35}, \mathcal{G}_{36}, \mathcal{G}_{37}, \mathcal{G}_{38}, \mathcal{G}_{40}\}. \tag{6}$$

We claim that $\mathcal{G}$ is not isomorphic to any graph in the set $\{\mathcal{G}_{34}, \mathcal{G}_{36}, \mathcal{G}_{37}, \mathcal{G}_{38}\}$. Indeed, consider the graph shown in Figure 2.

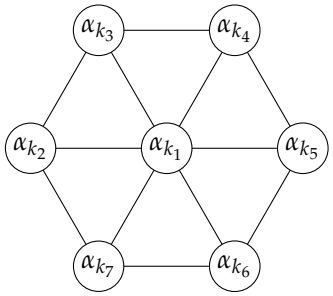

Figure 2. Graph `F}TJG` in graph6 format.

We will prove that this graph cannot be an induced subgraph of $\mathcal{G}$. Assume, to the contrary, that the graph `F}TJG` is an induced subgraph of $\mathcal{G}$. By (ii) of Lemma 8, the edge $(\alpha_{k_1}, \alpha_{k_2})$ belongs to a zero-sum triangle $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha')$, where $\alpha' \in N(\alpha_{k_1})$. Since α_{k_3} and α_{k_7} are the only vertices in $N(\alpha_{k_1})$ that are adjacent to α_{k_2}, we obtain that $\alpha' = \alpha_{k_3}$ or α_{k_7}. Without loss of generality, assume $\alpha' = \alpha_{k_3}$. Thus, $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3})$ is a zero-sum triangle. Now part (i) of Lemma 8 implies that neither $\triangle(\alpha_{k_1}, \alpha_{k_3}, \alpha_{k_4})$ nor $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_7})$ is a zero-sum triangle. Applying (ii) of Lemma 8, we obtain that the edges $(\alpha_{k_3}, \alpha_{k_4})$ and $(\alpha_{k_2}, \alpha_{k_7})$ belong to zero-sum triangles $\triangle(\alpha_{k_3}, \alpha_{k_4}, \tilde{\alpha})$ and $\triangle(\alpha_{k_2}, \alpha_{k_7}, \alpha'')$, respectively, and $\tilde{\alpha}, \alpha'' \notin N[\alpha_{k_1}] = \{\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_7}\}$.

Analogously, applying part (ii) of Lemma 8, we obtain that

$$\triangle(\alpha_{k_1}, \alpha_{k_4}, \alpha_{k_5}) \quad \text{and} \quad \triangle(\alpha_{k_1}, \alpha_{k_6}, \alpha_{k_7})$$

are zero-sum triangles, $\triangle(\alpha_{k_1}, \alpha_{k_5}, \alpha_{k_6})$ is not a zero-sum triangle and there exists a vertex α''' such that $\triangle(\alpha_{k_5}, \alpha_{k_6}, \alpha''')$ is a zero-sum triangle and $\alpha''' \notin N[\alpha_{k_1}]$.

Now, all obtained zero-sum triangles imply the relations

$$\begin{aligned}
&\alpha_{k_1} + \alpha_{k_2} + \alpha_{k_3} = 0,\\
&\alpha_{k_1} + \alpha_{k_4} + \alpha_{k_5} = 0,\\
&\alpha_{k_1} + \alpha_{k_6} + \alpha_{k_7} = 0,\\
&\alpha_{k_3} + \alpha_{k_4} + \tilde{\alpha} = 0,\\
&\alpha_{k_5} + \alpha_{k_6} + \alpha''' = 0,\\
&\alpha_{k_2} + \alpha_{k_7} + \alpha'' = 0.
\end{aligned} \tag{7}$$

By adding the first three relations in (7) and using the expressions for $\tilde{\alpha}, \alpha'', \alpha'''$ from the last three relations in (7), we obtain

$$3\alpha_{k_1} + (\alpha_{k_3} + \alpha_{k_4}) + (\alpha_{k_5} + \alpha_{k_6}) + (\alpha_{k_2} + \alpha_{k_7}) = 3\alpha_{k_1} - \tilde{\alpha} - \alpha''' - \alpha'' = 0.$$

If $\tilde{\alpha} = \alpha'' = \alpha'''$, the last equality gives $\alpha_{k_1} = \tilde{\alpha}$, contrary to $\tilde{\alpha} \notin N[\alpha_{k_1}]$. Otherwise, collect equal terms among $\tilde{\alpha}, \alpha'', \alpha'''$, all of which differ from α_{k_1}. The resulting relation has three or four distinct conjugates, with nonzero rational coefficients $(3, -2, -1)$ or $(3, -1, -1, -1)$, respectively. In both cases, the absolute value of the first coefficient equals the sum of the others' absolute values, so Lemma 3 gives a contradiction.This proves that the graph `F}TJG` cannot be an induced subgraph of $\mathcal{G}$.

We check with SageMath [28] that each graph in $\{\mathcal{G}_{34}, \mathcal{G}_{36}, \mathcal{G}_{37}, \mathcal{G}_{38}\}$ contains an induced subgraph isomorphic to `F}TJG`. Hence, $\mathcal{G}$ is not isomorphic to any of these graphs. This, together with (6), implies that $\mathcal{G}$ must be isomorphic to some graph in the following set

$$\{\mathcal{G}_{20}, \mathcal{G}_{21}, \mathcal{G}_{22}, \mathcal{G}_{23}, \mathcal{G}_{24}, \mathcal{G}_{35}, \mathcal{G}_{40}\}.$$

Next, we will prove that $\mathcal{G}$ is not isomorphic to any of the first five graphs in the set above, namely, $\{\mathcal{G}_{20}, \mathcal{G}_{21}, \mathcal{G}_{22}, \mathcal{G}_{23}, \mathcal{G}_{24}\}$.

The 16 vertices of each graph $\mathcal{G}_i$, $i \in \{20, 21, 22, 23, 24\}$, are labeled by the numbers $1, 2, \ldots, 16$. We will consider every such graph $\mathcal{G}_i$ separately. Assuming that $\mathcal{G}$ is isomorphic to $\mathcal{G}_i$, we will show that it leads to a contradiction. We will identify each vertex $v \in \{1, 2, \ldots, 16\}$ of $\mathcal{G}_i$ with α_v. Thus, instead of treating $\mathcal{G}$ and $\mathcal{G}_i$ as isomorphic graphs, we will treat them as coinciding graphs, i.e., $\mathcal{G} = \mathcal{G}_i$.

Assume that $\mathcal{G} = \mathcal{G}_{20}$. The induced subgraphs $\mathcal{G}_{20}(\alpha_1)$ and $\mathcal{G}_{20}(\alpha_6)$ are shown in Figure 3.

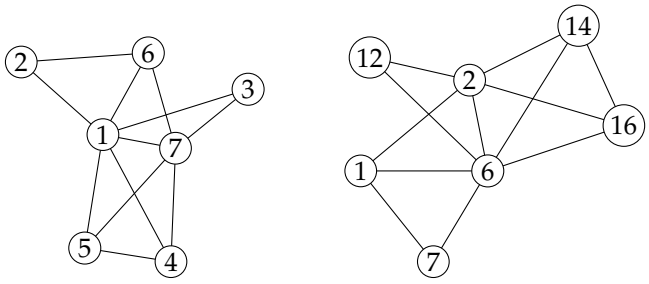

Figure 3. Induced subgraphs $\mathcal{G}_{20}(\alpha_1)$ and $\mathcal{G}_{20}(\alpha_6)$.

We have that $\alpha_2, \alpha_6 \in N(\alpha_1)$, $\deg_{\mathcal{G}_{20}(\alpha_1)}(\alpha_2) = 2$ and (α_2, α_6) is an edge in $\mathcal{G}_{20}$. Hence, part (iii) of Lemma 8 implies that $\triangle(\alpha_1, \alpha_2, \alpha_6)$ is a zero-sum triangle. Similarly, $\alpha_1, \alpha_7 \in N(\alpha_6)$, $\deg_{\mathcal{G}_{20}(\alpha_6)}(\alpha_7) = 2$ and (α_1, α_7) is an edge in $\mathcal{G}_{20}$. Again, applying part (iii) of Lemma 8, we obtain that $\triangle(\alpha_1, \alpha_6, \alpha_7)$ is a zero-sum triangle. Thus, the zero-sum triangles

$\triangle(\alpha_1, \alpha_2, \alpha_6)$ and $\triangle(\alpha_1, \alpha_6, \alpha_7)$ share two distinct vertices α_1 and α_6. This contradicts part (i) of Lemma 8. Thus, $\mathcal{G} \neq \mathcal{G}_{20}$.

Table 2 gives the witnesses for all five cases, using the vertex labels in Figures 3–7. Each ordered triple (v, u, w) means that u has degree 2 in $\mathcal{G}_i(v)$ and is adjacent there to v and w. Lemma 8 (iii) therefore forces the corresponding zero-sum triangle. In every row, the two forced triangles share an edge, contradicting Lemma 8 (i).

Table 2. Forced zero-sum triangles; vertex numbers denote conjugates.

Graph Indices	(v, u, w), First	(v, u, w), Second	Shared Edge
20, 21, 22	(1, 2, 6)	(6, 7, 1)	{1, 6}
23, 24	(1, 3, 6)	(6, 12, 3)	{3, 6}

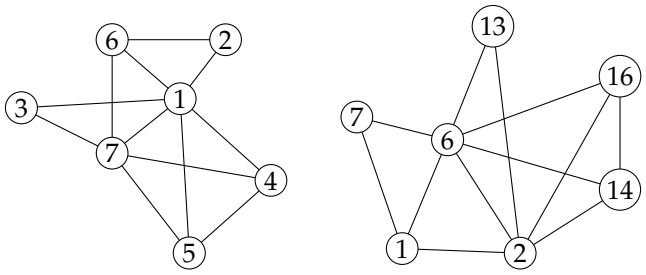

Figure 4. Induced subgraphs $\mathcal{G}_{21}(\alpha_1)$ and $\mathcal{G}_{21}(\alpha_6)$.

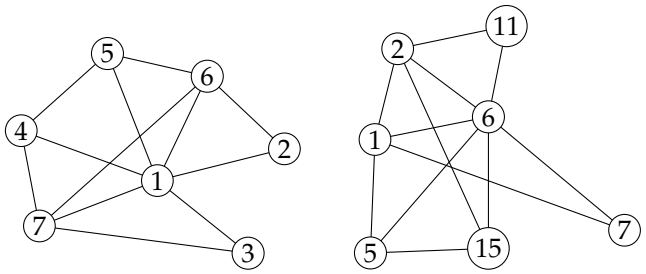

Figure 5. Induced subgraphs $\mathcal{G}_{22}(\alpha_1)$ and $\mathcal{G}_{22}(\alpha_6)$.

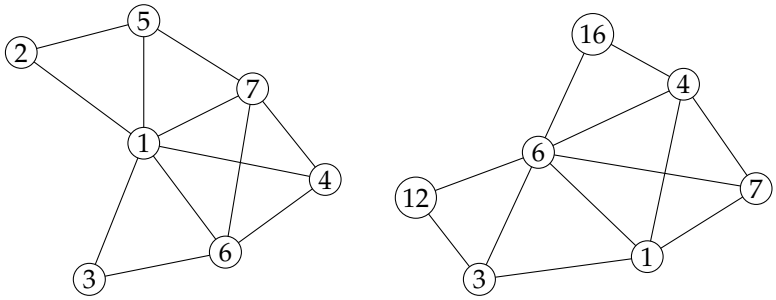

Figure 6. Induced subgraphs $\mathcal{G}_{23}(\alpha_1)$ and $\mathcal{G}_{23}(\alpha_6)$.

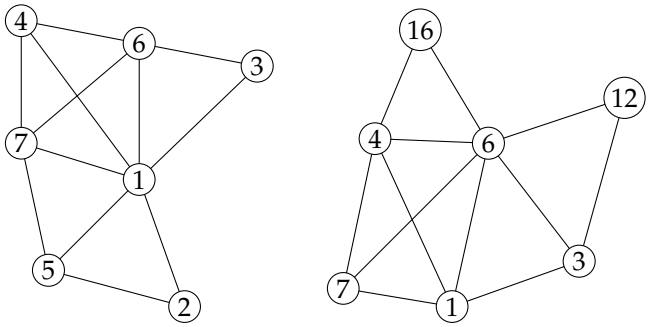

Figure 7. Induced subgraphs $\mathcal{G}_{24}(\alpha_1)$ and $\mathcal{G}_{24}(\alpha_6)$.

Hence, we are left with two cases: $\mathcal{G} = \mathcal{G}_{35}$ or $\mathcal{G} = \mathcal{G}_{40}$. To eliminate the first case, consider the graph shown in Figure 8.

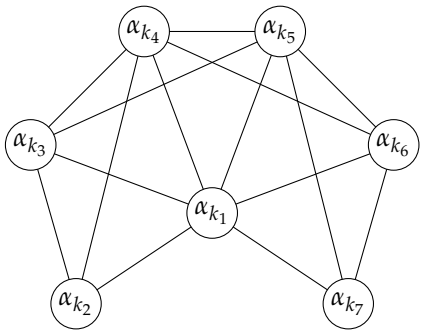

Figure 8. Graph `Fummw` in graph6 format.

We will prove that this graph cannot be an induced subgraph of $\mathcal{G}$. Assume, to the contrary, that it is. By (ii) of Lemma 8, the edge $(\alpha_{k_1}, \alpha_{k_2})$ belongs to a zero-sum triangle $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha')$, where $\alpha' \in N(\alpha_{k_1})$. Since α_{k_3} and α_{k_4} are the only vertices in $N(\alpha_{k_1})$ that are adjacent to α_{k_2}, we obtain that $\alpha' = \alpha_{k_3}$ or α_{k_4}. Thus, we have to consider two cases: (i) $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3})$ is a zero-sum triangle; (ii) $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_4})$ is a zero-sum triangle.

Case (i): $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3})$ is a zero-sum triangle. By part (ii) of Lemma 8, the edge $(\alpha_{k_1}, \alpha_{k_4})$ belongs to a zero-sum triangle $\triangle(\alpha_{k_1}, \alpha_{k_4}, \alpha'')$, where $\alpha'' \in N(\alpha_{k_1})$. We have that $\alpha'' \in \{\alpha_{k_2}, \alpha_{k_3}, \alpha_{k_5}, \alpha_{k_6}\}$, since α_{k_4} is not adjacent to α_{k_7}. Part (i) of Lemma 8 implies that neither $\triangle(\alpha_{k_1}, \alpha_{k_4}, \alpha_{k_2})$ nor $\triangle(\alpha_{k_1}, \alpha_{k_4}, \alpha_{k_3})$ is a zero-sum triangle. Hence, $\alpha'' \in \{\alpha_{k_5}, \alpha_{k_6}\}$.

Assume that $\alpha'' = \alpha_{k_5}$. Then $\triangle(\alpha_{k_1}, \alpha_{k_4}, \alpha_{k_5})$ and $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3})$ are zero-sum triangles, while $(\alpha_{k_2}, \alpha_{k_4})$ and $(\alpha_{k_3}, \alpha_{k_5})$ are edges of $\mathcal{G}$. This contradicts part (iv) of Lemma 8.

Assume that $\alpha'' = \alpha_{k_6}$, so that $\triangle(\alpha_{k_1}, \alpha_{k_4}, \alpha_{k_6})$ is a zero-sum triangle. Part (i) of Lemma 8 implies that $\triangle(\alpha_{k_1}, \alpha_{k_6}, \alpha_{k_7})$ is not a zero-sum triangle, since it shares an edge with the zero-sum triangle $\triangle(\alpha_{k_1}, \alpha_{k_4}, \alpha_{k_6})$. Hence, $\triangle(\alpha_{k_1}, \alpha_{k_5}, \alpha_{k_7})$ is a zero-sum triangle. Now, we have two zero-sum triangles $\triangle(\alpha_{k_1}, \alpha_{k_4}, \alpha_{k_6})$ and $\triangle(\alpha_{k_1}, \alpha_{k_5}, \alpha_{k_7})$, and $(\alpha_{k_4}, \alpha_{k_5})$ together with $(\alpha_{k_6}, \alpha_{k_7})$ are edges in $\mathcal{G}$. This contradicts part (iv) of Lemma 8.

Case (ii): $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_4})$ is a zero-sum triangle. Part (i) of Lemma 8 implies that neither $\triangle(\alpha_{k_1}, \alpha_{k_3}, \alpha_{k_2})$ nor $\triangle(\alpha_{k_1}, \alpha_{k_3}, \alpha_{k_4})$ is a zero-sum triangle, since they share an edge with the zero-sum triangle $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_4})$. Hence, $\triangle(\alpha_{k_1}, \alpha_{k_3}, \alpha_{k_5})$ is a zero-sum triangle. Now, we have two zero-sum triangles $\triangle(\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_4})$ and $\triangle(\alpha_{k_1}, \alpha_{k_3}, \alpha_{k_5})$, and $(\alpha_{k_2}, \alpha_{k_3})$ together with $(\alpha_{k_4}, \alpha_{k_5})$ are edges in $\mathcal{G}$. This contradicts part (iv) of Lemma 8.

This completes the proof that the graph `Fummw` cannot be an induced subgraph of $\mathcal{G}$. We check with SageMath [28] that the graph $\mathcal{G}_{35}$ contains an induced subgraph isomorphic to `Fummw`. Therefore, $\mathcal{G} \neq \mathcal{G}_{35}$, and we are left to consider the last option $\mathcal{G} = \mathcal{G}_{40}$. To do so, consider the graph shown in Figure 9.

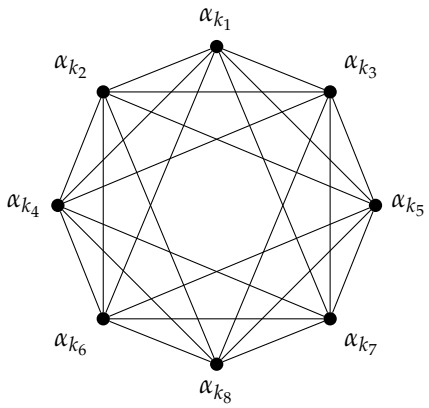

Figure 9. Graph `G~z\z{` in graph6 format.

We will prove that this graph cannot be an induced subgraph of $\mathcal{G}$. Assume, to the contrary, that it is. Part (iii) of Lemma 7 implies that there exist three distinct relations $\alpha_{k_1} + \alpha_i' + \alpha_i'' = 0$, $i \in \{1,2,3\}$, which correspond to three distinct zero-sum triangles $\triangle(\alpha_{k_1}, \alpha_i', \alpha_i'')$. Then part (ii) of Lemma 8 yields $\alpha_i', \alpha_i'' \in N(\alpha_{k_1})$ for every $i \in \{1,2,3\}$. Moreover, applying part (i) of Lemma 8, we obtain that any two of the three sets $\{\alpha_i', \alpha_i''\}$, $i \in \{1,2,3\}$, are disjoint. Hence,

$$\{\alpha_1', \alpha_1'', \alpha_2', \alpha_2'', \alpha_3', \alpha_3''\} = N(\alpha_{k_1}) = \{\alpha_{k_2}, \alpha_{k_3}, \alpha_{k_4}, \alpha_{k_5}, \alpha_{k_6}, \alpha_{k_7}\}.$$

Now, adding up the relations $\alpha_{k_1} + \alpha_i' + \alpha_i'' = 0$, $i \in \{1,2,3\}$, we obtain

$$3\alpha_{k_1} + \alpha_{k_2} + \alpha_{k_3} + \alpha_{k_4} + \alpha_{k_5} + \alpha_{k_6} + \alpha_{k_7} = 0. \tag{8}$$

Analogously, considering the three relations $\alpha_{k_8} + \tilde{\alpha}_i' + \tilde{\alpha}_i'' = 0$, $i \in \{1,2,3\}$, we obtain

$$3\alpha_{k_8} + \alpha_{k_2} + \alpha_{k_3} + \alpha_{k_4} + \alpha_{k_5} + \alpha_{k_6} + \alpha_{k_7} = 0.$$

This equality, together with (8), implies that $\alpha_{k_1} = \alpha_{k_8}$, which is a contradiction. Hence, the graph `G~z\z{` cannot be an induced subgraph of $\mathcal{G}$. We check with SageMath [28] that the graph $\mathcal{G}_{40}$ contains an induced subgraph isomorphic to `G~z\z{`. Therefore, $\mathcal{G} \neq \mathcal{G}_{40}$.

Finally, we have proved that $\mathcal{G} \neq \mathcal{G}_i$ for $i = 1, 2, \ldots 40$. Thus, no algebraic number of degree $d = 16$ satisfies the relation (2). □

Supplementary Materials: The following supporting information can be downloaded at https://www.mdpi.com/article/10.3390/math14193453/s1, Supplementary Files: Python source code and verification files for the computations reported in this article, including a README file with instructions for reproducing the computations.

Author Contributions: Conceptualization, Ž.B., P.D. and J.J.; methodology, Ž.B., P.D. and J.J.; formal analysis, Ž.B., P.D. and J.J.; investigation, Ž.B., P.D. and J.J.; writing—original draft preparation, Ž.B., P.D. and J.J.; writing—review and editing, Ž.B., P.D. and J.J. All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: Appendix A gives the execution instructions and expected results. The source census is the Zenodo dataset [21].

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Computational Verification

The standalone script `Supplementary/reproduce.py` (Supplementary Materials) contains all 40 graph6 strings in Table 1, every filtering and neighborhood test, and both final induced-subgraph tests. It prints graph indices, labeled neighborhoods, explicit embeddings, and assertions for all claimed outputs. It also checks that the graphs have order 16 and valency 6, are vertex-transitive, and are pairwise nonisomorphic.

The exact source is `alltrans16.tar` in [21], member `alltrans16_k=06.gz`. Its decompressed text has 40 lines and 880 bytes, including the final newline, and a SHA-256 checksum

`6df801e5f538cc9f406fab3e6fe1bba24744dc05b9fed7017166c6540dd8a36c.`

The supplied files are listed in `Supplementary/SHA256SUMS.txt` (Supplementary Materials) with their hashes. Graph $\mathcal{G}_i$ is line i of the text; manuscript vertex v is graph6 vertex $v-1$. Embeddings map the displayed figure labels to these manuscript vertex numbers.

Run `sage -python Supplementary/reproduce.py` (Supplementary Materials) from the project directory. The script prints its SageMath version and uses SageMath's subgraph-search routines. It can also be run with `python3 Supplementary/reproduce.py` (Supplementary Materials) when NetworkX is installed. The supplied runs used SageMath 10.9 with Python 3.14.7 and, independently, NetworkX 3.6.1 with Python 3.12.14. Every assertion passed under both backends, with identical results and witnesses. See `Supplementary/README.md` (Supplementary Materials) for the environment details.

The asserted sequence of surviving graph indices is

Test	Survivors
`FyTAG`, non-induced	20, 21, 22, 23, 24, 34, 35, 36, 37, 38, 40
Exclude induced `F}TJG`	20, 21, 22, 23, 24, 35, 40
Local tests in Table 2	35, 40
Final induced-subgraph tests	none

For the last two tests, the script verifies the induced embeddings $(1,2,4,6,7,5,3)$ of Figure 8 in $\mathcal{G}_{35}$ and $(1,2,3,4,5,6,7,16)$ of Figure 9 in $\mathcal{G}_{40}$. The complete transcripts `Supplementary/output_sage.txt` and `Supplementary/output_networkx.txt` (Supplementary Materials) include all intermediate witnesses and end with `ALL ASSERTIONS PASSED`.

References

1. Dubickas, A.; Smyth, C.; Stong, R. Polynomials with Three Distinct Zeros Summing to Zero: 11123. *Am. Math. Mon.* **2006**, *113*, 941–942. [CrossRef]
2. Baronėnas, Ž.; Drungilas, P.; Jankauskas, J. No three algebraic conjugates of degree sixteen sum to zero. *arXiv* **2026**, arXiv:2608.03583.
3. Kurbatov, V.A. Galoissche Erweiterungen vom Primzahlgrad und ihre primitiven Elemente. *Izv. Vyssh. Uchebn. Zaved. Mat.* **1977**, *176*, 61–66.
4. Baron, G.; Drmota, M.; Skałba, M. Polynomial relations between polynomial roots. *J. Algebra* **1995**, *177*, 827–846. [CrossRef]
5. Dixon, J.D. Polynomials with nontrivial relations between their roots. *Acta Arith.* **1997**, *82*, 293–302. [CrossRef]
6. Girstmair, K. Linear relations between roots of polynomials. *Acta Arith.* **1999**, *89*, 53–96. [CrossRef]
7. Girstmair, K. Linear dependence of zeros of polynomials and construction of primitive elements. *Manuscr. Math.* **1982**, *39*, 81–97. [CrossRef]
8. Girstmair, K. The Galois relation $x_1 = x_2 + x_3$ and Fermat over finite fields. *Acta Arith.* **2006**, *124*, 357–370. [CrossRef]
9. Girstmair, K. The Galois relation $x_1 = x_2 + x_3$ for finite simple groups. *Acta Arith.* **2007**, *127*, 301–303. [CrossRef]
10. Girstmair, K. The Galois relations $x_1 = x_2 + x_3$ and $x_1 = x_2x_3$ for certain solvable groups. *Ann. Sci. Math. Québec* **2008**, *32*, 171–174.
11. Lalande, F. La relation linéaire $a = b + c + \cdots + t$ entre les racines d'un polynôme. *J. Théorie Nr. Bordx.* **2007**, *19*, 473–484. [CrossRef]
12. Lalande, F. À propos de la relation galoisienne $x_1 = x_2 + x_3$. *J. Théorie Nr. Bordx.* **2010**, *22*, 661–673. . [CrossRef]
13. Dubickas, A.; Hare, K.G.; Jankauskas, J. No two non-real conjugates of a Pisot number have the same imaginary part. *Math. Comput.* **2017**, *86*, 935–950. [CrossRef]
14. Dubickas, A.; Jankauskas, J. Simple linear relations between conjugate algebraic numbers of low degree. *J. Ramanujan Math. Soc.* **2015**, *30*, 219–235.
15. Baronėnas, Ž.; Drungilas, P.; Jankauskas, J. Linear relations of four conjugates of an algebraic number. *Can. Math. Bull.* **2026**, *69*, 299–314. [CrossRef]
16. Dubickas, A.; Virbalas, P. Additive and multiplicative relations with algebraic conjugates. *Rev. Unión Mat. Argent.* **2026**, *69*, 143–154. [CrossRef]
17. Kitaoka, Y. Notes on the Distribution of Roots Modulo a Prime of a Polynomial. *Unif. Distrib. Theory* **2017**, *12*, 91–117. [CrossRef]
18. Holgado, Á.S. Nontriviality of the module of relations for degree 4 polynomials. *Acta Math. Hung.* **2025**, *176*, 236–243. [CrossRef]
19. Baronėnas, Ž.; Drungilas, P.; Jankauskas, J. No three algebraic conjugates of degree sixteen sum to zero. *arXiv* **2026**, arXiv:2606.06222.

20. McKay, B.D.; Royle, G.F. The transitive graphs with at most 26 vertices. *Ars Comb.* **1990**, *30*, 161–176.
21. Royle, G.; Holt, D. Vertex-transitive Graphs On Fewer Than 48 Vertices. 2020. Available online: https://zenodo.org/records/4010122 (accessed on 21 August 2026).
22. Smyth, C.J. Conjugate algebraic numbers on conics. *Acta Arith.* **1982**, *40*, 333–346. [CrossRef]
23. Dubickas, A. On the degree of a linear form in conjugates of an algebraic number. *Ill. J. Math.* **2002**, *46*, 571–585. [CrossRef]
24. Virbalas, P. Linear relations between three algebraic conjugates of degree twice a prime. *Glas. Mat.* **2025**, *60*, 229–242. [CrossRef]
25. Coolsaet, K.; D'hondt, S.; Goedgebeur, J. House of Graphs 2.0: A database of interesting graphs and more. *Discret. Appl. Math.* **2023**, *325*, 97–107. [CrossRef]
26. McKay, B.D.; Piperno, A. Practical graph isomorphism, II. *J. Symb. Comput.* **2014**, *60*, 94–112. [CrossRef]
27. McKay, B.D. Description of graph6, sparse6 and digraph6 Encodings. Available online: https://users.cecs.anu.edu.au/~bdm/data/formats.txt (accessed on 30 May 2026).
28. The Sage Developers. SageMath, the Sage Mathematics Software System (Version 10.6.0). 2025. Available online: https://www.sagemath.org (accessed on 24 August 2026).

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