Phys. Lett. B 880 (2026) 140760

Contents lists available at ScienceDirect

Physics Letters B

journal homepage: www.elsevier.com/locate/physletb

Letter

Conditions for boundedness from below of a Δ(54)-symmetric three-Higgs-doublet model

Darius Jurčiukonis [a,*], Luís Lavoura [b]

[a] Vilnius University, Institute of Theoretical Physics and Astronomy, Saulėtekio av. 3, Vilnius, 10257, Lithuania
[b] Universidade de Lisboa, Instituto Superior Técnico, CFTP, Av. Rovisco Pais 1, Lisboa, 1049-001, Portugal

ARTICLE INFO

Editor: Ryuichiro Kitano

Keywords:
3HDM
Three-Higgs-doublet
BFB
Bounded-from-below
Delta(54)
Orbit-space

ABSTRACT

We investigate the orbit space of the scalar potential of a Δ(54)-symmetric three-Higgs-doublet model. We find that, if the potential enjoys CP invariance, then its three-dimensional orbit space is a polytope; if the potential has no CP symmetry, then its four-dimensional orbit space has a boundary that is sometimes slightly concave, but seems never to be convex. Consequently, we conjecture necessary and sufficient conditions for the potential to be bounded from below; brute-force minimization of a large number of potentials affirms the accuracy of our conjecture. We list all possible charge-conserving and charge-breaking global minima of the potential.

1. Introduction

The Standard Model (SM) has gauge group $SU(2)\times U(1)$ and its scalar sector consists of just one $SU(2)$ doublet. An obvious extension of the SM consists in enlarging its scalar sector. Two-Higgs-doublet models have endured intense scrutiny [1]; in this paper we deal on three-Higgs-doublet models (3HDMs). Their scalar sector consists of three $SU(2)$ doublets Φ_k $(k=1,2,3)$, all of them with the same nonzero $U(1)$ hypercharge. We use an $SU(2)\times U(1)$ transformation to make the upper component of Φ_1 equal to zero, and thereafter to make both the lower component of Φ_1 and the upper component of Φ_2 non-negative real; one then has

$$\Phi_1 = \sqrt{N}\cos\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1a}$$

$$\Phi_2 = \sqrt{N}\sin\alpha\cos\beta \begin{pmatrix} \sin\gamma \\ e^{i\psi}\cos\gamma \end{pmatrix}, \tag{1b}$$

$$\Phi_3 = \sqrt{N}\sin\alpha\sin\beta \begin{pmatrix} e^{i\nu}\sin\delta \\ e^{i\xi}\cos\delta \end{pmatrix}, \tag{1c}$$

where

$$N := \sum_{k=1}^{3} \Phi_k^\dagger \Phi_k \tag{2}$$

and α, β, γ, and δ are angles of the first quadrant. We define $A_{kl} := \Phi_k^\dagger \Phi_l$. Evidently, $A_{lk} = A_{kl}^*$. Note that $N \geq 0$ and that $A_{kl}A_{lk} \geq 0$ $\forall l \neq k$. The scalar potential of a 3HDM is $V = V_2 + V_4$, where $V_2 = \sum_{k,l=1}^{3} \mu_{kl} A_{kl}$ and $V_4 = \sum_{k,l,m,n=1}^{3} \lambda_{klmn} A_{kl} A_{mn}$. This potential has many coefficients. Therefore, it is convenient to supplement 3HDMs with non-gauge symmetries, which reduce the number of coefficients in V. An example of one such symmetry is Δ(54), which is generated by the transformations[1]

$$\Phi_1 \to \Phi_2 \to \Phi_3 \to \Phi_1; \tag{3a}$$

$$\Phi_2 \to \omega\Phi_2, \quad \Phi_3 \to \omega^2\Phi_3, \tag{3b}$$

where $\omega := \exp(2i\pi/3)$.[2] Let us define

$$Q := A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33}, \tag{4a}$$

$$R := A_{12}A_{21} + A_{13}A_{31} + A_{23}A_{32}, \tag{4b}$$

[1] The transformations (3) generate the group Δ(27), which is smaller than Δ(54). However, since V is renormalizable, *i.e.* it has no terms with more than four doublets, a Δ(27)-invariant potential is automatically invariant under the transformation $\Phi_2 \leftrightarrow \Phi_3$ too, *cf.* Eq. (4), and therefore it is automatically Δ(54)-invariant.

[2] Notice that we do not specify the transformation rules, both under the gauge symmetry and under transformations (3), of the fermions present in a specific 3HDM. This is because here we just care about the scalar potential. Several phenomenological models with either Δ(27) or Δ(54) symmetry have already been constructed [2–6], and in those models fermions are of course present.

* Corresponding author.
E-mail addresses: darius.jurciukonis@tfai.vu.lt (D. Jurčiukonis), balio@cftp.tecnico.ulisboa.pt (L. Lavoura).

https://doi.org/10.1016/j.physletb.2026.140760
Received 20 May 2026; Received in revised form 3 July 2026; Accepted 13 July 2026
Available online 15 July 2026
0370-2693/© 2026 The Author(s). Published by Elsevier B.V. Funded by SCOAP³. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

$$P := A_{12}A_{13} + A_{21}A_{23} + A_{31}A_{32}. \tag{4c}$$

These three quantities, and also N in Eq. (2), are invariant under transformations (3). Both Q and R are real, while P is in general complex. The most general $\Delta(54)$-invariant potential is given by

$$V_2 = \mu N, \tag{5a}$$

$$V_4 = \lambda_1 N^2 + \lambda_2(R-Q) + 2\lambda_3(\Re P - R) + \lambda_4 Q + 2\sqrt{3}\,\lambda_5 \Im P, \tag{5b}$$

where $\lambda_1, \ldots, \lambda_5$ are real dimensionless coupling constants. Defining

$$x := \frac{R-Q}{N^2}, \quad y := \frac{2(\Re P - R)}{N^2}, \quad q := \frac{Q}{N^2}, \quad t := \frac{2\sqrt{3}\,\Im P}{N^2}, \tag{6}$$

and

$$\hat{\lambda} := \lambda_1 + \lambda_2 x + \lambda_3 y + \lambda_4 q + \lambda_5 t, \tag{7}$$

one has[3]

$$V = \mu N + \hat{\lambda} N^2. \tag{8}$$

We assume that μ is negative so that the minimum of V has nonzero values of the doublets. If $\mu, \lambda_1, \ldots, \lambda_5$, x, y, q, and t are kept fixed and one only lets N vary, then the minimum of the potential in Eq. (8) is given by $-\mu^2/(4\hat{\lambda})$. The task of finding the global minimum[4] of V thus reduces to the task of finding the smallest possible value of $\hat{\lambda}$. Note that $\lambda_1, \ldots, \lambda_5$ must be such that $\hat{\lambda}$ is non-negative for all possible x, y, q, and t; else, by infinitely increasing N one would make V in Eq. (8) tend to $-\infty$ and V would have no global minimum. There are thus two problems: keeping $\hat{\lambda} \geq 0$, and finding the minimum $\hat{\lambda}$; the first one is the problem of V being 'bounded from below' (BFB).

$\Delta(54)$ is a subgroup of a set of symmetry groups that one may impose on V, namely $\Delta(54) \times CP$, which occurs when $\lambda_5 = 0$; $[U(1) \times U(1)] \rtimes S_3$, which occurs when $\lambda_3 = \lambda_5 = 0$; $\Sigma(36)$, which occurs when $\lambda_4 = \lambda_5 = 0$; and $U(3)$, which occurs when $\lambda_3 = \lambda_4 = \lambda_5 = 0$. So, if one solves the above problems for a $\Delta(54)$-symmetric 3HDM, one automatically has their solution for $\Delta(54) \times CP$-invariant, $\Sigma(36)$-invariant, $[U(1) \times U(1)] \rtimes S_3$-invariant, and $U(3)$-invariant 3HDMs.

In order to solve our problems we must investigate the allowed orbit space, *i.e.* the set of quartets (x, y, q, t) that may result from arbitrary doublets (1); if the orbit space is either concave or a polytope, then the global minimum must be located at a vertex of the space [9].

There have been studies of the $\Delta(54)$-invariant scalar potential in the past [10–18]. Those studies affirmed that the minimum of V may lie at one of the four inequivalent orbit-space points

$$V_1 := (x, y, q, t) = (0,\ 0,\ 0,\ 0); \tag{9a}$$

$$V_2 := (x, y, q, t) = \left(0,\ 0,\ \frac{1}{3},\ 0\right); \tag{9b}$$

$$V_6 := (x, y, q, t) = \left(0,\ -1,\ \frac{1}{3},\ 1\right); \tag{9c}$$

$$V_8 := (x, y, q, t) = \left(0,\ -1,\ \frac{1}{3},\ -1\right). \tag{9d}$$

Those points are realized by the following sets of doublets [19]:

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \Phi_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } V_1; \tag{10a}$$

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for } V_2; \tag{10b}$$

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_3 = \begin{pmatrix} 0 \\ \omega \end{pmatrix} \text{ for } V_6; \tag{10c}$$

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_3 = \begin{pmatrix} 0 \\ \omega^2 \end{pmatrix} \text{ for } V_8. \tag{10d}$$

Furthermore, points $V_{1,2,6,8}$ are [10] the only possible minima of the $\Delta(54)$-invariant potential that conserve the $U(1)$ of electromagnetism, *i.e.* minima of V where one may set only the lower components of the Φ_k nonzero.[5] Three questions were left unanswered by all previous works:

1. What are the conditions on $\lambda_1, \ldots, \lambda_5$ that guarantee that V_4 is BFB?
2. What are the possible charge-breaking minima of V?
3. What are the conditions on $\lambda_1, \ldots, \lambda_5$ that guarantee that a specific point in orbit space is the global minimum of V?

Here we strive to answer questions 1–3; they have already been answered before for the cases of a $[U(1) \times U(1)] \rtimes S_3$-symmetric 3HDM [20] and of a $\Sigma(36)$-symmetric 3HDM [21], but neither for the case of a general $\Delta(54)$-symmetric 3HDM nor even for its CP-invariant version.

The outline of this letter is the following. In Section 2 we describe the polytope which is the convex hull[6] of the $\Delta(54)$-invariant scalar potential. In Section 3 we answer the three questions for a $\Delta(54)$-invariant 3HDM; we do the same in Section 4 for a $\Delta(54) \times CP$-invariant 3HDM.

2. The polytope

- Let
$$\Phi_k = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \Phi_l = \begin{pmatrix} c \\ d \end{pmatrix}, \tag{11}$$
where a, b, c, and d are complex numbers. Then,
$$\begin{aligned} A_{kk}A_{ll} - A_{kl}A_{lk} &= \left(|a|^2 + |b|^2\right)\left(|c|^2 + |d|^2\right) - \left|a^* c + b^* d\right|^2 \\ &= |ad - bc|^2, \end{aligned} \tag{12}$$
which is non-negative. Therefore $Q \geq R$, hence $x \leq 0$.
- Since $A_{kl}A_{lk} \geq 0$, $R \geq 0$. Hence, $x + q \geq 0$.
- We note that
$$\left|A_{12} + A_{23} + A_{31}\right|^2 = R + 2\Re P \geq 0. \tag{13}$$
Hence, $3(x+q) + y \geq 0$.
- We also note that
$$\left|A_{12} + \omega A_{23} + \omega^2 A_{31}\right|^2 = R - \Re P - \sqrt{3}\Im P, \tag{14a}$$
$$\left|A_{12} + \omega^2 A_{23} + \omega A_{31}\right|^2 = R - \Re P + \sqrt{3}\Im P \tag{14b}$$
are both non-negative. Therefore, $y \leq t \leq -y$.
- The parameterization (1) of the doublets gives
$$Q = N^2 \sin^2\alpha\left(\cos^2\alpha + \sin^2\alpha\cos^2\beta\sin^2\beta\right). \tag{15}$$
The maximum possible value of this quantity is $N^2/3$, which is achieved when $A_{11} = A_{22} = A_{33}$. Hence, $q \leq 1/3$.
- Defining $C_\theta = \cos^2\theta$ and $S_\theta = \sin^2\theta$ for $\theta \in \{\alpha, \beta, \gamma, \delta\}$, one has
$$\begin{aligned} Q - R = N^2 S_\alpha \Big\{ C_\alpha\left(C_\beta S_\gamma + S_\beta S_\delta\right) + S_\alpha C_\beta S_\beta \Big[C_\gamma S_\delta \\ + S_\gamma C_\delta - 2\sqrt{C_\gamma S_\gamma C_\delta S_\delta}\cos\left(\xi - \psi - \nu\right) \Big] \Big\}. \end{aligned} \tag{16}$$
Maximizing the quantity in the right-hand side of Eq. (16) as a function of α, β, γ, δ, and $\xi - \psi - \nu$, one finds that the maximum value of $Q - R$ is $N^2/4$, which is reached for a variety of configurations of the doublets. Thus, $x \geq -1/4$.[7]

[3] Note that an A_4-symmetric 3HDM [7,8] also has a potential of the form (8)—though with a different P and therefore a different shape of the orbit space.

[4] Throughout this letter, whenever we mention minima of potentials we have in mind their absolute or global minima, which are unique. We *never* refer to local minima.

[5] This means that the Φ_k mimic singlets of $SU(2)$, *i.e.* that $A_{kl}A_{lk} = A_{kk}A_{ll}\ \forall k \neq l$, *i.e.* that $x = 0$.

[6] The importance of finding the convex hulls of orbit spaces for deriving the BFB conditions has been emphasized in ref. [22].

[7] This bound is equivalent (after a change of variables) to a general 3HDM orbit-space bound derived in ref. [23].

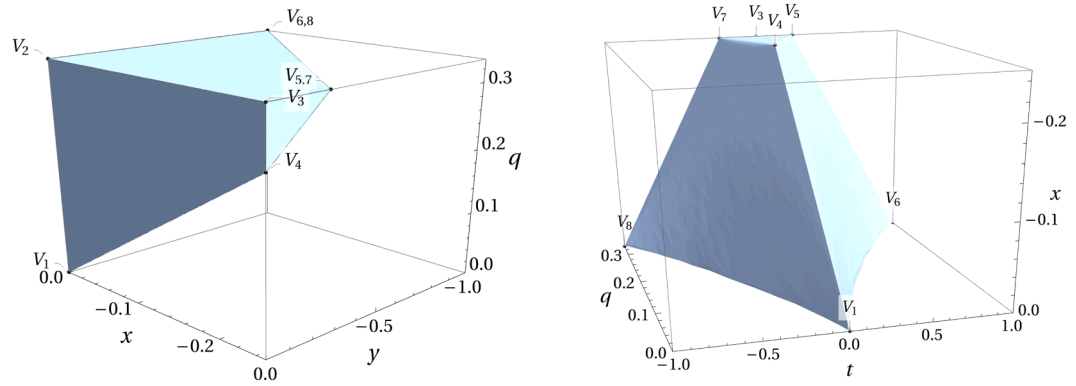

Fig. 1. Left panel: a perspective of the projection of the orbit space onto the (x, y, q) subspace. Right panel: same thing, but for a projection onto the (x, q, t) subspace. The projections of the eight vertices $V_{1,\ldots,8}$ of the four-dimensional polytope are displayed too (V_2 is hidden behind the volume in the right panel).

So, in the four-dimensional orbit space (x, y, q, t), the allowed region satisfies

$$-\frac{1}{4} \leq x \leq 0, \quad 3x + y + 3q \geq 0, \quad q \leq \frac{1}{3}, \quad y \leq t \leq -y. \tag{17}$$

Since inequalities (17) are linear in x, y, q, and t, they define a polytope in orbit space. That polytope is given in detail in https://github.com/jurciukonis/4D-polytope; here it suffices to state that it has eight vertices: those in Eq. (9) plus

$$V_3 := (x, y, q, t) = \left(-\frac{1}{4},\ 0,\ \frac{1}{3},\ 0\right); \tag{18a}$$

$$V_4 := (x, y, q, t) = \left(-\frac{1}{4},\ 0,\ \frac{1}{4},\ 0\right); \tag{18b}$$

$$V_5 := (x, y, q, t) = \left(-\frac{1}{4},\ -\frac{1}{4},\ \frac{1}{3},\ \frac{1}{4}\right); \tag{18c}$$

$$V_7 := (x, y, q, t) = \left(-\frac{1}{4},\ -\frac{1}{4},\ \frac{1}{3},\ -\frac{1}{4}\right). \tag{18d}$$

The vertices (18) have $x \neq 0$ and therefore they break electromagnetic charge, *i.e.* they can only be realized if not all three $SU(2)$ doublets have only one nonzero component, which moreover is the same for all three of them. Indeed, one reaches $V_{4,3,5,7}$ through

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \Phi_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for } V_4; \tag{19a}$$

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_2 = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix},\ \Phi_3 = \frac{1}{2}\begin{pmatrix} -\sqrt{3} \\ -1 \end{pmatrix} \text{ for } V_3; \tag{19b}$$

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_2 = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -\omega^2 \end{pmatrix},\ \Phi_3 = \frac{1}{2}\begin{pmatrix} -\sqrt{3} \\ -\omega^2 \end{pmatrix} \text{ for } V_5; \tag{19c}$$

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \Phi_2 = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -\omega \end{pmatrix},\ \Phi_3 = \frac{1}{2}\begin{pmatrix} -\sqrt{3} \\ -\omega \end{pmatrix} \text{ for } V_7. \tag{19d}$$

Thus: all points in orbit space are inside a polytope with eight vertices, and those vertices belong to the orbit space themselves.

3. Answer to the questions

One must investigate whether the orbit space coincides with the polytope, or else leaves much of its volume un-realized. In order to decide this we have randomly generated many sets of values to the parameters N, α, β, γ, δ, ψ, ν, and ξ of the doublets (1); we have then used the resulting doublets to compute x, y, q, and t. We have found that the orbit-space points thus obtained almost fill the entire polytope, but not quite; the allowed volume in orbit space appears to be slightly concave in directions that involve the parameter t. Note that in the preceding period we have used the phrase "appears to be" because we cannot visualize a four-dimensional space, so we are not fully sure of our conclusion. That conclusion is illustrated in Figs. 1 and 2, which display the projections of the four-dimensional orbit space onto selected three- and two-dimensional, respectively, subspaces.

Since the convex hull of the orbit space is a polytope and since the vertices of that polytope belong to the orbit space, the possible minima of V_4 are at those vertices. (It does not matter whether the orbit space is the full polytope or it is concave. Indeed, since V_4 is a *linear* function of the orbit-space parameters x, y, q, and t, V_4 has constant value in (three-dimensional) hyper-planes of that space. One such hyper-plane can only be tangent to a concave boundary of the orbit space at the *inner* side of that boundary, which implies that there are other hyper-planes, coresponding to lower values of V_4, that bisect orbit space—hence the point of tangency does not give the global minimum of V_4.) The questions posed at the end of the previous section have the following answers:

1. Define

$$\hat{\lambda}_1 := \lambda_1, \tag{20a}$$

$$\hat{\lambda}_2 := \lambda_1 + \frac{\lambda_4}{3}, \tag{20b}$$

$$\hat{\lambda}_3 := \lambda_1 - \frac{\lambda_2}{4} + \frac{\lambda_4}{3}, \tag{20c}$$

$$\hat{\lambda}_4 := \lambda_1 - \frac{\lambda_2}{4} + \frac{\lambda_4}{4}, \tag{20d}$$

$$\hat{\lambda}_5 := \lambda_1 - \frac{\lambda_2 + \lambda_3}{4} + \frac{\lambda_4}{3} + \frac{\lambda_5}{4}, \tag{20e}$$

$$\hat{\lambda}_6 := \lambda_1 - \lambda_3 + \frac{\lambda_4}{3} + \lambda_5, \tag{20f}$$

$$\hat{\lambda}_7 := \lambda_1 - \frac{\lambda_2 + \lambda_3}{4} + \frac{\lambda_4}{3} - \frac{\lambda_5}{4}, \tag{20g}$$

$$\hat{\lambda}_8 := \lambda_1 - \lambda_3 + \frac{\lambda_4}{3} - \lambda_5. \tag{20h}$$

   The potential is BFB if and only if all the $\hat{\lambda}_i$ $(i = 1, \ldots, 8)$ are non-negative.[8]
2. The possible charge-breaking minima of the potential are at the points (18) or, equivalently, at the orbits of the sets of doublets (19).
3. If one wants the global minimum of V to be at any particular vertex V_i, then one must choose $\lambda_2, \ldots, \lambda_5$ such that $\hat{\lambda}_i < \hat{\lambda}_j$ $\forall j \neq i$.

In order to check the correctness of our conclusion 1, we have randomly generated a data set comprising 10^6 quartic potentials V_4 with

[8] While our manuscript was being referred, we received a private communication from Josué Alexander Ibarra informing us that he had developed a formal proof of this result by using the Lean 4 proof assistant and artificial intelligence (Claude Code Fable 5 and Opus 4.8). Ibarra's work is presented in https://github.com/ibarrajo/ApplePie/blob/main/ApplePie/Hypotheses/Delta54ThreeHDMStability.lean.

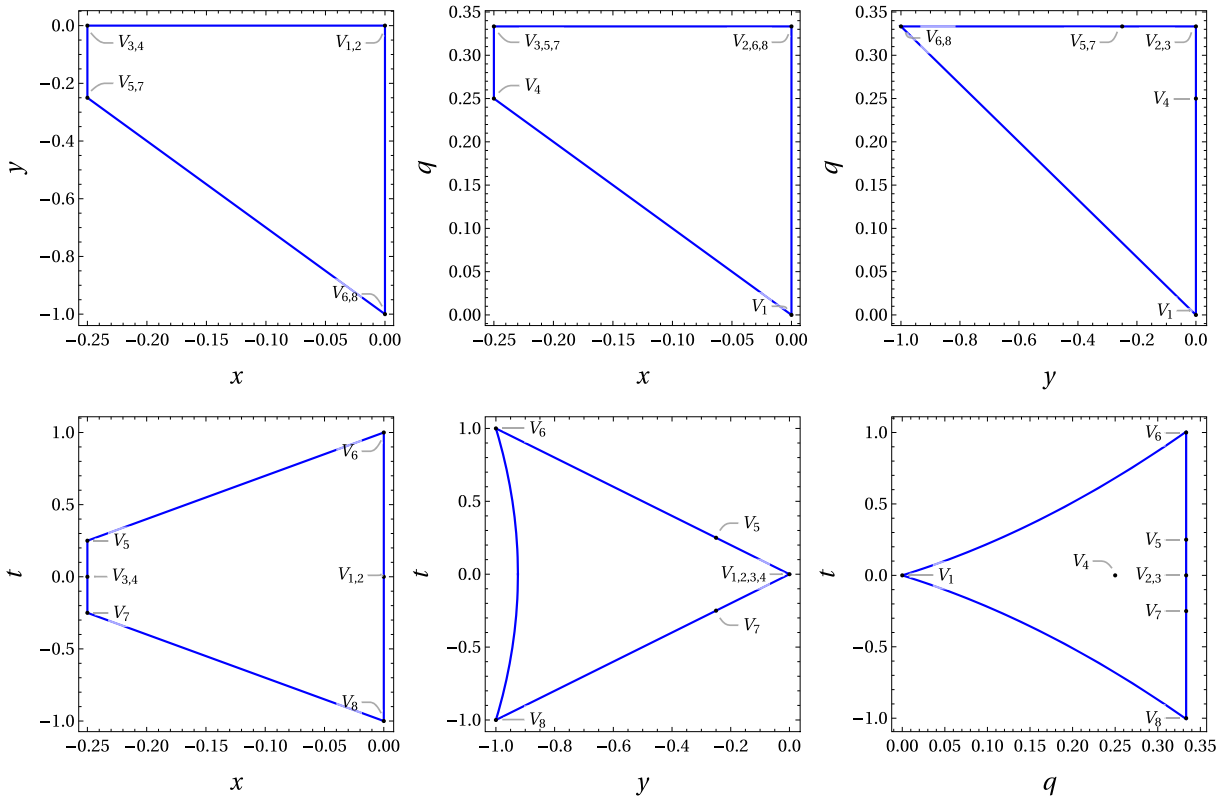

Fig. 2. Six projections of the orbit space onto two-dimensional subspaces. The projections of the vertices of the four-dimensional polytope are indicated too.

couplings in the ranges $\lambda_1 \in [0, 10]$ and $\lambda_{2-5} \in [-10, 10]$. We have minimized those V_4 by using brute force. Without loss of generality, we fixed

$$\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \Phi_2 = \begin{pmatrix} c \\ d + ie \end{pmatrix}, \ \Phi_3 = \begin{pmatrix} f + ig \\ h + ik \end{pmatrix} \tag{21}$$

and we minimized V_4 with respect to the real numbers c, d, e, f, g, h, k. The minimization procedure is described in ref. [24]; we have used the MATHEMATICA function NMinimize. The minimization identified 561 771 V_4 (out of the initial 1,000,000) with non-negative minimum. Applying our analytical BFB conditions $\hat{\lambda}_i \geq 0 \ \forall i = 1, \ldots, 8$, we found exactly the same 561 771 V_4. This brute-force test demonstrates that the analytical BFB conditions correctly identify all the BFB potentials.

4. The CP-invariant case

If the model has $\Delta(54) \times CP$ invariance, then the last term in the right-hand side of Eq. (5b) is absent. This implies that the orbit space (x, y, q) is three-dimensional instead of four-dimensional. The polytope with eight vertices $V_1, \ldots, V_8$ becomes a polytope with just six vertices $\tilde{V}_1, \ldots, \tilde{V}_6$ because, when one eliminates the parameter t, the vertices V_5 and V_7 merge, and the same happens to the vertices V_6 and V_8. Thus,

$$\tilde{V}_1 : \ (x, y, q) = (0, 0, 0); \tag{22a}$$

$$\tilde{V}_2 : \ (x, y, q) = \left(0, 0, \frac{1}{3}\right); \tag{22b}$$

$$\tilde{V}_3 : \ (x, y, q) = \left(-\frac{1}{4}, 0, \frac{1}{3}\right); \tag{22c}$$

$$\tilde{V}_4 : \ (x, y, q) = \left(-\frac{1}{4}, 0, \frac{1}{4}\right); \tag{22d}$$

$$\tilde{V}_5 : \ (x, y, q) = \left(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{3}\right); \tag{22e}$$

$$\tilde{V}_6 : \ (x, y, q) = \left(0, -1, \frac{1}{3}\right). \tag{22f}$$

The situation is simpler, not just because the orbit space is three-dimensional instead of four-dimensional, and because the polytope has six vertices instead of eight, but also because the orbit space *completely fills* the polytope, *i.e.* it has flat boundaries everywhere instead of it having some concave boundaries; this can be seen in the left panel of Fig. 1 and in the top panels of Fig. 2.

Corresponding to the vertices $\tilde{V}_1, \ldots, \tilde{V}_6$ we define

$$\tilde{\lambda}_i := \hat{\lambda}_i \quad \forall i \in \{1, 2, 3, 4\}, \tag{23a}$$

$$\tilde{\lambda}_5 := \lambda_1 - \frac{\lambda_2 + \lambda_3}{4} + \frac{\lambda_4}{3}, \tag{23b}$$

$$\tilde{\lambda}_6 := \lambda_1 - \lambda_3 + \frac{\lambda_4}{3}. \tag{23c}$$

Then:

1. The $\Delta(54) \times CP$-invariant scalar potential is bounded from below if and only if all the $\tilde{\lambda}_i$ $(i = 1, \ldots, 6)$ are non-negative.
2. The possible charge-breaking minima of the $\Delta(54) \times CP$-invariant potential are at the points (22c), (22d), and (22e).
3. If one wants the global minimum of the $\Delta(54) \times CP$-invariant potential to be at the vertex $\tilde{V}_i$, then one must choose $\lambda_2, \ldots, \lambda_5$ such that $\tilde{\lambda}_i < \tilde{\lambda}_j \ \forall j \neq i$.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Luis Lavoura reports financial support was provided by Portuguese Foundation for Science and Technology (FCT). If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

André Milagre helped us in the very early stages of this work. We thank Alex Ibarra for, during the referral process of our manuscript,

developing a formal proof of our result on boundedness from below, and also for calling our attention to ref. [23]. The work of D.J. received funding from the Research Council of Lithuania (LMT) under contract no S-CERN-24-2; part of the computations were performed using the infrastructure of the Lithuanian Particle Physics Consortium in the framework of agreement no VS-13 of Vilnius University with LMT. The work of L.L. received funding from the Portuguese Foundation for Science and Technology (FCT) through projects UIDB/00777/2020 and UIDP/00777/2020, and by the Recovery and Resilience Plan within the scope of investment RE-C06-i06, measure RE-C06-i06.m02, project 2024.01362.CERN.

Data availability

No data was used for the research described in the article.

References

[1] G.C. Branco, P.M. Ferreira, L. Lavoura, M.N. Rebelo, M. Sher, J.P. Silva, Theory and phenomenology of two-Higgs-doublet models, Phys. Reps. 516 (2012) 1–102. arXiv:1106.0034 [hep-ph]
[2] G. Bhattacharyya, I. de Medeiros Varzielas, P. Leser, Common origin of fermion mixing and geometrical CP violation, and its test through Higgs physics at the LHC, Phys. Rev. Lett. 109 (2012) 241603. arXiv:1210.0545 [hep-ph]
[3] I. de Medeiros Varzielas, D. Pidt, UV completions of flavour models and large θ_{13}, J. High Energy Phys. 2013 (2013) 65. arXiv:1211.5370 [hep-ph]
[4] I. de Medeiros Varzielas, D. Pidt, Towards realistic models of quark masses with geometrical CP violation, J. Phys. G: Nucl. Part. Phys. 41 (2014) 025004. arXiv:1307.0711 [hep-ph]
[5] G. Barreto, I. de Medeiros Varzielas, 3HDM with softly broken $\Delta(54)$ and $\Sigma(36)$, Eur. Phys. J. C 85 (2025) 1416. arXiv:2509.04369 [hep-ph]
[6] G. Barreto, I. de Medeiros Varzielas, Y.-L. Zhou, Domain walls from $\Sigma(36\times3)$, $\Delta(54)$ and $\Delta(27)$ potentials. arXiv:2603.04496 [hep-ph]
[7] I.P. Ivanov, F. Vazão, Yet another lesson on the stability conditions in multi-Higgs potentials, J. High Energy Phys. (2020) 104. arXiv:2006.00036 [hep-ph]
[8] I.P. Ivanov, N. Buskin, Bounded-from-below conditions for A_4-symmetric 3HDM, J. Phys. A: Math. Theor. 54 (2021) 325401. arXiv:2104.11428 [hep-ph]
[9] A. Degee, I.P. Ivanov, V. Keus, Geometric minimization of highly symmetric potentials, J. High Energy Phys. 2013 (2013) 125. arXiv:1211.4989 [hep-ph]
[10] I.P. Ivanov, C.C. Nishi, Symmetry breaking patterns in 3HDM, J. High Energy Phys. 2015 (2015) 21. arXiv:1410.6139 [hep-ph]
[11] M. Fallbacher, A. Trautner, Symmetries of symmetries and geometrical CP violation, Nucl. Phys. B 894 (2015) 136–160. arXiv:1502.01829 [hep-ph]
[12] I. de Medeiros Varzielas, D. Emmanuel-Costa, Geometrical spontaneous CP violation, Phys. Rev. D 84 (2011) 117901. arXiv:1106.5477 [hep-ph]
[13] I. de Medeiros Varzielas, S.F. King, C. Luhn, T. Neder, CP-odd invariants for multi-Higgs models: applications with discrete symmetry, Phys. Rev. D 94 (2016) 056007. arXiv:1603.06942 [hep-ph]
[14] I. de Medeiros Varzielas, S.F. King, C. Luhn, T. Neder, Spontaneous CP violation in multi-Higgs potentials with triplets of $\Delta(3n^2)$ and $\Delta(6n^2)$, J. High Energy Phys. 11 (2017) 136. arXiv:1706.07606 [hep-ph]
[15] I. de Medeiros Varzielas, S.F. King, C. Luhn, T. Neder, Minima of multi-Higgs potentials with triplets of $\Delta(3n^2)$ and $\Delta(6n^2)$, Phys. Lett. B 775 (2017) 303. arXiv:1704.06322 [hep-ph]
[16] I. de Medeiros Varzielas, D. Emmanuel-Costa, P. Leser, Geometrical CP violation from non-renormalisable scalar potentials, Phys. Lett. B 716 (2012) 193. arXiv:1204.3633 [hep-ph]
[17] J. Kalinowski, W. Kotlarski, M.N. Rebelo, I. de Medeiros Varzielas, 3HDM with $\Delta(27)$ symmetry and its phenomenological consequences, J. High Energy Phys. 02 (2023) 231. arXiv:2112.12699 [hep-ph]
[18] V. Keus, S.F. King, S. Moretti, Three-Higgs-doublet models: symmetries, potentials and Higgs boson masses, J. High Energy Phys. 01 (2014) 052. arXiv:1310.8253 [hep-ph]
[19] G.C. Branco, J.-M. Gerard, W. Grimus, Geometrical T-violation, Phys. Lett. B 136 (1984) 383–386.
[20] F.S. Faro, I.P. Ivanov, Boundedness from below in the U(1) × U(1) three-Higgs-doublet model, Phys. Rev. D 100 (2019) 035038. arXiv:1907.01963 [hep-ph]
[21] Y. Yang, I.P. Ivanov, Charge-breaking domain walls separating neutral vacua in multi-Higgs models, Phys. Rev. D 110 (2024) 015001. arXiv:2401.03264 [hep-ph]
[22] M. Heikinheimo, K. Kannike, F. Lyonnet, M. Raidal, K. Tuominen, H. Veermäe, Vacuum stability and perturbativity of SU(3) scalars, J. High Energy Phys. 2017 (2017) 014. arXiv:1707.08980 [hep-ph]
[23] I.P. Ivanov, C. Nishi, Properties of the general N-Higgs-doublet model. I. The orbit space, Phys. Rev. D 82 (2010) 015014. arXiv:1004.1799 [hep-ph]
[24] D. Jurčiukonis, L. Lavoura, A. Milagre, Assessing boundedness from below in the Z_2xZ_2-symmetric three-Higgs-doublet model: algorithm and machine learning, SciPost Phys. Core 9 (2026) 041. arXiv:2603.23590 [hep-ph]