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Multiplicative Dependence of Shifted Quadratic and Cubic Algebraic Numbers

Paulius Drungilas and Artūras Dubickas *

Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania; paulius.drungilas@mif.vu.lt
* Correspondence: arturas.dubickas@mif.vu.lt

Abstract

We study short multiplicative relations among additive translates of algebraic numbers. Every quadratic algebraic number admits infinitely many relations with three factors on each side and arbitrarily large integer shifts. For quadratic algebraic integers, we classify the minimum total length over integer tails and show that it is eventually five. We also construct infinitely many reduced relations of length eight for every cubic algebraic number using rational shifts, and prove that the integer-shift length of $\sqrt[3]{2}$ is exactly eight.

Keywords: multiplicative dependence; additive shifts; algebraic numbers; dependence length; Hurwitz zeta-function

MSC: 11R04; 11D57; 11M35

1. Introduction

Multiplicative relations among additive translates of an algebraic number lead to questions about products of split polynomials and the arithmetic restrictions on their roots. We study the shortest such relations when the shifts range over the integers, an integer tail, or the rational numbers.

Throughout, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ will be the sets of complex numbers, real numbers, rational numbers and integers, respectively. We will also use the notation

$$\mathbb{Z}_t = \{s \in \mathbb{Z} : s \geq t\} \quad (t \in \mathbb{Z}).$$

Let $M \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. We call the number z *M-dependent* if there exist integers $n \geq 1$, $m \geq 0$ and two distinct finite multisets $x_1, \ldots, x_n \in M$ and $y_1, \ldots, y_m \in M$ such that

$$\prod_{i=1}^{n}(z + x_i) = \prod_{j=1}^{m}(z + y_j).$$

The product on the right is understood as 1 when $m = 0$; repetitions within either multiset are permitted. If z is M-dependent, its *length of multiplicative dependence over M*, denoted by $\ell(z, M)$, is the smallest value of $n + m$ for which such a relation holds. (We set $\ell(z, M) = \infty$ if z is not M-dependent.)

For the algebraic irrational numbers and rational shifts considered below, every factor is nonzero. A relation is called *reduced* if no shift occurs on both sides; underlying sets are obtained by forgetting multiplicities. The definition of M-dependence and its length follows our previous paper [1]; see also the work by Drungilas and Klimavičius [2].

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For a fixed multiplicatively dependent tuple of nonzero algebraic numbers, van der Poorten and Loxton [3], with the corrections recorded there, bounded the exponents in a nontrivial multiplicative relation. After cancellation, the length studied here is the sum of the absolute values of these exponents, but the tuple itself is allowed to vary through translates of a fixed number. A complementary question concerns the distribution of dependent tuples. Pappalardi, Sha, Shparlinski and Stewart [4] obtained asymptotic counting results under degree and height restrictions, while Konyagin, Sha, Shparlinski and Stewart [5] studied density and covering radii. In particular, dependent rational vectors are dense in $\mathbb{R}^n$ for $n \geq 2$. These results concern the size and distribution of families of relations; our concern is the shortest relation within an individual translated family.

Common translations give another perspective. Dubickas and Sha [6] proved that every fixed tuple of pairwise distinct algebraic numbers is multiplicatively independent after every sufficiently large common integer translation, and related dependent integer pairs to Pillai-type equations. Vukusic and Ziegler [7] studied consecutive dependent triples: for a fixed integer $a > 1$, $a \notin \{2, 8\}$, only finitely many triples (a, b, c) of pairwise distinct integers greater than one have all three translates by $0, 1, 2$ multiplicatively dependent, and these triples are effectively determinable. Such finiteness is compatible with the tail-dependence considered below, since we allow the relative positions of the shifts to vary as their lower bound increases.

There is also a connection with identities between split polynomials. In the Prouhet–Tarry–Escott problem, Newton's identities relate matching initial coefficients to equal power sums of the roots; see Borwein [8] (Chapter 11). Here the difference of the two products of linear factors must instead be divisible by the minimal polynomial of the given algebraic number. Polynomial identities and Pell equations already play a central role in the paper in [9] on quadratic polynomial values and shifted quadratic algebraic integers.

An analytic motivation for $\mathbb{Z}_0$-dependence comes from the Hurwitz zeta-function, as explained by the authors in [1] (pp. 75–76). For this connection, take $0 < \alpha < 1$, and write

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \qquad (\operatorname{Re} s > 1).$$

For every collection of integers (k_n), satisfying $k_n = 0$ for each sufficiently large n, we have

$$\prod_{n \geq 0} (n + \alpha)^{k_n} = 1 \quad \Longleftrightarrow \quad \sum_{n \geq 0} k_n \log(n + \alpha) = 0.$$

Thus multiplicative independence of the shifts is equivalent to $\mathbb{Z}$-linear independence of the real logarithms that occur as Dirichlet exponents. Under a vertical translation $s \mapsto s + i\tau$, the corresponding phases are $u_n(\tau) = \exp(-i\tau \log(n + \alpha))$. Independence makes each finite phase vector dense in its full torus, whereas a multiplicative relation imposes the constraint $\prod_n u_n(\tau)^{k_n} = 1$. This is the arithmetic input into phase approximation and probabilistic models for value distribution; the general analytic framework is developed in Laurinčikas's book [10]. A relation by itself does not rule out universality, and dependence of the entire shift set is compatible with the existence of large independent finite subsets.

The latter distinction is already present in the history of zeros of the Hurwitz zeta-function. Davenport and Heilbronn [11] proved the existence of infinitely many zeros in $\operatorname{Re} s > 1$ for rational or transcendental $\alpha \in (0, 1)$, apart from $\alpha = 1/2$. Cassels [12] treated algebraic irrational parameters by finding large collections of independent shifts through prime-ideal divisibility. Worley [13] refined the independent-shift estimates and related them to large prime factors of polynomial values. Mine [14] (Remark 4.10) supplies a correction to the prime-ideal argument used in this line of work and explains how the affected proofs are recovered. This role of independence also appears in recent approxi-

mation theorems for short intervals. Laurinčikas [15] considers joint continuous shifts of Hurwitz zeta-functions under linear independence over $\mathbb{Q}$ of the logarithms $\log(m+\alpha_j)$. Laurinčikas and Šiaučiūnas [16] study joint discrete shifts, with an independence condition involving the weighted logarithms and 2π for approximation of arbitrary analytic target functions.

Multiplicative relations also determine the appropriate random phases for algebraic parameters. Laurinčikas and Steuding [17] proposed an explicit random model for a Hurwitz limit theorem. Mine [14] (Remark 1.3) identifies a gap in that construction and gives a corrected model. Its mixed integer moments detect exactly the multiplicative relations among the shifts. For algebraic $\alpha \in (0,1]$, this model yields weak convergence of vertical value distributions on each fixed line $\operatorname{Re} s > 1/2$ [14] (Theorems 1.1–1.2). Related approximation results address different regimes: Lee and Mishou [18] established joint denseness in the half-plane $\operatorname{Re} s > 1$ under field-independence hypotheses on the algebraic parameters, with both real and imaginary parts varying. Sourmelidis and Steuding [19] obtained effective simultaneous approximation of values and derivatives near $\operatorname{Re} s = 1$, subject to degree restrictions and a finite exceptional set depending on the approximation data. These qualified results do not settle the general universality problem for every fixed algebraic irrational parameter. Related mean-square estimates for the Hurwitz zeta-function in short intervals were obtained by Laurinčikas and Šiaučiūnas [20].

The arithmetic question posed in [1] (p. 76) is whether every algebraic number is $\mathbb{Z}_0$-dependent. Equivalently, this can be written in the form $\ell(\alpha, \mathbb{Z}_0) < \infty$. Our results address quantitative questions in degrees two and three but do not resolve that general question. In the quadratic integral case, the earlier bounds $\ell(\alpha, \mathbb{Z}) \leq 3$ and $\ell(\alpha, \mathbb{Z}_t) \leq 5$ come from [9], as recorded in [1] (p. 76). For shifted algebraic numbers, in [1] (Theorem 1) the authors proved that

$$\ell(\alpha, \mathbb{Z}_t) \leq 8 \tag{1}$$

for every quadratic algebraic number α and every integer t. Drungilas and Klimavičius [2] (pp. 7–8) subsequently gave a length-six example for the roots of $2X^2 + 3$.

Note that [1] (p. 77) also contains the lower bound

$$\ell(\alpha, \mathbb{Z}) \geq 2d + 2 \tag{2}$$

for any algebraic number α of degree d that is not an algebraic integer. Combined with the result in [9] (p. 226), where the opposite inequality $\ell(\alpha, \mathbb{Z}_t) \leq 4$ was established for any $\alpha \in \mathbb{Q}$ and any $t \in \mathbb{Z}$, this shows that $\ell(\alpha, \mathbb{Z}) = \ell(\alpha, \mathbb{Z}_t) = 4$ for each $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$ (the case when $d = 1$). For $d = 2$ the bound in (2) gives $\ell(\alpha, \mathbb{Z}) \geq 6$, while (1) implies that $\ell(\alpha, \mathbb{Z}) \leq \ell(\alpha, \mathbb{Z}_t) \leq 8$.

Our first result establishes the bound six in general and gives infinitely many relations of the same form above any prescribed fixed integer.

Theorem 1. *Let α be an algebraic number of degree 2 over $\mathbb{Q}$, and let*

$$f(X) = aX^2 + bX + c \in \mathbb{Z}[X]$$

be its primitive minimal polynomial, where $\gcd(a,b,c) = 1$. For every positive integer P, there exist integers

$$x_1, x_2, x_3, y_1, y_2, y_3 > P$$

such that

$$(\alpha + x_1)(\alpha + x_2)(\alpha + x_3) = (\alpha + y_1)(\alpha + y_2)(\alpha + y_3)$$

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and

$$\{x_1, x_2, x_3\} \neq \{y_1, y_2, y_3\}.$$

In fact, there are infinitely many such sextuples $(x_1, x_2, x_3, y_1, y_2, y_3)$.

Combined with (2) for $d = 2$ and the trivial inequality $\ell(\alpha, M) \leq \ell(\alpha, M')$ for $M' \subseteq M$, Theorem 1 implies the following corollary.

Corollary 1. *For any quadratic algebraic number* α *which is not an algebraic integer and any* $t \in \mathbb{Z}$, *we have* $\ell(\alpha, \mathbb{Z}) = \ell(\alpha, \mathbb{Z}_t) = 6$.

For quadratic algebraic integers, shorter relations are possible when negative shifts are allowed. The bound $\ell(\alpha, \mathbb{Z}) \leq 3$ is recorded in [1] (p. 76). The following result determines exactly when the minimum is two or three.

Theorem 2. *Let* α *be a quadratic algebraic integer with minimal polynomial* $X^2 + aX + b \in \mathbb{Z}[X]$. *Then*

$$\ell(\alpha, \mathbb{Z}) = \begin{cases} 2, & \text{if } a^2 - 4b - 4 \text{ is a perfect square,} \\ 3, & \text{otherwise.} \end{cases}$$

Restricting the shifts to $\mathbb{Z}_t$ leads to a different result for $\ell(\alpha, \mathbb{Z}_t)$. See Theorem 5 in Section 3, which gives a full arithmetic classification of the possible lengths and an explicit relation of length five. This classification yields the following uniform eventual value five previously established as an upper bound for quadratic algebraic integers [9] (p. 227).

Corollary 2. *Let* α *be a quadratic algebraic integer with minimal polynomial* $X^2 + aX + b \in \mathbb{Z}[X]$, *and put*

$$T = \max\left\{1, \left\lceil \frac{a+2}{2} \right\rceil \right\}.$$

Then $\ell(\alpha, \mathbb{Z}_t) = 5$ *for every integer* $t \geq T$.

For each fixed $e \in \{2, 3, 4\}$, the length e is also attained by quadratic algebraic integers.

Proposition 1. *For every positive integer* t *and every* $e \in \{2, 3, 4\}$, *there exists a quadratic algebraic integer* α *such that* $\ell(\alpha, \mathbb{Z}_t) = e$. *More precisely, the numbers*

$$\alpha_2 = \frac{\sqrt{5} - 1}{2} - t, \qquad \alpha_3 = \sqrt{2} - t, \qquad \alpha_4 = i - t$$

satisfy $\ell(\alpha_e, \mathbb{Z}_t) = e$ *for* $e = 2, 3, 4$.

For cubic algebraic numbers, in [1] (Theorem 2), the authors proved $\mathbb{Q}$-dependence with length at most eight. The next theorem gives infinitely many reduced relations of this length, with four factors on each side and pairwise distinct combined underlying sets of shifts.

Theorem 3. *Every cubic algebraic number* α *admits infinitely many nontrivial relations*

$$\prod_{i=1}^{4} (\alpha + x_i) = \prod_{i=1}^{4} (\alpha + y_i), \qquad x_i, y_i \in \mathbb{Q},$$

whose combined underlying sets of shifts $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$ *are pairwise distinct. Moreover, each relation has no shift common to its two sides. In particular, each cubic* α *is* $\mathbb{Q}$*-dependent, and* $\ell(\alpha, \mathbb{Q}) \leq 8$.

The restriction to integer shifts has also been studied for cubic algebraic integers. Drungilas and Klimavičius [2] (Theorem 1) constructed an infinite family with dependence length at most four. Behera and Ray [21] obtained another cubic family and studied $\mathbb{Z}$-dependence for algebraic integers satisfying certain cubic and quartic polynomials. For the real cube root of two, we determine the integer-shift length exactly.

Theorem 4. *For* $\alpha = \sqrt[3]{2}$, *we have* $\ell(\alpha, \mathbb{Z}) = 8$.

The proof combines congruence restrictions on the shifts with the coefficients of products reduced modulo $X^3 - 2$. Proposition 2 also gives an explicit relation for this number with positive integer shifts.

Section 2 proves Theorem 1. Section 3 treats quadratic algebraic integers, including the arithmetic classification and its consequences. Section 4 constructs the rational shift relations for cubic numbers. Section 5 proves the exact length for $\sqrt[3]{2}$ and gives the positive shift identity.

2. Quadratic Algebraic Numbers

The construction below produces two split cubic polynomials whose difference is a nonzero multiple of the prescribed quadratic polynomial, with all shifts exceeding the chosen bound.

Proof of Theorem 1. Replacing f with $-f$ if necessary, we can assume that $a > 0$. Choose an integer $R > P$ such that

$$G := f(-R) > 0, \qquad 2aR - b > 0,$$

and put

$$M = R + aG, \qquad L = a^2G + 2aR - b.$$

Then $M > P$ and $L > 0$. Also,

$$b - 2aM = -L - a^2G, \qquad f(-M) = (aL + 1)G.$$

Thus, for $\beta = \alpha + M$,

$$a\beta^2 - (L + a^2G)\beta + (aL + 1)G = 0. \tag{3}$$

Set

$$S = L + G + 2(a + 1)L(L + aG) > 0.$$

Since

$$S - (2aL + 1)(L + G) = 2L(L + a^2G),$$

expanding the two cubic products yields

$$\begin{aligned} &T(T + L + S)(T + G + S) - (T + L)(T + G)\big(T + 2(aL + 1)S\big) \\ &\quad = -2LS\big(aT^2 - (L + a^2G)T + (aL + 1)G\big). \end{aligned}$$

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Evaluating at $T = \beta$ and using (3) give the required equality with

$$(x_1, x_2, x_3) = (M,\ M + L + S,\ M + G + S),$$
$$(y_1, y_2, y_3) = (M + L,\ M + G,\ M + 2(aL + 1)S).$$

All six shifts exceed P. The first underlying set contains M, whereas every element of the second one exceeds M, so the sets are distinct. Finally, every sufficiently large integer R is admissible, and $x_1 = M > R$; hence there are infinitely many such sextuples. □

3. Quadratic Algebraic Integers

We first determine the minimum over all integer shifts and then classify the minimum when the shifts are bounded below.

Proof of Theorem 2. The equation $\alpha^2 + a\alpha + b = 0$ gives

$$\alpha(\alpha + a + 1) = \alpha - b.$$

This is a relation between multisets of integer shifts of cardinalities 2 and 1, respectively. These multisets are distinct, so $\ell(\alpha, \mathbb{Z}) \leq 3$.

A relation of total length 1 would have the form $\alpha + u = 1$ with $u \in \mathbb{Z}$, contrary to the irrationality of α. A relation of length 2 with one factor on each side would have the form $\alpha + u = \alpha + v$, which forces $u = v$ and is therefore excluded. Consequently, $\ell(\alpha, \mathbb{Z}) = 2$ if and only if there exist $u, v \in \mathbb{Z}$ such that

$$(\alpha + u)(\alpha + v) = 1.$$

Using the minimal polynomial, this equality is equivalent to

$$(u + v - a)\alpha + (uv - b - 1) = 0.$$

Since 1 and α are linearly independent over $\mathbb{Q}$, it holds if and only if

$$u + v = a, \qquad uv = b + 1.$$

If such integers exist, then

$$a^2 - 4b - 4 = (u + v)^2 - 4uv = (u - v)^2,$$

which is the square of an integer.

Conversely, suppose that $a^2 - 4b - 4 = k^2$ for some $k \in \mathbb{Z}$. Since $k^2 \equiv a^2 \pmod 4$, we have $k \equiv a \pmod 2$. Thus

$$u = \frac{a + k}{2}, \qquad v = \frac{a - k}{2}$$

are integers satisfying $u + v = a$ and $uv = b + 1$. They therefore yield $(\alpha + u)(\alpha + v) = 1$; hence $\ell(\alpha, \mathbb{Z}) = 2$.

If $a^2 - 4b - 4$ is not the square of an integer, no relation of length 2 exists. The preceding bounds then give $\ell(\alpha, \mathbb{Z}) = 3$. □

Theorem 5. *Let α be a quadratic algebraic integer with minimal polynomial $X^2 + aX + b \in \mathbb{Z}[X]$. Let $t > 0$ be an integer, and put*

$$\mathbb{Z}_t = \{s \in \mathbb{Z} : s \geq t\}, \qquad A = a - 2t, \qquad B = b - at + t^2.$$

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For each integer A, define the finite set

$$\mathcal{S}_A = \left\{ A^2 - A(u+v+w) + uv + uw + vw - 1 : \begin{array}{c} u,v,w \in \mathbb{Z}, \quad 0 \le u \le v \le w, \\ v + w \le A - 1 \end{array} \right\}.$$

Every triple in this definition satisfies $0 \le u \le v \le w \le A-1$; in particular, $\mathcal{S}_A = \varnothing$ for $A \le 0$. Consider the following conditions:

$$A \ge 0, \quad B + 1 \ge 0, \quad A^2 - 4B - 4 \text{ is the square of an integer;} \tag{C$_2$}$$

$$A \ge -1, \quad B \le \left\lfloor \frac{(A+1)^2}{4} \right\rfloor; \tag{C$_3$}$$

$$(A,B) = (0,1) \quad \text{or} \quad B \in \mathcal{S}_A. \tag{C$_4$}$$

Then α is $\mathbb{Z}_t$-dependent, and its length is given by

$$\ell(\alpha, \mathbb{Z}_t) = \begin{cases} 2, & \text{if (C}_2\text{) holds,} \\ 3, & \text{if (C}_3\text{) holds and (C}_2\text{) fails,} \\ 4, & \text{if (C}_4\text{) holds and (C}_3\text{) fails,} \\ 5, & \text{if both (C}_3\text{) and (C}_4\text{) fail.} \end{cases}$$

Thus the classification is determined by finite arithmetic tests on a, b, t.

Proof of Theorem 5. Put $\theta = \alpha + t$, whose minimal polynomial is $X^2 + AX + B$. Translation of the shifts identifies relations over $\mathbb{Z}_t$ for α with relations over $\mathbb{Z}_0$ for θ, preserving their lengths. All factors are nonzero since $\theta \notin \mathbb{Q}$. We call a relation with n factors on one side and m on the other one of type (n,m) and assume $n \ge m$. Irrationality excludes length one. Relations of types $(1,1)$ and $(2,2)$ are trivial: the difference of the two monic products has degree at most one, so vanishing at θ forces identical polynomials and hence identical multisets of shifts.

For $u, v, w \in \mathbb{Z}_0$, coefficient comparison gives

$$\begin{aligned} (\theta+u)(\theta+v) = 1 &\iff u + v = A, \quad uv = B + 1, \\ (\theta+u)(\theta+v) = \theta + w &\iff u + v = A + 1, \quad w = uv - B. \end{aligned}$$

The first system is solvable exactly under (C$_2$): its solutions are $(A \pm \sqrt{A^2 - 4B - 4})/2$, and an integral square root has the same parity as A. The second is solvable exactly under (C$_3$), since the largest product of two nonnegative integers with sum $A+1$ is $\lfloor (A+1)^2/4 \rfloor$. Moreover, the first system implies $B \le \lfloor A^2/4 \rfloor - 1$, so (C$_2$) implies (C$_3$).

Assume henceforth that (C$_3$) fails. To classify relations of length at most four, only types $(3,0)$, $(4,0)$ and $(3,1)$ remain. Write

$$N(s) = \mathrm{Nm}_{\mathbb{Q}(\theta)/\mathbb{Q}}(\theta + s) = s^2 - As + B.$$

For a relation of type $(3,1)$, order the shifts as $0 \le u \le v \le w$, and write $(\theta+u)(\theta+v)(\theta+w) = \theta + H$, where $H \ge 0$. Reduction using $\theta^2 = -A\theta - B$ shows that this is equivalent to

$$\begin{aligned} B &= A^2 - A(u+v+w) + uv + uw + vw - 1, \\ H &= u - (v + w - A)N(u). \end{aligned} \tag{4}$$

If $B \le 0$, irreducibility gives $B < 0$, and failure of (C$_3$) forces $A \le -2$. The positive conjugate of θ then exceeds $-A \ge 2$, excluding every one-sided relation. A relation of type

$(3,1)$ would give $B \geq A^2 - 1 > 0$ by (4), a contradiction. Thus no relation of length at most four exists in this case, and (C_4) also fails since $\mathcal{S}_A = \varnothing$.

Now let $B > 0$. For every $s \in \mathbb{Z}_0$ we have

$$N(s) - s = s^2 - (A+1)s + B > 0. \tag{5}$$

For $A \leq -2$ this follows from the signs of the terms; for $A \geq -1$ it follows from failure of (C_3) and $s(A+1-s) \leq \lfloor (A+1)^2/4 \rfloor$. Taking norms in a one-sided relation, (5) forces every shift to be zero and $B = 1$. If $\theta^3 = 1$, its quadratic minimal polynomial is $X^2 + X + 1$, which satisfies (C_3), a contradiction. If $\theta^4 = 1$, its quadratic minimal polynomial is $X^2 + 1$. Thus type $(3,0)$ is impossible, and type $(4,0)$ occurs exactly when $(A,B) = (0,1)$.

For type $(3,1)$, (5) and (4) exclude $v + w - A \geq 1$, since then $H < 0$. If $v + w = A$, the first equation gives $vw = B + 1$, implying (C_2) and hence (C_3), again a contradiction. Therefore $v + w \leq A - 1$, and (4) gives $B \in \mathcal{S}_A$. Conversely, a triple in the definition of $\mathcal{S}_A$ satisfies (4) with

$$H = u + (A - v - w)N(u) > 0$$

by (5) and hence yields a relation of type $(3,1)$. This proves the stated criteria for lengths two, three and four.

Finally, choose $R \in \mathbb{Z}_0$ so large that $h = 2R - A \geq 1$ and $N = R^2 - AR + B \geq 1$. Put $\beta = \theta + R$, $k = N + h + 1$ and $S = (N+1)k^2 - k + 1$. Then $\beta^2 - h\beta + N = 0$, and the identity

$$\begin{aligned}(T+1)(T+N+1)(T+Nk) - T(T+S)\\ = \big(T + (N+1)k\big)\big(T^2 - hT + N\big)\end{aligned}$$

at $T = \beta$ gives a relation of total length five. All its shifts of θ are nonnegative, since $R \geq 0$ and $N, k, S > 0$; its two multisets have different cardinalities. Consequently, when (C_3) and (C_4) both fail, the length is five, completing the classification. □

Proof of Corollary 2. Let $t \geq T$, and put $A = a - 2t$ and $B = b - at + t^2$.

The definition of T gives

$$t \geq T \geq \left\lceil \frac{a+2}{2} \right\rceil \geq \frac{a+2}{2}, \qquad A = a - 2t \leq -2.$$

Thus (C_3) fails. Moreover, $\mathcal{S}_A = \varnothing$, and $(A,B) \neq (0,1)$, so (C_4) also fails. Theorem 5 therefore gives $\ell(\alpha, \mathbb{Z}_t) = 5$. □

Proof of Proposition 1. Note that each α_e is a quadratic algebraic integer. Moreover, for any algebraic number γ and any positive integer t, we have $\ell(\gamma, \mathbb{Z}_t) = \ell(\gamma + t, \mathbb{Z}_0)$. Hence, it suffices to prove

$$\ell\left(\frac{\sqrt{5}-1}{2}, \mathbb{Z}_0\right) = 2, \qquad \ell(\sqrt{2}, \mathbb{Z}_0) = 3, \qquad \ell(i, \mathbb{Z}_0) = 4.$$

Put $\theta = (\sqrt{5} - 1)/2$. The identity

$$\theta(\theta + 1) = 1$$

gives a nontrivial relation of total length two with nonnegative integer shifts. Length one is impossible because θ is irrational, proving the first equality.

Next, the identity

$$\sqrt{2}(\sqrt{2} + 1) = \sqrt{2} + 2$$

gives length at most three. Every allowable factor $\sqrt{2}+u$, with $u \in \mathbb{Z}_0$, exceeds 1. Hence no nonempty product of such factors equals 1. Since a relation with one factor on each side is necessarily trivial, no nontrivial relation of total length at most two exists. This proves the second equality.

Finally, $i^4 = 1$ gives a relation of total length four. If

$$\prod_{j=1}^{k}(i+u_j) = 1, \qquad u_j \in \mathbb{Z}_0,$$

then taking squared absolute values yields

$$\prod_{j=1}^{k}(1+u_j^2) = 1.$$

Thus every $u_j = 0$, forcing $i^k = 1$, which is impossible for $1 \le k \le 3$. A relation of type $(1,1)$ is trivial. Up to interchanging the two sides, the only remaining possibility of total length at most three is

$$(i+u)(i+v) = i+w, \qquad u,v,w \in \mathbb{Z}_0.$$

Comparing imaginary and real parts gives $u+v=1$ and $w = uv-1$. Nonnegative integers with sum 1 have product 0, so $w=-1$, a contradiction. Therefore $\ell(i,\mathbb{Z}_0) = 4$, completing the proof. □

4. Rational Shifts of Cubic Algebraic Numbers

We use a rational parameter to obtain a family of identities for a depressed cubic.

Proof of Theorem 3. Let a be the coefficient of X^2 in the monic minimal polynomial of α. Then $\beta = \alpha + a/3$ has an irreducible minimal polynomial of the form

$$f(X) = X^3 + pX + r \in \mathbb{Q}[X].$$

For each positive integer h with $D = p + 3h^2 \neq 0$, put

$$u = \frac{r+2h^3}{D}, \qquad v = \frac{r-2h^3}{D}.$$

Direct expansion gives

$$(X+h)^2(X-2h)(X+v) - (X-h)^2(X+2h)(X+u) = (v-u)f(X).$$

Evaluating at $X = \beta$ gives a relation with four factors on each side, all nonzero because β is irrational. Since $v-u = -4h^3/D \neq 0$, a common rational root of the two quartics would be a rational root of f, contradicting irreducibility. Thus the relation has no common shift and is nontrivial. Adding $a/3$ to every shift gives the required relation for α.

There are infinitely many admissible positive integers h, and each combined underlying set contains $a/3+h$. These sets therefore assume infinitely many distinct values, since a finite family of finite sets has finite union. Choosing one relation for each distinct set proves the theorem. □

5. The Real Cube Root of Two

Proof of Theorem 4. The identity $\alpha(\alpha-1)(\alpha+1) = \alpha^3 - \alpha = 2 - \alpha = -(\alpha-2)$ gives

$$\alpha^2(\alpha-1)^2(\alpha+1)^2 = (\alpha-2)^2, \tag{6}$$

which is a nontrivial multiplicative relation of total length eight. Suppose that for $\alpha = \sqrt[3]{2}$ there is a nontrivial relation

$$\prod_{i=1}^{n}(\alpha + x_i) = \prod_{j=1}^{m}(\alpha + y_j), \qquad x_i, y_j \in \mathbb{Z}, \tag{7}$$

where $n \geq m$ and $n + m < 8$.

Since $\mathrm{Nm}(\alpha + k) = k^3 + 2$, a one-sided relation ($m = 0$) would require every shift to be -1: this is the only integer k with $k^3 + 2 \in \{1, -1\}$. But $0 < \alpha - 1 < 1$, so such a relation is impossible. Thus $m \geq 1$. The 2-adic and 3-adic valuations of $k^3 + 2$ are respectively one for $k \equiv 0 \pmod 2$ and $k \equiv 1 \pmod 3$ and zero otherwise. Taking norms in (7) therefore shows that the two sides have equal numbers of even shifts and equal numbers of shifts congruent to 1 modulo 3.

Normalize each even-shift factor using $\alpha + 2h = \alpha(1 + h\alpha^2)$, and cancel the common power of α. Under the homomorphism

$$\mathbb{Z}[\alpha] \longrightarrow \mathbb{F}_2[T]/(T^2), \qquad \alpha \longmapsto T,$$

the normalized even factors become 1, and odd-shift factors become $1 + T$. Comparing linear coefficients gives $n \equiv m \pmod 2$.

Similarly, put $\beta = \alpha + 1$, and use

$$\beta^3(\alpha - 1) = 3, \qquad \alpha + 1 + 3h = \beta\big(1 + h\beta^2(\alpha - 1)\big).$$

After canceling the common power of β, apply

$$\mathbb{Z}[\alpha] \longrightarrow \mathbb{F}_3[T]/(T^2), \qquad \alpha \longmapsto T - 1.$$

Both homomorphisms respect $\alpha^3 = 2$. The normalized factors for shifts congruent to $0, 1, 2$ modulo 3 become $-(1 - T), 1, 1 + T$, respectively. Let d_r be the number of $x_i \equiv r \pmod 3$ minus the number of $y_j \equiv r \pmod 3$. Together with the norm calculation, comparison of constant and linear coefficients gives

$$d_1 = 0, \qquad d_0 \equiv 0 \pmod 2, \qquad d_2 - d_0 \equiv 0 \pmod 3. \tag{8}$$

Set $P(X) = \prod_i (X + x_i)$ and $Q(X) = \prod_j (X + y_j)$. These are distinct monic polynomials, and $X^3 - 2$ divides $P - Q$. If $n = m$, then $n \leq 3$, and $\deg(P - Q) \leq 2$, a contradiction. Since $n + m$ is even and $m \geq 1$, only the possibilities

$$(n, m) = (3, 1), \quad (4, 2), \quad (5, 1)$$

remain. Write e_j for the elementary symmetric sums of $x_1, \ldots, x_n$.

For the first two pairs, $d_0 + d_2 = 2$, so (8) gives $d_0 \equiv 4 \pmod 6$. The bound $-m \leq d_0 \leq n$ excludes $(3, 1)$ and leaves $d_0 = -2$ or 4 for $(4, 2)$. If $d_0 = 4$, all four x_i are 0 modulo 3. If $d_0 = -2$, both y_j are 0 modulo 3, and no x_i is; since $d_1 = 0$, all four x_i are 2 modulo 3. Hence $e_2 \equiv \binom{4}{2} r^2 \equiv 0 \pmod 3$ for $r = 0$ or 2. However,

$$P(X) \equiv e_2X^2 + (e_3+2)X + e_4 + 2e_1 \pmod{X^3-2},$$

and the monic quadratic Q requires $e_2 = 1$, a contradiction.

Finally, let $(n,m) = (5,1)$. Reduction modulo X^3-2 gives

$$e_3 = -2, \qquad e_4 + 2e_1 = 1.$$

Since the two sides have equally many even shifts, at least four of the x_i are odd. Write these as $1+2u_i$ for $1 \le i \le 4$, and put $x_5 = z$ and $U = u_1+u_2+u_3+u_4$. Expansion modulo 4 gives

$$\begin{aligned} e_3 &\equiv 2z + 2(z+1)U, \\ e_4 + 2e_1 &\equiv 1 + 2z + 2(z+1)U \pmod 4. \end{aligned}$$

Thus $e_4 + 2e_1 \equiv e_3 + 1 \pmod 4$, contradicting the required values 1 and -2. All shorter relations are excluded. □

The integer-shift relation in (6) involves two positive shifts (by $+1$), four negative shifts (by -1 and -2), and two zero shifts. The following nontrivial identity uses only positive shifts and implies that $\ell(\sqrt[3]{2}, \mathbb{Z}_1) \le 20$.

Proposition 2. *The number $\alpha = \sqrt[3]{2}$ is $\mathbb{Z}_1$-dependent. More precisely,*

$$\begin{aligned} &(\alpha+1)(\alpha+3)^2(\alpha+4)(\alpha+15)^2(\alpha+22)^2(\alpha+32)(\alpha+44)(\alpha+1572) \\ &\quad = (\alpha+2)^4(\alpha+37)(\alpha+235)^2(\alpha+322)(\alpha+936). \end{aligned}$$

Proof. Put

$$\begin{aligned} P(X) &= (X+1)(X+3)^2(X+4)(X+15)^2(X+22)^2(X+32)(X+44)(X+1572), \\ Q(X) &= (X+2)^4(X+37)(X+235)^2(X+322)(X+936). \end{aligned}$$

We compute both remainders modulo X^3-2 using integer arithmetic. If a partial product has remainder $A+BX+CX^2$, then adjoining a factor $X+n$ gives

$$\begin{aligned} (A+BX+CX^2)(X+n) &= nA + (A+nB)X + (B+nC)X^2 + CX^3 \\ &\equiv (nA+2C) + (A+nB)X + (B+nC)X^2 \pmod{X^3-2}. \end{aligned}$$

Thus the coefficient triple is updated by

$$(A,B,C) \longmapsto (nA+2C,\ A+nB,\ B+nC). \tag{9}$$

For example, the first four factors on the two sides give

$$\begin{aligned} (X+1)(X+3)^2(X+4) &= X^4 + 11X^3 + 43X^2 + 69X + 36 \\ &\equiv 58 + 71X + 43X^2, \\ (X+2)^4 &= X^4 + 8X^3 + 24X^2 + 32X + 16 \\ &\equiv 32 + 34X + 24X^2 \pmod{X^3-2}. \end{aligned}$$

Starting with these triples, we apply (9) to each remaining factor, in the order displayed in P and Q. The following tables record the coefficients after adjoining the factors in the first column. A squared factor means that the recurrence is applied twice with the same value of n.

Factors of P just adjoined	A	B	C
$(X+1)(X+3)^2(X+4)$	58	71	43
$(X+15)^2$	15,772	17,801	11,863
$(X+22)^2$	8,713,194	9,333,378	6,540,708
$X+32$	291,903,624	307,381,290	218,636,034
$X+44$	13,281,031,524	13,816,680,384	9,927,366,786

Factors of Q just adjoined	A	B	C
$(X+2)^4$	32	34	24
$X+37$	1232	1290	922
$(X+235)^2$	68,906,460	71,821,134	51,524,982
$X+322$	22,290,930,084	23,195,311,608	16,662,865,338

For the final factor $X+1572$ of P, the three calculations are

$$\begin{aligned} 1572 \cdot 13281031524 + 2 \cdot 9927366786 &= 20897636289300, \\ 13281031524 + 1572 \cdot 13816680384 &= 21733102595172, \\ 13816680384 + 1572 \cdot 9927366786 &= 15619637267976. \end{aligned}$$

For the final factor $X+936$ of Q, they are

$$\begin{aligned} 936 \cdot 22290930084 + 2 \cdot 16662865338 &= 20897636289300, \\ 22290930084 + 936 \cdot 23195311608 &= 21733102595172, \\ 23195311608 + 936 \cdot 16662865338 &= 15619637267976. \end{aligned}$$

Consequently, both products have the remainder

$$20897636289300 + 21733102595172X + 15619637267976X^2.$$

Since $\alpha^3 = 2$, evaluating these reductions at $X = \alpha$ gives $P(\alpha) = Q(\alpha)$. All shifts are positive integers, and no shift occurs on both sides. There are eleven factors in P and nine in Q, counting multiplicities, so the relation is nontrivial and has total length $11 + 9 = 20$. □

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