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https://orcid.org/0000-0003-0641-1751

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Simona Jokubauskienė

Multidimensional Compound Poisson and Signed Compound Poisson Approximations

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Academic Consultant – Prof. Habil. Dr. Vydas Čekanavičius (Vilnius University, Natural Sciences, Mathematics, N 001).

The dissertation will be defended in a public meeting of the Dissertation Defence Panel:
Chairman – Prof. Dr. Jurgita Markevičiūtė (Vilnius University, Natural Sciences , Mathematics, N 001).
Members:
Prof. Dr. Ričardas Krikštolaitis (Vytautas Magnus University, Natural Sciences, Mathematics, N 001);
Prof. Dr. Neelesh Shankar Upadhye (Indian Institute of Technology Madras, Natural Sciences, Mathematics, N 001);
Assoc. Prof. Dr. Viktor Skorniakov (Vilnius University, Natural Sciences, Mathematics, N 001);
Prof. Dr. Jonas Šiaulys (Vilnius University, Natural Sciences, Mathematics, N 001).

The dissertation shall be defended at a public meeting of the Dissertation Defence Panel at 10:00 a.m. on 28th of October 2026 in Room 201 of the Vilnius University Faculty of Mathematics and Informatics. Address: Naugarduko St. 24-201, LT-03225, Vilnius, Lithuania Tel. +370 5 219 3050; e-mail: mif@mif.vu.lt

https://doi.org/10.15388/vu.thesis.1002
https://orcid.org/0000-0003-0641-1751

VILNIAUS UNIVERSITETAS

Simona Jokubauskienė

Daugiamatės sudėtinės Puasono ir ženklo sudėtinės Puasono aproksimacijos

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Mokslinis konsultantas – prof. habil. dr. Vydas Čekanavičius (Vilniaus universitetas, gamtos mokslai, matematika, N 001).

Gynimo taryba:
Pirmininkė – prof. dr. Jurgita Markevičiūtė (Vilniaus universitetas, gamtos mokslai, matematika, N 001).
Nariai:
prof. dr. Ričardas Krikštolaitis (Vytauto Didžiojo universitetas, gamtos mokslai, matematika, N 001);
prof. dr. Neelesh Shankar Upadhye (Indijos technologijų institutas Madrase, Indija, gamtos mokslai, matematika, N 001);
doc. dr. Viktor Skorniakov (Vilniaus universitetas, gamtos mokslai, matematika, N 001);
prof. dr. Jonas Šiaulys (Vilniaus universitetas, gamtosmokslai, matematika, N 001).

Disertacija ginama viešame Gynimo tarybos posėdyje 2026 m. spalio mėn. 28 d. 10 val. Vilniaus universiteto, Matematikos ir informatikos fakulteto 201 auditorijoje. Adresas: (Naugarduko g. 24-201, LT-03225, Vilnius, Lietuva), tel. +370 5 219 3050; el. mif@mif.vu.lt

Notation and abbreviations

$\mathbb{R}^d$	d-dimentional real space
$\mathbb{R}$	the set of real numbers
$\mathbb{N}^d$	d-dimentional natural number space
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}^d$	d-dimentional integer space
$\mathbb{Z}$	the set of integer numbers
$\mathbb{Z}_+^d$	$\mathbb{N}^d \cup \{\vec{0}\}$
ch.f.	characteristic function
CP	compound Poisson
SCP	signed compound Poisson
i.i.d.	independent and identically distributed
r.v.	random variable
r.v.s.	random vectors
π_λ	Poisson r.v. with parameter λ
$\mathscr{L}(\vec{X})$	distribution of r.v.s. $\vec{X}$
$I_{\vec{a}}$	distribution concentrated at $\vec{a}$
I	$I_{\vec{0}}$
$Bi(n,p)$	Binomial distribution with parameters n, p
$Pois(\lambda)$	Poisson distribution with parameter λ
Θ	any measure satisfying $\|\Theta\| \leqslant 1$
$\exp\{M\}$	exponent of measure M
$M * V$	convolution of measures M and V
F^{*n}	n-fold convolution of F
$\operatorname{supp} M$	support of a measure M
$\widehat{M}(t)$	Fourier transform of measure M
$\|M\|_{TV}$	the total variation of a measure M
$\|M\|_\infty$	local (point) norm of a measure M
$\|M\|_r$	ℓ_r, $(r \geqslant 1)$, norm of a measure M
$\mathscr{D}_d$	the class of infinitely divisible d-dimentional distributions
$\mathscr{F}_d$	the class of all d-dimentional distributions
$\mathscr{F}_d^S$	the class of symmetric d-dimentional distributions
$\mathscr{F}_d^Z$	the class of d-dimentional distributions taking values in $\mathbb{Z}^d$
i	imaginary unit: $\mathrm{i}^2 = -1$

$Re(z)$	real part of a complex number z
$Im(z)$	imaginary part of a complex number z
$\lfloor x \rfloor$	integer part of x
$\lceil x \rceil$	fractional part of x
C	an absolute constant
LHS, RHS	the left-hand side, the right-hand side
$\asymp$	asymptotically equal
$\square$	end of a proof

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INTRODUCTION

Research topic and actuality

Compound Poisson (CP) approximations have a broad range of theoretical and practical applications, for a comprehensive review see [8, 16, 30, 31, 32, 34, 39, 40, 59, 63, 68] and references therein. It was shown that CP approximations can be useful in actuarial mathematics, biology, word counts, extreme value theory, etc. CP approximations also play significant role in theoretical problems of Probability Theory. However, the absolute majority of research was done for one-dimensional cases. The multidimensional CP approximations are comparatively little researched. In this Thesis we estimate the accuracy of multidimensional CP and signed compound Poisson (SCP) approximations to the sum of random vectors. The choice of approximations (accompanying laws, Hipp-type approximation, Skellam distribution) was determined by the needs of Actuarial Mathematics and theoretical problem of the best infinitely divisible approximation for the sums of independent random vectors (the first uniform Kolmogorov theorem). We discuss both topics below.

CP approximations and Actuarial models. Let p_i be probability that risk i produces a claim and $V_i \in \mathscr{F}_d$ be the conditional distribution of the claims in risk i. Then claims amount has distribution $(1-p_i)I + p_iV_i$ and *aggregate claims distribution* for portfolio of n claims is the sum of independent claims and its distribution can be written as a convolution of individual claims distributions

$$\prod_{i=1}^{*n}((1-p_i)I + p_iV_i). \tag{0.1}$$

It is natural to assume that p_i are small (claims are rare). If we assume that the amounts of claims are independent and identically distributed, then $V_i \equiv V$.

Aggregate claims distribution plays very important role in Actuarial Mathematics, see discussion on p. 3 of [81]. However, to calculate aggregate claims distribution is not always easy; even for V_i concentrated on integers. Therefore, approximations with somewhat better applicability are used. In [81], Ch. 10 it is shown that accompanying CP distributions and Hipp [46] and Kornya [52] SCP approximations can be calculated via modified De Pril recursion.

In [81], Ch. 14.1 it is argued that multivariate $V_i \in \mathscr{F}_d$ are also important actuarial models, since they represent multiple and related claims. For example, V_i can represent 3-dimensional claims distribution for health; home and

car insurances.

Application of CP and SCP approximations in Actuarial Science typically consists of two parts:

a) estimation of the accuracy of approximation in some metric;

b) algorithms for numerical calculation of characteristics of interest.

Monograph [81] deals with calculation algorithms. Our results are related to estimation of the accuracy of approximation. Remarkably, though all recursion algorithms in [81] deal with local probabilities, the majority of approximation results are proved for the total variation metric. There is obvious lack of local estimates.

CP approximations and uniform Kolmogorov problems. CP approximations play crucial role in solutions of the so-called Kolmogorov problems [51] about the best possible accuracy of approximation of the sum of independent random variables by the whole set of infinitely divisible distributions. One-dimensional case is completely solved, see the monograph by Arak and Zaitsev [4]. Multidimensional case has solution for the second Kolmogorov problem about the best infinitely divisible approximation for sums of non-identically distributed independent vectors in metrics weaker that uniform one, see [86]. The first multivariate uniform Kolmogorov theorem has only a partial solution in uniform metric [64]. Known results for uniform metric are given in Chapter 2.

Symmetric distributions. A significant part of Thesis is devoted to approximations of symmetric lattice vectors. On one hand, the obtained results can be treated as a research in Kolmogorov's problem for symmetric vectors. On the other hand, as shown in [1, 2, 3, 17, 58, 67] symmetric CP distributions can be related to research in Finance, Medicine, Sports and Science. Although symmetric lattice distributions do not have an obvious actuarial interpretation, it is mentioned in [81] (see Ch. 12) that random vectors taking negative values might be applied to model negative severities.

Aims and goals

1. To improve the accuracy of the Kolmogorov approximation of a sum of shifted independent identically distributed random vectors by constructing an asymptotic expansion to the accompanying CP law.

2. To investigate structural simplifications to Hipp-type approximations for rare events. To obtain estimates for the short asymptotic expansion and for the multivariate Skellam approximation in total variation.

3. To obtain general local, ℓ_r norm and marginal non-uniform estimates for the Hipp-type approximations to the sums of independent lattice random vectors.

4. To derive a total variation estimate of the distribution of the sum of independent, identically distributed symmetric lattice vectors, using the accompanying CP law and a Hipp-type SCP measure for non-rare events.

5. To obtain general local, ℓ_r norm and marginal non-uniform estimates for the Hipp-type approximations to the sums of independent symmetric lattice random vectors.

6. To obtain general local, ℓ_r norm, total variation norm, and marginal non-uniform estimates for the Skellam approximation to the sums of independent symmetric random vectors.

7. To propose a partial solution to the first Kolmogorov problem for a sequence of independent, identically distributed, lattice two-dimensional random vectors with finite fourth-order moments.

8. To obtain recursive relations for two-dimensional Hipp-type approximations.

Novelty

The thesis presents estimates for (shifted) accompanying CP approximations and Hipp-type SCP approximations to the distribution of the sum of random vectors, extending one-dimensional results to the multivariate case, which is comparatively little researched. Key contributions include improved estimates in local, Kolmogorov, total variation, and ℓ_r ($r \geqslant 2$) norms. In addition, non-uniform, local, marginal non-uniform local estimates are derived for the first time. Estimates in total variation are derived under weaker assumptions. These advances fill critical gaps in actuarial models of multivariate aggregate claims distributions and contribute toward solving the multidimensional uniform Kolmogorov problem for the best infinitely divisible approximation of sums of independent random vectors. All results are new. Some of them are new even in one-dimensional case.

Main results

Taking into account the correspondence between characteristic functions and distributions we can illustrate the main research topics of this Thesis in the following way. Let f_j be the characteristic function of r.v. $\vec{X}_j$ (the precise definition is given in the next chapter). The characteristic function of the sum of independent r.v.s. $S_n = \vec{X}_1 + \ldots + \vec{X}_n$ is equal to $f_1 f_2 \cdot \ldots \cdot f_n$. We can write formally

$$f_j = \mathrm{e}^{\ln f_j} = \exp\left\{ \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(f_j - 1)^m}{m} \right\}.$$

Therefore,

$$\prod_{j=1}^{n} f_j \approx \exp\left\{ \sum_{j=1}^{n} \sum_{m=1}^{k} \frac{(-1)^{m+1}(f_j - 1)^m}{m} \right\},$$

where Fourier transform on the RHS corresponds to the so-called Hipp-type approximation. Note that if $k = 1$, then the Hipp-type approximation is the accompanying CP distribution, but for $k > 1$ we have the signed CP measure. The main results of this Thesis deal with the accuracy of Hipp-type CP approximations in various metrics. For symmetric r.v.s. we also investigate possibility to replace $\exp\{f_j - 1\}$ by $\exp\{g_j - 1\}$, where g_j has simpler structure than f_j.

The accuracy of the uniform approximation of a sum of shifted independent identically distributed random vectors by the accompanying compound Poisson law is improved by choosing a suitable asymptotic expansion (Theorem 2.2.1, Corollary 2.2.1). The impact of shifting on convolutions with a CP distribution is considered in a multidimensional case (Theorem 2.2.2, Corollary 2.2.2). The closeness of two CP distributions with shifted compounding distributions is estimated in Kolmogorov's norm (Theorem 2.2.3). The successive convolutions of shifted distributions are estimated (Theorem 2.2.4), which demonstrates that if the shifting of the initial distribution is allowed, then the estimate holds for the scheme of triangular arrays with a constant that does not depend on the distribution.

Some structural simplifications to Hipp-type approximations are presented, and the total variation estimate is obtained under the assumption that random vectors take zero values with quite large probability (Theorem 3.2.1). Closeness of accompanying CP approximation to multivariate Skellam distribution is estimated in total variation norm (Theorem 3.2.2) under assumption that distributions are concentrated on a finite number of hyper-spaces.

Distribution of the sum of independent lattice vectors is approximated by

accompanying CP distribution (Corollary 4.3.1) and Hipp-type SCP measure (Theorem 4.3.1, Theorem 4.3.4). The accuracy of approximation is estimated in point and ℓ_r, $r \geqslant 2$ metrics. It is shown that the smallness of estimates directly depends on the dimension of random vectors. First-order asymptotic expansions to CP approximation and SCP approximation are considered (Theorem 4.3.2, Theorem 4.3.5). The estimate extends the corresponding asymptotical result for r.v.s. Upper bounds are established for non-uniform local estimates (Theorem 4.3.3), generalizing and extending known one-dimensional estimates. General lower bounds are proved for local and ℓ_r metrics (Theorems 4.3.6–4.3.8). Particularly, it is proved that upper and lower bounds have the same order of accuracy in the scheme of sequences.

The distribution of the sum of independent identically distributed symmetric lattice vectors is approximated by the accompanying CP law (Theorem 5.2.1, Theorem 5.2.1*) and Hipp-type SCP measure (Theorem 5.2.2) under significantly weakened assumptions. The accuracy of approximation is estimated in the total variation metric. The first-order asymptotic expansion is constructed (Theorem 5.2.3).

A distribution of the sum of independent lattice vectors having symmetric distributions is approximated by the Hipp-type SCP measures (Theorem 6.2.1, Theorem 6.2.2). A non-exponential asymptotic expansion is considered (Theorem 6.2.3). The upper and lower bounds are established for the point metric and ℓ_r, $r \geqslant 1$, metrics (Theorem 6.2.7). Marginal non-uniform local estimates are obtained (Theorem 6.2.4–6.2.6).

The sum of independent identically distributed random lattice vectors is approximated by the difference of two independent Poisson vectors (Theorem 7.4.2, Theorem 7.4.1, Theorem 7.4.2*). Local estimates and estimates in ℓ_r metric are also considered (Theorems 7.11–7.13).

A sequence of independent identically distributed lattice two-dimensional random vectors having finite moments of 4'th order are considered. CP approximation is constructed (Theorems 8.2.1–8.2.4). The obtained results are closely related to the first uniform Kolmogorov theorem.

The assumptions for the Hipp-type CP approximation of two-dimentional lattice vectors in total variation are weakened (Theorem 9.1.1). The non-identically distributed symmetric r.v.s. are approximated (Theorem 9.2.1). Practical applicability of Hipp-type approximations is considered (Proposition 9.3.1).

Statements presented for defence

1. The accuracy of the uniform approximation of a sum of shifted independent identically distributed random vectors by the accompanying compound Poisson law can be improved by constructing a suitable asymptotic expansion.

2. It is possible to simplify structurally the Hipp-type approximation by constructing asymptotic expansion under the assumption that random vectors take zero values with large probabilities. If, in addition, random vectors are concentrated on a finite number of lines, then the accompanying CP approximation is close in total variation to the difference of two independent Poisson vectors (Skellam distribution).

3. Known assumptions for local Hipp-type approximation to the sum of independent lattice r.v.s. can be significantly weakened. The abstract lower bound theorem shows that in many cases the upper bound estimates presented in this Thesis are of the correct order. Marginal non-uniform local estimates and estimates in $\ell_r, r \geqslant 2$, metric can be obtained under additional assumptions of finite moments.

4. The total variation estimate of the distribution of the sum of independent identically distributed symmetric lattice vectors by the accompanying CP law and Hipp-type SCP measure can be derived under significantly weakened assumptions.

5. The structure of the total variation accompanying CP approximation of the sum of independent identically distributed random lattice vectors can be simplified to Skellam distribution if random vectors are concentrated on a finite number of lines. In addition, local estimates and estimates in ℓ_r metric can be proved.

6. In the scheme of sequences, there exists CP distribution, which in total variation approximates the sum of n independent, identically distributed, lattice two-dimensional random vectors with the accuracy of the order $O(n^{-1})$.

7. The assumptions for the Hipp-type CP approximations in total variation can be weakened for two-dimensional lattice vectors.

8. 'Probabilities' of the second order two-dimensional Hipp-type approximation satisfy simple recursive relations.

Methods

In this Thesis, properties of convolutions, metrics and the characteristic function method are the main tools used for the proofs.

Publications

1. Čekanavičius V., Jokubauskienė S. Asymptotics in multivariate Le Cam's theorem, Journal of theoretical probability, 2025, vol. 38, no. 4, p. 1–15.
2. Čekanavičius V., Jokubauskienė S. Multidimensional compound Poisson approximations for symmetric distributions, Modern Stoch. Theory Appl., 2026, 1–23.
3. Jokubauskienė S., Čekanavičius V. Multivariate Hipp-type compound Poisson approximations for lattice distributions. I, Lithuanian Mathematical Journal, 2025, vol. 65, no. 3, p. 337–356.
4. Jokubauskienė S., Čekanavičius V. Multivariate Hipp-type compound Poisson approximations for lattice distributions. II, Lithuanian Mathematical Journal, 2026 (accepted).

Conferences

1. Jokubauskienė S. Daugiamatės sudėtinės Puasono aproksimacijos, *Informatikos fakulteto seminaras, VDU*, September 26, 2025, Kaunas, Lithuania.
2. Jokubauskienė S. Hipp's approximation for d-dimentional lattice vectors, *1st international Vilnius conference on Statistics and its applications*, August 28, 2025, Vilnius, Lithuania.
3. Jokubauskienė S. Multivariate Hipp-type compound Poisson approximations for lattice distributions, *28th International Congress on Insurance: Mathematics and Economics*, July 3, 2025, Tartu, Estonia.
4. Jokubauskienė S. Simetrinių diskrečiųjų daugiamačių vektorių sudėtinės Puasono aproksimacijos, *66-oji Lietuvos matematikų draugijos konferencija*, June 26, 2025, Klaipėda, Lithuania.
5. Jokubauskienė S. Daugiamatės sudėtinės Puasono aproksimacijos, *Finansų ir draudimo matematikos seminaras, VU*, May 12, 2025, Vilnius, Lithuania.

6. Jokubauskienė S. Local Hipp's Approximation for d-Dimensional Lattice Vectors, *International Conference on Probability Theory and Number Theory*, September 16, 2024, Palanga, Lithuania.

7. Jokubauskienė S. Netolygus įvertis Hipp'o teoremoje dvimačiu atveju, *65-oji Lietuvos matematikų draugijos konferencija*, June 27, 2024, Šiauliai, Lithuania.

Structure of the thesis

Chapter 1 provides the necessary notation and general facts about CP and SCP measures.

Chapter 2 is devoted to asymptotics for the shifted accompanying CP law. No assumptions apart from the independence of r.v.s. are used in this chapter. The accuracy of approximation is estimated in the Kolmogorov metric. This chapter provides an overview of known results, presents new results for defense (4 theorems), and includes their proofs.

Chapter 3 deals with estimates in total variation for sums of r.v.s. taking value zero with large probabilities (the case of rare events). This chapter contains an overview of known results alongside new results presented for defense (2 theorems), including their proofs.

In Chapter 4, local estimates, non-uniform local estimates, estimates in ℓ_r metric, and lower estimates are considered for r.v.s. concentrated on integers. This chapter comprises an overview of known results as well as new results presented for defense (9 theorems, 1 proposition), along with their proofs, and provides some insight for open problems.

Chapter 5 focuses on estimates in the total variation metric for symmetric distributions and contains an overview of known results alongside new results presented for defense (14 theorems), including their proofs.

Chapter 6 is devoted to the approximation of a distribution of the sum of independent lattice vectors having symmetric distributions. The upper and lower bounds are established for the point metric and ℓ_r, $r \geqslant 1$, metrics. Marginal non-uniform local estimates are obtained. This chapter provides an overview of known results, presents new results for defense (8 theorems), and includes their proofs.

In Chapter 7, the sum of independent identically distributed random lattice vectors is approximated by the difference of two independent Poisson vectors. The chapter contains approximation in total variation, local estimates, and es-

timates in ℓ_r, $r \geqslant 1$, metric. This chapter comprises an overview of known results as well as new results presented for defense (6 teorems), along with their proofs.

Chapter 8 deals with constructing a CP approximation that matches the first three moments of the initial distribution. This chapter provides an overview of known results, presents new results for defense (4 theorems), and includes their proofs.

Chapter 9 considers the two-dimentional case and provides some insight for possible solutions to open problems from previous chapters, presents new results for defense (2 theorems, 1 propositions), and includes their proofs.

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1 PRELIMINARIES

1.1 Notation

The set of all integers is denoted by $\mathbb{Z}$, natural numbers are denoted by $\mathbb{N}$=$\{1,2,\dots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$; $\mathbb{Z}^d$ is cartesian product of sets $\mathbb{Z}\times\mathbb{Z}\times\ldots\times\mathbb{Z}$. For $\vec{x},\vec{y}\in\mathbb{R}^d$ we write $\vec{x}\leqslant\vec{y}$ if $x_i \leqslant y_i$, for each $i=1,\dots,d$. For $\vec{x},\vec{t}\in\mathbb{R}^d$ we denote their scalar product $(\vec{x},\vec{t}) := x_1t_1 + x_2t_2 + \ldots + x_dt_d$. To make notation shorter we also write

$$\sum_{\vec{x}} = \sum_{\vec{x}\in\mathbb{Z}^d\setminus\{\vec{0}\}}.$$

Let $\mathscr{L}(\vec{X})$ denote the distribution of a random vector (r.v.s.) $\vec{X}$. Let $\mathscr{F}_d$ denote the set of all d-dimensional distributions.

By $\mathscr{M}_d$ we denote the set of all finite (signed) d-dimensional measures. Obviously, $\mathscr{F}_d \subset \mathscr{M}_d$, $d=1,2,\dots$. The measure of set A is denoted by $M\{A\}$, $M\{\vec{x}\} := M\{\{\vec{x}\}\}$.

Whenever, we write that distribution F (measure M) is concentrated on $\mathbb{Z}^d$ we mean that support of F (M) can not be a subset of a lattice, which is not a full-rank lattice. Moreover, we assume that there does not exist natural numbers $j_1, j_2, \dots, j_d$, such that $\operatorname{supp} F \subset \{j_1\mathbb{Z}\times j_2\mathbb{Z}\times\ldots\times j_d\mathbb{Z}\}$ and $j_1+j_2+\ldots+j_d<d$. For example, two dimensional distributions concentrated on $\{(0,0);(1,1)\}$ or $\{(0,0);(1,0);(2,0)\}$ are not considered in this Thesis.

Let $I_{\vec{a}}$ be the distribution concentrated at $\vec{a}\in\mathbb{R}^d$, $I = I_{\vec{0}}$, where $\vec{0} = (0,0,\dots,0)\in\mathbb{R}^d$.

Fourier transform. If $M\in\mathscr{M}_d$, then its Fourier transform is

$$\widehat{M}(t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} M\{\mathrm{d}x\}.$$

If M is concentrated on $\mathbb{Z}^d$, then

$$\widehat{M}(\vec{t}) = \sum_{\vec{m}\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(\vec{m},\vec{t})} M\{\vec{m}\}. \tag{1.1}$$

If $F\in\mathscr{F}_d$, then $\widehat{F}(\vec{t})$ is the characteristic function (ch.f.) of F. To make notation shorter further we usually omit dependence on $\vec{t}$ and write $\widehat{M},\widehat{F}$ instead of $\widehat{M}(\vec{t}),\widehat{F}(\vec{t})$.

Convolutions of measures. Let $M,V \in \mathscr{M}_d$, then their convolution, for any d-dimensional Borel set is defined as

$$(M*V)\{A\} = \int_{\mathbb{R}^d} M\{A-\vec{x}\}V\{\mathrm{d}\vec{x}\}.$$

If M and V are concentrated on $\mathbb{Z}^d$, then

$$(M*V)\{\vec{m}\} = \sum_{\vec{j}\in\mathbb{Z}^d} M\{\vec{m}-\vec{j}\}V\{\vec{j}\}.$$

For any independent r.v.s. $\vec{X}$ and $\vec{Y}$ the distribution of the sum is the convolution of the distributions: $\mathscr{L}(\vec{X}+\vec{Y}) = \mathscr{L}(\vec{X})*\mathscr{L}(\vec{Y})$. It is known that $\widehat{M*V} = \widehat{M}\widehat{V}$.

Exponent of a measure. Let $M \in \mathscr{M}_d$, then

$$\exp\{M\} = \sum_{j\geqslant 0} \frac{M^{*j}}{j!}.$$

Here powers M^{*j}, are understood in the convolution sense, $M^{*0} = I$.

Convolution of two exponential measures is also exponential measure:

$$\exp\{M\}*\exp\{V\} = \exp\{M+V\}.$$

Fourier transform of exponential measure is exponential of Fourier transform:

$$\widehat{\exp\{M\}} = \exp\{\widehat{M}\}.$$

Compound Poisson distribution. Let π_λ denote Poisson random variable with parameter λ, that is $\mathbb{P}(\pi_\lambda = m) = \mathrm{e}^{-\lambda}\lambda^m/m!$ $(m\in\mathbb{Z}_+)$.

Let $\vec{X},\vec{X}_1,\vec{X}_2,\ldots$ be independent identically distributed random vectors (i.i.d. r.v.s.) with $\mathscr{L}(\vec{X}) = F$, independent of π_λ. Then we say that

$$\vec{Y} = \sum_{j=0}^{\pi_\lambda} \vec{X}_j$$

has compound Poisson (CP) distribution with parameter λ and compounding

distribution F. We have

$$\mathscr{L}(\vec{Y}) = \exp\{\lambda(F-I)\}$$

and ch.f. of $\vec{Y}$ is equal to $\exp\{\lambda(\widehat{F}(\vec{t})-1)\}$. Convolution of two CP distributions is also CP distribution. Indeed,

$$\exp\{\lambda(F-I)+\gamma(G-I)\} = \exp\{(\lambda+\gamma)(H-I)\},$$
$$H = \frac{\lambda}{\lambda+\gamma}F + \frac{\gamma}{\lambda+\gamma}G \in \mathscr{F}_k.$$

Note that all CP distributions are infinitely divisible. A well-known Khintchin's [50] result states that the class of all infinitely divisible distributions $\mathscr{D}$ coincides with the class of weak limits of compound Poisson distributions. As can be seen from characteristic function CP distribution can also be written in the following way

$$\exp\{\lambda(F-I)\} = \sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda} F^{*j}}{j!}. \tag{1.2}$$

CP distribution $\exp\{F-I\}$ is called *accompanying distribution* to $F \in \mathscr{F}_d$.

Signed compound Poisson measure. If $F \in \mathscr{F}_d$, but $\lambda < 0$, then $\exp\{\lambda(F-I)\}$ is called signed compound measure (SCP). By SCP we also call all exponential measures $\exp\left\{\sum_{j=1}^{\infty}\lambda_j(F_j-I)\right\}$, where $F_j \in \mathscr{F}_d$ and at least one $\lambda_j < 0$.

Norms. Let $M \in \mathscr{M}_d$. Then Kolmogorov and total variation norms are resp. defined as

$$\|M\|_K = \sup_{\vec{x}\in\mathbb{R}_d} |M\{(-\infty,x_1]\times(-\infty,x_2]\times\ldots\times(-\infty,x_d]\}|,$$
$$\|M\|_{TV} = \int_{\mathbb{R}^d} 1|M\{d\vec{x}\}| = M^+\{\mathbb{R}^d\} + M^-\{\mathbb{R}^d\}.$$

Here $M = M^+ - M^-$ is the Hahn decomposition of M. Note that the total variation distance for $F, G \in \mathscr{F}^d$ is one half of the total variation norm:

$$d_{var}(F,G) := \sup_B |F\{B\} - G\{B\}| = \frac{1}{2}\|F-G\|_{TV},$$

where supremum is taken over all d-dimensional Borel sets. For $M \in \mathscr{M}$ concentrated on $\mathbb{Z}^d$ we use the following notation for norms

$$\begin{aligned} \|M\|_{TV} &= \sum_{\vec{x}\in\mathbb{Z}^d} |M\{\vec{x}\}|, \quad \|M\|_{\infty} = \sup_{\vec{x}\in\mathbb{Z}^d} |M\{\vec{x}\}|, \\ \|M\|_{r} &= \left(\sum_{\vec{x}\in\mathbb{Z}^d} |M\{\vec{x}\}|^r\right)^{1/r}, \quad (r \geqslant 1). \end{aligned}$$

Scheme of series and scheme of sequences. In general, we investigate random vectors that form triangular arrays (scheme of series):

$$\begin{aligned} &\vec{X}_{1,1}\,, \\ &\vec{X}_{2,1}\,,\ \vec{X}_{2,2}\,, \\ &\vec{X}_{3,1}\,,\ \vec{X}_{3,2}\,,\ \vec{X}_{3,3}\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\vec{X}_{n,1}\,,\ \vec{X}_{n,2}\,,\ \ldots\ldots\ldots,\ \vec{X}_{n,n}\,, \\ &S_{n,n} = \vec{X}_{n,1} + \vec{X}_{n,2} + \ldots + \vec{X}_{n,n}. \end{aligned}$$

i.e., distribution of $\mathscr{L}(X_{n,i})$, $(i = 1,\ldots,n)$ can depend on the number of series n.

Scheme of sequences menas that $S_n = \vec{X}_1 + \vec{X}_2 + \ldots + \vec{X}_n$, where $\vec{X}_1, \vec{X}_2, \ldots$ have distributions $\mathscr{L}(\vec{X}_i)$ do not depending on n.

Obviously, the scheme of sequences is partial case of the scheme of series. Double index makes all notation unnecessary complicated, therefore we always use notation $S_n, \vec{X}_1, \ldots, \vec{X}_n$ and, if not stated otherwise, consider triangular arrays of random vectors.

Infinite divisibility. We say that distribution $F \in \mathscr{F}_d$ is infinitely divisible if, for any natural n, there exist distribution G, such that $F = G^{*n}$. In terms or r.v.s., $\vec{X}$ is infinitely divisible if, for any n, it can be written as the sum of i.i.d. r.v.s. Obviously any CP distribution is infinitely divisible. Concept of infinite divisibility can be extended in a natural way to signed measures. All SCP measures then are infinitely divisible.

Other notation. The same symbol C denotes different absolute constants. If ambiguities can arise we supply C with indices. Symbols $C(F)$, C_F, C_X denote constants that depend on a distribution $\mathscr{L}(X)$ or F.

As usual, $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$. We write $f(n) = O(g(n))$ if $f(n)/g(n) \leqslant C < \infty$ for all large enough n. Multiplication is superior to division.

For any $x \in \mathbb{R}$ let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integer and the fractional parts of x.

Symbol Θ may denote any measure with total variation $\|\Theta\|_{TV} \leqslant 1$;

We use θ to denote all real or complex numbers, satisfying $|\theta| \leqslant 1$. For example, $a = b + \theta 3\sin^2 t$ means that $|a-b| \leqslant 3\sin^2 t$.

1.2 CP and SCP approximations

The general idea leading to CP and SCP approximations can be expressed in the following way: let $F \in \mathscr{F}_d$, then we can write formally (for shortness omitting the dependence on $\vec{t}$)

$$\widehat{F}^n = \mathrm{e}^{n\ln(1+(\widehat{F}-1))} = \exp\left\{ n(\widehat{F}-1) - \frac{n(\widehat{F}-1)^2}{2} + \frac{n(\widehat{F}-1)^3}{3} + \dots \right\}.$$

Leaving the first summand in the exponent leads to accompanying CP distribution $\exp\{n(F-I)\}$. By leaving longer part of logarithmic series in the exponent we get Hipp-type SCP approximation:

$$\exp\left\{ n\sum_{k=1}^{s} \frac{(-1)^{k+1}(F-I)^{*k}}{k} \right\}.$$

The natural way to construct asymptotic expansions is to use exponential series. For example,

$$\mathrm{e}^{n(F-I)} * \left(I - \frac{n(F-I)^{*2}}{2} + \dots \right); \quad \mathrm{e}^{n(F-I) - \frac{n(F-I)^{*2}}{2}} * \left(I + \frac{n(F-I)^{*3}}{3} + \dots \right).$$

Accompanying law is by far the most popular CP approximation investigated in numerous papers. It seems that Presman [65] (for the binomial random variable), Hipp [46] (general case) were the first to use SCP approximations based on idea outlined in above.

Total variation of Hipp's approximation is always finite. Indeed

$$\left\| \exp\left\{ n\sum_{k=1}^{s} (-1)^{k+1}(F-I)^{*k}/k \right\} \right\|_{TV} \leqslant \exp\left\{ n\sum_{k=1}^{s} \|F-I\|_{TV}^{k}/k \right\}$$

$$\leqslant \exp\left\{ n \sum_{k=1}^{s} 2^k/k \right\} \leqslant \exp\{2^{s+1} n\}.$$

Slightly different approach for rare events (i.e., r.v.s. taking value zero with large probabilities) was used by Kornya [52] for $\widehat{F} = (1-p) + p\widehat{V}$ truncated version of the following formal expansion

$$\begin{aligned} \widehat{F} &= \exp\left\{ \ln(1-p) + \ln(1+p\widehat{V}/(1-p)) \right\} \\ &= \exp\left\{ \sum_{j=1}^{\infty} \frac{(-1)^j p^j}{j(1-p)^j} (\widehat{V}^j - 1) \right\}. \end{aligned}$$

Hipp and Kornya approaches can be used for one-dimensional and d-dimensional distributions.

Some other approaches were applied only for construction of one-dimensional approximations. If X with distribution $F \in \mathscr{F}_1$ takes only non-negative integer values, then we can formally write

$$\widehat{F}^n(t) = \exp\left\{ \kappa_1 (e^{it} - 1) + \frac{\kappa_2}{2} (e^{it} - 1)^2 + \frac{\kappa_3}{3} (e^{it} - 1)^3 + \dots \right\} \tag{1.3}$$

Poisson approximation is obtained by leaving the first summand in the exponent. By leaving longer part of the series in the exponent we get SCP approximation.

Franken [35] was arguably the first to use (1.3) to construct Poisson approximation with the same mean for a general case of integer-valued random variables. Herrmann [44] and Kruopis [54, 55] were the first to use SCP approximations obtained from (1.3). In [23] another method for construction of SCP with matching factorial cumulants was proposed for sequences of one-dimensional random variables.

2 ASYMPTOTICS IN LE CAM-PRESMAN THEOREM

One of the most general theoretical problems in Probability Theory is the closeness of a sum of independent identically distributed random variables to the set of all infinitely divisible distributions. Kolmogorov [51] proved that *no condition* is needed for the existence of a good infinitely divisible approximation. Subsequently, this result was called the first uniform Kolmogorov theorem. Le Cam [57] proved that accompanying Compound Poisson (CP) distributions can be used for approximation. The problem of the best possible infinitely divisible approximation for any sum of independent random variables, in one-dimensional case, was solved by Arak, who used CP distributions with very complicated structure; see for the history of the problem [4] or [30], Ch. 10. However, very little is known about multidimensional version of Kolmogorov's problem. The best general result so far is a multidimensional version of Le Cam's result proved by Presman [64]. This chapter discusses possible improvements to multidimensional Le Cam's theorem via asymptotics and proves some related results.

2.1 Known results

In the early eighties, the correct accuracy of estimate in one-dimensional Kolmogorov's first uniform theorem was obtained by Arak, who proved that

$$C_1 n^{-2/3} \leqslant \sup_{\mathsf{F}\in\mathscr{F}_1} \inf_{\mathsf{D}\in\mathscr{D}_1} \|\mathsf{F}^{*n} - \mathsf{D}\|_K \leqslant C_2 n^{-2/3}. \tag{2.1}$$

Here $\mathscr{D}_1$ is the set of all one-dimensional infinitely divisible distributions; see [4] for a comprehensive history of the problem and its solution. Le Cam [57] proved an intermediate result, showing that suitable centered distributions can be approximated by the convolution of accompanying CP distributions with accuracy $Cn^{-1/3}$. Presman [64] extended Le Cam's result to the multidimensional case, proving that there exists an absolute constant depending only on the dimension of space $C(d)$ such that

$$\sup_{\mathsf{F}\in\mathscr{F}_d} \inf_{\vec{\alpha}\in\mathbb{R}^d} \|\mathsf{F}^{*n} - I_{-n\vec{\alpha}} * \exp\{n(\mathsf{F} * I_{\vec{\alpha}} - I)\}\|_K \leqslant C(d) n^{-1/3}. \tag{2.2}$$

Estimate (2.2) is very general, since there is no assumption on the existence of finite moments or the structure of F. Also, there is no assumption that F^{*n} has a limit distribution as $n \to \infty$. The only tool is the suitable shifting of

the initial distribution. Note that, since the Kolmogorov norm is invariant to shifting, estimate (2.2) is equivalent to the approximation of $(\mathsf{F} * I_{\vec{\alpha}})^{*n}$ by the convolution of accompanying laws. Some multivariate extensions of (2.2) for random sums are proved in [26].

So far, there is no multidimensional analogue of Arak's inequality (2.1), and (2.2) is the best available general estimate. On the other hand, for symmetric distributions and some of their subsets, complete analogues of one-dimensional results are proved, see [42, 41, 87] and references therein. Particularly, it is proved that for any symmetric d-dimensional distribution F_S

$$\|\mathsf{F}_S^{*n} - \exp\{n(\mathsf{F}_S - I)\}\|_K \leqslant C(d)n^{-1/2} \tag{2.3}$$

and the accuracy is $C(d)n^{-1}$ if, in addition, F_S has non-negative characteristic function. For CP approximation to the sums of r.v.s. taking zero with large probabilities, see [78].

The most general multidimensional SCP approximation was obtained in [22]: for any $\mathsf{F} \in \mathscr{F}_d$ it was proved that

$$\inf_{\mathsf{V}\in\mathscr{F}_d}\inf_{\vec{\alpha}\in\mathbb{R}^d} \|\mathsf{F}^{*n} - I_{-n\vec{\alpha}} * \exp\{n(\mathsf{F}*I_{\vec{\alpha}} - I) - n(\mathsf{V}-I)^{*2}/2\}\|_K \leqslant C(d)n^{-1/2}.$$

Estimates for successive convolutions of distributions are closely related to CP approximations, see [84, 85, 89]. In [89] Zaitsev showed that for arbitrary $\tilde{\mathsf{F}} \in \mathscr{F}_d$, which is not concentrated on hyper-space of $\mathbb{R}^d$, the following estimate holds

$$\|\tilde{\mathsf{F}}^{*n} - \tilde{\mathsf{F}}^{*(n+1)}\|_K \leqslant C(d,\tilde{\mathsf{F}})n^{-1/2}. \tag{2.4}$$

Note that, unlike (2.2), estimate (2.4) is not uniform with respect to some subset of $\mathscr{F}_d$, since constant $C(d,\tilde{\mathsf{F}})$ depends on $\tilde{\mathsf{F}}$.

2.2 Results

First, we show that asymptotic expansion can improve the estimate (2.2). What is asymptotic expansion in such a general setting? Arguably, the most general rules for constructing such an expansion require that all n'th power convolutions be of infinitely divisible distributions and only finite convolutions of other

distributions might be used. Suppose, $m \in \mathbb{N}$, $\vec{\alpha} \in \mathbb{R}^d$ and let

$$\Delta_{0,n}(\vec{\alpha}, \mathsf{F}) = 0, \quad \Delta_{m,n}(\vec{\alpha}, \mathsf{F}) := \sum_{j=1}^{m} \frac{1}{j!} \left(\frac{-n}{2} \right)^j (\mathsf{F} * I_{\vec{\alpha}} - I)^{*2j}.$$

Theorem 2.2.1. *Let $m \in \{0, 1, 2, \dots\}$, $n \in \mathbb{N}$. Then*

$$\sup_{\mathsf{F} \in \mathscr{F}_d} \inf_{\vec{\alpha} \in \mathbb{R}^d} \| \mathsf{F}^{*n} - I_{-n\vec{\alpha}} * \exp\{n(\mathsf{F} * I_{\vec{\alpha}} - I)\} * (I + \Delta_{m,n}(\vec{\alpha}, \mathsf{F})) \|_K \leqslant \frac{C(d,m)}{n^{(m+1)/(2m+3)}}. \quad (2.5)$$

As follows from the proof below, each m requires different shifting, that is, $\vec{\alpha} = \vec{\alpha}(m)$. If $m = 0$, then (2.5) coincides with (2.2). For $m = 1$, we get the following corollary.

Corollary 2.2.1. *Let $n \in \mathbb{N}$. Then*

$$\sup_{\mathsf{F} \in \mathscr{F}_d} \inf_{\vec{\alpha} \in \mathbb{R}^d} \left\| \mathsf{F}^{*n} - I_{-n\vec{\alpha}} * \exp\{n(\mathsf{F} * I_{\vec{\alpha}} - I)\} * \left(I - \frac{n^2 (\mathsf{F} * I_{\vec{\alpha}} - I)^{*2}}{2} \right) \right\|_K \leqslant \frac{C(d)}{n^{2/5}}. \quad (2.6)$$

As shown in [4], p. 181, even for $d = 1$ estimate in (2.2) can not be better than $Cn^{-1/3}$. Can estimate (2.5) be improved remains an open problem. We conjecture that the order of approximation in (2.5) is correct. To some extent the following example supports this conjecture.

Example 2.2.1. *Let us consider one-dimensional Binomial distribution $Bi(n, p_B)$, $p_B = n^{-1/5}$. Then, there exists absolute constants c, C, such that for all $n \geqslant C$*

$$\left\| Bi(n, p_B) - \mathrm{e}^{np_B(I_1 - I)} * \left(I - \frac{np_B^2 (I_1 - I)^{*2}}{2} \right) \right\|_K \geqslant \frac{c}{n^{2/5}}. \quad (2.7)$$

Proof of (2.7) is carried out in Section 2.4. We got lower bound of the same order as upper bound in (2.6). On the other hand, we did not used any shifting. Thus, to prove our conjecture, one needs to show that, for this example, no shifting can improve the accuracy of approximation.

Next, we consider the impact of shifting on convolutions with a CP distribution. We formulate multidimensional versions of Theorem 3.1 and Corollary 3.1 from [21].

Theorem 2.2.2. *For any $a > 0$, $k \in \mathbb{N}$,*

$$\sup_{\mathsf{F}\in\mathscr{F}_d}\inf_{\vec{\alpha}\in\mathbb{R}^d}\left\|(\mathsf{F}*I_{\vec{\alpha}}-I)^{*k}*\exp\{a(\mathsf{F}*I_{\vec{\alpha}}-I)\}\right\|_K \leqslant C(d,k)a^{-k/2-k/(2k+2)}.$$

Corollary 2.2.2. *For any $a > 0$*

$$\sup_{\mathsf{F}\in\mathscr{F}_d}\inf_{\vec{\alpha}\in\mathbb{R}^d}\left\|(\mathsf{F}*I_{\vec{\alpha}}-I)*\exp\{a(\mathsf{F}*I_{\vec{\alpha}}-I)\}\right\|_K \leqslant C(d)a^{-3/4}.$$

Comparing Corollary 2.2.2 with (A.16), we see that shifting improves the order of approximation by a factor $n^{-1/4}$. Theorem 2.2.2 permits us to estimate the closeness of two CP distributions with shifted compounding distributions.

Theorem 2.2.3. *For any $a, b > 0$, $m \in \mathbb{N}$,*

$$\sup_{\mathsf{F}\in\mathscr{F}_d}\inf_{\vec{\alpha}\in\mathbb{R}^d}\left\|\exp\{a(\mathsf{F}*I_{\vec{\alpha}}-I)\}-\exp\{b(\mathsf{F}*I_{\vec{\alpha}}-I)\}\right\|_K \leqslant \frac{C(d)|a-b|}{\max(a,b)^{3/4}}, \tag{2.8}$$

$$\sup_{\mathsf{F}\in\mathscr{F}_d}\inf_{\vec{\alpha}\in\mathbb{R}^d}\left\|\exp\{a(\mathsf{F}*I_{\vec{\alpha}}-I)\}*\left(\exp\{b(\mathsf{F}*I_{\vec{\alpha}}-I)\}-\sum_{j=0}^{m}\frac{b^j(\mathsf{F}*I_{\vec{\alpha}}-I)^{*j}}{j!}\right)\right\|_K \leqslant \frac{C(d,m)b^{m+1}}{a^{m/2+(2m+3)/(2m+4)}}. \tag{2.9}$$

Estimates (2.8) and (2.9) are multidimensional versions of Lemma C.12 [30] and Corollary 3.2 [21].

Finally we formulate the following analogue of (2.4) for shifted distributions.

Theorem 2.2.4. *For any $n \in \mathbb{N}$*

$$\sup_{\mathsf{F}\in\mathscr{F}_d}\inf_{\vec{\alpha}\in\mathbb{R}^d}\left\|(\mathsf{F}*I_{\vec{\alpha}})^{*n}-(\mathsf{F}*I_{\vec{\alpha}})^{*(n+1)}\right\|_K \leqslant \frac{C(d)}{\sqrt{n}}.$$

Theorem 2.2.4 complements Zaitsev's estimate for sequences of r.v.s. (2.4). It demonstrates that, if shifting of initial distribution is allowed, then analogue of (2.4) holds not only for sequences of r.v.s. but also for the scheme of triangular arrays with constant $C(d)$ which does not depend on distribution F.

2.3 Auxiliary results

Let $\mathscr{H}_0$ be the set of all rectangular prisms in $\mathbb{R}^d$. For the set $B \in \mathscr{H}_0$ let $(\partial B)_\tau$ be the τ-neighborhood of the boundary of B. For a finite signed measure $\tilde{\mathsf{M}}$ set

$$\Gamma_{\mathscr{H}_0}(\tilde{\mathsf{M}}, \tau) = \sup_{B \in \mathscr{H}_0} \tilde{\mathsf{M}}\{(\partial B)_\tau\}.$$

Let $\mathsf{W} \in \mathscr{F}_d$ have density

$$\mathsf{w}(\vec{x}) = \beta_d (\|\vec{x}\|_2^{-d/2} \mathrm{I}_{d/2}(\|\vec{x}\|_2/4))^4,$$

where β_d is given by the normalizing condition, and $\mathrm{I}_{d/2}$ the Bessel function of order $d/2$.

Lemma 2.3.1. *Let* V *be finite signed measure and* $\mathsf{V}\{\mathbb{R}^d\} = 0$. *Let* $\tau = \tau(d)$ *be such that* $\mathsf{W}\{\|\vec{x}\|_2 < \tau\} = 3/4$. *Then*

$$\|\mathsf{V}\|_K \leqslant 2\|\mathsf{W} * \mathsf{V}\|_K + \frac{3}{2} \min(\Gamma_{\mathscr{H}_0}(\mathsf{V}^+, 2\tau); \Gamma_{\mathscr{H}_0}(\mathsf{V}^-, 2\tau)).$$

Here V^+ *and* V^- *denote the positive and negative variations of* V.

Proof. Lemma 2.3.1 is essentially Lemma 2 from [64]. One should observe that $\mathsf{V} = \mathsf{V}^+ - \mathsf{V}^-$ and $\mathsf{V}^+\{\mathbb{R}^d\} = \mathsf{V}^-\{\mathbb{R}^d\} = \|\mathsf{V}\|_{TV}/2$. Therefore, one can write

$$\|\mathsf{V}\|_K = \mathsf{V}^+\{\mathbb{R}^d\} \Big\| \frac{\mathsf{V}^+}{\mathsf{V}^+\{\mathbb{R}^d\}} - \frac{\mathsf{V}^-}{\mathsf{V}^+\{\mathbb{R}^d\}} \Big\|_K,$$

and apply Lemma 2 from [64], since

$$\frac{\mathsf{V}^+}{\mathsf{V}^+\{\mathbb{R}^d\}}; \ \frac{\mathsf{V}^-}{\mathsf{V}^+\{\mathbb{R}^d\}} \in \mathscr{F}_d.$$

□

Lemma 2.3.2. *Let* W *be defined as above, and let the distribution* M *be concentrated inside the unit sphere,* $\int \vec{x}\mathsf{M}(\mathrm{d}\vec{x}) = \vec{0}$. *Then, for any* $m \in \mathbb{N}$, $a, b > 0$,

$$\begin{aligned}
\big\|\mathsf{W} * (\mathsf{M}^{*m} - \mathrm{e}^{m(\mathsf{M}-I)})\big\|_{TV} &\leqslant \frac{C(d)}{m+1}, \\
\big\|\mathsf{W} * (\mathrm{e}^{a(\mathsf{M}-I)} - \mathrm{e}^{b(\mathsf{M}-I)})\big\|_{TV} &\leqslant \frac{C(d)|a-b|}{b}, \qquad (2.10) \\
\big\|\mathsf{W} * (\mathsf{M} - I) * \mathrm{e}^{a(\mathsf{M}-I)}\big\|_{TV} &\leqslant \frac{C(d)}{a}. \qquad (2.11)
\end{aligned}$$

Proof. For $a,b \in \mathbb{N}$, the first two estimates are given in Lemma 4, [64]. The third estimate can be proved in the following way: let $a > 1$ and let $\lfloor a \rfloor$, $\lceil a \rceil$ resp. denote integer and fractional parts of a. Then, taking into account that the total variation of any distribution equals 1, and applying an analogue of (2.10) for integer numbers (see [64], Lemma 4), we prove (2.11) in the following way

$$\begin{aligned}
\left\|\mathsf{W} * (\mathsf{M}-I) * \mathrm{e}^{a(\mathsf{M}-I)}\right\|_{TV} &\leqslant \left\|\mathsf{W} * (\mathsf{M}-I) * \mathrm{e}^{\lfloor a \rfloor(\mathsf{M}-I)}\right\|_{TV} \left\|\mathrm{e}^{\lceil a \rceil(\mathsf{M}-I)}\right\|_{TV} \\
&\leqslant \left\|\mathsf{W} * \mathsf{M} * (\mathrm{e}^{\lfloor a \rfloor(\mathsf{M}-I)} - \mathsf{M}^{*\lfloor a \rfloor})\right\|_{TV} \\
&\quad + \left\|\mathsf{W} * (\mathsf{M}^{*(\lfloor a \rfloor+1)} - \mathrm{e}^{(\lfloor a \rfloor+1)(\mathsf{M}-I)})\right\|_{TV} \\
&\quad + \left\|\mathsf{W} * (\mathrm{e}^{(\lfloor a \rfloor+1)(\mathsf{M}-I)} - \mathrm{e}^{\lfloor a \rfloor(\mathsf{M}-I)})\right\|_{TV} \\
&\leqslant \frac{\|\mathsf{M}\|_{TV} C(d)}{\lfloor a \rfloor + 1} + \frac{C(d)}{\lfloor a \rfloor + 2} + \frac{C(d)}{\lfloor a \rfloor} \leqslant \frac{C(d)}{\lfloor a \rfloor} \leqslant \frac{C(d)}{a}.
\end{aligned}$$

If $0 < a \leqslant 1$, then

$$\left\|\mathsf{W} * (\mathsf{M}-I) * \mathrm{e}^{a(\mathsf{M}-I)}\right\|_{TV} \leqslant 2 \leqslant \frac{2}{a}.$$

The proof of (2.10) for non-integers a,b is similar. Let $a \geqslant b$, then applying (2.11) and (A.30) we get

$$\begin{aligned}
\left\|\mathsf{W} * (\mathrm{e}^{a(\mathsf{M}-I)} - \mathrm{e}^{b(\mathsf{M}-I)})\right\|_{TV} &= \left\|W * \mathrm{e}^{b(\mathsf{M}-I)} * \left(\mathrm{e}^{(a-b)(\mathsf{M}-I)} - I\right)\right\|_{TV} \\
&\leqslant C(a-b)\left\|\mathsf{W} * (\mathsf{M}-I) * \mathrm{e}^{b(\mathsf{M}-I)}\right\|_{TV} \leqslant \frac{C(d)(a-b)}{b}.
\end{aligned}$$

Now let $a < b \leqslant 2a$. Then, repeating the above argument, we get

$$\left\|\mathsf{W} * (\mathrm{e}^{a(\mathsf{M}-I)} - \mathrm{e}^{b(\mathsf{M}-I)})\right\|_{TV} \leqslant \frac{C(d)|a-b|}{a} \leqslant \frac{2C(d)|a-b|}{b}.$$

If $2a < b$, then

$$\frac{b-a}{b} = 1 - \frac{a}{b} \geqslant \frac{1}{2}$$

and

$$\begin{aligned}
\left\|\mathsf{W} * (\mathrm{e}^{a(\mathsf{M}-I)} - \mathrm{e}^{b(\mathsf{M}-I)})\right\|_{TV} &\leqslant \left\|\mathsf{W} * \mathrm{e}^{a(\mathsf{M}-I)}\right\|_{TV} + \left\|\mathsf{W} * \mathrm{e}^{b(\mathsf{M}-I)}\right\|_{TV} \\
&= 2 \leqslant \frac{4|a-b|}{b}.
\end{aligned}$$

Lemma is proved. □

The following lemma is a version of Lemma 3 from [64] for $\mathscr{H}_0$.

Lemma 2.3.3. *Let the random vector $\vec{\xi} = (\xi_1, \dots, \xi_d)$ correspond to distribution N, where $\mathbb{P}(\xi_i \leqslant x_i^-) \geqslant \tilde{\alpha}$, $\mathbb{P}(\xi_i \geqslant x_i^+) \geqslant \tilde{\alpha}$, $x_i^+ - x_i^- \geqslant l$ for $i = 1, \dots, d$. Then, for $\tau > l$,*

$$\Gamma_{\mathscr{H}_0}(\mathsf{N}^{*m}, \tau) < Cd\tau l(\tilde{\alpha}(m+1))^{-1/2}.$$

We need some additional lemmas.

Lemma 2.3.4. *Let $\mathsf{F} \in \mathscr{F}_d$ and $\tilde{\mathsf{V}}$ be a measure of finite variation, $\tau > 0$. Then*

$$\sup_{B \in \mathscr{H}_0} (\mathsf{F} * \tilde{\mathsf{V}})^+ \{(\partial B)_\tau\} \leqslant \|\tilde{\mathsf{V}}\|_{TV} \Gamma_{\mathscr{H}_0}(\tilde{\mathsf{V}}, \tau).$$

Proof. Let $A = \operatorname{supp}(\mathsf{F} * \mathsf{M})^+$. Then

$$\begin{aligned}
\sup_{B \in \mathscr{H}_0} (\mathsf{F} * \tilde{\mathsf{V}})^+ \{(\partial B)_\tau\} &= \sup_{B \in \mathscr{H}_0} (\mathsf{F} * \tilde{\mathsf{V}}) \{(\partial B)_\tau \cap A\} \\
&\leqslant \sup_{B \in \mathscr{H}_0} \int_{\mathbb{R}^d} \mathsf{F}\{(\partial B)_\tau \cap A - \vec{y}\} |\tilde{\mathsf{V}}\{\mathrm{d}\vec{y}\}| \\
&\leqslant \sup_{B \in \mathscr{H}_0} \int_{\mathbb{R}^d} \mathsf{F}\{(\partial B)_\tau - \vec{y}\} |\tilde{\mathsf{V}}\{\mathrm{d}\vec{y}\}| \\
&\leqslant \Gamma_{\mathscr{H}_0}(\mathsf{F}, \tau) \int_{\mathbb{R}^d} 1 \cdot |\tilde{\mathsf{V}}\{\mathrm{d}\vec{y}\}| = \|\tilde{\mathsf{V}}\|_{TV} \Gamma_{\mathscr{H}_0}(\mathsf{F}, \tau).
\end{aligned}$$

□

Lemma 2.3.5. *Let assumptions of Lemma 2.3.3 hold. Then, for any $\tau > l$ and any $a > 0$, $n \in \mathbb{N}$, $p \in (0, 1/2)$.*

$$\begin{aligned}
\Gamma_{\mathscr{H}_0}\left(\mathrm{e}^{a(\mathsf{N}-I)}, \tau\right) &\leqslant \frac{C(d\tau l + 1)}{\sqrt{a\alpha}}, \\
\Gamma_{\mathscr{H}_0}(((1-p)I + p\mathsf{N})^{*n}, \tau) &\leqslant \frac{C(d\tau l + 1)}{\sqrt{\alpha np}}.
\end{aligned}$$

Proof. By the definition of exponential measure and Lemma 2.3.3

$$\begin{aligned}
\Gamma_{\mathscr{H}_0}\left(\mathrm{e}^{a(\mathsf{N}-I)}, \tau\right) &\leqslant \mathrm{e}^{-a} + \sum_{j=1}^{\infty} \frac{a^j \mathrm{e}^{-a}}{j!} \Gamma_{\mathscr{H}_0}(\mathsf{N}^{*j}, \tau) \\
&\leqslant \mathrm{e}^{-a} + \frac{Cd\tau l}{\sqrt{\alpha}} \sum_{j=1}^{\infty} \frac{a^j \mathrm{e}^{-a}}{j!\sqrt{(j+1)}}
\end{aligned}$$

$$\leqslant \quad \mathrm{e}^{-a} + \frac{Cd\tau l}{\sqrt{a\alpha}} \leqslant \frac{C(d\tau l+1)}{\sqrt{a\alpha}}.$$

Similarly

$$\begin{aligned}\Gamma_{\mathscr{H}_0}(((1-p)I+p\mathsf{N})^{*n},\tau) &\leqslant \sum_{j=0}^{n}\binom{n}{j}(1-p)^{n-j}p^j\Gamma_{\mathscr{H}_0}(\mathsf{N}^{*j},\tau)\\ &\leqslant \frac{C(d\tau l+1)}{\sqrt{\alpha}}\sum_{j=0}^{n}\binom{n}{j}\frac{(1-p)^{n-j}p^j}{\sqrt{j+1}} \leqslant \frac{C(d\tau l+1)}{\sqrt{\alpha np}}.\end{aligned}$$

□

The above lemmas allow us to prove the following result.

Lemma 2.3.6. *Let* $\mathsf{M}^\star$ *have mean zero and be concentrated inside the rectangular prism* $[x_1^-;x_1^+]\times[x_2^-;x_2^+]\times\ldots\times[x_d^-;x_d^+]$ *with* $\max_{1\leqslant i\leqslant d}|x_i^+-x_i^-| = d^{-1/2}$. *Let* N *be defined as in Lemma 2.3.3 with* $\tilde{\alpha} = (2d)^{-1}$, $l = d^{-1/2}$. *Set* $p \leqslant 1/2$, $j,n \in \mathbb{N}$. *Then*

$$\left\|(\mathsf{M}^\star - I)^{*j} * \mathrm{e}^{a(1-p)(\mathsf{M}^\star-I)+ap(\mathsf{N}-I)}\right\|_K \leqslant \frac{C(d,j)}{a^{j/2}\sqrt{ap}} \tag{2.12}$$

and

$$\left\|((1-p)\mathsf{M}^\star + p\mathsf{N})^{*n} - \mathrm{e}^{n(1-p)(\mathsf{M}^\star-I)} * ((1-p)I+p\mathsf{N})^{*n}\right\|_K \leqslant \frac{C(d)}{\sqrt{np}}. \tag{2.13}$$

Proof. Presman [64] proved that the LHS of (2.13) is less or equal to

$$\frac{C(d)}{n(1-p)} + \frac{C(d)\sqrt{p}}{\sqrt{n(1-p)}} + \frac{C(d)}{\sqrt{np}}$$

for $p = n^{-1/3}$. However, it can be easily verified that Presman's proof holds for arbitrary $p \in (0,1)$. Since we chose $p \leqslant 1/2$, estimate (2.13) follows from Presman's estimate. It remains to prove (2.12).

The total variation of any distribution is 1. Therefore $\|\mathrm{e}^{ap(\mathsf{N}-I)}\|_{TV}=1$ and applying Lemma 2.3.1 we get

$$\begin{aligned}\left\|(\mathsf{M}^\star - I)^{*j} * \mathrm{e}^{aq(\mathsf{M}^\star-I)+ap(\mathsf{N}-I)}\right\|_K &\leqslant 2\left\|\mathsf{W} * (\mathsf{M}^\star - I)^{*j} * \mathrm{e}^{aq(\mathsf{M}^\star-I)}\right\|_{TV}\\ &+\frac{3}{2}\left\|(\mathsf{M}^\star - I)^{*j} * \mathrm{e}^{aq(\mathsf{M}^\star-I)}\right\|_{TV}\Gamma_{\mathscr{H}_0}\left(\mathrm{e}^{ap(\mathsf{N}-I)},2\tau\right).\end{aligned} \tag{2.14}$$

From (A.15) and (A.18) it follows that

$$\left\|(\mathsf{M}^\star - I)^{*j} * \mathrm{e}^{aq(\mathsf{M}^\star - I)}\right\|_{TV} \leqslant \frac{C(j)}{a^{j/2}}. \tag{2.15}$$

Similarly, applying (2.11) and (A.18), we prove that

$$\begin{aligned}\left\|\mathsf{W} * (\mathsf{M}^\star - I)^{*j} * \mathrm{e}^{aq(\mathsf{M}^\star - I)}\right\|_{TV} &\leqslant \left\|\mathsf{W} * (\mathsf{M}^\star - I) * \mathrm{e}^{(aq/2)(\mathsf{M}^\star - I)}\right\|_{TV} \\ &\times \left\|(\mathsf{M}^\star - I)^{*(j-1)} * \mathrm{e}^{(aq/2)(\mathsf{M}^\star - I)}\right\|_{TV} \leqslant \frac{C(d)}{a^{(j+1)/2}}. \end{aligned} \tag{2.16}$$

From Lemma 2.3.5 we get

$$\Gamma_{\mathscr{H}_0}\left(\mathrm{e}^{ap(\mathsf{N}-I)}, 2\tau\right) \leqslant \left(1 + \frac{\tau}{l}\right)\Gamma_{\mathscr{H}_0}\left(\mathrm{e}^{ap(\mathsf{N}-I)}, 2l\right) \leqslant \frac{C(d)}{\sqrt{ap}}. \tag{2.17}$$

Substituting estimates (2.15), (2.16), (2.17) into (2.14) we complete the proof of (2.12). □

2.4 Proofs

Proof of Theorem 2.2.1. Without loss of generality, we can assume that $n > 4^{2m+1}$. Indeed, if $n \leqslant 4^{2m+1}$, then

$$\begin{aligned}&\left\|\mathsf{F}^{*n} - I_{-n\vec{\alpha}} * \exp\{n(\mathsf{F} * I_{\vec{\alpha}} - I)\} * (I + \Delta_{m,n}(\vec{\alpha}, \mathsf{F}))\right\|_K \\ &\quad\leqslant 1 + \|\exp\{n(\mathsf{F} * I_{\vec{\alpha}} - I)\}\|_{TV}\left(1 + \sum_{j=1}^{m}\left(\frac{n}{2}\right)^j \|(\mathsf{F} * I_{\vec{\alpha}} - I)\|_{TV}^{2j}\right) \\ &\quad\leqslant 1 + \left(1 + \sum_{j=1}^{m}\left(\frac{4^{2m+1}}{2}\right)^j 2^{2j}\right) \leqslant C(m) \leqslant \frac{C(m)4^{m+1/2}}{\sqrt{n}} \\ &\quad\leqslant \frac{C(m)4^{m+1/2}}{n^{(m+1)/(2m+3)}}.\end{aligned}$$

Let us assume that $\mathsf{F} \in \mathscr{F}_d$ is absolutely continuous. Then in [64] it was shown that there exists $\vec{\alpha}$, such that $\bar{\mathsf{F}} = \mathsf{F} * I_{\vec{\alpha}} = (1-p)\mathsf{M} + p\mathsf{N}$, where M satisfies the conditions of Lemma 2.3.2 and N satisfies the conditions of Lemma 2.3.3 with $\tilde{\alpha} = (2d)^{-1}$, $l = d^{-1/2}$. Note that in [64] it was assumed that $p = n^{-1/3}$, however, it can be easily checked that the same decomposition holds for any $p \in (0,1)$. Let $p = n^{-1/(2m+3)}$. From (2.13), (A.15) and (A.23)

it follows that

$$\begin{aligned}
&\left\| \mathrm{e}^{n(1-p)(\mathsf{M}-I)} * ((1-p)I+p\mathsf{N})^{*n} \right. \\
&\qquad \left. -\mathrm{e}^{n(1-p)(\mathsf{M}-I)} * \mathrm{e}^{np(\mathsf{N}-I)-np^2(\mathsf{N}-I)^{*2}/2} \right\|_K \\
&\leqslant \left\| ((1-p)I+p\mathsf{N})^{*n} - \exp\left\{ np(\mathsf{N}-I) - \frac{np^2(\mathsf{N}-I)^{*2}}{2} \right\} \right\|_{TV} \\
&\leqslant \left\| ((1-p)I+pI_1)^{*n} - \exp\left\{ np(I_1-I) - \frac{np^2(I_1-I)^{*2}}{2} \right\} \right\|_{TV} \\
&\leqslant \frac{Cp^2}{\sqrt{np}} \leqslant \frac{C}{\sqrt{np}} = \frac{C}{n^{(m+1)/(2m+3)}}.
\end{aligned}$$

Here $I_1 \in \mathscr{F}_1$ and for one-dimensional distributions we applied (A.23) with $d=1$, $m=2$, $p_i = p$. Combining the last estimate with (2.13), we obtain the following estimate

$$\left\| \bar{\mathsf{F}}^{*n} - \exp\left\{ n(\bar{\mathsf{F}}-I) - \frac{np^2(\mathsf{N}-I)^{*2}}{2} \right\} \right\|_K \leqslant \frac{C(d,m)}{n^{(m+1)/(2m+3)}}.$$

Next, we use (A.27), (A.28), and, once more, switch to one-dimensional distributions:

$$\begin{aligned}
&\left\| \mathrm{e}^{n(1-p)(\mathsf{M}-I)+np(\mathsf{N}-I)} * \left(\exp\left\{ -\frac{np^2(\mathsf{N}-I)^{*2}}{2} \right\} \right. \right. \\
&\qquad \left. \left. - \sum_{j=0}^{m} \frac{(-np^2(\mathsf{N}-I)^{*2})^{*j}}{2^j j!} \right) \right\|_K \\
&\leqslant \left\| \mathrm{e}^{np(\mathsf{N}-I)} * \left(\exp\left\{ -\frac{np^2(\mathsf{N}-I)^{*2}}{2} \right\} - \sum_{j=0}^{m} \frac{(-np^2(\mathsf{N}-I)^{*2})^{*j}}{2^j j!} \right) \right\|_{TV} \\
&\leqslant \left\| \mathrm{e}^{np(I_1-I)} * \left(\exp\left\{ -\frac{np^2(I_1-I)^{*2}}{2} \right\} - \sum_{j=0}^{m} \frac{(-np^2(I_1-I)^{*2})^{*j}}{2^j j!} \right) \right\|_{TV}
\end{aligned}$$

The last estimate is for $I_1, I \in \mathscr{F}_1$. Observe that by our assumptions $p \leqslant 1/4$. Next we apply (A.30), (A.24) and (A.18) obtaining

$$\left\| \mathrm{e}^{np(I_1-I)} * \left(\exp\left\{ -\frac{np^2(I_1-I)^{*2}}{2} \right\} - \sum_{j=0}^{m} \frac{(-np^2(I_1-I)^{*2})^{*j}}{2^j j!} \right) \right\|_{TV}$$

$$
\begin{aligned}
&\leqslant C(m)\left\| \mathrm{e}^{np(I_1-I)} * \left(\frac{-np^2(I_1-I)^{*2}}{2}\right)^{m+1} \right. \\
&\quad \left. * \int_0^1 \mathrm{e}^{-\tau np^2(I_1-I)^{*2}/2}(1-\tau)\mathrm{d}\tau \right\|_{TV} \\
&\leqslant C(m)(np^2)^{m+1} \left\| \exp\{0.1np(I_1-I)\} * (I_1-I)^{2(m+1)} \right\|_{TV} \\
&\leqslant C(m)p^{m+1} = \frac{C(m)}{n^{(m+1)/(2m+3)}}.
\end{aligned}
$$

Collecting the above estimates, we get

$$
\begin{aligned}
&\left\| \exp\left\{ n(\bar{\mathsf{F}}-I) - \frac{np^2(\mathsf{N}-I)^{*2}}{2} \right\} \right. \\
&\qquad \left. -\exp\left\{ n(\bar{\mathsf{F}}-I) \right\} * \sum_{j=0}^{m} \frac{(-np^2(\mathsf{N}-I)^{*2})^{*j}}{2^j j!} \right\|_K \leqslant \frac{C(d,m)}{n^{(m+1)/(2m+3)}}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \left(I + \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2}(\bar{\mathsf{F}}-I)^{*2} \right)^{*j} \right) \\
&\qquad -\mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \left(I + \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{np^2}{2}(\mathsf{N}-I)^{*2} \right)^{*j} \right) \\
&= \mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \left(\sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2}(\bar{\mathsf{F}}-I)^{*2} \right)^{*j} - \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{np^2}{2}(\mathsf{N}-I)^{*2} \right)^{*j} \right) \\
&= \mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2} \right)^{j} \left((\bar{\mathsf{F}}-I)^{*2j} - p^{2j}(\mathsf{N}-I)^{*2j} \right) \\
&= \mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2} \right)^{j} \left[\left(q(\mathsf{M}-I) + p(\mathsf{N}-I) \right)^{*2j} - p^{2j}(\mathsf{N}-I)^{*2j} \right] \\
&= \mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2} \right)^{j} \left[\sum_{i=0}^{2j} \binom{2j}{i} (q(\mathsf{M}-I))^{*i} * (p(\mathsf{N}-I))^{*(2j-i)} \right. \\
&\qquad \left. -p^{2j}(\mathsf{N}-I)^{*2j} \right] \\
&= \mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2} \right)^{j} \sum_{i=1}^{2j} \binom{2j}{i} (q(\mathsf{M}-I))^{*i} * (p(\mathsf{N}-I))^{*(2j-i)} \\
&= \mathrm{e}^{nq(\mathsf{M}-I)+np(\mathsf{N}-I)} * \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2} \right)^{j} q(\mathsf{M}-I)
\end{aligned}
$$

$$
\begin{aligned}
&* \sum_{i=1}^{2j} \binom{2j}{i} (q(\mathsf{M}-I))^{*(i-1)} * (p(\mathsf{N}-I))^{*(2j-i)} \\
= \quad & \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2} \right)^{j} \left(q(\mathsf{M}-I) * \exp\left\{ \frac{nq}{2}(\mathsf{M}-I) + \frac{np}{2}(\mathsf{N}-I) \right\} \right) \\
& * \sum_{i=1}^{2j} \binom{2j}{i} \left((q(\mathsf{M}-I))^{*(i-1)} * \exp\left\{ \frac{nq}{2}(\mathsf{M}-I) \right\} \right) \\
& * \left((p(\mathsf{N}-I))^{*(2j-i)} * \exp\left\{ \frac{np}{2}(\mathsf{N}-I) \right\} \right).
\end{aligned}
$$

Therefore, applying Lemma 2.3.6 and estimate (2.15) we get

$$
\begin{aligned}
&\left\| \mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \left(I + \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{n}{2} (\bar{\mathsf{F}}-I)^{*2} \right)^{*j} \right) \right. \\
&\qquad \left. - \mathrm{e}^{n(\bar{\mathsf{F}}-I)} * \left(I + \sum_{j=1}^{m} \frac{1}{j!} \left(-\frac{np^2}{2} (\mathsf{N}-I)^{*2} \right)^{*j} \right) \right\|_{K} \\
&\leqslant \quad \sum_{j=1}^{m} \frac{1}{j!} \left(\frac{n}{2} \right)^{j} \left\| q(\mathsf{M}-I) * \exp\left\{ \frac{nq}{2}(\mathsf{M}-I) + \frac{np}{2}(\mathsf{N}-I) \right\} \right\|_{TV} \\
&\qquad \times \sum_{i=1}^{2j} \binom{2j}{i} \left\| (q(\mathsf{M}-I))^{*(i-1)} * \exp\left\{ \frac{nq}{2}(\mathsf{M}-I) \right\} \right\|_{TV} \\
&\qquad \times \left\| p(\mathsf{N}-I))^{*(2j-i)} * \exp\left\{ \frac{np}{2}(\mathsf{N}-I) \right\} \right\|_{TV} \\
&\leqslant \quad C(m) \frac{1}{\sqrt{np}}.
\end{aligned}
$$

If F is not continuous, then we can use the same argument as in [64], that is, we apply the above proof for $\Phi_\sigma * \mathsf{F}$, where Φ_σ is a normal distribution with mean zero and covariance matrix $\sigma \mathrm{I}$ (I is a unit matrix). Then we take limit, when $\sigma \to 0$, see for details [64]. □

Proof of Theorem 2.2.2. Let us assume that $\mathsf{F} \in \mathscr{F}_d$ is absolutely continuous. Then in [64] it was shown that there exists $\vec{\alpha}$, such that $\bar{\mathsf{F}} = \mathsf{F} * I_{\vec{\alpha}} = (1-p)\mathsf{M} + p\mathsf{N}$, where M satisfies the conditions of Lemma 2.3.2 and N satisfies the conditions of Lemma 2.3.3 with $\alpha = (2d)^{-1}$, $l = d^{-1/2}$. Note that in [64] it was assumed that $p = n^{-1/3}$, however, it can be easily checked that the same decomposition holds for any $p \in (0,1)$. Let $p = n^{-1/(k+1)}$. Applying (2.12)

and (A.18) we get

$$\begin{aligned}\left\|(\bar{\mathsf{F}}-I)^{*k}*\mathrm{e}^{a(\bar{\mathsf{F}}-I)}\right\|_K &\leqslant \sum_{j=1}^{k}\binom{k}{j}q^j\|(\mathsf{M}-I)^{*j}*\mathrm{e}^{(a/2)(\bar{\mathsf{F}}-I)}\|_K \\ &\quad\times p^{k-j}\left\|(\mathsf{N}-I)^{*(k-j)}*\mathrm{e}^{(pa/2)(\mathsf{N}-I)}\right\|_{TV} \\ &\quad+p^k\left\|(\mathsf{N}-I)^{*k}*\mathrm{e}^{ap(\mathsf{N}-I)}\right\|_{TV} \\ &\leqslant C(d,k)\left(\frac{1}{a^{k/2}\sqrt{ap}}+\frac{p^{k/2}}{a^{k/2}}\right)=\frac{C(d,k)}{a^{k/2+k/(2k+2)}}.\end{aligned}$$

If F is not continuous, then we apply the above proof for $\Phi_\sigma * \mathsf{F}$, where Φ is a normal distribution with mean zero and covariance matrix σI , where I is a unit matrix. Then we take the limit as $\sigma \to 0$. □

Proof of Theorem 2.2.3. Proof of (2.8) is practically the same as the proof of Lemma C.12 from [30]. For completeness, we repeat it there. Without loss of generality, we may assume that $a \leqslant b$. Let $\bar{\mathsf{F}} = \mathsf{F} * I_{\vec{\alpha}}$. Then

$$\begin{aligned}\left\|\mathrm{e}^{a(\bar{\mathsf{F}}-I)}-\mathrm{e}^{b(\bar{\mathsf{F}}-I)}\right\|_K &\leqslant |a-b|\|\bar{\mathsf{F}}-I\|_{TV}\int_0^1\left\|\mathrm{e}^{(a\tau+b(1-\tau))(\bar{\mathsf{F}}-I)}\right\|_{TV}\mathrm{d}\tau \\ &\leqslant 2(b-a), \qquad (2.18)\end{aligned}$$

which completes the proof in the case $a \leqslant b \leqslant 1$.

Next, consider the case $a < 1 \leqslant b \leqslant 2$. From (2.18) we deduce

$$\left\|\mathrm{e}^{a(\bar{\mathsf{F}}-I)}-\mathrm{e}^{b(\bar{\mathsf{F}}-I)}\right\|_K \leqslant 2(b-a) \leqslant 4(b-a)b^{-1} \leqslant 4(b-a)b^{-3/4}.$$

Now consider the case $a < 1 < 2 < b$. Then

$$\left\|\mathrm{e}^{a(\bar{\mathsf{F}}-I)}-\mathrm{e}^{b(\bar{\mathsf{F}}-I)}\right\|_K \leqslant 1 \leqslant 2(b-1)b^{-1} \leqslant 2(b-a)b^{-1} \leqslant 2(b-a)b^{-3/4}.$$

Let now $1 \leqslant a \leqslant b \leqslant 1.5a$. Then by Corollary 2.2.2

$$\begin{aligned}\inf_{\vec{\alpha}\in\mathbb{R}}\left\|\mathrm{e}^{a(\bar{\mathsf{F}}-I)}-\mathrm{e}^{b(\bar{\mathsf{F}}-I)}\right\|_K &\leqslant (b-a)\inf_{\vec{\alpha}\in\mathbb{R}}\left\|(\bar{\mathsf{F}}-I)*\mathrm{e}^{a(\bar{\mathsf{F}}-I)}\right\|_K \\ &\leqslant C(d)(b-a)a^{-3/4} \leqslant C(d)(b-a)b^{-3/4}.\end{aligned}$$

Finally, let $1 \leqslant a < 1.5a < b$. Then

$$\left\|\mathrm{e}^{a(\bar{\mathsf{F}}-I)}-\mathrm{e}^{b(\bar{\mathsf{F}}-I)}\right\|_K \leqslant 2 \leqslant 6(b-a)b^{-1} \leqslant 6(b-a)b^{-3/4}.$$

The proof of (2.8) is complete. For the proof of (2.9) observe that by (A.30) and (A.27)

$$\begin{aligned}
&\left\| \mathrm{e}^{a(\bar{\mathsf{F}}-I)} * \left(\mathrm{e}^{b(\bar{\mathsf{F}}-I)} - \sum_{j=0}^{m} \frac{b^j (\bar{\mathsf{F}}-I)^{*j}}{j!} \right) \right\|_K \\
&\qquad \leqslant \frac{b^{m+1}}{m!} \left\| \mathrm{e}^{a(\bar{\mathsf{F}}-I)} * (\bar{\mathsf{F}}-I)^{*(m+1)} \right\|_K \int_0^1 \left\| \mathrm{e}^{b\tau(\bar{\mathsf{F}}-I)} \right\|_{TV} (1-\tau)^m \mathrm{d}\tau \\
&\qquad = \frac{b^{m+1}}{(m+1)!} \left\| \mathrm{e}^{a(\bar{\mathsf{F}}-I)} * (\bar{\mathsf{F}}-I)^{*(m+1)} \right\|_K .
\end{aligned}$$

It remains to apply Corollary 2.2.2. □

Proof of Theorem 2.2.4. Similarly to the proof of Theorem 2.2.2 we assume that $\mathsf{F} \in \mathscr{F}_d$ is absolutely continuous and $\bar{\mathsf{F}} = \mathsf{F} * I_{\vec{a}} = (1-p)\mathsf{M} + p\mathsf{N}$, where M satisfies the conditions of Lemma 2.3.2 and N satisfies the conditions of Lemma 2.3.3 with $\alpha = (2d)^{-1}$, $l = d^{-1/2}$ and $p = C \in (0, 1/2]$. Then

$$\begin{aligned}
\|\bar{\mathsf{F}}^{*n} - \bar{\mathsf{F}}^{*(n+1)}\|_K &\leqslant \left\| \bar{\mathsf{F}}^{*n} - \mathrm{e}^{n(1-p)(\mathsf{M}-I)} * \left((1-p)I + p\mathsf{N}\right)^{*n} \right\|_K \\
&+ \left\| \mathrm{e}^{n(1-p)(\mathsf{M}-I)} * \left(\left((1-p)I + p\mathsf{N}\right)^{*n} - \left((1-p)I + p\mathsf{N}\right)^{*(n+1)} \right) \right\|_K \\
&+ \left\| \left((1-p)I + p\mathsf{N}\right)^{*(n+1)} * \left(\mathrm{e}^{n(1-p)(\mathsf{M}-I)} - \mathrm{e}^{(n+1)(1-p)(\mathsf{M}-I)} \right) \right\|_K \\
&+ \left\| \left((1-p)I + p\mathsf{N}\right)^{*(n+1)} * \mathrm{e}^{(n+1)(1-p)(\mathsf{M}-I)} - \bar{\mathsf{F}}^{*(n+1)} \right\|_K . \qquad (2.19)
\end{aligned}$$

By (2.13) the first and last summands on RHS of (2.19) are majorized by $C(d)n^{-1/2}$. From (A.15) and (A.22) it follows that

$$\begin{aligned}
&\left\| \mathrm{e}^{n(1-p)(\mathsf{M}-I)} * \left(\left((1-p)I + p\mathsf{N}\right)^{*n} - \left((1-p)I + p\mathsf{N}\right)^{*(n+1)} \right) \right\|_K \\
&\qquad \leqslant \left\| \mathrm{e}^{n(1-p)(\mathsf{M}-I)} \right\|_{TV} \left\| \left(\left((1-p)I + p\mathsf{N}\right)^{*n} - \left((1-p)I + p\mathsf{N}\right)^{*(n+1)} \right) \right\|_{TV} \\
&\qquad \leqslant \left\| \left(\left((1-p)I + pI_1\right)^{*n} - \left((1-p)I + pI_1\right)^{*(n+1)} \right) \right\|_{TV} \\
&\qquad \leqslant C(np)^{-1/2} \leqslant Cn^{-1/2},
\end{aligned}$$

where $I_1 \in \mathscr{F}_1$ and the last estimate follows from the fact that we assumed $p = C \in (0, 1/2]$. Applying Lemmas 2.3.1, 2.3.5 and (2.10) and the fact that total variation of any distribution is equal to 1, we obtain

$$\begin{aligned}
&\left\| \left((1-p)I + p\mathsf{N}\right)^{*(n+1)} * \left(\mathrm{e}^{n(1-p)(\mathsf{M}-I)} - \mathrm{e}^{(n+1)(1-p)(\mathsf{M}-I)} \right) \right\|_K \\
&\qquad \leqslant \left\| \left((1-p)I + p\mathsf{N}\right)^{*(n+1)} \right\|_{TV}
\end{aligned}$$

$$\times \left\| \mathsf{W} * (\mathrm{e}^{n(1-p)(\mathsf{M}-I)} - \mathrm{e}^{(n+1)(1-p)(\mathsf{M}-I)}) \right\|_{TV} + Cn^{-1/2}$$
$$\leqslant \quad Cn^{-1/2}.$$

Substituting all above estimates into (2.19) we complete the proof of Theorem. If F is not continuous, then we can use the same arguing as in the proof of Theorem 2.2.2 by replacing F with $\Phi_\sigma * \mathsf{F}$ and taking limit, when $\sigma \to 0$. □

Proof of Estimate (2.7). Let $p = n^{-1/5}$,

$$\begin{aligned} J &= Bi(n,p) - \mathrm{e}^{np(I_1-I)} * (I - np^2(I_1-I)^{*2}/2), \\ z &= \exp\left\{\frac{\mathrm{i}t}{h\sqrt{np}}\right\} - 1, \quad U(t) = (1+pz)^n - \mathrm{e}^{npz}\left(1 - \frac{np^2z^2}{2}\right). \end{aligned}$$

Here $h \geqslant 1$ is absolute constant, the value of which will be chosen later. From Lemma A.1.8 it follows that

$$\begin{aligned} \|J\|_K &\geqslant C\left|\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2} U(t) \mathrm{e}^{-\mathrm{i}t\sqrt{np}/h} \mathrm{d}t\right| \\ &\geqslant Cp^2\left|\int_{-\infty}^{\infty} t^4 \mathrm{e}^{-t^2/2} \mathrm{d}t\right| - \tilde{I}_1 - \tilde{I}_2 - \tilde{I}_3 - \tilde{I}_4, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} \tilde{I}_1 &= C\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\left|(1+pz)^n - \mathrm{e}^{npz - np^2z^2/2}\right| \mathrm{d}t, \\ \tilde{I}_2 &= C\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\left|\mathrm{e}^{npz}\left(\mathrm{e}^{-np^2z^2/2} - 1 + \frac{np^2z^2}{2} - \frac{n^2p^4z^4}{8}\right)\right| \mathrm{d}t, \\ \tilde{I}_3 &= C\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\frac{n^2p^4}{8}\left|\mathrm{e}^{npz}\right|\left|z^4 - \left(\frac{\mathrm{i}t}{h\sqrt{np}}\right)^4\right| \mathrm{d}t; \\ \tilde{I}_4 &= \frac{Cp^2}{h^4 8}\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\left|\mathrm{e}^{npz - \mathrm{i}t\sqrt{np}/h} - 1\right| \mathrm{d}t. \end{aligned}$$

We will assume that $p \leqslant 1/2$, since we need estimate for large n only. Observe that

$$Re\, z = -2\sin^2\frac{t}{2h\sqrt{np}}, \quad |z|^2 = 4\sin^2\frac{t}{2h\sqrt{np}} = -2Re\, z = 2|Re\, z|.$$

By (A.9)

$$|1+pz| \leqslant \exp\{p(1-p)Re\, z\} \leqslant \exp\left\{\frac{p}{2}Re\, z\right\},$$

$$pRe\,z+\frac{p^2|z|^2}{2}=pRe\,z-p^2Re\,z=p(1-p)Re\,z\leqslant\frac{p}{2}Re\,z.$$

Therefore,

$$\left|1+pz-\mathrm{e}^{pz-p^2z^2/2}\right|\leqslant Cp^3|z|^3\leqslant Cp^3|Re\,z|^{3/2}.$$

Note that $|\exp\{-p|z|\}|\leqslant \mathrm{e}^{2p}\leqslant C$. Collecting the above estimates and applying simple estimate $a\mathrm{e}^{-a}<1$, $a>0$, we prove that

$$\begin{aligned}\left|(1+pz)^n-\mathrm{e}^{npz-np^2z^2/2}\right|&\leqslant Cn\mathrm{e}^{(n-1)pRe\,z/2}p^3|Re\,z|^{3/2}\\ &\leqslant\quad C\mathrm{e}^{npRe\,z}np^3|Re\,z|^3\leqslant\frac{Cnp^3}{np\sqrt{np}}=Cn^{-4/5}.\end{aligned}$$

Therefore, $\tilde{I}_1\leqslant Cn^{-4/5}$.

Similarly

$$\begin{aligned}\tilde{I}_2&\leqslant C\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}\mathrm{e}^{npRe\,z+np^2|z|^2/2}n^3p^6|z|^6\mathrm{d}t\\ &\leqslant Cn^3p^6\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}\mathrm{e}^{npRe\,z/2}|Re\,z|^3\mathrm{d}t\leqslant\frac{Cn^3p^6}{n^3p^3}=Cn^{-3/5}.\end{aligned}$$

Next observe that by (A.47)

$$|z|\leqslant\frac{|t|}{h\sqrt{np}},\qquad\left|z-\frac{\mathrm{i}t}{h\sqrt{np}}\right|\leqslant\frac{t^2}{2h^2np}$$

so that

$$\begin{aligned}\left|z^4-\left(\frac{\mathrm{i}t}{h\sqrt{np}}\right)^4\right|&\leqslant\left|z-\frac{\mathrm{i}t}{h\sqrt{np}}\right|\left(|z|+\frac{|t|}{h\sqrt{np}}\right)\left(|z|^2+\frac{t^2}{h^2np}\right)\\ &\leqslant\frac{C|t|^5}{h^5(np)^{5/2}}\end{aligned}$$

and

$$\tilde{I}_3\leqslant\frac{Cn^2p^2}{h^5(np)^{5/2}}\leqslant Cn^{-4/5}.$$

For estimation of I_4 observe that

$$\mathrm{e}^{np(\mathrm{e}^{\mathrm{i}t}-1-\mathrm{i}t)}=\mathbb{E}\mathrm{e}^{\mathrm{i}t(X-\mathbb{E}X)},$$

where $X \sim Pois(np)$. Therefore

$$\begin{aligned}\left|e^{np(e^{it}-1-it)}-1\right| &\leqslant \mathbb{E}\left|e^{it(X-\mathbb{E}X)}-1\right| \leqslant \mathbb{E}|X-\mathbb{E}X||t| \\ &\leqslant \sqrt{VarX}|t| = \sqrt{np}|t|\end{aligned}$$

and

$$\tilde{I}_4 \leqslant \frac{Cp^2}{h^5} = \frac{Cn^{-2/5}}{h^5}.$$

Collecting estimates for I_i and substituting them into (2.20) we get

$$\|J\|_K \geqslant \frac{C_{01}}{n^{2/5}h^4} - \frac{C_{02}}{n^{2/5}h^5} - \frac{C_{03}}{n^{3/5}},$$

for some absolute constants C_{01}, C_{02}, C_{03}. If we take $h = 2C_{02}/C_{01}$, then

$$\|J\|_K \geqslant \frac{1}{n^{2/5}}\left(\frac{C_{01}^5}{32C_{02}^4} - \frac{C_{03}}{n^{1/5}}\right) \geqslant \frac{C}{n^{2/5}}$$

if $n \geqslant C_{04}$ and absolute constant C_{04} is sufficiently large. □

3 APPROXIMATION IN TOTAL VARIATION

Total variation is by far the most popular metric used for distributions having sufficient probabilistic mass at zero. In this chapter we present some structural simplifications to known Hipp-type approximations.

3.1 Known results

If random vectors take value $\vec{0}$ with large probabilities, then natural approach is to sum estimates for individual summands. In this case proofs for one-dimensional and multi-dimensional cases are virtually the same. We present a few such results.

Khinchin's estimate and related results. Let $\vec{\zeta}_1, \vec{\zeta}_2, \ldots, \vec{\zeta}_n$ be independent d-dimensional r.v.s., satisfying $\mathbb{P}(\vec{\zeta}_i = \vec{0}) = p_{i0} > 0$. Then

$$\mathscr{L}(\vec{\zeta}_i) = p_{i0} I + (1-p_{i0})\mathsf{V}_i, \quad \mathsf{V}_i = \mathscr{L}(\vec{\zeta}_i | \vec{\zeta}_i \neq \vec{0})$$

and

$$\mathscr{L}(\vec{S}_n) = \mathscr{L}(\vec{\zeta}_1 + \ldots + \vec{\zeta}_n) = \prod_{i=1}^{*n} (p_{i0} I + (1-p_{i0})\mathsf{V}_i).$$

The well known estimate, which is usually attributed to Khinchin or Doeblin, states that

$$\left\| \mathscr{L}(\vec{S}_n) - \prod_{i=1}^{*n} \exp\{(1-p_{i0})(\mathsf{V}_i - I)\} \right\|_{TV} \leqslant \sum_{i=1}^{n} (1-p_{i0})^2. \tag{3.1}$$

In [24] Khinchin's result was extended to so-called Bergström's asymptotic expansions. Particularly it was proved that for any $n \geqslant 1$

$$\begin{aligned} \left\| \mathscr{L}(\vec{S}_n) - \mathsf{G}_1 \right\|_{TV} &\leqslant \frac{8}{3} \left(\sum_{j=1}^{n} (1-p_{j0})^2 \right)^{3/2} + 2 \left(\sum_{j=1}^{n} (1-p_{j0})^2 \right)^2, \\ \left\| \mathscr{L}(\vec{S}_n) - \mathsf{G}_1' \right\|_{TV} &\leqslant 2 \left(\sum_{j=1}^{n} (1-p_{j0})^2 \right)^2, \end{aligned}$$

where

$$\mathsf{G}_1 = \prod_{i=1}^{*n} \exp\{(1-p_{i0})(\mathsf{V}_i - I)\} * \left(I - \frac{1}{2} \sum_{i=1}^{n} (1-p_{i0})(\mathsf{V}_i - I)^{*2} \right),$$

$$\begin{aligned}\mathsf{G}'_1 &= \prod_{i=1}^{*n}\exp\{(1-p_{i0})(\mathsf{V}_i-I)\}+\sum_{k=1}^{n}\prod_{j\neq k}^{*n}\exp\{(1-p_{j0})(\mathsf{V}_j-I)\}\\ &\quad *(p_{k0}I+(1-p_{k0})\mathsf{V}_k-\exp\{(1-p_{k0})(\mathsf{V}_k-I)\}).\end{aligned}$$

Hipp's SCP approximation. Let us consider d-dimensional independent r.v.s. $\vec{\zeta}_i=(\zeta_{i1},\zeta_{i2},\ldots,\zeta_{id})$, $(i=1,\ldots,n)$ having distributions $\mathsf{F}_1,\ldots,\mathsf{F}_n$. Let $\vec{S}_n=\sum_{i=1}^n\vec{\zeta}_i$ and let

$$\mathsf{H}_{i,k}=\exp\left\{\sum_{j=1}^{k}(-1)^{j+1}(\mathsf{F}_i-I)^{*j}/j\right\},\quad (i=1,\ldots,n). \tag{3.2}$$

Observe, that the case $k=1$ corresponds to accompanying CP distribution considered in (3.1). Unlike G_1 and G'_1, which correspond to the more usual asymptotic expansion, Hipp's SCP approximation can be viewed as asymptotic expansion in the exponent. Denote

$$\mathbf{F}=\mathscr{L}(\vec{S}_n)=\prod_{i=1}^{*n}\mathsf{F}_i,\quad \mathbf{H}_{\mathrm{n,k}}=\prod_{i=1}^{*n}\mathsf{H}_{i,k},\quad q_i=\mathbb{P}(\vec{\zeta}_i=\vec{0}). \tag{3.3}$$

If $\min_i q_i>1/2$, then

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n,k}}\|_{TV}\leqslant\exp\left\{2\sum_{i=1}^{n}(1-q_i)^{k+1}/(k+1)(2q_i-1)\right\}-1. \tag{3.4}$$

Estimate (3.4) in one-dimensional case was proved by Hipp in [46]. However, replacing one-dimensional total variation norm by multivariate total variation norm it is easy to check that the same estimate holds for $\mathsf{F}_i\in\mathscr{F}_d$. Observe that in (3.4) the only assumption is sufficiently large probabilities q_i. Other than that, F_i can have arbitrary distributions. The accuracy of approximation in (3.4) is $O\left(\sum_{i=1}^n(1-q_i)^{k+1}\right)$. If $1-q_i=n^{-1/2}$, then the order of accuracy is $O(n^{-(k-1)/2})$.

If $d=1$ and random variables take non-negative integer values, then De Pril's modified recursive algorithms can be used to calculate 'probabilities' of $\mathbf{H}_{\mathrm{n,s}}$, see [81] Ch. 14. In [76] analogues of (3.4) were proved for Kolmogorov and stop-loss metrics, see also [45].

Estimates for mixtures of distributions. Roos [70, 77, 78] significantly improved Hipp's results for the case when F_i are mixtures of distributions

$\mathsf{U}_1, \mathsf{U}_2, \ldots$. More precisely, let for $i = 1, \ldots, n$

$$\mathsf{F}_i = (1-p_i)I + p_i \sum_{m=1}^{\infty} q_{i,m}\mathsf{U}_m; \tag{3.5}$$

$$\sum_{m=1}^{\infty} q_{i,m} = 1, \quad \lambda_m = \sum_{j=1}^{n} p_j q_{j,m}, \quad \mathsf{U}_m \in \mathscr{F}_d; \tag{3.6}$$

$$g(z) = 2\sum_{j=2}^{\infty} \frac{z^{j-2}}{j!}(j-1) = \frac{2\mathrm{e}^z}{z^2}(\mathrm{e}^{-z}-1+z), \quad z \neq 0; \tag{3.7}$$

$$\alpha_0 = \sum_{i=1}^{n} g(2p_i)p_i^2 \min\left(2^{-3/2}\sum_{m=1}^{\infty}\frac{q_{i,m}^2}{\lambda_m}; 1\right),$$

$$\beta_0 = \sum_{i=1}^{n} p_i^2 \min\left(\sum_{m=1}^{\infty}\frac{q_{i,m}^2}{\lambda_m}; 1\right).$$

Further it was assumed that all $\lambda_m > 0$. In [70] it was proved that without any restrictions

$$\left\| \prod_{i=1}^{*n} \mathsf{F}_i - \exp\left\{\sum_{i=1}^{n}(\mathsf{F}_i - I)\right\}\right\|_{TV} \leqslant 17.6\beta_0. \tag{3.8}$$

If $\alpha_0 \leqslant 1/(2\mathrm{e})$, then

$$\left\| \prod_{i=1}^{*n} \mathsf{F}_i - \exp\left\{\sum_{i=1}^{n}(\mathsf{F}_i - I)\right\}\right\|_{TV} \leqslant \frac{2\alpha_0}{1-2\alpha_0\mathrm{e}}. \tag{3.9}$$

In [78] for the case $\mathsf{F}_i = (1-p_i)I + p_i\sum_{m=1}^{d} q_{i,m}\mathsf{U}_m$, this result was improved to

$$\left\| \prod_{i=1}^{*n} \mathsf{F}_i - \exp\left\{\sum_{i=1}^{n}(\mathsf{F}_i - I)\right\}\right\|_{TV} \leqslant 15.6\beta_1;$$

$$\left\| \prod_{i=1}^{*n} \mathsf{F}_i - \exp\left\{\sum_{i=1}^{n}(\mathsf{F}_i - I)\right\}\right\|_{TV} \leqslant \frac{2\alpha_1}{1-2^{3/2}\alpha_1},$$

where the second estimate holds for $\alpha_1 < 2^{-3/2}$ and the first estimate holds without restrictions. Here

$$\alpha_1 = \sum_{i=1}^{n} g(2p_i)p_i^2 \sum_{m=1}^{d} q_{i,m} \min\left\{\frac{q_{i,m}}{2^{3/2}\lambda_m}; 2\right\},$$

$$\beta_1 = \sum_{i=1}^{n} p_i^2 \sum_{m=1}^{d} q_{i,m} \min\left(q_{i,m}(\lambda_m)^{-1}; 1\right).$$

The main difference among above estimates is dependence on d and smallness of constants.

Estimates for Hipp approximation $\mathbf{H}_{\mathrm{n,k}}$ were considered by Roos in [74] for the case of distributions, concentrated on coordinate vectors. In [77] those estimates were extended to the case, when F_i has density with respect of some measure Q. We formulate the version of Theorem 3 from [77] for the setting defined in (3.5), (3.6). Let

$$\beta_k(x) = \sum_{i=1}^{n} p_i^{k+1} \min\left(x \sum_{m=1}^{\infty} \frac{q_{i,m}^2}{\lambda_m}; 1 \right)^{(k+1)/2},$$

$$\hat{p} = \max_{1 \leqslant j \leqslant n} p_j; \quad A_1 + A_2 = 1;$$

$$V_k(x) = \left(1 - (1-x) \exp\left\{ \sum_{m=1}^{k} x^m/m \right\} \right) \frac{k+1}{x^{k+1}};$$

$$c_1(k) = \begin{cases} (k+1)2^{-5/2}, & \text{for odd } k, \\ (k+1)2^{1/2(k+1)-5/2}, & \text{for even } k; \end{cases}$$

$$c_2(k,\hat{p}) = \frac{\mathrm{e}2^k \lceil k/2-1 \rceil ! V_k(2\hat{p})}{\sqrt{2\pi}(k+1)}, \quad c_3(k,\hat{p}) = \frac{\mathrm{e}2^{k+1} V_k(2\hat{p})}{k+1},$$

$$c_4(k,\hat{p}) = 4\mathrm{e} \sum_{m=2}^{k} \frac{(2\hat{p})^{m-2}}{m};$$

$$\hat{C}(k) = \frac{2c_2(k,\hat{p})}{\left(1 - c_3(k,\hat{p}) \hat{p}^{k-1} \beta_1(2^{-3/2} A_2^{-1})\right)^{\lceil k/2 \rceil} \left(1 - c_4(k,\hat{p}) \beta_1(2^{-3/2} A_1^{-1})\right)}.$$

From Theorem 3 [77] it follows that if

$$c_3(k,\hat{p})(\hat{p})^{k-1} \beta_1(2^{-3/2} A_2^{-1}) < 1; \quad c_4(k,\hat{p}) \beta_1(2^{-3/2} A_1^{-1}) < 1 \tag{3.10}$$

then

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_{TV} \leqslant \hat{C}(k) \beta_k(c_1(k) A_2^{-1}). \tag{3.11}$$

Observe that estimate (3.11) is very general, since no assumption about distributions $\mathsf{U}_1, \mathsf{U}_2, \ldots$ are made. Moreover, assumption (3.10) can be fulfilled for various constants A_1, A_2 and their choice can be used to minimize RHS of estimate (3.11). On the other hand, checking of (3.10) is not that simple.

Example 3.1.1. *To get better idea of the accuracy of above estimates, let*

$$\mathsf{F}_i = q_i I + \sum_{m=1}^{d} p_{i,m} I_{\bar{e}_m} \quad i = 1, \ldots,$$

where $q_i = \mathsf{F}_i\{\vec{0}\}$, $\bar{e}_j = (0,\dots,0,1,0,\dots,0)$, $j = 1,\dots,d$. *In [74] it was shown that, if*

$$\sum_{r=1}^{d} \max_{1\leqslant j\leqslant n} p_{j,r} \leqslant \frac{1}{4},$$

then there exists $C(k)$ *such that*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_{TV} \leqslant C(k) \times \sum_{i=1}^{n} \left(\min \left\{ \sum_{r=1}^{d} p_{i,r}^2 \left(\sum_{j=1}^{n} p_{j,r} \right)^{-1}; \left(\sum_{r=1}^{d} p_{i,r} \right)^2 \right\} \right)^{(k+1)/2}. \tag{3.12}$$

If all probabilities do not depend on n, then the order of approximation in (3.11) is $O(n^{-(k-1)/2})$.

Example 3.1.2. *Let* F_i *be concentrated on integers. Then one can replace in (3.5), (3.6) distributions* U_m *by* $I_{\vec{x}}$, $\vec{x} \in \mathbb{N}^d$ *so that*

$$\mathsf{F}_i = q_i I + \sum_{\vec{x}\in\mathbb{N}^d} p_i(\vec{x}) I_{\vec{x}}, \quad p_i(\vec{x}) = \mathsf{F}_i\{\vec{x}\}.$$

In this case,

$$\beta_k(x) = \sum_{i=1}^{n} \min \left(x \sum_{\vec{x}} p_i^2(\vec{x}) \left(\sum_{j=1}^{n} p_i(\vec{x}) \right)^{-1}; (1-q_i)^2 \right)^{(k+1)/2} \tag{3.13}$$

and

$$\beta_1(x) \leqslant x^{(k+1)/2} \max_{1\leqslant i\leqslant n} (\max_{\vec{x}} p_i(\vec{x}))^{k+1}.$$

Therefore, if $\min_{1\leqslant i\leqslant n} \mathsf{F}_i\{\vec{0}\} \to 1$, *then assumptions (3.10) will be fulfilled for large n and*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_{TV} = O(\beta_1(1))$$

with $\beta_1(1)$ *defined by (3.13).*

Though constants in (3.11) are given explicitly, they depend on the choice of A_1, A_2. To get the idea of the magnitude of constants, we cite one-dimensional result from [28], p. 239. Let $Bi(n, p_B) = ((1p_B)I + p_B I_1)^{*n} \in \mathscr{F}_1$ be the Binomial distribution. Then 2-order Hipp approximation is equal to $\mathbf{H}_{\mathrm{n},2} = \exp\{np_B(I_1 - I) - np_B^2(I_1 - I)^{*2}/2\}$. It was proved in Corollary 14.4,

[28] that if $p_B \to 0$, $np_B \to \infty$, then

$$\lim_{n\to\infty} \frac{\|Bi(n,p_B) - \mathbf{H}_{\mathrm{n},2}\|_{TV}}{p_B^{3/2}\sqrt{n}} = 0.503\ldots$$

Estimates following from one-dimensional estimates. For any $\mathsf{F} \in \mathscr{F}_d$ and $p_1,\ldots,p_n \leqslant 1/4$ consequently applying (A.15) and (3.12) (with $d=1$) we get

$$\left\| \prod_{i=1}^{*n} ((1-p_i)I + p_i\mathsf{F}) - \exp\left\{ \sum_{j=1}^{k} (-1)^{j+1} p_i^j (\mathsf{F}-I)^{*j}/j \right\} \right\|_{TV}$$
$$\leqslant C(k) \sum_{i=1}^{n} p_i^{k+1} \min\left(1; \left(\sum_{i=1}^{n} p_i\right)^{-(k+1)/2}\right). \qquad (3.14)$$

We see that for total variation norm the cases of interest are when F is replaced by F_i. Moreover, if $p_i = p$, then we get the accuracy $O(p^{(k+1)/2} n^{-(k-1)/2})$.

3.2 Results

All results of this Chapter are obtained under assumption that random vectors take zero-value with quite large probability. It is well known, that for approximation by accompanying CP distribution no such restriction is needed. However, in the next example we show that to ensure meaningful approximation for $k \geqslant 3$, $\mathsf{F}_i\{\vec{0}\}$ can't be very close to 0.

Example 3.2.1. *Let us consider one dimensional Binomial distribution* $Bi(n,p_B)$, $\widehat{Bi}(n,p_B|t) = (1+p_B(\mathrm{e}^{\mathrm{i}t}-1))^n$ *and SCP approximation* $D^{*n} = \mathbf{H}_{\mathrm{n},2}$, *which in this case has Fourier transform*

$$\widehat{D}^n(t) = \exp\left\{ np_B(\mathrm{e}^{\mathrm{i}t}-1) - \frac{np_B^2(\mathrm{e}^{\mathrm{i}t}-1)^2}{2} + \frac{np_B^3(\mathrm{e}^{\mathrm{i}t}-1)^3}{3} \right\}.$$

If $p_B = 12/13$, *then*

$$\begin{aligned} \|Bi(n,p_B) - D^{*n}\|_{TV} &\geqslant \sup_t |\widehat{Bi}(n,p_B|t) - \widehat{D}^n(t)| \\ &\geqslant \left|\widehat{Bi}\left(n,p_B\middle|\frac{\pi}{6}\right) - \widehat{D}^n\left(\frac{\pi}{6}\right)\right| \\ &\geqslant \left|\widehat{D}^n\left(\frac{\pi}{6}\right)\right| - 1 \geqslant \mathrm{e}^{0.0009n} - 1. \end{aligned} \qquad (3.15)$$

Therefore, in this case,

$$\|Bi(n,p_B)-D^{*n}\|_{TV}\to\infty,\quad n\to\infty,$$

so, obviously, D^{*n} is useless as approximation.

3.2.1 First order asymptotics to $\mathbf{H}_{\mathrm{n,2}}$

On one hand, the larger k, the better accuracy of approximation by $\mathbf{H}_{\mathrm{n,k}}$. On the other hand, for large k, applicability of $\mathbf{H}_{\mathrm{n,k}}$ is doubtful due to vast amount of required computing resources. Therefore, one of the main tasks is structural simplification of approximation. In [30], p. 95 it was demonstrated that, if $d=1$, $\mathsf{F}_i=I_1$, then $\mathbf{H}_{\mathrm{n,2}}\{j\}$ and its first order asymptotics can be expressed through Hermite polynomials and their third order forward differences, respectively. No such formulas are available for $\mathbf{H}_{\mathrm{n,3}}\{j\}$. Therefore, it is meaningful to replace $\mathbf{H}_{\mathrm{n,3}}$ by the first order asymptotics to $\mathbf{H}_{\mathrm{n,2}}$ retaining, at least for some cases, the same order of accuracy.

Let

$$\mathfrak{u}(x,m)=\left(\min\left\{\frac{4c_1(m)}{x}\sum_{r=1}^{\infty}\frac{(p_iq_{i,r})^2}{\lambda_r};4\left(\sum_{r=1}^{\infty}p_iq_{i,r}\right)^2\right\}\right)^{(m+1)/2} \tag{3.16}$$

and

$$\mathscr{A}_2:=\frac{1}{3}\sum_{j=1}^{n}(\mathsf{F}_j-I)^{*3}.$$

In the following theorem we estimate the closeness of $\mathbf{H}_{\mathrm{n,3}}$ and $\mathbf{H}_{\mathrm{n,2}}*(I+\mathscr{A}_2)$. Closeness of $\mathbf{F}$ and $\mathbf{H}_{\mathrm{n,2}}*(I+\mathscr{A}_2)$ then can be estimated by application of the triangle inequality and (3.11). Theorem 3.2.1 is new even for one-dimensional case $d=1$.

Theorem 3.2.1. *Let F_i be defined by (3.5), $A_1,A_2\in(0,1)$; $A_1+A_2=1$. If $c_4(3,\hat{p})\beta_1(2^{-3/2}A_1)<1$, then*

$$\begin{aligned}&\|\mathbf{H}_{\mathrm{n,3}}-\mathbf{H}_{\mathrm{n,2}}*(I+\mathscr{A}_2)\|_{TV}\\ &\qquad\leqslant\ \frac{1}{9(1-c_4(3,\hat{p})\beta_1(2^{-3/2}A_1))}\left(\sum_{i=1}^{n}\mathfrak{u}(A_2/2,2)\right)^2.\end{aligned}$$

Example 3.2.2. *Let* F_i *be from Example 3.1.1. If*

$$\sum_{r=1}^{d} \max_{1\leqslant j\leqslant n} p_{j,r} \leqslant \frac{1}{4},$$

then there exists C such that

$$\begin{aligned}\|\mathbf{H}_{\mathrm{n},3} - \mathbf{H}_{\mathrm{n},2} * (I + \mathscr{A}_2)\|_{TV} \\ \leqslant \quad C\left(\sum_{i=1}^{n} \min\left\{\left(\sum_{r=1}^{d} p_{i,r}^2\left(\sum_{j=1}^{n} p_{j,r}\right)^{-1}\right)^{3/2}; \left(\sum_{r=1}^{d} p_{i,r}\right)^{3}\right\}\right)^{2}.\end{aligned}$$

If all probabilities do not depend on n, then the order of approximation in (3.11) is $O(n^{-1})$*. Taking into account Example 3.1.1 and applying the triangle inequality, then we get*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},3}\|_{TV} = O(n^{-1}), \quad \|\mathbf{F} - \mathbf{H}_{\mathrm{n},2} * (I + \mathscr{A})\|_{TV} = O(n^{-1});$$

i.e. the order of accuracy is the same for 3-order Hipp-type approximation and for the first order asymptotics to the 2-order Hipp approximation.

3.2.2 Skellam approximation. General case

Next we consider simplification of CP approximation $\mathbf{H}_{\mathrm{n},1}$ (i.e. accompanying law), when all F_i are concentrated on finite number of hyper-spaces. More precisely let $\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_N$ be non-identical non-zero vectors from $\mathbb{R}^d$. Let for $i = 1, 2, \ldots, n$; $p_i \in (0,1)$ and

$$\mathsf{F}_i = (1 - p_i)I + p_i \sum_{m=1}^{N} q_{i,m}\mathsf{U}_m; \tag{3.17}$$

$$\sum_{m=1}^{N} q_{i,m} = 1, \quad \operatorname{supp}\mathsf{U}_m \subset \{k\vec{y}; k \in \mathbb{Z}\setminus\{0\}\}. \tag{3.18}$$

Similarly to [55] we define right-hand and left-hand factorial moments for U_m:

$$\begin{aligned}&\nu_{1,m}^{+} := \sum_{k=1}^{\infty} k\mathsf{U}_m\{k\vec{y}_m\}, \quad \nu_{1,m}^{-} := \sum_{k=1}^{\infty} k\mathsf{U}_m\{-k\vec{y}_m\},\\ &\nu_{2,m}^{+} := \sum_{k=2}^{\infty} k(k-1)\mathsf{U}_m\{k\vec{y}_m\}, \quad \nu_{2,m}^{-} := \sum_{k=2}^{\infty} k(k-1)\mathsf{U}_m\{-k\vec{y}_m\}.\end{aligned}$$

In one dimensional case, Skellam distribution is the distribution of the dif-

ference of two independent Poisson random variables. We extend this definition to N such differences. More precisely we define multivariate Skellam distribution as

$$Sk := \exp\left\{ \sum_{m=1}^{N} \nu_{1,m}^{+} \sum_{i=1}^{n} p_i q_{i,m} (I_{\vec{y}_m} - I) + \sum_{m=1}^{N} \nu_{1,m}^{-} \sum_{i=1}^{n} p_i q_{i,m} (I_{-\vec{y}_m} - I) \right\}.$$

Theorem 3.2.2. *If* $\mathsf{U}_m\{\vec{y}_m\} > 0$ *and* $\mathsf{U}_m\{-\vec{y}_m\} > 0$, $m = 1,2,\ldots,N$, *then*

$$\begin{aligned} &\left\| \exp\left\{ \sum_{i=1}^{n} (\mathsf{F}_i - I) \right\} - Sk \right\|_{TV} \\ &\qquad \leqslant \frac{3}{4\mathrm{e}} \sum_{m=1}^{N} \sum_{i=1}^{n} p_i q_i \left(\nu_{2,m}^{+} \left(\sum_{k=1}^{n} p_k q_{k,m} \mathsf{U}_m\{\vec{y}_m\} \right)^{-1} \right. \\ &\qquad \left. + \nu_{2,m}^{-} \left(\sum_{k=1}^{n} p_k q_{k,m} \mathsf{U}_m\{-\vec{y}_m\} \right)^{-1} \right). \end{aligned} \tag{3.19}$$

Theorem's estimate can be combined with Roos' estimates (3.8), (3.9).

Example 3.2.3. *Let us consider i.i.d. two-dimensional r.v.s.* $\vec{X}, \vec{X}_1, \ldots, \vec{X}_n$, *such that*

$$\begin{aligned} &\mathbb{P}(\vec{X} = (0,0)) = p(0,0) \in (0.5;1); \\ &\mathbb{P}(\vec{X} = (1,0)) = p(1,0); \qquad \mathbb{P}(\vec{X} = (2,0)) = p(2,0); \\ &\mathbb{P}(\vec{X} = (0,1)) = p(0,1); \qquad \mathbb{P}(\vec{X} = (0,2)) = p(0,2); \\ &\mathbb{P}(\vec{X} = (-1,0)) = p(-1,0); \quad \mathbb{P}(\vec{X} = (0,-1)) = p(0,-1); \\ &\mathbb{P}(\vec{X} = (1,1)) = p(1,1); \qquad \mathbb{P}(\vec{X} = (2,2)) = p(2,2). \end{aligned}$$

Then taking $\vec{y}_1 = (1,0)$, $\vec{y}_2 = (0,1)$, $\vec{y}_3 = (1,1)$ *we can write (3.17) and (3.18) in the following way*

$$\begin{aligned} \mathsf{U}_1 &= \frac{p(1,0) I_{(1,0)}}{a} + \frac{p(2,0) I_{(2,0)}}{a} + \frac{p(-1,0) I_{(-1,0)}}{a}, \\ a &= p(1,0) + p(2,0) + p(-1,0); \\ \mathsf{U}_2 &= \frac{p(0,1) I_{(0,1)}}{b} + \frac{p(0,2) I_{(0,2)}}{b} + \frac{p(0,-1) I_{(0,-1)}}{b}, \\ b &= p(0,1) + p(0,2) + p(0,-1); \\ \mathsf{U}_3 &= \frac{p(1,1) I_{(1,1)}}{c} + \frac{p(2,2) I_{(2,2)}}{c}, \quad c = p(1,1) + p(2,2). \end{aligned}$$

Then

$$q_{i,1}=\frac{a}{a+b+c};\quad q_{i,2}=\frac{b}{a+b+c};\quad q_{i,3}=\frac{c}{a+b+c},$$
$$p_i=a+b+c=1-p(0,0)$$

and

$$\nu^{+}_{1,1}=\frac{p(1,0)+2p(2,0)}{a};\quad \nu^{-}_{1,1}=\frac{p(-1,0)}{a};\quad \mathsf{U}_1\{\vec{y}_1\}=\frac{p(1,0)}{a},\quad \textit{etc.}$$

Then from Theorem 3.2.2 it follows that

$$\|\exp\{n(\mathscr{L}(\vec{X})-I)\}-Sk\|_{TV}\leqslant\frac{3}{2\mathrm{e}}\left(\frac{p(2,0)}{p(1,0)}+\frac{p(0,2)}{p(0,1)}+\frac{p(2,2)}{p(1,1)}\right),$$

i.e. estimate is meaningful only if $p(2,0)=o(p(1,0))$ *and* $p(0,2)=o(p(0,1))$, $p(2,2)=o(p(1,1))$.

Finally observe that if $\mathsf{U}_m\{k\vec{y}_m\}=0$, $k=1,2,\ldots$; then Skellam distribution becomes Poisson distribution

$$Pois=\exp\left\{\sum_{m=1}^{N}\nu^{+}_{1,m}\sum_{i=1}^{n}p_iq_{i,m}(I_{\vec{y}_m}-I)\right\}$$

and (3.19) estimates closeness of accompanying distribution and Poisson law.

3.3 Auxiliary results

In the next two lemmas we use the same notation as in Theorem 3.2.1.

Lemma 3.3.1. *Let* F_i *be defined by (3.5), (3.6),* $i=1,\ldots,n$, $m=0,1,2,\ldots$, $x\in(0,\infty)$. *Then*

$$\left\|(\mathsf{F}_i-I)^{*(m+1)}*\exp\left\{x\sum_{j=1}^{n}(\mathsf{F}_j-I)\right\}\right\|_{TV}\leqslant\mathfrak{u}(x,m).\tag{3.20}$$

where $\mathfrak{u}(x,m)$ *is defined by (3.16).*

Lemma 3.3.1 is essentially estimate (4) from [74], Lemma 1. In [74] estimate (3.20) was proved for the case when U_m were degenerate unit distributions concentrated on coordinate axes, i.e. $\mathsf{U}_m=I_{\vec{e}_m}$, $m=1,2,\ldots,d$ and Lemma 1 from [70] played crucial role int he proof. However (A.15) allows to switch in Lemma 1 [70] (an consequently in Lemma 1 from [74])

to $\mathsf{F}_i = (1-p_i)I + p_i \sum_{m=1}^{N} q_{i,m}\mathsf{U}_m$ for any number N of arbitrary multidimensional distributions U_m. Inequalities for F_i defined in (3.5), (3.6) then are obtained as limit cases, when $N \to \infty$. Note that estimate (3.20) in this general form was already used by Roos in [77].

Lemma 3.3.2. *Let F_i, $i = 1,\dots,n$, be defined by (3.5) and (3.6); $k \in \{2,3,\dots\}$, $\tau_2,\tau_3,\dots,\tau_k \in [0,1]$ and $c_4(k,\hat{p})\beta_1(2^{-3/2}A) < 1$, $A \in (0,1)$,. Then*

$$\begin{aligned} T(A,k) &:= \left\| \exp\left\{ A\sum_{i=1}^{n}(\mathsf{F}_i - I) + \sum_{i=1}^{n}\sum_{m=2}^{k}\frac{\tau_m(-1)^{m+1}}{m}(\mathsf{F}_i - I)^{*m} \right\} \right\|_{TV} \\ &\leqslant \frac{1}{1 - c_4(k,\hat{p})\beta_1(2^{-3/2}A)}. \end{aligned}$$

Proof. Lemma 3.3.2 is a generalized version of estimate (5) from Lemma 1, [74]. For completeness we repeat the proof from [74]. Application of (3.20) and Stirling's formula allows to write

$$\begin{aligned} T(A,k) &\leqslant \left\| \exp\left\{ A\sum_{i=1}^{n}(\mathsf{F}_i - I) \right\} \right\|_{TV} \\ &\quad + \left\| \sum_{j=1}^{\infty}\frac{1}{j!}\left(\sum_{i=1}^{n}\sum_{m=2}^{k}\frac{(-1)^{m+1}}{m}\tau_m(\mathsf{F}_i - I)^{*m}\exp\left\{ \frac{A}{j}\sum_{i=1}^{n}(\mathsf{F}_i - I) \right\} \right)^{*j} \right\|_{TV} \\ &\leqslant 1 + \sum_{j=1}^{\infty}\frac{1}{j!}\left(\sum_{i=1}^{n}\frac{\|F_i - I\|_{TV}^{m-2}}{m} \right)^{j} \left\| (\mathsf{F}_i - I)^{*2} * \exp\left\{ \frac{A}{j}\sum_{i=1}^{n}(\mathsf{F}_i - I) \right\} \right\|_{TV}^{j} \\ &\leqslant 1 + \sum_{j=1}^{\infty}\frac{j^j}{j!}\left(\frac{c_4(k,\hat{p})\beta_1(2^{-3/2}A^{-1})}{\mathrm{e}} \right)^{j} \leqslant \frac{1}{1 - c_4(k,\hat{p})\beta_1(2^{-3/2}A)}. \end{aligned}$$

□

3.4 Proofs

Proof of Theorem 3.2.1. Observe that

$$\begin{aligned} \left\| \mathbf{H}_{\mathrm{n},3} - \mathbf{H}_{\mathrm{n},2} * (I + \mathscr{A}_2) \right\|_{TV} &= \left\| \mathbf{H}_{\mathrm{n},2} * (\mathrm{e}^{\mathscr{A}_2} - I - \mathscr{A}_2) \right\|_{TV} \\ &= \left\| \mathbf{H}_{\mathrm{n},2} * (\mathscr{A}_2)^{*2} * \int_0^1 \mathrm{e}^{\tau\mathscr{A}_2}(1-\tau)\mathrm{d}\tau \right\|_{TV} \\ &= \left\| \exp\left\{ \mathscr{A}_2 \sum_{i=1}^{n}(\mathsf{F}_i - I) \right\} * (\mathscr{A}_2)^{*2} * \int_0^1 \exp\left\{ \mathscr{A}_1 \sum_{i=1}^{n}(\mathsf{F}_i - I) \right. \right. \end{aligned}$$

$$
\begin{aligned}
&\quad -\frac{1}{2}\sum_{i=1}^{n}(\mathsf{F}_i-I)^{*2}+\frac{\tau}{3}\sum_{i=1}^{n}(\mathsf{F}_i-I)^{*3}\Big\}(1-\tau)\,\mathrm{d}\tau\Big\|_{TV}\\
&\leqslant \Big\|\exp\Big\{\frac{\mathscr{A}_2}{2}\sum_{i=1}^{n}(\mathsf{F}_i-I)\Big\}*\mathscr{A}_2\Big\|_{TV}^{2}\int_0^1\Big\|\exp\Big\{\mathscr{A}_1\sum_{i=1}^{n}(\mathsf{F}_i-I)\\
&\quad -\frac{1}{2}\sum_{i=1}^{n}(\mathsf{F}_i-I)^{*2}+\frac{\tau}{3}\sum_{i=1}^{n}(\mathsf{F}_i-I)^{*3}\Big\}\Big\|_{TV}(1-\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Lemma 3.3.1 and Lemma 3.3.2 complete the proof. □

Proof of Theorem 3.2.2. Bergström identity (A.32) allows to write for $j\geqslant 2$:

$$
I_{j\vec{y}}-I=I_{\vec{y}}^{*j}-I=j(I_{\vec{y}}-I)+(I_{\vec{y}}-I)^{*2}*\sum_{l=2}^{j}(l-1)I_{(j-l)\vec{y}}.
$$

Therefore,

$$
\begin{aligned}
\mathsf{U}_m-I &= \sum_{j=1}^{\infty}\mathsf{U}_m\{j\vec{y}_m\}j(I_{\vec{y}_m}-I)+\sum_{j=1}^{\infty}\mathsf{U}_m\{-j\vec{y}_m\}j(I_{-\vec{y}_m}-I)\\
&\quad +\sum_{j=2}^{\infty}\mathsf{U}_m\{j\vec{y}_m\}\sum_{l=2}^{j}(l-1)I_{(j-l)\vec{y}_m}\\
&\quad +\sum_{j=2}^{\infty}\mathsf{U}_m\{-j\vec{y}_m\}\sum_{l=2}^{j}(l-1)I_{-(j-l)\vec{y}_m}\\
&= \nu_{1,m}^{+}(I_{\vec{y}_m}-I)+\nu_{1,m}^{-}(I_{-\vec{y}_m}-I)\\
&\quad +\frac{\nu_{2,m}^{+}}{2}(I_{\vec{y}_m}-I)^{*2}*\Theta_m^{+}+\frac{\nu_{2,m}^{-}}{2}(I_{-\vec{y}_m}-I)^{*2}*\Theta_m^{-},
\end{aligned}
$$

where $\Theta_m^{+},\Theta_m^{-}$ are measures satisfying $\|\Theta_m^{+}\|_{TV},\|\Theta_m^{-}\|_{TV}\leqslant 1$. Therefore,

$$
\mathsf{F}_i-I=A_i+B_i,\quad D_i=\exp\{A_i\},
$$

where

$$
\begin{aligned}
A_i &= p_i\sum_{m=1}^{N}q_{i,m}\nu_{1,m}^{+}(I_{\vec{y}_m}-I)+p_i\sum_{m=1}^{N}q_{i,m}\nu_{1,m}^{-}(I_{-\vec{y}_m}-I),\\
B_i &= p_i\sum_{m=1}^{N}q_{i,m}\nu_{2,m}^{+}(I_{\vec{y}_m}-I)^{*2}*\Theta_m^{+}\\
&\quad +p_i\sum_{m=1}^{N}q_{i,m}\nu_{2,m}^{-}(-I_{\vec{y}_m}-I)^{*2}*\Theta_m^{-}.
\end{aligned}
$$

Observe that (A.30) allows to write for any finite measures V, M

$$\mathrm{e}^{M+V} - \mathrm{e}^{V} = \mathrm{e}^{V} * (\mathrm{e}^{M} - I) = M * \mathrm{e}^{V} * \int_0^1 \mathrm{e}^{\tau M} \mathrm{d}\tau.$$

Therefore

$$\begin{aligned} \exp\Big\{ \sum_{i=1}^{n} (\mathsf{F}_i - I) \Big\} - \prod_{i=1}^{*n} D_i &= \sum_{i=1}^{n} B_i * \exp\Big\{ \sum_{j=1}^{n} A_j \Big\} \\ &\quad * \int_0^1 \exp\Big\{ \tau \sum_{j=1}^{n} B_j \Big\} \mathrm{d}\tau. \end{aligned}$$

It is obvious that $\nu_{1,m}^{+} \geqslant \mathsf{U}_m\{\vec{y}_m\}$, $\nu_{1,m}^{-} \geqslant \mathsf{U}_m\{-\vec{y}_m\}$. Therefore, we can write

$$\begin{aligned} \exp\Big\{ \sum_{j=1}^{n} A_j \Big\} * \exp\Big\{ \tau \sum_{j=1}^{n} B_j \Big\} &= \exp\Big\{ (1-\tau) \sum_{j=1}^{n} A_j + \tau \sum_{j=1}^{n} (\mathsf{F}_j - I) \Big\} \\ &= \exp\Big\{ \sum_{j=1}^{n} p_j \sum_{m=1}^{N} q_{j,m} \mathsf{U}_m\{\vec{y}_m\} (I_{\vec{y}_m} - I) \\ &\quad + \sum_{j=1}^{n} p_j \sum_{m=1}^{N} q_{j,m} \mathsf{U}_m\{-\vec{y}_m\} (I_{-\vec{y}_m} - I) \Big\} * \tilde{M}_\tau, \end{aligned}$$

for some CP distribution $\tilde{M}_\tau$. Taking into account that $\|\tilde{M}_\tau\|_{TV} = 1$ we obtain

$$\begin{aligned} &\Big\| \exp\Big\{ \sum_{i=1}^{n} (\mathsf{F}_i - I) \Big\} - Sk \Big\|_{TV} \\ &\leqslant \frac{1}{2} \sum_{i=1}^{n} \sum_{m=1}^{N} q_{i,m} \nu_{2,m}^{+} \Big\| (I_{\vec{y}_m} - I)^{*2} \\ &\quad * \exp\Big\{ \sum_{k=1}^{n} p_k q_{k,m} \mathsf{U}_m\{\vec{y}_m\} (I_{\vec{y}_m} - I) \Big\} \Big\|_{TV} \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{m=1}^{N} q_{i,m} \nu_{2,m}^{-} \Big\| (I_{-\vec{y}_m} - I)^{*2} \\ &\quad * \exp\Big\{ \sum_{k=1}^{n} p_k q_{k,m} \mathsf{U}_m\{\vec{y}_m\} (-I_{\vec{y}_m} - I) \Big\} \Big\|_{TV}. \end{aligned}$$

To complete Theorem's proof it suffices to apply (A.15) switching to one-dimensional estimates and to use estimate (A.17). □

4 LOCAL ESTIMATES AND ESTIMATES IN ℓ_r METRIC FOR LATTICE DISTRIBUTIONS

The accuracy of Hipp-type approximation to aggregate claims distribution is usually estimated for the total variation metric. However, in practice, Panjer recursion is used to calculate local probabilities, see [81], Ch. 10. Therefore, it seems natural to estimate the accuracy of approximation for the local norm. No analogue of (A.15) exists for the local norm, therefore the case of identically distributed r.v.s. can't be reduced to one-dimensional case. Closely related to local estimates are estimates in ℓ_r, $r \geqslant 2$, metric.

4.1 Known results

Accompanying law is by far the most popular multivariate CP approximation, see, for example, [56, 70, 74, 78, 86, 87, 90] and the references therein. For dependent summands see Theorem 6.8 in [62]. Note also that accompanying law can be generalized as an operator on semigroups, see [13]. The general approximation $H_{i,k}$, $(k > 2)$ defined in (3.2) was introduced by Hipp in [46]. Earlier Presman [65] used the partial case of $H_{i,2}$ for approximation of the binomial distribution. Convolutions of accompanying laws with $H_{i,2}$ were investigated in [19, 22]. Kornya [52] introduced similar to Hipp's SCP approximation for rare events taking zero with large probabilities. Hipp and Presman considered one-dimensional distributions. Roos [75, 77] extended and generalized Hipp's results to the multidimensional case. However, so far the emphasis was on total variation or uniform metrics. To the best of our knowledge, no general local estimates for multivariate Hipp-type approximations are known. It must also be mentioned related research: multivariate discrete normal approximation [11, 12] and closeness of successive convolutions [89, 90].

Local estimate for approximation by multivariate Poisson distribution and asymptotic expansions was obtained in [73]. Let $\vec{X}_1^0, \vec{X}_2^0, \ldots, \vec{X}_n^0$ be independent random vectors (r.v.s.) concentrated on coordinate axes, that is $\mathbb{P}(\vec{X}_i^0 = \bar{e}_m) = p_{i,m}^0$, $(i = 1, 2, \ldots, n)$, $(m = 1, 2, \ldots, d)$; where $\bar{e}_m = (0, \ldots, 0, 1, 0, \ldots, 0)$. Let $\vec{S}_n^0 = \vec{X}_1^0 + \vec{X}_2^0 + \ldots + \vec{X}_n^0$. Let $\vec{\pi}_n$ have multivariate Poisson distribution:

$$\mathscr{L}(\vec{\pi}_n) = \exp\left\{ \sum_{m=1}^{d} \lambda_\pi(m)(I_{\bar{e}_m} - I) \right\}, \quad \lambda_\pi(m) = \sum_{i=1}^{n} p_{i,m}^0.$$

Let

$$\gamma_\pi(m) = 2\sum_{i=1}^{n} \left(p_{i,m}^0\right)^2 \min\left(\frac{1}{2\lambda_\pi(m)}, \mathrm{e}\right), \quad c_0 = \frac{1}{2\sqrt{2}}\left(1+\sqrt{\frac{\pi}{8}}\right).$$

Roos [69] equation (26), proved that if $\sum_{m=1}^{d}\sqrt{2\gamma_\pi(m)} < 1$, then

$$\begin{aligned}\|\mathscr{L}(\vec{S}_n^0) - \mathscr{L}(\vec{\pi}_n)\|_\infty \;\leqslant\; & 2c_0^d \prod_{m=1}^{d}\sqrt{2\min\left(\frac{1}{2\lambda_\pi(m)}, \mathrm{e}\right)} \\ & \times\left(\sum_{k=1}^{d}\sqrt{\gamma_\pi(k)}\right)^2\left(1-\sum_{k=1}^{d}\sqrt{2\gamma_\pi(k)}\right)^{-1}. \quad (4.1)\end{aligned}$$

Let $p_{i,m}^0 = p$. Then $2\gamma_\pi(m) = \min(2p; 4\mathrm{e}np^2)$. Let $\gamma_\pi(m) = p$, then (4.1) becomes

$$\|\mathscr{L}(\vec{S}_n^0) - \mathscr{L}(\vec{\pi}_n)\|_\infty \leqslant 2c_0^d d^2\left(1-d\sqrt{2p}\right)^{-1}(np)^{-d/2}p,$$

under assumption that $p < 1/2d^2$. Thus, we see that if a space dimension is large, then p must be small. To some extent this is natural requirement, since necessarily $p < 1/d$. However, $1/2d^2 << 1/d$. Assumptions that random vectors are concentrated on coordinate axes and take just $d+1$ values are also quite restrictive.

In [29] the analogue of $\vec{S}_n^0$ was considered under assumption of 1-dependence of random vectors. For independent r.v.s. the following result follows. Let

$$V = qI + \sum_{m=1}^{d} p_m I_{\bar{e}_m}, \qquad G = \exp\left\{(V-I) - \frac{(V-I)^{*2}}{2}\right\}.$$

As shown in [29], if

$$\max_{1\leqslant j\leqslant d} p_j \leqslant \frac{1}{144d}, \quad np_j \geqslant 1, \quad j = 1,\ldots,d; \qquad (4.2)$$

then

$$\begin{aligned}\frac{C_1(d)(1-q)^{3/2}}{\sqrt{n}}\prod_{j=1}^{d}\frac{1}{\sqrt{np_j}} &\leqslant \|V^{*n} - G^{*n}\|_\infty \\ &\leqslant \frac{C_2(d)(1-q)^{3/2}}{\sqrt{n}}\prod_{j=1}^{d}\frac{1}{\sqrt{np_j}}. \qquad (4.3)\end{aligned}$$

In [29] it was also shown that estimates in ℓ_r metric are similar to local estimates. Particularly, if (4.2) is satisfied, then

$$\begin{aligned}\frac{C_3(d)(1-q)^{3/2}}{\sqrt{n}}\prod_{m=1}^{d}\frac{1}{(np_m)^{(r-1)/2r}} &\leqslant \|V^{*n}-G^{*n}\|_r \\ &\leqslant \frac{C_3(d)(1-q)^{3/2}}{\sqrt{n}}\prod_{m=1}^{d}\frac{1}{(np_m)^{(r-1)/2r}},\end{aligned}$$

where upper bound holds for $r \geqslant 2$ and lower bound holds for $r \geqslant 1$. Assumptions (4.2) are quite restrictive – random vector takes just one non-zero value on each coordinate axis with very small probability. In [53] Theorem 9.1 local estimates were proved under assumption of equality factorial moments, however these estimates were very rough, essentially replacing estimate for the sums of r.v.s. by the sum of local estimates for each summand.

4.2 Setting

Let us consider d-dimensional independent r.v.s. $\vec{X}_i = (X_{i1}, X_{i2}, \ldots, X_{id})$, $(i = 1, \ldots, n)$ taking values in $\mathbb{Z}^d$ and having distributions $F_1, \ldots, F_n \in \mathscr{F}_d^Z$. Our aim is to approximate distribution of $\vec{S}_n = \sum_{i=1}^n \vec{X}_i$ by Hipp-type SCP measures. We recall definition (4.4) from previous chapter. Let

$$H_{i,k} = \exp\left\{\sum_{j=1}^{k}(-1)^{j+1}(F_i - I)^{*j}/j\right\}, \quad (i = 1, \ldots, n).$$

so that

$$\widehat{H}_{i,k} = \exp\left\{\sum_{j=1}^{k}(-1)^{j+1}(\widehat{F}_i - 1)^{j}/j\right\}$$

and denote

$$\mathbf{F} = \mathscr{L}(\vec{S}_n) = \prod_{i=1}^{*n} F_i, \quad \mathbf{H}_{\mathrm{n,k}} = \prod_{i=1}^{*n} H_{i,k}, \tag{4.4}$$

We recall that $\vec{x} = (x_1, x_2, \cdots, x_d)$. To make notation shorter we write

$$\sum_{\vec{x}} = \sum_{\vec{x}\in\mathbb{Z}^d\setminus\{\vec{0}\}}, \quad q_i = \mathbb{P}(\vec{X}_i = \vec{0}), \quad p_i(\vec{x}) = \mathbb{P}(\vec{X}_i = \vec{x}).$$

Let $i = 1, 2, \ldots, n$. Next, we define quantities used for estimating the accuracy

of approximation. Let $k \in \mathbb{N}$

$$c_{i,1} = c_{i,2} = q_i, \quad c_{i,k} = q_i - \frac{2(1-q_i)^2\left(1-(2(1-q_i))^{k-2}\right)}{2q_i - 1}, \quad (k \geqslant 3), \tag{4.5}$$

and

$$\lambda_k(\vec{x}) = \max\left(1; \sum_{i=1}^{n} c_{i,k} p_i(\vec{x})\right), \Lambda_{i,k} = \sum_{\vec{x}} \frac{p_i(\vec{x})}{\sqrt{\lambda_k(\vec{x})}}, L_{i,k} = \sum_{\vec{x}} \frac{p_i(\vec{x})}{\lambda_k(\vec{x})}. \tag{4.6}$$

The following notation is used for probabilities concentrated on coordinate axes and related quantities. Let $k, m \in \mathbb{N}$

$$\begin{aligned} &p^0_{i,1}(j) = p_i(j,0,\ldots,0), \quad p^0_{i,d}(j) = p_i(0,\ldots,0,j), \\ &p^0_{i,m}(j) = p_i(0,\ldots,0,j,0,\ldots,0), \end{aligned} \tag{4.7}$$

$$\lambda^0_{m,k} = \max\left(1; \sum_{i=1}^{n} c_{i,k} \sum_{j \in \mathbb{Z}\setminus\{0\}} p^0_{i,m}(j)\right), \quad \omega_{d,k} = \prod_{m=1}^{d} \lambda^0_{m,k}, \tag{4.8}$$

$$\widehat{\Psi}_k = \exp\left\{\frac{1}{2}\sum_{i=1}^{n} c_{i,k}(Re\,\widehat{F}_i - 1)\right\}, \tag{4.9}$$

and

$$\begin{aligned} b_1 &= (\mathrm{e}^2+1)/4, \quad b_2 = 8.5, \quad \text{if } k \geqslant 3, \text{ then} \\ b_k &= \frac{(-1)^k k!}{\left(8/3 + 2^{k-1}\right)^{k+1}}\left(\mathrm{e}^{8/3+2^{k-1}}\right. \\ &\quad \left. \times \sum_{m=0}^{k} \frac{(-1)^m \left(8/3+2^{s-1}\right)^m}{m!} - 1\right). \end{aligned} \tag{4.10}$$

Remark 4.2.1. *If $k \geqslant 3$ and $c_{i,k} > 0$, then $q_i > 1/2$ and $c_{i,k} \leqslant q_i$. Observe also, that if $q_i \geqslant C_0 > 2/3$, then $c_{i,k} \geqslant (3C_0-2)/(2C_0-1)$. Note also that $\widehat{\Psi}_1 = \widehat{\Psi}_2$, $\lambda_1(\vec{x}) = \lambda_2(\vec{x})$ and $\omega_{d,1} = \omega_{d,2}$.*

Indeed, if $c_{i,k} > 0$, then $q_i \geqslant 1/2$ ($k \geqslant 3$). Let $q_i \leqslant 1/2$, then

$$\begin{aligned} 0 < c_{i,k} &= 1 - (1-q_i)(1 + (2(1-q_i)) + \ldots + (2(1-q_i))^{k-2}) \\ &\leqslant 1 - (1-q_i)(1+2(1-q_i)) = 1 - (1-q_i)(3-2q_i) \\ &\leqslant 1 - (1-1/2)(3-2/2) = 0. \end{aligned}$$

and we get contradiction. If $q_i > 1/2$, then $c_{i,k}$ is equal to the difference of q_i and some non-negative quantity. Therefore, $c_{i,k} \leqslant q_i$.

Remark 4.2.2. *Observe that, if $0 < q_i < 1$, then we can write F_i in the form similar to (0.1):*

$$F_i = \sum_{\vec{x}\in\mathbb{Z}_+^d} p_i(\vec{x}) I_{\vec{x}} = q_i I + (1-q_i)V_i, \qquad V_i := \sum_{\vec{x}\neq\vec{0}} \frac{p_i(\vec{x})}{1-q_i} I_{\vec{x}} \in \mathscr{F}_d^Z.$$

4.3 Results

4.3.1 Estimates in local metric

The following result generalizes (4.3) for $k > 1$ and holds under much weaker assumptions.

Theorem 4.3.1. *Let $k \in \mathbb{N}$ and let $F_i \in \mathscr{F}_d^Z$, $0 < q_i < 1$ and $c_{i,k} > 0$; $i = 1, \ldots, n$. Then*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_\infty \leqslant 6^d b_k \left(\frac{2(k+1)}{\mathrm{e}}\right)^{(k+1)/2} \omega_{d,k}^{-1/2} \sum_{i=1}^n \Lambda_{i,k}^{k+1}. \qquad (4.11)$$

It is obvious, that estimate (4.11) is meaningful only if all $\lambda_k(\vec{x}) > 0$ and $\lambda_{m,k}^0 > 0$. In the case, when $\lambda_k(\vec{x}) = 0$ or $\lambda_{m,k}^0 = 0$ it is assumed that the RHS of (4.11) is equal to ∞, that is, we have trivial estimate.

Corollary 4.3.1. *Let $0 < q_i < 1$. Then*

$$\begin{aligned}
\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1}\|_\infty &\leqslant \frac{6^d(\mathrm{e}^2+1)}{\mathrm{e}} \prod_{m=1}^d \left(\sum_{l=1}^n q_l \sum_{j\neq 0} p_{l,m}^0(j)\right)^{-1/2} \\
&\quad \times \sum_{i=1}^n \left(\sum_{\vec{x}\neq\vec{0}} p_i(\vec{x}) \left(\sum_{j=1}^n q_j p_j(\vec{x})\right)^{-1/2}\right)^2; \\
\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_\infty &\leqslant 6^d \cdot 27.9 \prod_{m=1}^d \left(\sum_{l=1}^n q_l \sum_{j\neq 0} p_{l,m}^0(j)\right)^{-1/2} \\
&\quad \times \sum_{i=1}^n \left(\sum_{\vec{x}\neq\vec{0}} p_i(\vec{x}) \left(\sum_{j=1}^n q_j p_j(\vec{x})\right)^{-1/2}\right)^3.
\end{aligned}$$

Observe, that there is no assumption of non-negativeness of vectors. Thus, Theorem 4.3.1 contains more general result than is needed for modeling of aggregate claims distribution.

Example 4.3.1. *Let $\vec{X}_i$, $(i = 1,\dots,n)$ be i.i.d. two-dimensional vectors concentrated at four points; $\mathscr{L}(\vec{X}) = F \in \mathscr{F}_2$,*

$$F\{(1,0)\} = p(1,0), \quad F\{(0,1)\} = p(0,1),$$
$$F\{(1,1)\} = p(1,1), \quad F\{(0,0)\} = q \in (0,1).$$

Then, observing that

$$1-q \leqslant \left(\sqrt{p(1,0)} + \sqrt{p(0,1)} + \sqrt{p(1,1)}\right)^2 \leqslant 9(1-q)$$

we get

$$\omega_{2,1}^{-1/2} = \omega_{2,2}^{-1/2} = \frac{1}{nq\sqrt{p(1,0)p(0,1)}}, \quad \Lambda_{i,1} = \Lambda_{i,2} = \sqrt{\frac{(1-q)}{nq}},$$
$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1}\|_\infty \leqslant \frac{C}{n} \cdot \frac{1-q}{q^2\sqrt{p(1,0)p(0,1)}},$$
$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_\infty \leqslant \frac{C}{n\sqrt{n}} \cdot \frac{(1-q)^{3/2}}{q^{5/2}\sqrt{p(1,0)p(0,1)}}.$$

Indeed

$$\begin{aligned}
&\left(\sqrt{p(1,0)} + \sqrt{p(0,1)} + \sqrt{p(1,1)}\right)^2 \\
&\quad\leqslant \left(3\max(\sqrt{p(1,0)}, \sqrt{p(0,1)}, \sqrt{p(1,1)})\right)^2 \\
&\quad= 9\,(\max(p(1,0); p(0,1); p(1,1)) \leqslant 9(p(1,0) + p(0,1) + p(1,1)) \\
&\quad= 9(1-q);
\end{aligned}$$

$$\begin{aligned}
&\left(\sqrt{p(1,0)} + \sqrt{p(0,1)} + \sqrt{p(1,1)}\right)^2 \geqslant \max\left(p(1,0); p(0,1); p(1,1)\right) \\
&\quad= \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)\max(p(1,0); p(0,1); p(1,1)) \\
&\quad\geqslant \frac{p(1,0)}{3} + \frac{p(0,1)}{3} + \frac{p(1,1)}{3} = \frac{1-q}{3}.
\end{aligned}$$

If $p(1,0) = p(0,1) = p(1,1) = p$, *then*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1}\|_\infty \leqslant \frac{C}{nq^2}, \qquad \|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_\infty \leqslant \frac{C\sqrt{p}}{n\sqrt{n}\,q^{5/2}}.$$

Theorem 4.3.1 provides good estimate, when r.v.s. take values on coordinate axes with large probabilities. The following example shows that sometimes better result than (4.11) is possible.

Example 4.3.2. *Let* $\vec{X}_i$, $(i = 1,\dots,n)$, $n \geqslant 4$, *be the same as in Example 4.3.1 and let* $p(1,0) = 0.2$, $p(1,1) = 0.1$, $p(0,1) = n^{-1}$. *From example 4.3.1 we see that* $\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_\infty \leqslant Cn^{-1}$. *However, it is possible 'to move' part of* $p(1,1)$ *to one axis, while subtracting some of probabilistic mass from* $p(1,0)$, *obtaining estimate*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_\infty \leqslant Cn^{-3/2}. \tag{4.12}$$

Remark 4.3.1. *In example 4.4.1 the advantage was taken over magnitude of probabilities. There is still an open problem of more precise general estimate, when all distribution is almost degenerate (say* $p(1,0) = p(0,1) = n^{-1}$, $p(1,1) = 0.5$*).*

Remark 4.3.2. *For the proof of Theorem 4.3.1 we multiple times use one-dimensional Le Cam's inequality (which is a version of the Kolmogorov-Rogozin inequality, see Theorem 2.1 in [4] or (E.2) in [30]) for concentration function omitting all probabilities that are not concentrated on coordinate axes, see Lemma A.1.4 below. For recent research on one-dimensional Kolmogorov-Rogozin inequality see [61]. There is a multidimensional analogue of Kolmogorov-Rogozin inequality, see [33], Proposition 1. However at best it ensures additional factor of the order* $O(n^{-1/2})$. *Meanwhile, as can be seen from the definition of* $\omega_{d,k}$, *for local estimates we expect a factor depending on the dimension of space, roughly of the order* $O(n^{-d/2})$.

For better illustration of the accuracy of approximation, let us consider the case, when random vectors are identically distributed and concentrated on coordinate axes. More precisely, let

$$\widehat{F}_{(0)}(\vec{t}) = q + p_1\widehat{F}_1^{(0)}(t_1) + p_2\widehat{F}_2^{(0)}(t_2) + \ldots + p_d\widehat{F}_d^{(0)}(t_d),$$

where $\widehat{F}_k^{(0)}(t_k) = \sum_{j=-\infty, j\neq 0}^{\infty} F_k^{(0)}\{j\}\mathrm{e}^{\mathrm{i}jt_k}$. Then

$$\mathbf{H}_{\mathrm{n,k}}^{(0)} = H_k^{*n} = \exp\left\{ n \sum_{j=1}^{k} \frac{(-1)^{j+1}(F_{(0)} - I)^{*j}}{j} \right\}.$$

Then the following corollary of Theorem 4.3.1 follows.

Corollary 4.3.2. *Let $q \geqslant C_0 > 2/3$, then*

$$\|F_{(0)}^{*n} - H_k^{*n}\|_\infty \leqslant C(d,k)(1-q)^{(k+1)/2} n^{-(k+d-1)/2}$$

As it follows from one-dimensional estimate (A.23), in this case

$$\|F_{(0)}^{*n} - H_k^{*n}\|_{TV} \leqslant C(d,k)(1-q)^{(k+1)/2} n^{-(k-1)/2},$$

that is, local estimate is smaller by factor $n^{-d/2}$, where d is the dimention of space.

Approximation $\mathbf{H}_{\mathrm{n,k}}$ can be viewed as asymptotic in the exponent. Next we consider first order asymptotic expansions to CP approximation $\mathbf{H}_{\mathrm{n},1}$ and SCP approximation $\mathbf{H}_{\mathrm{n},2}$, which have more standard forms. Let

$$\mathscr{A}_1 = -\frac{1}{2}\sum_{i=1}^{n}(F_i - I)^{*2}, \qquad \mathscr{A}_2 = \frac{1}{3}\sum_{i=1}^{n}(F_i - I)^{*3}.$$

Theorem 4.3.2. *Let $0 < q_i < 1$, $(i = 1, \ldots, n)$. Then*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1} * (I + \mathscr{A}_1)\|_\infty \leqslant 6^d C \omega_{d,2}^{-1/2} \left(\sum_{i=1}^{n} \Lambda_{i,2}^3 + \left(\sum_{i=1}^{n} \Lambda_{i,2}^2 \right)^2 \right). \qquad (4.13)$$

If $1/2 < q_i < 1$, $(i = 1, 2, \ldots, n)$, then

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2} * (I + \mathscr{A}_2)\|_\infty \leqslant 6^d C \omega_{d,3}^{-1/2} \left(\sum_{i=1}^{n} \Lambda_{i,3}^4 + \left(\sum_{i=1}^{n} \Lambda_{i,3}^3 \right)^2 \right).$$

Estimate (4.13) extends corresponding asymptotical result from [73] for r.v.s., concentrated on $d+1$ point to the general case of lattice r.v.s. However, it must be noted that results in [73] contain explicit constants, not only explicit dependence on the dimension of space.

Example 4.3.3. *Let $\vec{X}_i$, $(i=1,\dots,n)$ be the same as in Example 4.3.1. Then*

$$\begin{aligned}\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_\infty &\leqslant \frac{C}{n}\cdot\frac{(1-q)^{3/2}}{q^3\sqrt{p(1,0)p(0,1)}} \qquad (\text{if } 0<1<q),\\ \|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}*(I+\mathscr{A}_2)\|_\infty &\leqslant \frac{C}{n^2}\cdot\frac{(1-q)^2}{(2q-1)^4\sqrt{p(1,0)p(0,1)}} \quad (\text{if } q>1/2).\end{aligned}$$

If $p(1,0)=p(0,1)=p(1,1)=p<1/6$, then

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_\infty \leqslant \frac{C\sqrt{p}}{n\sqrt{n}}, \qquad \|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}*(I+\mathscr{A}_2)\|_\infty \leqslant \frac{Cp}{n^2(1-6p)^4}.$$

Comparing estimates in Example 4.3.1 and Example 4.3.3, we see that approximation by $\mathbf{H}_{\mathrm{n},2}$ is by factor $n^{-1/2}$ better than asymptotics to $\mathbf{H}_{\mathrm{n},1}$. However, in practice, asymptotic expansion in the exponent is much more difficult to calculate.

4.3.2 Non-uniform local estimates

We are unaware about any non-uniform local result for sums of random vectors (r.v.s.).

For marginal moments we use the following notation:

$$\begin{aligned}&\mu_{i,j}=\sum_{\vec{x}}x_j p_i(\vec{x}), \quad \mu_{i,jm}=\sum_{\vec{x}}x_j x_m p_i(\vec{x}), \quad \mu_j=\sum_{i=1}^n\mu_{i,j},\\ &\beta_{i,j}=\sum_{\vec{x}}|x_j|p_i(\vec{x}), \quad \beta_{i,jm}=\sum_{\vec{x}}|x_j||x_m|p_i(\vec{x}),\\ &\beta^{(2)}_{i,j}=\sum_{\vec{x}}x_j^2 p_i(\vec{x}); \quad \beta^{(4)}_{i,j}=\sum_{\vec{x}}x_j^4 p_i(\vec{x}).\end{aligned}$$

To include the case $x_j=\mu_j$, we write $|x_j-\mu_j|$ on the LHS of the estimate rather than writing $|x_j-\mu_j|^{-1}$ of the RHS.

Theorem 4.3.3. *Let $k\in\mathbb{N}$, $F_i\in\mathscr{F}_d^Z$, $0<q_i<1$ and $c_{i,k}>0$ for $i=1,2,\dots,n$. Let, for $j,m\in\{1,2,\dots,d\}$, $\beta^{(2)}_{i,j}<\infty$. Then for any $\vec{x}\in\mathbb{Z}^d$,*

$$\begin{aligned}\big|x_j-\mu_j\big|\,|\mathbf{F}\{\vec{x}\}-\mathbf{H}_{\mathrm{n},\mathrm{k}}\{\vec{x}\}| &\leqslant C(d,k)\omega_{d,k}^{-1/2}\sum_{i=1}^n\beta_{i,j}\Lambda_{i,k}^k\\ &+C(d,k)\omega_{d,k}^{-1/2}\sum_{i=1}^n\Lambda_{i,k}^{k+1}\sum_{l=1}^n\sqrt{\beta^{(2)}_{l,j}L_{l,k}},\end{aligned}$$

and

$$(x_j-\mu_j)^2\,|\mathbf{F}\{\vec{x}\}-\mathbf{H}_{\mathrm{n,k}}\{\vec{x}\}| \leqslant C(d,k)\omega_{d,k}^{-1/2}\Bigg\{\sum_{i=1}^{n}\Lambda_{i,k}^{k+1}\sum_{m=1}^{n}\beta_{m,j}^{(2)} \times\left(1+\sum_{l=1}^{n}L_{l,k}\right)+\sum_{i=1}^{n}\left(\beta_{i,j}^{(2)}\Lambda_{i,k}^{k}+\beta_{i,j}^{2}\Lambda_{i,k}^{k-1}\right) +\sum_{i=1}^{n}\beta_{i,j}\Lambda_{i,k}^{k}\sum_{m=1}^{n}\sqrt{\beta_{m,j}^{(2)}L_{m,k}}\Bigg\}.$$

If $\beta_{i,j}^{(2)},\beta_{i,m}^{(2)}<\infty,\beta_{i,jm}<\infty$, $(i=1,\ldots,n)$, *then*

$$\begin{aligned}
&|x_j-\mu_j||x_m-\mu_m|\,|\mathbf{F}\{\vec{x}\}-\mathbf{H}_{\mathrm{n,k}}\{\vec{x}\}| \\
&\leqslant\ C(d,k)\omega_{d,k}^{-1/2}\Bigg\{\sum_{i=1}^{n}\Lambda_{i,k}^{k+1}\sum_{l=1}^{n}\left(\beta_{l,jm}+\beta_{l,j}\beta_{l,m}\right) \\
&\quad+\sum_{l_1=1}^{n}\sqrt{\beta_{l_1,j}^{(2)}L_{l_1,k}}\left(\sum_{i=1}^{n}\beta_{i,m}\Lambda_{i,k}^{k}+\sum_{i=1}^{n}\Lambda_{i,k}^{k+1}\sum_{l_2=1}^{n}\sqrt{\beta_{l_2,m}^{(2)}L_{l_2,k}}\right) \\
&\quad+\sum_{i=1}^{n}\left(\beta_{i,j}\beta_{i,m}\Lambda_{i,k}^{k-1}+\beta_{i,jm}\Lambda_{i,k}^{k}\right)+\sum_{i=1}^{n}\beta_{i,j}\Lambda_{i,k}^{k}\sum_{l=1}^{n}\sqrt{\beta_{l,m}^{(2)}L_{l,k}}\Bigg\}.
\end{aligned}$$

Example 4.3.4. *Let* $\vec{X}_i$, $(i=1,\ldots,n)$ *be i.i.d. two-dimensional vectors concentrated at four points;* $\mathscr{L}(\vec{X})=F\in\mathscr{F}_2$,

$$\begin{aligned}
&F\{(1,0)\}=p(1,0), \quad F\{(0,1)\}=p(0,1),\\
&F\{(1,1)\}=p(1,1), \quad F\{(0,0)\}=q\in(0,1).
\end{aligned}$$

Then $\mu_1=n(p(1,0)+p(1,1))$, $\mu_2=n(p(0,1)+p(1,1))$ *and*

$$\begin{aligned}
|j-\mu_1|\,|\mathbf{F}\{(j,m)\}-\mathbf{H}_{\mathrm{n},2}\{(j,m)\}| &\leqslant \frac{C(1-q)}{nq^3}\sqrt{\frac{p(1,0)+p(0,1)}{p(1,0)p(0,1)}},\\
(j-\mu_1)^2\,|\mathbf{F}\{(j,m)\}-\mathbf{H}_{\mathrm{n},2}\{(j,m)\}| &\leqslant \frac{C(1-q)}{\sqrt{n}\,q^{7/2}}\sqrt{\frac{p(1,0)+p(1,1)}{p(1,0)p(0,1)}},\\
|j-\mu_1||m-\mu_2|\,|\mathbf{F}\{(j,m)\}-\mathbf{H}_{\mathrm{n},2}\{(j,m)\}| &\leqslant \frac{C\sqrt{1-q}}{\sqrt{n}\,q^3}\\
&\quad\times\sqrt{\frac{p(1,1)+p(1,0)p(0,1)}{p(1,0)p(0,1)}}.
\end{aligned}$$

Example 4.3.5. *Let* $d=1$, *and let* $\mathbf{F}=Bi(n,p_B)$ *be Binomial distribution. If* $p_B\leqslant C<1/3$, $np_B\geqslant 1$. *Then combining Theorem 1 from [48] and Theorem*

4.3.3 we get

$$\begin{aligned}\left(1+\frac{|m-np_B|}{\sqrt{np_B}}\right)|Bi(n,p_B)\{m\}-\mathbf{H}_{\mathrm{n,k}}\{m\}| &\leqslant C(k)\left(\frac{p_B}{n}\right)^{k/2},\\ \left(1+\frac{(m-np_B)^2}{np_B}\right)|Bi(n,p_B)\{m\}-\mathbf{H}_{\mathrm{n,k}}\{m\}| &\leqslant C(k)\left(\frac{p_B}{n}\right)^{k/2}.\end{aligned}$$

For $k=1,2$ these estimates coincide with corresponding estimates from [20]. For $k\geqslant 3$ they are new even in one-dimensional case.

4.3.3 Estimates in ℓ_r metric

Closely related to local estimates are estimates in ℓ_r, $r\geqslant 2$ metric. The case $r=1$ corresponds to estimates in total variation. Notably, for the total variation, if the dimension of space d grows, it might have an impact on constants, but apart from that, dimension has no impact on the order of accuracy, see [74]. We demonstrate, that this is not the case for $r\geqslant 2$ and the estimates are very similar to the estimates in point metric.

Theorem 4.3.4. *Let $k\in\mathbb{N}$, $r\geqslant 2$ and let $F_i\in\mathscr{F}_d^Z$, $0<q_i<1$ and $c_{i,k}>0$; $i=1,\dots,n$. Then*

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n,k}}\|_r\leqslant 6^{d(r-1)/r}b_k\left(\frac{2(k+1)}{\mathrm{e}}\right)^{(k+1)/2}\omega_{d,k}^{-(r-1)/2r}\sum_{i=1}^n\Lambda_{i,k}^{k+1}. \tag{4.14}$$

Corollary 4.3.3. *Let $q\geqslant C_0>2/3$, $r\geqslant 2$ and assumptions of Corollary 4.3.2 hold. Then*

$$\|F_{(0)}^{*n}-H_k^{*n}\|_r\leqslant C(d,k)(1-q)^{(k+1)/2}n^{-(k+d-1-d/r)/2}.$$

Similar to local estimates we can consider short asymptotic expansions.

Theorem 4.3.5. *Let $0<q_i<1$, $(i=1,\dots,n)$, $r\geqslant 2$. Then*

$$\begin{aligned}\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_r &\leqslant 6^{d(r-1)/r}C\omega_{d,2}^{-(r-1)/2r}\\ &\quad\times\left(\sum_{i=1}^n\Lambda_{i,2}^3+\left(\sum_{i=1}^n\Lambda_{i,2}^2\right)^2\right).\end{aligned} \tag{4.15}$$

If $1/2 < q_i < 1$, $(i = 1,2,\dots,n)$, *then*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2} * (I + \mathscr{A}_2)\|_r \leqslant 6^{d(r-1)/r} C \omega_{d,3}^{-(r-1)/2r} \times \left(\sum_{i=1}^{n} \Lambda_{i,3}^4 + \left(\sum_{i=1}^{n} \Lambda_{i,3}^3 \right)^2 \right). \tag{4.16}$$

Estimates (4.14), (4.15), (4.16) contain factor $\omega_{d,k}^{-(r-1)/2r}$, $k = 2,3$. It was already mentioned in chapter Preliminaries that local norm can be viewed as limit for ℓ_r, $r \geqslant 2$, metric, as $r \to \infty$. Observe that, $\omega_{d,k}^{-(r-1)/2r} \to \omega_{d,k}^{-1/2}$, as $r \to \infty$. Therefore, estimate (4.11) can be viewed as limit case of (4.14).

4.3.4 Lower estimates

What can be said about the accuracy of upper estimates from [48]? We will show that, at least in some cases, the accuracy of approximation is of correct order. To this end we prove some lower estimates. It must be noted that lower estimates also include lower estimates for the total variation norm (ℓ_1 norm).

Let $k \geqslant 2$, $i, j = 1,2,\dots,n$ and for $\vec{h} = (h_1, h_2, \dots, h_d)$

$$B_i = \sum_{m=1}^{d} \frac{\beta_{i,m}}{h_m}, \tag{4.17}$$

$$\begin{aligned} \varepsilon_{d,k}(\vec{h}) &= C_1(d,k) \sum_{i=1}^{n} B_i^{k+1} \\ &\quad - C_2(d,k) \left(\sum_{i=1}^{n} B_i^{k+1} \left(\sum_{j=1}^{n} \sum_{l=1}^{d} \frac{\beta_{j,l}^{(2)}}{h_l^2} + \sum_{j=1}^{n} B_j^2 \right) \right. \\ &\quad \left. + \sum_{i=1}^{n} B_i^k \sum_{l=1}^{d} \frac{\beta_{i,l}^{(2)}}{h_l^2} + \left(\sum_{i=1}^{n} B_i^{k+1} \right)^2 + \sum_{i=1}^{n} B_i^{k+2} \right). \end{aligned} \tag{4.18}$$

The following theorem is abstract in a sense that it allows for arbitrary choice of $\vec{h}$ and does not exclude trivial case, when RHS estimate is negative.

Theorem 4.3.6. *Let* $\vec{X}_i$ *be concentrated on* $\mathbb{Z}_+^d$ *and let* $\beta_{i,m}^{(2)} < \infty$ *for* $i = 1,2,\dots,n$; $m = 1,2,\dots,d$. *If* $c_{i,k+1} > 0$, $(i = 1,2,\dots,n)$ *for some* $s \in \{2,3,4,\dots\}$ *then there exist constants* $C_1(d,k), C_2(d,k)$ *such that, for any* $h_1, h_2, \dots, h_d \geqslant 1$, *the following estimates hold*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_\infty \geqslant \varepsilon_{d,k}(\vec{h}) \prod_{m=1}^{d} h_m^{-1},$$

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n,k}}\|_r \;\geqslant\; 5^{-d(r-1)/r}\varepsilon_{d,k}(\vec{h})\prod_{m=1}^{d}h_m^{-(r-1)/r},\quad r\geqslant 1.$$

In the case of CP approximation ($k=1$), non-negativeness of random vectors can be omitted.

Theorem 4.3.7. *Let* $0<q_i<1$, $\beta_{i,m}^{(2)}<\infty$, $(i=1,\dots,n)$, $(m=1,2,\dots,d)$. *Then, there exist constants* $C_1(d)$, $C_2(d)$, *such that for any* $h_1,h_2,\dots,h_d\geqslant 1$

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}\|_\infty \;\geqslant\; \bar{\varepsilon}_{d,1}(\vec{h})\prod_{m=1}^{d}h_m^{-1},$$

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}\|_r \;\geqslant\; 5^{-d(r-1)/r}\bar{\varepsilon}_{d,1}(\vec{h})\prod_{m=1}^{d}h_m^{-(r-1)/r},\quad r\geqslant 1.$$

Here

$$\begin{aligned}\bar{\varepsilon}_{d,1}(\vec{h}) \;=\; & C_1(d)\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\frac{\mu_{i,m}}{h_m}\right)^2 - C_2(d)\left(\sum_{i=1}^{n}B_i^2\sum_{j=1}^{n}\sum_{l=1}^{d}\frac{\beta_{j,l}^{(2)}}{h_l^2}\right.\\ & \left.+\sum_{i=1}^{n}B_i\sum_{l=1}^{d}\frac{\beta_{i,l}^{(2)}}{h_l^2}+\left(\sum_{i=1}^{n}B_i^2\right)^2+\sum_{i=1}^{n}B_i^3\right).\end{aligned}$$

Typically $h_m=a\sqrt{\lambda_{m,k}^0}$ or $h_m=a\left(\sum_{i=1}^{n}\operatorname{var}X_{im}\right)^{1/2}$, where constant $a\geqslant 1$ is chosen to ensure non-negativeness of the RHS of estimates. We illustrate application of Theorem 4.3.6 to the case when all random vectors are bounded and concentrated on coordinate axes (that is all $p_i(\vec{x})=0$ except $p_{i,m}^0(j)$, $i=1,\dots,n$, $m=1,2,\dots,d$; $j\in\mathbb{Z}^+$).

Theorem 4.3.8. *Let* $\vec{X}_i$ *be concentrated on nonnegative coordinate axes and let* $\max_{i,j}X_{ij}\leqslant K$. *Let* $q_i\geqslant C_0>2/3$, $(i=1,2,\dots,n)$. *Then, for* $s\in\mathbb{N}$ *there exist constants* $C(d,k,K)$ *and* $C(d,k,K,r)$ *such that*

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n,k}}\|_\infty \;\geqslant\; C(d,k,K)\,\omega_{d,k}^{-1/2}\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\frac{\mu_{i,m}}{\sqrt{\lambda_{m,k}^0}}\right)^{k+1},$$

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n,k}}\|_r \;\geqslant\; C(d,k,K,r)\,\omega_{d,k}^{-(r-1)/2r}\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\frac{\mu_{i,m}}{\sqrt{\lambda_{m,k}^0}}\right)^{k+1},\ r\geqslant 1.$$

If $k=1$, *then assumption* $q_i\geqslant C_0>2/3$ *can be replaced by* $q_i\geqslant C_0>0$.

Example 4.3.6. *Let $\bar{e}_m = (0,\dots,1,\dots,0)$ denote m-th coordinate vector in $\mathbb{R}^d$, $m = 1,2,\dots,d$. Let $\mathbb{P}(\vec{X}_i = \bar{e}_m) = p^0_{i,m}$, $p^0_{i,1} + p^0_{i,2} + \dots + p^0_{i,d} = 1 - q_i$, $\bar{\lambda}(m) = \sum_{j=1}^n p^0_{j,m}$. If $k \geqslant 2$, $q_i \geqslant 2/3$, and $\bar{\lambda}(m) \geqslant 1$, $(i = 1,\dots,n)$, then*

$$\begin{aligned}
\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_\infty &\geqslant C(d,s)\omega_{d,k}^{-1/2} \sum_{i=1}^n \left(\sum_{m=1}^d \frac{p^0_{i,m}}{\sqrt{\bar{\lambda}(m)}} \right)^{k+1}, \\
\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_r &\geqslant C(d,k,r)\omega_{d,k}^{-(r-1)/2r} \sum_{i=1}^n \left(\sum_{m=1}^d \frac{p^0_{i,m}}{\sqrt{\bar{\lambda}(m)}} \right)^{k+1}, \ r \geqslant 1.
\end{aligned}$$

Observe that

$$\begin{aligned}
\left(\sum_{m=1}^d \frac{p^0_{i,m}}{\sqrt{\bar{\lambda}(m)}} \right)^{k+1} &= \left(\left(\sum_{m=1}^d \frac{p^0_{i,m}}{\sqrt{\bar{\lambda}(m)}} \right) \right)^{(k+1)/2} \\
&\geqslant \left(\sum_{m=1}^d \frac{(p^0_{i,m})^2}{\bar{\lambda}(m)} \right)^{(k+1)/2}.
\end{aligned}$$

Therefore, combining this estimate with Theorem 3 from [77], we get the following result. If $k \geqslant 2$, $q_i \geqslant 2/3$, and $\bar{\lambda}(m) \geqslant 1$, $(i = 1,\dots,n)$, then

$$\begin{aligned}
C_1(d,k) \left(\sum_{m=1}^d \frac{(p^0_{i,m})^2}{\bar{\lambda}(m)} \right)^{(k+1)/2} &\leqslant \|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_{TV} \\
&\leqslant C_2(k) \left(\sum_{m=1}^d \frac{(p^0_{i,m})^2}{\bar{\lambda}(m)} \right)^{(k+1)/2}.
\end{aligned}$$

Next we consider an example, when significant part of probabilistic mass is concentrated not on coordinate axes.

Example 4.3.7. *Let $\vec{X}_i$, $(i = 1,\dots,n)$ be the same as in Example 4.3.1 and let $p(1,0) = p(0,1) = p(1,1) = p$, $np \geqslant 1$. Let $k = 2$. Then by choosing $h_1 = h_2 = a\sqrt{np}$ for some absolute constants $C, \tilde{C}$, we get*

$$\varepsilon_{2,2}(\vec{h}) \geqslant \frac{C}{a^3} \frac{\sqrt{p}}{n\sqrt{n}} (1 - \tilde{C}a^{-1}).$$

By choosing a to be sufficiently large absolute constant, we see that

$$\mu_{i,1} = \beta^{(2)}_{i,1} = p(1,0) + p(1,1) = 2p,$$

$$\mu_{i,2} = \beta_{i,2}^{(2)} = p(0,1) + p(1,1) = 2p;$$
$$\beta_{i,1}^{(2)} + \beta_{i,2}^{(2)} \leqslant Cp, \quad B_i = \frac{4\sqrt{p}}{a\sqrt{n}}.$$

Thus, we derive the estimate

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_{\infty} \geqslant \frac{C\sqrt{p}}{n\sqrt{n}},$$

which is of the same order as upper bound given in example 4.3.1.

4.3.5 Estimates in the scheme of sequences.

For the sequences of random vectors (i.e. when all probabilities do not depend on the number of vectors n) we established correct order of Hipp's approximation. More precisely the following result holds.

Theorem 4.3.9. *Let $\vec{X}, \vec{X}_1, \vec{X}_2, \ldots$ be a sequence of independent identically distributed lattice random vectors, satisfying assumption of Theorem 4.3.8. Let $\lambda_{m,k}^0 > 0$, $m = 1, \ldots, d$ and $r \geqslant 2$. Then there exist constants $C_i(d,k,X)$, $(i = 1,2,3,4)$ depending on d,k and distribution of $\vec{X}$, such that*

$$\begin{aligned} C_1(d,k,X)\, n^{-(d+k-1)/2} &\leqslant \|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_{\infty} \leqslant C_2(d,k,X)\, n^{-(d+k-1)/2}, \\ C_3(d,k,X)\, n^{-(d+k-1)/2 - d/2r} &\leqslant \|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_{r} \\ &\leqslant C_4(d,k,X)\, n^{-(d+k-1)/2 - d/2r}. \end{aligned}$$

The statement of Theorem 6.2.8 follows directly from Theorems 1 and 3 from [48] and 4.3.8. As can be seen from examples 4.3.6 and 4.3.7 similar results hold for the scheme of triangular arrays (scheme of series), when sufficient probabilistic mass is concentrated on coordinate axes.

4.3.6 Remarks about the smallness of zero probability

The natural problem dealing with Hipp-type approximations is assumptions needed for their applicability. Do we necessarily need to impose some restrictions on r.v.s.? On one hand, it is easy to check, that $\mathbf{H}_{\mathrm{n,k}}$ always has finite variation: $\|\mathbf{H}_{\mathrm{n,k}}\|_{TV} \leqslant \exp\{nk2^k\}$. On the other hand, the next example shows that, at least for $k \geqslant 3$, r.v.s. should take value $\vec{0}$ with not very small probability.

Example 4.3.8. *Let us consider one dimensional binomial distribution, denoted by $Bi(n, p_B)$, $\widehat{Bi}(n, p_B|t) = (1 + p_B(\mathrm{e}^{\mathrm{i}t} - 1))^n$ and SCP approximation*

$D^{*n} = \mathbf{H}_{n,3}$, *which in this case has Fourier transform*

$$\widehat{D}^n(t) = \exp\left\{np(e^{it}-1) - \frac{np^2(e^{it}-1)^2}{2} + \frac{np^3(e^{it}-1)^3}{3}\right\}.$$

If $p = 12/13$, *then*

$$\begin{aligned}\|Bi(n,p_B) - D^{*n}\|_{TV} &\geqslant \sup_t |\widehat{Bi}(n,p_B|t) - \widehat{D}^n(t)| \\ &\geqslant \left|\widehat{Bi}\left(n,p_B\middle|\frac{\pi}{6}\right) - \widehat{D}^n\left(\frac{\pi}{6}\right)\right| \geqslant \left|\widehat{D}^n\left(\frac{\pi}{6}\right)\right| - 1 \geqslant e^{0.0009n} - 1. \quad (4.19)\end{aligned}$$

Therefore, in this case, $\|Bi(n,p_B) - D^{*n}\|_{TV} \to \infty$, *as* $n \to \infty$, *and, obviously,* D^{*n} *is useless as approximation.*

In actuarial models such situation (very large probability of occurrence of insurance event) is practically impossible. However from mathematical point of view, an interesting problem is the possibility to replace Hipp's approximation with similar structured SCP measure, guaranteeing good accuracy of approximation even if r.v.s. take value zero with small probabilities. We give one possible solution of this problem, when $k = 3$ and initial distribution is concentrated on coordinate axes. More precisely, let $F \in \mathscr{F}_d^Z$ and let

$$\widehat{F}(\vec{t}) = q + \sum_{m=1}^{d} p_m \widehat{F}_m(t_m) = 1 + \sum_{m=1}^{d} p_m(\widehat{F}_m(t_m) - 1).$$

We also assume that $F_m\{\vec{0}\} = 0$. Set $u_m := \widehat{F}_m(t_m) - 1$, $\bar{u}_m := \widehat{F}_m(-t_m) - 1$. We replace

$$\begin{aligned}(\widehat{F}-1)^3 &= \left(\sum_{m=1}^{d} p_m u_m\right)^3 \\ &= \sum_{m=1}^{d} p_m^2 u_m^3 + 3\sum_{m\neq j}^{d} p_m^2 p_j u_m^2 u_j + 6\sum_{m<j<l}^{d} p_m p_j p_l u_m u_j u_l\end{aligned}$$

with

$$\begin{aligned}\widehat{F}_{3\star} - 1 &= \sum_{m=1}^{d} p_m^3 \bar{u}_m |u_m|^2 + 3\sum_{m\neq j}^{d} p_m^2 p_j \bar{u}_j |u_m|^2 \\ &\quad + \frac{3}{2}\sum_{m<j<l}^{d} p_m p_j p_l \left(u_m u_j u_l + \bar{u}_m \bar{u}_j u_l + \bar{u}_m u_j \bar{u}_l + u_m \bar{u}_j \bar{u}_l\right).\end{aligned}$$

Let $H_{3\star}$ be SCP measure with Fourier transform

$$\widehat{H}_{3\star} = \exp\left\{(\widehat{F}-1) - \frac{(\widehat{F}-1)^2}{2} + \frac{(\widehat{F}_{3\star}-1)}{3}\right\}.$$

The following proposition gives an answer to the question about the accuracy of approximation by $H_{3\star}$.

Proposition 4.3.10. *Let $q \in (0,1)$. Then*

$$\|F^{*n} - H_{3\star}^{*n}\|_\infty \leqslant \frac{C(d)(1-q)}{n^{1+d/2}q^{2+d/2}} \prod_{m=1}^{d} \frac{1}{\sqrt{p_m}}. \tag{4.20}$$

Thus we see, that accuracy meaningful, even if q slowly tends to zero. Moreover $H_{3\star}^{*n}$, in many cases, is a significant improvement over $\mathbf{H}_{\mathrm{n},2}$ which, in this case, is equal to $\exp\{n(F-I) - n(F-I)^{*2}/2\}$. Indeed, Theorem 4.3.1 gives the estimate

$$\|F^{*n} - H_2^{*n}\|_\infty \leqslant \frac{C(d)(1-q)^{3/2}}{n^{1/2+d/2}q^{3/2+d/2}} \prod_{m=1}^{d} \frac{1}{\sqrt{p_m}}. \tag{4.21}$$

If all $p_m = O(1)$ (and this case includes example considered in (4.19), then the accuracy of approximation in (4.20) is $O(n^{-1-d/2})$ vs $O(n^{-1/2-d/2})$ in (4.21). Proposition's proof is given in subsection 4.6.

4.4 Discussion and open problems

For triangular arrays of random vectors (when probabilities can depend on the number of vectors n) upper and lower bounds differ when random vectors take values on coordinate axes with small probabilities fast tending to zero. We discuss this problem in more detail.

Example 4.4.1. *Let $\vec{X}_i$, $(i = 1,\dots,n)$, $n \geqslant 4$, be the same as in Example 4.3.1 and let $p(1,0) = 0.2$, $p(1,1) = 0.1$, $p(0,1) = n^{-1}$. Then $\mu_{i,1} = \beta_{i,1}^{(2)} = p(1,0) + p(1,1) = 0.3$, $\mu_{i,2} = \beta_{i,2}^{(2)} = 0.1 + n^{-1}$; $\lambda_{1,2}^0 = 0.2$; $\lambda_{2,2}^0 = n^{-1}$. Let $h_1 = h_2 = a\sqrt{n}$. Then $B_i \leqslant a^{-1}n^{-1/2}$ and $\beta_{j,l}^{(2)}/h_l^2 \leqslant a^{-2}n^{-1}$. Simple calculation than gives*

$$\varepsilon_{2,2}(\vec{h}) \geqslant Ca^{-1}n^{-1/2}(1 - C/a) \geqslant Cn^{-1/2},$$

if to choose a sufficiently large. Then

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_\infty \geqslant Cn^{-3/2}, \quad \|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_r \geqslant C(r)n^{-3/2+1/r},\ r \geqslant 1. \tag{4.22}$$

From example 4.3.1 we see that

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_{\infty} \leqslant Cn^{-1},$$

that is, lower bound and upper bounds have different orders. For this example a sharper estimate in Lemma 4.5.2 allows to prove that

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_{\infty} \leqslant Cn^{-3/2}, \quad \|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_{r} \leqslant C(r)n^{-3/2+1/r},\ r \geqslant 2, \tag{4.23}$$

that is, obtain upper bounds of the same order as lower bounds. The proof of (4.23) is given in subsection 4.6.

In example 4.4.1 the advantage was taken over magnitude of probabilities. There is still an open problem of more precise general estimate for

$$I = \frac{1}{(2\pi)^d} \int\limits_{-\pi}^{\pi} \dots \int\limits_{-\pi}^{\pi} \widehat{\Psi}_k \,\mathrm{d}t_1 \dots \mathrm{d}t_d,$$

where $\widehat{\Psi}_k$ is defined in (4.9). In Lemma 4.5.2 we use one dimensional Le Cam's inequality (A.6) for concentration function (which is a special case of Kolmogorov-Rogozin inequality) omitting all probabilities that are not concentrated on coordinate axes. There is a multidimensional analogue of Kolmogorov-Rogozin inequality, see [33], Proposition 1 (though not Le Cam's inequality). However at best it ensures additional factor of the order $O(n^{-1/2})$. Meanwhile, as can be seen from the definition of $\omega_{d,k}$, for local estimates we expect a factor depending on the dimension of space, roughly of the order $O(n^{-d/2})$. Observe that by formula of inversion (A.40), we have $I = \mathbb{P}(\vec{Y} = \vec{0})$ for lattice random vector $\vec{Y}$, which has CP distribution with the characteristic function $\widehat{\Psi}_k(\vec{t})$. Therefore, one can use some known facts about estimates for multidimensional probabilities. As one possibility for identically distributed vectors we cite the result of Ushakov [82]

Theorem 4.4.1. *([82]) Let $\vec{X}_1, \dots, \vec{X}_n$ have an identical lattice distribution which is not concentrated on any proper subspace of the space $\mathscr{H}$, and let the dimensionality of $\mathscr{H}$ be not less than $d \geqslant 1$. We denote by $\vec{e}_1, \dots, \vec{e}_d$ any set of linearly independent vectors of $\mathscr{H}$, and let $\mathscr{H}_1, \dots, \mathscr{H}_d$ be arbitrary straight lines such that $\mathscr{H}_i$ is parallel to $\vec{e}_i$. We put $\bar{\alpha}_i = \mathbb{P}(\vec{X}_1 \in \mathscr{H}_i)$, $p_i = sup_{\vec{x} \in \mathscr{H}_i} \mathbb{P}(\vec{X}_1 = \vec{x})$ (if some of the straight lines $\mathscr{H}_i$ intersect, then a mass concentrated at the point of intersection can be distributed in an arbitrary*

manner between the straight lines intersecting at this point). Then

$$\sup_{\vec{x}\in\mathscr{H}} \mathbb{P}(\vec{S}_n=\vec{x}) \leqslant \pi^{-d/2}\left(\prod_{i=1}^{d}(\bar{\alpha}_i-p_i)(n+1)\right)^{-1/2} \times\left(1+c(d)\left((n+1)\prod_{i=1}^{d}(\bar{\alpha}_i-p_i)\right)^{-1/2}\right),$$

where $c(d)$ is a constant that depends only on d (we can put $c(d)=(2^d-1)/\lfloor(d-1)/2\rfloor!$).

Particularly Ushakov's theorem allows to move system of coordinates to a position, when coordinate axes have sufficient probabilistic mass.

Ushakov's theorem is not so effective, when each line contains just two points with non-zero probabilities, since after choosing a line $\mathscr{H}_i$ we use the sum of probabilities of its points minus the maximal one. So if there is a line with only two corresponding probabilities – the large one and the small one, $\alpha_i - p_i$ results in the smaller one.

Sometimes, another Ushakov's result can be used to improve upper bound.

Theorem 4.4.2. *[82]. Let $\vec{X}_1,\ldots,\vec{X}_n$, be independent random vectors assuming values in separable Hilbert space $\mathscr{H}$ (finite- or infinite-dimensional). Let $\vec{S}_n=\vec{X}_1+\ldots+\vec{X}_n$ and $\sup_{\vec{x}\in\mathscr{H}}\mathbb{P}(\vec{X}_i=\vec{x})\leqslant p_i$, $i=1,\ldots,n$. Then*

$$\sup_{\vec{x}\in\mathscr{H}}\mathbb{P}(\vec{S}_n=\vec{x})\leqslant C_1\left(\sum_{i=1}^{n}(1-p_i)p_i^{-2}\right)^{-1/2}. \tag{4.24}$$

Example 4.4.2. *Let in example 4.3.1 $p(1,0)=p(0,1)=n^{-1}$, $p(1,1)=0.2$; $n\geqslant 5$. Then the accuracy of approximation in Theorem 4.3.1 is*

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_{\infty}\leqslant\frac{C}{\sqrt{n}}, \tag{4.25}$$

i.e. up to constant the same estimate holds for the total variation. Note that we can not use the same approach as in example 4.4.1, since both probabilities and coordinate axes are small. However, applying Theorem 4.4.2 we can prove that

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_{\infty}\leqslant\frac{C}{n}. \tag{4.26}$$

The proof of (4.26) is given in subsection 4.6. However, arguing as in example 4.4.1 we get lower bound $d_{\infty}(\mathbf{F};\mathbf{H}_{\mathrm{n},2})\geqslant Cn^{-3/2}$, i.e. there is still

difference in orders of accuracy for upper and lower bounds. The correct order of approximation remains an open problem.

4.5 Auxiliary results

The following estimates hold for ch.f. $\widehat{F}_i$ (we recall that $\widehat{F}_i$ stands for $\widehat{F}_i(\vec{t})$).

Lemma 4.5.1. *Let $k \in \mathbb{N}$; $F_i \in \mathscr{F}_d^Z$, $0 < q_i < 1$ and $c_{i,k} > 0$; $i = 1,\dots,n$, $\vec{t} \in \mathbb{R}^d$. Then*

$$\widehat{F}_i - 1 = \sum_{\vec{x}} p_i(\vec{x})(\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} - 1), \quad |\widehat{F}_i - 1| \leqslant 2\sum_{\vec{x}} p_i(\vec{x}) \left|\sin\frac{(\vec{x},\vec{t})}{2}\right|, \tag{4.27}$$

$$Re\,\widehat{F}_i - 1 = -2\sum_{\vec{x}} p_i(\vec{x}) \sin^2\frac{(\vec{x},\vec{t})}{2}, \quad |\widehat{F}_i - 1|^2 \leqslant 2\big(1 - Re\,\widehat{F}_i\big), \tag{4.28}$$

$$|\widehat{F}_i| \leqslant \exp\big\{q_i(Re\,\widehat{F}_i - 1)\big\}, \quad |\widehat{H}_{i,k}| \leqslant \exp\big\{c_{i,k}(Re\,\widehat{F}_i - 1)\big\}. \tag{4.29}$$

Proof. The first three relations follow directly from (1.1):

$$\begin{aligned}
\widehat{F}_i - 1 &= \sum_{\vec{x}} p_i(\vec{x})\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} - \sum_{\vec{x}} p_i(\vec{x}) = \sum_{\vec{x}} p_i(\vec{x})(\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} - 1);\\
|\widehat{F}_i - 1| &\leqslant \sum_{\vec{x}} p_i(\vec{x}) \left|\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} - 1\right| = 2\sum_{\vec{x}} p_i(\vec{x}) \left|\sin\frac{(\vec{x},\vec{t})}{2}\right|;\\
Re\,\widehat{F}_i - 1 &= Re\left\{\sum_{\vec{x}} p_i(\cos(\vec{x},\vec{t}) + i\sin(\vec{x},\vec{t}))\right\} - 1\\
&= \sum_{\vec{x}} p_i\cos(\vec{x},\vec{t}) - 1 = \sum_{\vec{x}} p_i\cos(\vec{x},\vec{t}) - \sum_{\vec{x}} p_i\\
&= \sum_{\vec{x}} p_i(\cos(\vec{x},\vec{t}) - 1) = -2\sum_{\vec{x}} p_i(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}.
\end{aligned}$$

The last estimate of (4.28) can be proved exactly as in one-dimensional case:

$$\begin{aligned}
|\widehat{F}_i|^2 &= \left|Re\,\widehat{F}_i + Im\,\widehat{F}_i\right|^2 = (Re\,\widehat{F}_i)^2 + (Im\,\widehat{F}_i)^2 \leqslant 1,\\
|\widehat{F}_i - 1|^2 &= (Re\,\widehat{F}_i - 1)^2 + (Im\,\widehat{F}_i)^2 \leqslant (Re\,\widehat{F}_i - 1)^2 + 1 - (Re\,\widehat{F}_i)^2\\
&= 2(1 - Re\,\widehat{F}_i).
\end{aligned}$$

Observe that

$$(1-q_i)(Re\,\widehat{V}_i - 1) = q_i + (1-q_i)Re\,\widehat{V}_i - 1 = q_i + p_i Re\,\widehat{V}_i - 1 = Re\,\widehat{F}_i - 1,$$

and $1+x \leqslant e^x$. Then

$$\begin{aligned}
|\widehat{F}_i|^2 &= |q_i+(1-q_i)\widehat{V}_i|^2 = (q_i+(1-q_i)Re\,\widehat{V}_i)^2+(1-q_i)^2(Im\,\widehat{V}_i)^2 \\
&= q_i^2+2q_i(1-q_i)Re\,\widehat{V}_i+(1-q_i)^2((Re\,\widehat{V}_i)^2+(Im\,\widehat{V}_i)^2) \\
&\leqslant q_i^2+2q_i(1-q_i)Re\,\widehat{V}_i+(1-q_i)^2 = 1+2q_i(1-q_i)(Re\,\widehat{V}_i-1) \\
&= 1+2q_i(Re\,\widehat{F}_i-1) \leqslant \exp\{2q_i(Re\,\widehat{F}_i-1)\},
\end{aligned}$$

and the first estimate of (4.29) follows.

Next observe that

$$|\widehat{F}_i-1| = |q_i+(1-q_i)\widehat{V}_i-1| = |-(1-q_i)+(1-q_i)\widehat{V}_i| = (1-q_i)|\widehat{V}_i-1|.$$

Then

$$\begin{aligned}
|\widehat{H}_{i,k}| &\leqslant \exp\left\{|\widehat{F}_i-1|+\sum_{j=2}^{k}|\widehat{F}_i-1|^j/j\right\} \\
&\leqslant \exp\left\{(Re\,\widehat{F}_i-1)+\sum_{j=2}^{k}(1-q_i)^j|\widehat{V}_i-1|^j/j\right\}. \qquad (4.30)
\end{aligned}$$

Note that $|\widehat{V}_i-1| \leqslant |\widehat{V}_i|+1 \leqslant 2$. Then by (4.28)

$$\begin{aligned}
\sum_{j=2}^{k}(1-q_i)^j|\widehat{V}_i-1|^j/j &= (1-q_i)^2|\widehat{V}_i-1|^2\sum_{j=2}^{k}(1-q_i)^{j-2}|\widehat{V}_i-1|^{j-2}/j \\
&\leqslant (1-q_i)^2|\widehat{V}_i-1|^2\sum_{j=2}^{k}(2(1-q_i))^{j-2}/j \\
&\leqslant \frac{(1-q_i)^2|\widehat{V}_i-1|^2}{2}\sum_{j=1}^{k-1}(2(1-q_i))^{j-1} \\
&\leqslant (1-q_i)^2(1-Re\,\widehat{V}_i)\frac{1-(2(1-q_i))^{k-1}}{1-2(1-q_i)} \\
&= (1-q_i)(1-Re\,\widehat{F}_i)\frac{1-(2(1-q_i))^{k-1}}{1-2(1-q_i)}.
\end{aligned}$$

Substituting the last estimate into (4.30) we complete the proof of lemma. □

In the next lemma we use $\widehat{\Psi}_k, \omega_{d,k}$ from (9.1) and (4.9).

Lemma 4.5.2. *Let $c_{i,k} > 0$, $0 < q_i < 1$, $i = 1, 2, \ldots, n$. Then*

$$\frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \widehat{\Psi}_k \, \mathrm{d}t_1 \ldots \mathrm{d}t_d \leqslant 6^d \omega_{d,k}^{-1/2}.$$

Proof. We have

$$Re\, \widehat{F}_i - 1 = -2 \sum_{\vec{x}} p_i(\vec{x}) \sin^2 \frac{(\vec{x}, \vec{t})}{2} \leqslant -2 \sum_{m=1}^{d} \sum_{j \in \mathbb{Z} \setminus \{0\}} p_{i,m}^0(j) \sin^2 \frac{j t_m}{2}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \widehat{\Psi}_k \, \mathrm{d}t_1 \ldots \mathrm{d}t_d \\
&\quad \leqslant \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \exp\left\{ \frac{1}{2} \sum_{i=1}^{n} c_{i,k} (Re\, \widehat{F}_i - 1) \right\} \mathrm{d}t_1 \ldots \mathrm{d}t_d \\
&\quad \leqslant \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \exp\left\{ \sum_{i=1}^{n} c_{i,k} \sum_{m=1}^{d} \sum_{j \in \mathbb{Z} \setminus \{0\}} p_{i,m}^0(j) \sin^2 \frac{j t_m}{2} \right\} \mathrm{d}t_m \\
&\quad \leqslant \prod_{m=1}^{d} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{ - \sum_{i=1}^{n} c_{i,k} \sum_{j \in \mathbb{Z} \setminus \{0\}} p_{i,m}^0(j) \sin^2 \frac{j t_m}{2} \right\} \mathrm{d}t_m \right) \\
&\quad = \prod_{m=1}^{d} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{ - \sum_{j \in \mathbb{Z} \setminus \{0\}} \sum_{i=1}^{n} c_{i,k} p_{i,m}^0(j) \sin^2 \frac{j t_m}{2} \right\} \mathrm{d}t_m \right) \\
&\quad \leqslant \prod_{m=1}^{d} 5.62 \left(\sum_{j \in \mathbb{Z} \setminus \{0\}} \sum_{i=1}^{n} c_{i,k} p_{i,m}^0(j) \right)^{-1/2} \leqslant 6^d \omega_{d,k}^{-1/2},
\end{aligned}$$

by (A.8). □

Next we estimate the difference of $\widehat{F}_i$ and $\widehat{H}_{i,k}$. We recall that b_k is defined by (4.9) and (4.10).

Lemma 4.5.3. *For $k \geqslant 1$ and all $\vec{t} \in \mathbb{R}^d$*

$$|\widehat{F}_i - \widehat{H}_{i,k}| \leqslant b_k |\widehat{H}_{i,k}| |\widehat{F}_i - 1|^{k+1}.$$

Proof. From (A.51) it follows that for any (complex) x, such that $|x| \leqslant 2$,

$$\left|(1+x)\exp\left\{\sum_{j=1}^{k}(-1)^j\frac{x^j}{j}\right\}-1\right| \leqslant |x|^{k+1}\int_0^1 \tau^k \exp\left\{\tau\sum_{j=1}^{k}\frac{2^j}{j}\right\}\mathrm{d}\tau. \qquad (4.31)$$

For $k = 1,2,3$ we resp. have

$$\int_0^1 \tau \mathrm{e}^{2\tau}\mathrm{d}\tau = \frac{\mathrm{e}^2+1}{4}, \int_0^1 \tau^2\mathrm{e}^{4\tau}\mathrm{d}\tau \leqslant 8.5, \int_0^1 \tau^3\mathrm{e}^{(20/3)\tau}\mathrm{d}\tau \leqslant 78.4. \qquad (4.32)$$

For $k \geqslant 4$

$$\sum_{j=1}^{k}\frac{2^j}{j} \leqslant \frac{20}{3}+\frac{2^4}{4}\sum_{j=4}^{k}2^{j-4} = \frac{8}{3}+2^{k-1}, \int_0^1 \tau^k \mathrm{e}^{(8/3+2^{k-1})\tau}\mathrm{d}\tau = b_k. \qquad (4.33)$$

Indeed, for any $A > 0$,

$$\begin{aligned}\int \tau^k \mathrm{e}^{A\tau}\mathrm{d}\tau &= \mathrm{e}^{A\tau}\sum_{j=1}^{k+1}\frac{(-1)^{j+1}}{A^j}\tau^{k-j+1}k^{(j-1)} \\ &= \frac{\mathrm{e}^{A\tau}(-1)^k k!}{A^{k+1}}\sum_{m=0}^{k}\frac{(-1)^m A^m}{m!}\tau^m.\end{aligned}$$

and

$$\int_0^1 \tau^k \mathrm{e}^{A\tau}\mathrm{d}\tau = \frac{(-1)^k k!}{A^{k+1}}\left(\mathrm{e}^A\sum_{m=0}^{k}\frac{(-1)^m A^m}{m!}-1\right).$$

Observe that

$$\begin{aligned}|\widehat{F}_i - \widehat{H}_{i,k}| &= \left|\widehat{H}_{i,k}\left(\frac{\widehat{F}_i}{\widehat{H}_{i,k}}-1\right)\right| \\ &= |\widehat{H}_{i,k}|\left|(1+(\widehat{F}_i-1))\exp\left\{\sum_{j=1}^{k}(-1)^j(\widehat{F}_i-1)^j/j\right\}-1\right|\end{aligned}$$

and (4.31) can be applied. Combining (4.31) with (4.32) and (4.33) we complete the proof of Lemma. □

Lemma 4.5.4. *Let* $0 < q_i < 1$, $(i = 1,2,\ldots,n)$. *Then, for any* $\vec{t} \in \mathbb{R}^d$

$$|\widehat{\mathbf{H}}_{\mathrm{n},2} - \widehat{\mathbf{H}}_{\mathrm{n},1}(1+\widehat{\mathscr{A}_1})| \leqslant C\widehat{\Psi}_2\left(\sum_{i=1}^{n}\Lambda_{i,2}^2\right)^2,$$

If $q_i > 1/2$, $(i = 1,2,\dots,n)$, then

$$|\widehat{\mathbf{H}}_{\mathrm{n},3} - \widehat{\mathbf{H}}_{\mathrm{n},2}(1+\widehat{\mathscr{A}_2})| \leqslant C\widehat{\Psi}_3 \left(\sum_{i=1}^{n} \Lambda_{i,3}^3\right)^2.$$

Proof. Applying estimate $x\mathrm{e}^{-x^2} \leqslant 1/\sqrt{2\mathrm{e}} \leqslant C$, $x > 0$, we get

$$\begin{aligned}
|\widehat{F}_i - 1|\widehat{\Psi}_2^{1/4} &\leqslant 2\sum_{\vec{x}} p_i(\vec{x}) \left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \exp\left\{-\frac{1}{4}\sum_{j=1}^{n} q_j \sum_{\vec{y}} p_j(\vec{y}) \sin^2\frac{(\vec{y},\vec{t})}{2}\right\} \\
&\leqslant 2\sum_{\vec{x}} p_i(\vec{x}) \left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \exp\left\{-\frac{1}{4}\sum_{j=1}^{n} q_j p_j(\vec{x}) \sin^2\frac{(\vec{x},\vec{t})}{2}\right\} \\
&\leqslant C\sum_{\vec{x}\neq\vec{0}} \frac{p_i(\vec{x})}{\sqrt{\lambda_2(\vec{x})}}.
\end{aligned}$$

Applying (A.48) and arguing as in the proof of (4.29) we get

$$\begin{aligned}
\left|\widehat{\mathbf{H}}_{\mathrm{n},2} - \widehat{\mathbf{H}}_{\mathrm{n},1}(1+\widehat{\mathscr{A}_1})\right| &= \left|\widehat{\mathbf{H}}_{\mathrm{n},1}\right| \left|\prod_{i=1}^{n} \exp\left\{-\frac{1}{2}(\widehat{F}_i - 1)^2\right\} - (1+\widehat{\mathscr{A}_1})\right| \\
&= \left|\widehat{\mathbf{H}}_{\mathrm{n},1}\right| \left|\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(\widehat{F}_i - 1)^2\right\} - \left(1 - \frac{1}{2}\sum_{j=1}^{n}(\widehat{F}_j - 1)^2\right)\right| \\
&\leqslant \left|\widehat{\mathbf{H}}_{\mathrm{n},1}\right| \frac{1}{4}\left(\sum_{i=1}^{n}|\widehat{F}_i - 1|^2\right)^2 \int_0^1 \exp\left\{\frac{1}{2}\tau\sum_{j=1}^{n}|\widehat{F}_j - 1|^2\right\}(1-\tau)\mathrm{d}\tau \\
&\leqslant \frac{1}{4}\widehat{\Psi}_2^2\left(\sum_{i=1}^{n}|\widehat{F}_i - 1|^2\right)^2 \int_0^1 (1-\tau)\mathrm{d}\tau \\
&\leqslant C\widehat{\Psi}_2\left(\sum_{i=1}^{n}\left(|\widehat{F}_i - 1|\widehat{\Psi}_2^{1/4}\right)^2\right)^2 \leqslant C\widehat{\Psi}_2\left(\sum_{i=1}^{n}\Lambda_{i,2}^2\right)^2.
\end{aligned}$$

Proof of the second estimate is very similar and, therefore, omitted. □

For the proof of Theorem 4.3.3 we need estimates for partial derivatives.

Lemma 4.5.5. *Let $j \in \{1,2,\dots,d\}$ and let $\beta_{i,j}^{(2)} \leqslant \infty$, $(i = 1,2,\dots,n)$. Then*

$$\left|\frac{\partial}{\partial t_j}\left(\widehat{F}_i \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\right)\right|, \left|\frac{\partial}{\partial t_j}\left(\widehat{H}_{i,k}\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\right)\right| \leqslant C\sqrt{\beta_{i,j}^{(2)}}$$

$$\times \left(\sum_{\vec{x}\neq\vec{0}} p_i(\vec{x}) \sin^2 \frac{(\vec{x},\vec{t})}{2} \right)^{1/2},$$

$$\left| \frac{\partial^2}{\partial t_j^2} \left(\widehat{F}_i \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}} \right) \right| \leqslant C\beta_{i,j}^{(2)}.$$

Proof. Recalling that $|\mathrm{e}^{\mathrm{i}a}| = 1$, for any $a \in \mathbb{R}$, we write

$$\begin{aligned}
\left| \frac{\partial}{\partial t_j} \left(\widehat{F}_i \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}} \right) \right| &= \left| \sum_{\vec{x}\in\mathbb{Z}^d} p_i(\vec{x}) \frac{\partial}{\partial t_j} \left(\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}} \right) \right| \\
&= \left| \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}} \sum_{\vec{x}\in\mathbb{Z}^d} (x_j - \mu_{i,j}) p_i(\vec{x}) \mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} \right| \\
&= \left| \sum_{\vec{x}\in\mathbb{Z}^d} (x_j - \mu_{i,j}) p_i(\vec{x}) + \sum_{\vec{x}\neq\vec{0}} (x_j - \mu_{i,j}) p_i(\vec{x}) (\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} - 1) \right| \\
&= \left| \sum_{\vec{x}\neq\vec{0}} (x_j - \mu_{i,j}) p_i(\vec{x}) (\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} - 1) \right| \\
&\leqslant \sum_{\vec{x}\neq\vec{0}} |x_j - \mu_{i,j}| p_i(\vec{x}) |\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} - 1| \\
&= 2 \sum_{\vec{x}\neq\vec{0}} |x_j - \mu_{i,j}| p_i(\vec{x}) \left| \sin \frac{(\vec{x},\vec{t})}{2} \right| \\
&\leqslant 2 \left(\sum_{\vec{x}\neq\vec{0}} (x_j - \mu_{i,j})^2 p_i(\vec{x}) \right)^{1/2} \left(\sum_{\vec{x}\neq\vec{0}} p_i(\vec{x}) \sin^2 \frac{(\vec{x},\vec{t})}{2} \right)^{1/2} \\
&\leqslant 2\sqrt{\beta_{i,j}^{(2)}} \left(\sum_{\vec{x}\neq\vec{0}} p_i(\vec{x}) \sin^2 \frac{(\vec{x},\vec{t})}{2} \right)^{1/2}.
\end{aligned}$$

For the last estimate we used Hölder's inequality. Similarly

$$\begin{aligned}
&\left| \frac{\partial}{\partial t_j} \left(\widehat{H}_{i,k} \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}} \right) \right| \\
&\quad = \left| \widehat{H}_{i,k} \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}} \frac{\partial}{\partial t_j} \left(-\mathrm{i}t_j\mu_{i,j} + \sum_{l=1}^{k} \frac{(-1)^{l+1}}{l} (\widehat{F}_i - 1)^l \right) \right| \\
&\quad = \left| \widehat{H}_{i,k} \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}} \left(\frac{\partial \widehat{F}_i}{\partial t_j} - \mathrm{i}\mu_{i,j} + \sum_{l=2}^{k} (-1)^{l+1} (\widehat{F}_i - 1)^{l-1} \frac{\partial \widehat{F}_i}{\partial t_j} \right) \right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant |\widehat{H}_{i,k}|\left|\frac{\partial \widehat{F}_i}{\partial t_j}-\mathrm{i}\mu_{i,j}\right|+C\left|\frac{\partial \widehat{F}_i}{\partial t_j}\right|\left|\widehat{F}_i-1\right| \\
&\leqslant C\sum_{\vec{x}\neq\vec{0}}|x_j|p_i(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|+C\beta_{i,j}\sum_{\vec{x}\neq\vec{0}}p_i(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \\
&\leqslant C\sqrt{\beta_{i,j}^{(2)}}\left(\sum_{\vec{x}\neq\vec{0}}p_i(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right)^{1/2},
\end{aligned}
$$

and

$$
\left|\frac{\partial^2}{\partial t_j^2}\left(\widehat{F}_i\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\right)\right|\leqslant\sum_{\vec{x}\neq\vec{0}}(x_j-\mu_{i,j})^2p_i(\vec{x})=\beta_{i,j}^{(2)}-\mu_{i,j}^2\leqslant\beta_{i,j}^{(2)}.
$$

□

Lemma 4.5.6. *Let $k,n\in\mathbb{N}$, $j\in\{1,2,\ldots,d\}$ and let $\beta_{i,j}<\infty$, $(i=1,2,\ldots,n)$. Then*

$$
\begin{aligned}
\left|\frac{\partial}{\partial t_j}\left(\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right)\right)\right| &\leqslant C(k)\beta_{i,j}|\widehat{F}_i-1|^k, \\
\left|\frac{\partial^2}{\partial t_j^2}\left(\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right)\right)\right| &\leqslant C(k)\left(\beta_{i,j}^{(2)}|\widehat{F}_i-1|^k+\beta_{i,j}^2|\widehat{F}_i-1|^{k-1}\right).
\end{aligned}
$$

Proof. Let denote

$$
B_{\tau,i}:=\sum_{l=1}^{k}(-1)^{l+1}(\widehat{F}_i-1)^l(1-\tau^l)/l-\mathrm{i}t_j\mu_{i,j}.
$$

Similarly to the proof of Lemma 4.5.3 we can write

$$
\begin{aligned}
\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right) &= \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\widehat{H}_{i,k}\left(\frac{\widehat{F}_i}{\widehat{H}_{i,k}}-1\right) \\
&= \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\widehat{H}_{i,k}\left((1+(\widehat{F}_i-1))\exp\left\{\sum_{l=1}^{k}\frac{(-1)^l}{l}(\widehat{F}_i-1)^l\right\}-1\right) \\
&= \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\widehat{H}_{i,k}\int_0^1\exp\left\{\sum_{l=1}^{k}\frac{(-\tau)^l}{l}(\widehat{F}_i-1)^l\right\}(-1)^{k+2}(\widehat{F}_i-1)^{k+1}\tau^k\mathrm{d}\tau. \\
&= (\widehat{F}_i-1)^{k+1}\int_0^1\mathrm{e}^{B_{\tau,i}}(-1)^k\tau^k\mathrm{d}\tau.
\end{aligned}
$$

Note that $|B_{\tau,i}| \leqslant C(k)$. Therefore

$$\begin{aligned}
&\left|\frac{\partial}{\partial t_j}\left(\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right)\right)\right| \\
&\quad\leqslant \left|(k+1)(\widehat{F}_i-1)^k\frac{\partial\widehat{F}_i}{\partial t_j}\int_0^1 \mathrm{e}^{B_{\tau,i}}(-1)^k\tau^k\mathrm{d}\tau\right| \\
&\qquad+\left|(\widehat{F}_i-1)^{k+1}\int_0^1 \mathrm{e}^{B_{\tau,i}}(-1)^k\tau^k\frac{\partial B_{\tau,i}}{\partial t_j}\mathrm{d}\tau\right| \\
&\quad\leqslant C(k)\left|\frac{\partial\widehat{F}_i}{\partial t_j}\right||\widehat{F}_i-1|^k+C(k)|\widehat{F}_i-1|^{k+1}\int_0^1\left|\frac{\partial B_{\tau,i}}{\partial t_j}\right|\mathrm{d}\tau. \qquad (4.34)
\end{aligned}$$

We have

$$\left|\frac{\partial\widehat{F}_i}{\partial t_j}\right| = \left|\sum_{\vec{x}} x_j p_i(\vec{x})\right| \leqslant \sum_{\vec{x}}|x_j|p_i(\vec{x}) = \beta_{i,j}, \qquad (4.35)$$

$$\left|\frac{\partial B_{\tau,i}}{\partial t_j}\right| \leqslant C(k)\left|\frac{\partial\widehat{F}_i}{\partial t_j}\right| \leqslant C(k)\beta_{i,j}.$$

Substituting these estimates into (4.34) and using trivial estimate $|\widehat{F}_i-1|^{k+1}\leqslant 2|\widehat{F}_i-1|^k$ we obtain the first estimate. Similarly

$$\left|\frac{\partial^2\widehat{F}_i}{\partial t_j^2}\right| = \left|\sum_{\vec{x}} x_j^2 p_i(\vec{x})\right| = \beta_{i,j}^{(2)}, \left|\frac{\partial^2\widehat{B}_{\tau,i}}{\partial t_j^2}\right| \leqslant C(k)\left(\beta_{i,j}^2+\beta_{i,j}^{(2)}\right) \leqslant C(k)\beta_{i,j}^{(2)},$$

where the last estimate follows from Hölder's inequality. Next observe that

$$\begin{aligned}
\frac{\partial^2}{\partial t_j^2}\left(\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right)\right) &= k(k+1)(\widehat{F}_i-1)^{k-1} \\
&\quad\times\left(\frac{\partial\widehat{F}_i}{\partial t_j}\right)^2\int_0^1\mathrm{e}^{B_{\tau,i}}(-1)^k\tau^k\mathrm{d}\tau \\
&\quad+(k+1)(\widehat{F}_i-1)^k\frac{\partial^2\widehat{F}_i}{\partial t_j^2}\int_0^1\mathrm{e}^{B_{\tau,i}}(-1)^k\tau^k\mathrm{d}\tau \\
&\quad+2(k+1)(\widehat{F}_i-1)^k\frac{\partial\widehat{F}_i}{\partial t_j}\int_0^1\mathrm{e}^{B_{\tau,i}}\frac{\partial B_{\tau,i}}{\partial t_j}(-1)^k\tau^k\mathrm{d}\tau \\
&\quad+(\widehat{F}_i-1)^{k+1}\int_0^1\mathrm{e}^{B_{\tau,i}}\left(\frac{\partial B_{\tau,i}}{\partial t_j}\right)^2(-1)^k\tau^k\mathrm{d}\tau
\end{aligned}$$

$$+(\widehat{F}_i-1)^{k+1}\int_0^1 \mathrm{e}^{B_{\tau,i}}\left(\frac{\partial^2 B_{\tau,i}}{\partial t_j^2}\right)(-1)^k\tau^k\mathrm{d}\tau.$$

Therefore

$$\begin{aligned}
\left|\frac{\partial^2}{\partial t_j^2}\left(\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right)\right)\right| &\leqslant C(k)|\widehat{F}_i-1|^{k-1}\left|\frac{\partial\widehat{F}_i}{\partial t_j}\right|^2\\
&\quad +C(k)|\widehat{F}_i-1|^k\left|\frac{\partial^2\widehat{F}_i}{\partial t_j^2}\right|+C(k)|\widehat{F}_i-1|^k\left|\frac{\partial\widehat{F}_i}{\partial t_j}\right|\int_0^1\left|\frac{\partial\widehat{B}_{\tau,i}}{\partial t_j}\right|\mathrm{d}\tau\\
&\quad +C(k)|\widehat{F}_i-1|^{k+1}\int_0^1\left|\frac{\partial\widehat{B}_{\tau,i}}{\partial t_j}\right|^2\mathrm{d}\tau+C(k)|\widehat{F}_i-1|^{k+1}\int_0^1\left|\frac{\partial^2\widehat{B}_{\tau,i}}{\partial t_j^2}\right|\mathrm{d}\tau\\
&\leqslant\ C(k)\left(|\widehat{F}_i-1|^{k-1}\beta_{i,j}^2+|\widehat{F}_i-1|^k\beta_{i,j}^{(2)}\right).
\end{aligned}$$

Lemma's proof is complete. □

Lemma 4.5.7. *Let* $k,n\in\mathbb{N}$, $j,m\in\{1,2,\ldots,d\}$ *and* $\beta_{i,j},\beta_{i,m},\beta_{i,jm}<\infty$, $(i=1,2,\ldots,n)$. *Then*

$$\begin{aligned}
\left|\frac{\partial^2}{\partial t_j\partial t_m}\left(\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}-\mathrm{i}t_m\mu_{i,m}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right)\right)\right| &\leqslant C(k)\Big(|\widehat{F}_i-1|^k\beta_{i,jm}\\
&\quad +|\widehat{F}_i-1|^{k-1}\beta_{i,m}\beta_{i,j}\Big).
\end{aligned}$$

Proof. Similarly to the proof of Lemma L4.8 we introduce auxiliary notation

$$\tilde{B}_{\tau,i}:=\sum_{l=1}^k(-1)^{l+1}(\widehat{F}_i-1)^l(1-\tau^l)/l-\mathrm{i}t_j\mu_{i,j}-\mathrm{i}t_m\mu_{i,m}.$$

Then

$$\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}-\mathrm{i}t_m\mu_{i,m}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right)=(\widehat{F}_i-1)^{k+1}\int_0^1\mathrm{e}^{\tilde{B}_{\tau,i}}(-1)^k\tau^k\mathrm{d}\tau,$$

and

$$\begin{aligned}
&\left|\frac{\partial^2}{\partial t_j\partial t_m}\left(\mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}-\mathrm{i}t_m\mu_{i,m}}\left(\widehat{F}_i-\widehat{H}_{i,k}\right)\right)\right|\\
&\qquad\leqslant\ k(k+1)|\widehat{F}_i-1|^{k-1}\left|\frac{\partial\widehat{F}_i}{\partial t_j}\right|\left|\frac{\partial\widehat{F}_i}{\partial t_m}\right|\int_0^1\mathrm{e}^{|\tilde{B}_{\tau,i}|}\tau^k\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+(k+1)|\widehat{F}_i-1|^k\left|\frac{\partial^2\widehat{F}_i}{\partial t_j\partial t_m}\right|\int_0^1 \mathrm{e}^{|\tilde{B}_{\tau,i}|}\tau^k\mathrm{d}\tau\\
&+(k+1)|\widehat{F}_i-1|^k\left|\frac{\partial\widehat{F}_i}{\partial t_j}\right|\int_0^1 \mathrm{e}^{|\tilde{B}_{\tau,i}|}\left|\frac{\partial\tilde{B}_{\tau,i}}{\partial t_m}\right|\tau^k\mathrm{d}\tau\\
&+(k+1)|\widehat{F}_i-1|^k\left|\frac{\partial\widehat{F}_i}{\partial t_m}\right|\int_0^1 \mathrm{e}^{|\tilde{B}_{\tau,i}|}\left|\frac{\partial\tilde{B}_{\tau,i}}{\partial t_j}\right|\tau^k\mathrm{d}\tau\\
&+|\widehat{F}_i-1|^{k+1}\int_0^1 \mathrm{e}^{|\tilde{B}_{\tau,i}|}\left|\frac{\partial\tilde{B}_{\tau,i}}{\partial t_j}\right|\left|\frac{\partial\tilde{B}_{\tau,i}}{\partial t_m}\right|\tau^k\mathrm{d}\tau\\
&+|\widehat{F}_i-1|^{k+1}\int_0^1 \mathrm{e}^{|\tilde{B}_{\tau,i}|}\left|\frac{\partial^2\tilde{B}_{\tau,i}}{\partial t_j\partial t_m}\right|\tau^k\mathrm{d}\tau. \qquad (4.36)
\end{aligned}
$$

It is easy to check that $|\tilde{B}_{\tau,i}|\leqslant C(k)$, therefore, only estimates for derivatives are needed. Estimate (4.35), trivial estimate $|\widehat{F}_i-1|\leqslant 2$ and direct calculation allow to obtain

$$
\begin{aligned}
\left|\frac{\partial\tilde{B}_{\tau,i}}{\partial t_j}\right| &\leqslant C(k)\beta_{i,j}, \quad \left|\frac{\partial\tilde{B}_{\tau,i}}{\partial t_m}\right|\leqslant C(k)\beta_{i,m}, \qquad (4.37)\\
\left|\frac{\partial^2\widehat{F}_i}{\partial t_j\partial t_m}\right| &\leqslant \sum_{\vec{x}}|x_j||x_m|p(\vec{x})=\beta_{i,jm}. \qquad (4.38)
\end{aligned}
$$

Next, taking into account (4.35) and (4.37) we prove that

$$
\begin{aligned}
\left|\frac{\partial^2\tilde{B}_{\tau,i}}{\partial t_j\partial t_m}\right| &\leqslant \sum_{l_1=1}^{k}|\widehat{F}_i-1|^{l_1-1}\left|\frac{\partial^2\widehat{F}_i}{\partial t_j\partial t_m}\right|\\
&\quad+\sum_{l_2=2}^{k}(l_2-1)|\widehat{F}_i-1|^{l_2-2}\left|\frac{\partial\widehat{F}_i}{\partial t_j}\right|\left|\frac{\partial\widehat{F}_i}{\partial t_m}\right|\\
&\leqslant \beta_{i,jm}\sum_{l_1=1}^{k}2^{l_1-1}+\beta_{i,j}\beta_{i,m}\sum_{l_2=2}^{k}(l_2-1)2^{l_2-2}\\
&\leqslant C(k)(\beta_{i,jm}+\beta_{i,j}\beta_{i,m}). \qquad (4.39)
\end{aligned}
$$

Substituting (4.37), (4.38) and (4.39) into (4.36) we complete the proof of lemma. □

The next few lemmas are needed for lower estimates.

Let $\vec{\mu}=(\mu_1,\mu_2,\ldots,\mu_d)$, $\mu_j=\sum_{i=1}^n\mu_{i,j}$, and let

$$
J(\vec{t}) \quad = \quad \exp\{-\mathrm{i}(\vec{\mu},\vec{t})\}\left(\prod_{i=1}^{n}\widehat{F}_i(\vec{t})-\prod_{i=1}^{n}\widehat{H}_{i,k}(\vec{t})\right)
$$

$$
\begin{aligned}
&= J_1(\vec{t}) + J_2(\vec{t}) + J_3(\vec{t}) + J_4(\vec{t}) + J_5(\vec{t}),\\
J_1(\vec{t}) &= \frac{(-1)^{k+2}}{k+1} \sum_{i=1}^{n} \left(\mathrm{i} \sum_{\vec{x}} p_i(\vec{x}) (\vec{x},\vec{t}) \right)^{k+1}\\
&= \frac{(-1)^{k+2} \mathrm{i}^{k+1}}{k+1} \sum_{i=1}^{n} \left(\sum_{m=1}^{d} \mu_{i,m} t_m \right)^{k+1},\\
J_2(\vec{t}) &= J_1(\vec{t}) \left(\exp\{-\mathrm{i}(\vec{\mu},\vec{t})\} \prod_{j=1}^{n} \widehat{H}_{j,k}(\vec{t}) - 1 \right),\\
J_3(\vec{t}) &= \exp\{-\mathrm{i}(\vec{\mu},\vec{t})\} \prod_{i=1}^{n} \widehat{H}_{i,k}(\vec{t}) \frac{(-1)^{k+2}}{k+1}\\
&\quad \times \sum_{j=1}^{n} \left((\widehat{F}_j(\vec{t}) - 1)^{k+1} - \left(\mathrm{i} \sum_{\vec{x}} p_j(\vec{x}) (\vec{x},\vec{t}) \right)^{k+1} \right),\\
J_4(\vec{t}) &= \exp\{-\mathrm{i}(\vec{\mu},\vec{t})\} \left(\prod_{i=1}^{n} \widehat{H}_{i,k+1}(\vec{t}) - \prod_{i=1}^{n} \widehat{H}_{i,k}(\vec{t}) \right.\\
&\quad \left. \times \left(1 + \frac{(-1)^{k+2}}{k+1} \sum_{j=1}^{n} (\widehat{F}_j(\vec{t}) - 1)^{k+1} \right) \right),\\
J_5(\vec{t}) &= \exp\{-\mathrm{i}(\vec{\mu},\vec{t})\} \left(\prod_{i=1}^{n} \widehat{F}_i(\vec{t}) - \prod_{i=1}^{n} \widehat{H}_{i,k+1}(\vec{t}) \right).
\end{aligned}
$$

Lemma 4.5.8. *Let for some* $k \in \mathbb{N}$, $\beta_{i,m}^{(2)} < \infty$, $c_{i,k+1} > 0$, $(i = 1,2,\ldots,n)$; $(m = 1,2,\ldots,d)$. *Then for any* $\vec{t} \in \mathbb{R}^d$,

$$
\begin{aligned}
|J_2(\vec{t})| &\leqslant C(d,k) \sum_{i=1}^{n} \left(\sum_{m=1}^{d} \beta_{i,m} |t_m| \right)^{k+1}\\
&\quad \times \sum_{j=1}^{n} \left(\sum_{l=1}^{d} \beta_{j,l}^{(2)} t_l^2 + (2^{k-1} - 1) \left(\sum_{l=1}^{d} \beta_{j,l} t_l \right)^2 \right), \qquad (4.40)\\
|J_3(\vec{t})| &\leqslant C(d,k) \sum_{i=1}^{n} \left(\sum_{m=1}^{d} \beta_{i,m} |t_m| \right)^{k} \sum_{l=1}^{d} \beta_{i,l}^{(2)} t_l^2, \qquad (4.41)\\
|J_4(\vec{t})| &\leqslant C(d,k) \left(\sum_{i=1}^{n} \left(\sum_{m=1}^{d} \beta_{i,m} |t_m| \right)^{k+1} \right)^2, \qquad (4.42)\\
|J_5(\vec{t})| &\leqslant C(d,k) \sum_{i=1}^{n} \left(\sum_{m=1}^{d} \beta_{i,m} |t_m| \right)^{k+2}. \qquad (4.43)
\end{aligned}
$$

Proof. Similarly to the proof of Lemma 4.5.3 we show that

$$Re\,(\widehat{F}_i-1)+\sum_{j=2}^{k+1}|\widehat{F}_i-1|^j/j<0,\quad |\widehat{H}_{i,k}|\leqslant 1,\quad |\widehat{H}_{i,k+1}|\leqslant 1. \tag{4.44}$$

Observe also that

$$\begin{aligned}
|\widehat{F}_i-1| &\leqslant 2\sum_{\vec{x}} p_i(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \leqslant \sum_{\vec{x}} p_i(\vec{x})|(\vec{x},\vec{t})| \\
&\leqslant \sum_{\vec{x}} p_i(\vec{x})\sum_{m=1}^{d}|x_m||t_m| = \sum_{m=1}^{d}\beta_{i,m}|t_m|, \qquad (4.45)\\
\sum_{\vec{x}} p_i(\vec{x})(\vec{x},\vec{t}) &= \sum_{\vec{x}} p_i(\vec{x})\sum_{m=1}^{d}x_m t_m = \sum_{m=1}^{d} t_m\mu_{i,m}, \qquad (4.46)\\
\sum_{\vec{x}} p_i(\vec{x})|(\vec{x},\vec{t})|^2 &\leqslant d^2\sum_{\vec{x}} p_i(\vec{x})\sum_{m=1}^{d}x_m^2 t_m^2 = d^2\sum_{m=1}^{d}\beta_{i,m}^{(2)}t_m^2. \qquad (4.47)
\end{aligned}$$

First we estimate $J_2(\vec{t})$. Taking into account (A.47) and (4.47) we get

$$\begin{aligned}
\left|\sum_{j=1}^{n}(\widehat{F}_j-1)-\mathrm{i}(\vec{\mu},\vec{t})\right| &\leqslant \sum_{j=1}^{n}\sum_{\vec{x}} p_j(\vec{x})\left|\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})}-1-\mathrm{i}(\vec{x},\vec{t})\right| \\
&\leqslant C\sum_{j=1}^{n}\sum_{\vec{x}} p_j(\vec{x})|(\vec{x},\vec{t})|^2 \leqslant C(d)\sum_{j=1}^{n}\sum_{m=1}^{d}\beta_{j,m}^{(2)}t_m^2. \qquad (4.48)
\end{aligned}$$

Noting that $|\widehat{F}_j-1|\leqslant 2$ and taking into account (4.46), (A.48), (4.44), (4.45) and (4.48) we obtain

$$\begin{aligned}
|J_2(\vec{t})| &\leqslant C(d,k)\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\beta_{i,m}|t_m|\right)^{k+1} \\
&\quad\times\left|\sum_{j=1}^{n}(\widehat{F}_j-1)-\mathrm{i}(\vec{\mu},\vec{t})+\sum_{j=1}^{n}\sum_{l=2}^{k}(-1)^{l+1}(\widehat{F}_j-1)^l/l\right| \\
&\leqslant C(d,k)\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\beta_{i,m}|t_m|\right)^{k+1} \\
&\quad\times\left(\sum_{j=1}^{n}\sum_{l=1}^{d}\beta_{j,l}^{(2)}t_l^2+\sum_{j=1}^{n}|\widehat{F}_j-1|^2\sum_{l=2}^{k}2^{l-2}/l\right)
\end{aligned}$$

$$\begin{aligned} &\leqslant C(d,k)\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\beta_{i,m}|t_m|\right)^{k+1} \\ &\quad\times\left(\sum_{j=1}^{n}\sum_{l=1}^{d}\beta_{j,l}^{(2)}t_l^2+(2^{k-1}-1)\sum_{j=1}^{n}\left(\sum_{l=1}^{d}\beta_{j,l}|t_l|\right)^2\right). \end{aligned}$$

Estimate (4.40) is proved. Next applying (4.45), (4.48) and (4.46) and noting that $|\mu_{i,m}|\leqslant\beta_{i,m}$ we prove that

$$\begin{aligned} &\sum_{i=1}^{n}\left|(\widehat{F}_i(\vec{t})-1)^{k+1}-\left(\mathrm{i}\sum_{\vec{x}}p_i(\vec{x})(\vec{x},\vec{t})\right)^{k+1}\right| \\ &\qquad\leqslant \sum_{i=1}^{n}\left|\widehat{F}_i(\vec{t})-1-\mathrm{i}\sum_{\vec{x}}p_i(\vec{x})(\vec{x},\vec{t})\right|\sum_{j=1}^{k+1}|\widehat{F}_i-1|^{j-1} \\ &\qquad\quad\times\left|\mathrm{i}\sum_{\vec{x}}p_i(\vec{x})(\vec{x},\vec{t})\right|^{k+1-j} \\ &\qquad\leqslant C(d,k)\sum_{i=1}^{n}\sum_{l=1}^{d}\beta_{i,l}^{(2)}t_l^2\left(\sum_{m=1}^{d}\beta_{i,m}|t_m|\right)^k. \end{aligned}$$

Estimate (4.41) is proved. Next observe that, for any complex or real b, from (A.48) it follows that $|\mathrm{e}^b-1-b|\leqslant b^2\mathrm{e}^{|b|}/2$. Therefore, by taking

$$b=\frac{(-1)^{k+2}}{k+1}\sum_{j=1}^{n}(\widehat{F}_j-1)^{k+1}$$

and employing the first estimate in (4.44) we have

$$\begin{aligned} |J_4(\vec{t})| &\leqslant \frac{1}{2}\left(\frac{1}{k+1}\sum_{j=1}^{n}|\widehat{F}_j-1|^{k+1}\right)^2\prod_{i=1}^{n}\exp\left\{Re\,\widehat{F}_i-1+\sum_{l=2}^{k+1}\frac{|\widehat{F}_i-1|^l}{l}\right\} \\ &\leqslant \frac{1}{2}\left(\frac{1}{k+1}\sum_{j=1}^{n}|\widehat{F}_j-1|^{k+1}\right)^2. \end{aligned}$$

Applying (4.45) we arrive at the estimate for $J_4(\vec{t})$ in (4.42). Next observe that by conditioning $|\widehat{H}_{i,s+1}|\leqslant 1$, and therefore similarly to the proof of Lemma 4.5.3

$$\left|\prod_{i=1}^{n}\widehat{F}_i-\prod_{i=1}^{n}\widehat{H}_{i,k+1}\right| \leqslant \sum_{i=1}^{n}|\widehat{F}_i-\widehat{H}_{i,k+1}|\prod_{j=1}^{i-1}|\widehat{F}_j|\prod_{j=i+1}^{n}|\widehat{H}_{j,k+1}|$$

$$\leqslant \quad C(d,k)\sum_{i=1}^{n}|\widehat{F}_i-1|^{k+2}.$$

By applying (4.45) we obtain the required estimate in (4.43) for $J_5(\vec{t})$. □

4.6 Proofs

Proof of Theorem 4.3.1. For each expresion assuming that $\prod_{j=k}^{m}=1$, when $k>m$ and applying (4.29) we get

$$\begin{aligned}\left|\prod_{i=1}^{n}\widehat{F}_i-\prod_{i=1}^{n}\widehat{H}_{i,k}\right| &\leqslant \sum_{i=1}^{n}|\widehat{F}_i-\widehat{H}_{i,k}|\prod_{j=1}^{i-1}|\widehat{F}_j|\prod_{m=i+1}^{n}|\widehat{H}_{m,k}|\\ &\leqslant \quad b_k\sum_{i=1}^{n}|\widehat{H}_{i,k}||\widehat{F}_i-1|^{k+1}\prod_{j=1}^{n}\exp\left\{c_{j,k}(Re\,\widehat{F}_j-1)\right\}\\ &\leqslant \quad b_k\sum_{i=1}^{n}|\widehat{F}_i-1|^{k+1}\exp\left\{\sum_{j=1}^{n}c_{j,k}(Re\,\widehat{F}_j-1)\right\}\\ &\leqslant \quad b_k\widehat{\Psi}_k^2\sum_{i=1}^{n}|\widehat{F}_i-1|^{k+1}. \end{aligned}\tag{4.49}$$

Next, applying estimate $xe^{-x^2}\leqslant 1/\sqrt{2e}$, $x>0$, we get

$$\begin{aligned}\widehat{\Psi}_k|\widehat{F}_i-1|^{k+1}=\widehat{\Psi}_k\left(2\sum_{\vec{x}}p_i(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|\right)^{k+1}&\\ \leqslant \quad \left(2\sum_{\vec{x}}p_i(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|\right.&\\ \left.\times\exp\left\{\frac{-1}{k+1}\sum_{j=1}^{n}\sum_{\vec{y}}c_{j,k}p_j(\vec{y})\sin^2\frac{(\vec{y},\vec{t})}{2}\right\}\right)^{k+1}&\\ \leqslant \quad 2^{k+1}\left(\sum_{\vec{x}}p_i(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|\right.&\\ \left.\times\exp\left\{\frac{-1}{k+1}\sum_{j=1}^{n}c_{j,k}p_j(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right\}\right)^{k+1}&\\ \leqslant \quad \left(\frac{2(k+1)}{e}\right)^{(k+1)/2}\left(\sum_{\vec{x}\neq\vec{0}}\frac{p_i(\vec{x})}{\sqrt{\lambda_k(\vec{x})}}\right)^{k+1}.&\end{aligned}\tag{4.50}$$

Collecting estimates (4.49), (4.50) and applying (A.36) and Lemma 4.5.2

we complete the proof of (4.11). □

Proof of Theorem 4.3.2. By the triangle inequality

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_\infty \leqslant \|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_\infty + \|\mathbf{H}_{\mathrm{n},2}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_\infty. \quad (4.51)$$

From Theorem 4.3.1 we have

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_\infty \leqslant 6^d C\omega_{d,2}^{-1/2}\sum_{i=1}^n \Lambda_{i,2}^3.$$

Lemma 4.5.4, (A.36) and Lemma 4.5.2 allow to prove that

$$\begin{aligned}
&\|\mathbf{H}_{\mathrm{n},2}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_\infty \\
&\quad\leqslant \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}|\widehat{\mathbf{H}}_{\mathrm{n},2}-\widehat{\mathbf{H}}_{\mathrm{n},1}*(1+\widehat{\mathscr{A}_1})|\,\mathrm{d}t_1\dots\mathrm{d}t_d \\
&\quad\leqslant \frac{C}{(2\pi)^d}\left(\sum_{i=1}^n\Lambda_{i,2}^2\right)^2\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\widehat{\Psi}_k\,\mathrm{d}t_1\dots\mathrm{d}t_d \\
&\quad\leqslant C6^d\omega_{d,2}^{-1/2}\left(\sum_{i=1}^n\Lambda_{i,2}^2\right)^2.
\end{aligned}$$

Substituting these estimates into (4.51) we complete the proof of first estimate in Theorem 4.3.2. The proof of the second estimate is very similar and, therefore, omitted. □

Proof of Theorem 4.3.3. Let

$$f_i := \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\widehat{F}_i, \qquad g_i := \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}}\widehat{H}_{i,k}.$$

Then

$$\begin{aligned}
\left|\frac{\partial}{\partial t_j}\left(\prod_{i=1}^n f_i-\prod_{i=1}^n g_i\right)\right| \;\leqslant\; &\sum_{i=1}^n\left|\frac{\partial f_i}{\partial t_j}\right|\left|\prod_{m\neq i}^n f_m-\prod_{m\neq i}^n g_m\right| \\
&+\sum_{i=1}^n\left|\frac{\partial}{\partial t_j}(f_i-g_i)\prod_{m\neq i}^n g_m\right|.
\end{aligned}$$

Due to the fact that $c_{i,k}\leqslant q_i\leqslant 1$ and $|Re\,\widehat{F}_i-1|\leqslant 2$ we have the following

simple inequality

$$\begin{aligned} 1 &= \exp\{c_{i,k}(Re\,\widehat{F}_i-1)\}\exp\{c_{i,k}(1-Re\widehat{F}_i)\} \\ &\leqslant \mathrm{e}^2\exp\{c_{i,k}(Re\,\widehat{F}_i-1)\}. \end{aligned} \tag{4.52}$$

Then similarly to the proof of Theorem 1 from [48]:

$$\begin{aligned} \left|\prod_{m\neq i}^{n} f_m-\prod_{m\neq i}^{n} g_m\right| &= \left|\prod_{m\neq i}^{n}\widehat{F}_m-\prod_{m\neq i}^{n}\widehat{H}_{m,k}\right| \\ &\leqslant \sum_{m\neq i}^{n}|\widehat{F}_m-\widehat{H}_{m,k}|\prod_{l\neq i,m}^{n}\exp\{c_{l,k}(Re\,\widehat{F}_l-1)\} \\ &\leqslant \mathrm{e}^4\widehat{\Psi}_k^2\sum_{m\neq i}^{n}|\widehat{F}_m-\widehat{H}_{m,k}|\leqslant C(k)\widehat{\Psi}_k^2\sum_{m\neq i}^{n}|\widehat{F}_m-1|^{k+1} \\ &\leqslant C(k)\widehat{\Psi}_k^2\sum_{m=1}^{n}|\widehat{F}_m-1|^{k+1}\leqslant C(k)\widehat{\Psi}_k^{3/2}\sum_{m=1}^{n}\Lambda_{m,k}^{k+1}. \end{aligned} \tag{4.53}$$

The same simple inequality $x\mathrm{e}^{-x}\leqslant 1/\mathrm{e}$, $x>0$ and Lemma 4.5.5 allows to prove

$$\begin{aligned} \widehat{\Psi}_k^{1/2}\left|\frac{\partial f_i}{\partial t_j}\right| &\leqslant C(k)\sqrt{\beta_{i,j}^{(2)}}\left(\widehat{\Psi}_k\sum_{\vec{x}}p_i(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right)^{1/2} \\ &\leqslant C(k)\sqrt{\beta_{i,j}^{(2)}}\left(\sum_{\vec{x}}p_i(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right. \\ &\qquad\left.\times\exp\left\{-\sum_{j=1}^{n}c_{j,k}\sum_{\vec{x}}p_i(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right\}\right)^{1/2} \\ &\leqslant C(k)\sqrt{\beta_{i,j}^{(2)}}\left(\sum_{\vec{x}}p_i(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right. \\ &\qquad\left.\times\exp\left\{-\sum_{j=1}^{n}c_{j,k}p_i(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right\}\right)^{1/2} \\ &\leqslant C(k)\sqrt{\beta_{i,j}^{(2)}}\left(\sum_{\vec{x}}\frac{p_i(\vec{x})}{\lambda_k(\vec{x})}\right)^{1/2}=C(k)\sqrt{\beta_{i,j}^{(2)}L_{i,k}}. \end{aligned} \tag{4.54}$$

Applying (4.52) and (4.29) we obtain

$$\prod_{m\neq i}^{n}|g_m|=\prod_{m\neq i}^{n}|\widehat{H}_{m,k}|\leqslant\prod_{m\neq i}\exp\left\{c_{m,k}(Re\,\widehat{F}_m-1)\right\}\leqslant\mathrm{e}^2\widehat{\Psi}_k^2.$$

Taking into account Lemma 4.5.6 and (4.50) we can write

$$\begin{aligned}\sum_{i=1}^{n}\left|\frac{\partial}{\partial t_j}(f_i-g_i)\right|\prod_{m\neq i}^{n}|g_m| &\leqslant C(k)\widehat{\Psi}_k^2\sum_{i=1}^{n}\beta_{i,j}|\widehat{F}_i-1|^k\\ &\leqslant C(k)\widehat{\Psi}_k\sum_{i=1}^{n}\beta_{i,j}\Lambda_{i,k}^k.\end{aligned}$$

Collecting above estimates we get

$$\begin{aligned}\left|\frac{\partial}{\partial t_j}\left(\prod_{i=1}^{n}f_i-\prod_{i=1}^{n}g_i\right)\right| &\leqslant C(k)\widehat{\Psi}_k\Bigg\{\sum_{i=1}^{n}\beta_{i,j}\Lambda_{i,k}^k\\ &\quad+\sum_{i=1}^{n}\Lambda_{i,k}^{k+1}\sum_{l=1}^{n}\sqrt{\beta_{l,j}^{(2)}L_{l,k}}\Bigg\}.\end{aligned}\tag{4.55}$$

Applying (A.37) and Lemma 4.5.2 we prove the first estimate in Theorem 4.3.3. Similarly we get

$$\begin{aligned}\left|\frac{\partial^2}{\partial t_j^2}\left(\prod_{i=1}^{n}f_i-\prod_{i=1}^{n}g_i\right)\right| &\leqslant C(k)\Bigg\{\sum_{k=1}^{n}\Lambda_{k,k}^{k+1}\sum_{i=1}^{n}\left|\frac{\partial^2 f_i}{\partial t_j^2}\right|\widehat{\Psi}_k\\ &\quad+\widehat{\Psi}_k^2\sum_{i=1}^{n}\left|\frac{\partial^2}{\partial t_j^2}(f_i-g_i)\right|+\widehat{\Psi}_k^2\sum_{i=1}^{n}\left|\frac{\partial f_i}{\partial t_j}\right|\sum_{m\neq i}\left|\frac{\partial}{\partial t_j}(f_m-g_m)\right|\\ &\quad+\widehat{\Psi}_k^2\sum_{i=1}^{n}\left|\frac{\partial}{\partial t_j}(f_i-g_i)\right|\sum_{m\neq i}\left|\frac{\partial g_m}{\partial t_j}\right|\\ &\quad+\widehat{\Psi}_k^{3/2}\sum_{l=1}^{n}\Lambda_{l,k}^{k+1}\sum_{i=1}^{n}\left|\frac{\partial f_i}{\partial t_j}\right|\sum_{m\neq i}\left|\frac{\partial f_m}{\partial t_j}\right|\Bigg\}.\end{aligned}\tag{4.56}$$

Taking into account Lemmas 4.5.1, 4.5.3, 4.5.5, 4.5.6 and arguing similarly to the proof of (4.50) we obtain

$$\widehat{\Psi}_k\left|\frac{\partial^2}{\partial t_j^2}(f_i-g_i)\right|\leqslant C(k)\beta_{i,j}^{(2)}\Lambda_{i,k}^k+C(k)\beta_{i,j}^2\Lambda_{i,k}^{k-1},\tag{4.57}$$

$$\widehat{\Psi}_k^{1/2}\left|\frac{\partial f_i}{\partial t_j}\right|\leqslant C\sqrt{\beta_{i,j}^{(2)}L_{i,k}},\tag{4.58}$$

$$\widehat{\Psi}_k^{1/2}\left|\frac{\partial}{\partial t_j}(f_i-g_i)\right|\leqslant C(k)\beta_{i,j}\Lambda_{i,k}^k,\ \widehat{\Psi}_k^{1/2}\left|\frac{\partial g_i}{\partial t_j}\right|\leqslant C\sqrt{\beta_{i,j}^{(2)}L_{i,k}}.\tag{4.59}$$

Substituting above estimates into (4.56) and applying (A.38) and Lemma 4.5.2 we prove the second estimate in Theorem 4.3.3. The third estimate in Theorem

4.3.3 is proved very similarly. We give a sketch of the proof. Set

$$\tilde{f}_i := \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}-\mathrm{i}t_m\mu_{i,m}}\widehat{F}_i, \qquad \tilde{g}_i := \mathrm{e}^{-\mathrm{i}t_j\mu_{i,j}-\mathrm{i}t_m\mu_{i,m}}\widehat{H}_{i,k}.$$

Applying (4.52) we get

$$\begin{aligned}\left|\frac{\partial^2}{\partial t_j\partial t_m}\left(\prod_{i=1}^n\tilde{f}_i-\prod_{i=1}^n\tilde{g}_i\right)\right| &\leqslant C(k)\widehat{\Psi}_k\sum_{l=1}^n\Lambda_{l,k}^{k+1}\sum_{i=1}^n\left|\frac{\partial^2\tilde{f}_i}{\partial t_j\partial t_m}\right| \\ &+C(k)\sum_{i=1}^n\left|\frac{\partial\tilde{f}_i}{\partial t_j}\right|\left|\frac{\partial}{\partial t_m}\left(\prod_{l\neq i}\tilde{f}_l-\prod_{l\neq i}\tilde{g}_l\right)\right| \\ &+C(k)\widehat{\Psi}_k^2\sum_{i=1}^n\left|\frac{\partial^2}{\partial t_j\partial t_m}(\tilde{f}_i-\tilde{g}_i)\right| \\ &+C(k)\widehat{\Psi}_k^2\sum_{i=1}^n\left|\frac{\partial}{\partial t_j}(\tilde{f}_i-\tilde{g}_i)\right|\sum_{l=1}^n\left|\frac{\partial\tilde{g}_l}{\partial t_m}\right|. \qquad (4.60)\end{aligned}$$

We have

$$\begin{aligned}\left|\frac{\partial^2\tilde{f}_i}{\partial t_j\partial t_m}\right| &\leqslant \left|\sum_{\vec{x}}(x_j-\mu_{i,j})(x_m-\mu_{i,m})p_i(\vec{x})\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})-\mathrm{i}t_j\mu_{i,j}-\mathrm{i}t_m\mu_{i,m}}\right| \\ &\leqslant \sum_{\vec{x}}(|x_j|+\beta_{i,j})(|x_m|+\beta_{i,m})p_i(\vec{x}) = \beta_{i,jm}+3\beta_{i,j}\beta_{i,m} \\ &\leqslant 3(\beta_{i,jm}+\beta_{i,j}\beta_{i,m}).\end{aligned}$$

From Lemma 4.5.7 and Lemma 4.5.1 we get

$$\begin{aligned}\widehat{\Psi}_k\left|\frac{\partial^2}{\partial t_j\partial t_m}(\tilde{f}_i-\tilde{g}_i)\right| &\leqslant C(k)\beta_{i,jm}\widehat{\Psi}_k|\widehat{F}_i-1|^k \\ &\quad +C(k)\beta_{i,j}\beta_{i,m}\widehat{\Psi}_k|\widehat{F}_i-1|^{k-1} \\ &\leqslant C(k)\beta_{i,jm}\Lambda_{i,k}^k+C(k)\beta_{i,j}\beta_{i,m}\Lambda_{i,k}^{k-1}.\end{aligned}$$

We substitute the last two estimates into (4.60). It is easy to see, that estimates (4.53), (4.54), (4.55), (4.59) also hold for $\tilde{f}_i,\tilde{g}_i$ and can be substituted into (4.60). It remains to use (A.39) and Lemma 4.5.2. □

Proof of Theorem 4.3.4. Let $\widehat{M}=\prod_{i=1}^n\widehat{F}_i-\prod_{i=1}^n\widehat{H}_{i,k}$. From (4.49) and (4.50) it follows that

$$|\widehat{M}|\leqslant\widehat{\Psi}_k^2 b_k\sum_{i=1}^n|\widehat{F}_i-1|^{k+1}\leqslant\widehat{\Psi}_k b_k\left(\frac{2(k+1)}{\mathrm{e}}\right)^{(k+1)/2}\sum_{i=1}^n\Lambda_{i,k}^{k+1}.$$

Repeating the proof of Lemma 4.5.2 we obtain

$$\frac{1}{(2\pi)^d}\int\limits_{-\pi}^{\pi}\cdots\int\limits_{-\pi}^{\pi}\widehat{\Psi}_k^2\,\mathrm{d}t_1\cdots\mathrm{d}t_d\leqslant\left(3\sqrt{2}\right)^d\omega_{d,k}^{-1/2}.$$

Indeed

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left\{-2\sum_{j\in\mathbb{Z}\setminus\{0\}}\sum_{i=1}^{n}c_{i,k}p_{i,m}^0(j)\sin^2\frac{jt_m}{2}\right\}\mathrm{d}t_m\\ \leqslant\ \min\left(1;\frac{6}{\sqrt{2\lambda_{m,k}^0}}\right)\leqslant 3\sqrt{2}\min\left(1;\frac{1}{\sqrt{\lambda_{m,k}^0}}\right)\\ =\ \frac{3\sqrt{2}}{\sqrt{\max(1,\lambda_{m,k}^0)}}.\end{aligned}$$

Substituting above estimates into (A.35) we prove estimate

$$\|M\|_2\leqslant\left(3\sqrt{2}\right)^{d/2}\omega_{d,k}^{-1/4}b_k\left(\frac{2(k+1)}{\mathrm{e}}\right)^{(k+1)/2}\sum_{i=1}^{n}\Lambda_{i,k}^{k+1}. \tag{4.61}$$

Combining (4.61) with Theorem 4.3.1 and simple estimate

$$\|M\|_r\leqslant\left(\|M\|_\infty\right)^{(r-2)/r}\left(\|M\|_2\right)^{2/r} \tag{4.62}$$

we prove (4.14). □

Proof of Theorem 4.3.5. The proof of Theorem 4.3.5 is very similar to the proof of Theorem 4.3.2. By the triangle inequality

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_r\leqslant\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_r+\|\mathbf{H}_{\mathrm{n},2}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_r. \tag{4.63}$$

From Theorem 4.3.4 we have

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_r\leqslant 6^{d(2r-3)/2r}b_2\left(\frac{6}{\mathrm{e}}\right)^{3/2}\omega_{d,2}^{-(r-1)/2r}\sum_{i=1}^{n}\Lambda_{i,2}^3.$$

Consequently applying Lemma 4.5.4, Lemma A.4.1 and Lemma 4.5.2 we

prove that

$$\begin{aligned}\|\mathbf{H}_{\mathrm{n},2}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_\infty &\leqslant 6^d C\omega_{d,2}^{-1/2}\left(\sum_{i=1}^n \Lambda_{i,2}^2\right)^2,\\ \|\mathbf{H}_{\mathrm{n},2}-\mathbf{H}_{\mathrm{n},1}*(I+\mathscr{A}_1)\|_2 &\leqslant C(3\sqrt{2})^{d/2}\omega_{d,2}^{-1/4}\left(\sum_{i=1}^n \Lambda_{i,2}^2\right)^2.\end{aligned}$$

For $r\geqslant 2$, the proof of the first estimate in Theorem 4.3.5 is completed by application of (6.20), (4.14), the triangle inequality and (4.63). The proof of the second estimate is very similar and, therefore, omitted. □

Proof of Theorem 4.3.6. Let $\vec{t}_{\mathbf{h}}=(t_1/h_1;t_2/h_2;\dots;t_d/h_d)$ and let $J(\vec{t}),J_1(\vec{t}),\dots,J_5(\vec{t})$ be defined as in Lemma 4.5.8. Note that due to assumptions $\mu_{i,m}=\beta_{i,m}$. Let

$$V(\vec{\mu},\vec{h})=\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\prod_{m=1}^d \psi_m(t_m)J(\vec{t}_{\mathbf{h}})\,\mathrm{d}t_1\mathrm{d}t_2\dots\mathrm{d}t_d.$$

Then

$$\begin{aligned}|V(\vec{\mu},\vec{h})| &\geqslant \left|\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\prod_{m=1}^d \psi_m(t_m)J_1(\vec{t}_{\mathbf{h}})\mathrm{d}t_1\dots\mathrm{d}t_d\right|\\ &\quad-\sum_{l=2}^5\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\prod_{m=1}^d|\psi_m(t_m)||J_l(\vec{t}_{\mathbf{h}})|\mathrm{d}t_1\dots\mathrm{d}t_d\\ &=: \frac{1}{k+1}|\mathbf{I}_1|-\sum_{l=2}^5\mathbf{I}_l.\end{aligned}\tag{4.64}$$

Observe that, for any $j\in\{1,2,\dots\}$,

$$\int_{-\infty}^{\infty}t^{2j+1}\mathrm{e}^{-t^2/2}\mathrm{d}t=0,\quad \int_{-\infty}^{\infty}t^{2j}\mathrm{e}^{-t^2/2}\mathrm{d}t=(2j-1)!!\sqrt{2\pi}.$$

Moreover

$$\left(\sum_{m=1}^d\frac{\mu_{i,m}}{h_m}\right)^{k+1}\leqslant d^{k+1}\left(\max_m\frac{\mu_{i,m}}{h_m}\right)^{k+1}\leqslant d^{k+1}\sum_{m=1}^d\frac{\mu_{i,m}^{k+1}}{h_m^{k+1}}.\tag{4.65}$$

We have

$$\left(\sum_{m=1}^{d}\mu_{i,m}\frac{t_m}{h_m}\right)^{k+1}=\sum_{m=1}^{d}\mu_{i,m}^{k+1}\frac{t_m^{k+1}}{h_m^{k+1}}+\dots$$

Let k be odd. Then we choose $\psi_m(t_m)=\mathrm{e}^{-t_m^2/2}$, $m=1,\dots,d$, and after integration and application of (4.65) we obtain

$$\begin{aligned}|\mathbf{I}_1| &\geqslant (1\cdot 3\cdot 5\cdot\ldots\cdot k)(2\pi)^{d/2}\sum_{i=1}^{n}\sum_{m=1}^{d}\frac{\mu_{i,m}^{k+1}}{h_m^{k+1}}\\ &\geqslant \frac{(1\cdot 3\cdot 5\cdot\ldots\cdot k)(2\pi)^{d/2}}{d^{k+1}}\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\frac{\mu_{i,m}}{h_m}\right)^{k+1}.\end{aligned}$$

Now let k be even. Then by choosing $\psi_m(t_m)=t_m\mathrm{e}^{-t_m^2/2}$, $\psi_l(t_l)=\mathrm{e}^{-t_l^2/2}$, $l,m=1,2,\dots,d$, $l\neq m$, we get

$$\begin{aligned}|\mathbf{I}_1| &\geqslant (1\cdot 3\cdot\ldots\cdot(k+1))(2\pi)^{d/2}\sum_{i=1}^{n}\left(\max_m\frac{\mu_{i,m}}{h_m}\right)^{k+1}\\ &\geqslant \frac{(1\cdot 3\cdot 5\cdot\ldots\cdot(k+1))(2\pi)^{d/2}}{d^{k+1}}\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\frac{\mu_{i,m}}{h_m}\right)^{k+1}.\end{aligned}$$

Thus, for any $k\in\mathbb{N}$, we have

$$|\mathbf{I}_1|\geqslant C_1(d,k)\sum_{i=1}^{n}\left(\sum_{m=1}^{d}\frac{\mu_{i,m}}{h_m}\right)^{k+1}. \tag{4.66}$$

Substituting (4.66) and estimates for $J(\vec{t}_{\mathbf{h}})$ from Lemma 4.5.8 into (4.64) and applying Lemma A.4.3 we complete the proof of Theorem 4.3.6. □

Proof of Theorem 4.3.7. The proof is very similar to the proof of Theorem 4.3.6. The only difference is more elaborate estimate for $|\mathbf{I}_1|$. We have

$$J_1(\vec{t}_{\mathbf{h}})=\frac{1}{2}\sum_{m=1}^{d}t_m^2\sum_{i=1}^{n}\left(\frac{\mu_{i,m}}{h_m}\right)^2+\sum_{m=1}^{d}\sum_{l=m+1}^{d}t_mt_l\sum_{i=1}^{n}\frac{\mu_{i,m}\mu_{i,l}}{h_mh_l}.$$

Let $\psi_m=\mathrm{e}^{-t_m^2/2}$, $m=1,\dots,d$. Then after integration we get

$$|\mathbf{I}_1|=(2\pi)^{d/2}\sum_{m=1}^{d}\sum_{i=1}^{n}\left(\frac{\mu_{i,m}}{h_m}\right)^2.$$

Next, by choosing all possible versions of $\psi_m(t_m) = t_m \mathrm{e}^{-t_m^2/2}$; $\psi_l(t_l) = t_l \mathrm{e}^{-t_l^2/2}$, $\psi_j(t_j) = \mathrm{e}^{-t_j^2/2}$, $j \neq m, l$ we arrive at the estimate

$$|\mathbf{I}_1| = (2\pi)^{d/2} \max_{m<l} \left| \sum_{i=1}^{n} \frac{\mu_{i,m}\mu_{i,l}}{h_m h_l} \right| \geqslant \frac{(2\pi)^{d/2}}{d(d-1)} \sum_{m<l}^{d} \left| \sum_{i=1}^{n} \frac{\mu_{i,m}\mu_{i,l}}{h_m h_l} \right|.$$

Therefore

$$\begin{aligned}
|\mathbf{I}_1| &\geqslant \frac{(2\pi)^{d/2}}{d(d-1)} \max \left(\sum_{m=1}^{d} \sum_{i=1}^{n} \left(\frac{\mu_{i,m}}{h_m} \right)^2 ; \sum_{m<l}^{d} \left| \sum_{i=1}^{n} \frac{\mu_{i,m}\mu_{i,l}}{h_m h_l} \right| \right) \\
&\geqslant \frac{(2\pi)^{d/2}}{d(d-1)} \left(\frac{1}{2} \sum_{m=1}^{d} \sum_{i=1}^{n} \left(\frac{\mu_{i,m}}{h_m} \right)^2 + \frac{1}{2} \sum_{m<l}^{d} \left| \sum_{i=1}^{n} \frac{\mu_{i,m}\mu_{i,l}}{h_m h_l} \right| \right) \\
&\geqslant \frac{(2\pi)^{d/2}}{2d(d-1)} \left| \sum_{m=1}^{d} \sum_{i=1}^{n} \left(\frac{\mu_{i,m}}{h_m} \right)^2 + \sum_{m<l}^{d} \sum_{i=1}^{n} \frac{\mu_{i,m}\mu_{i,l}}{h_m h_l} \right| \\
&= \frac{(2\pi)^{d/2}}{2d(d-1)} \sum_{i=1}^{n} \left(\sum_{m=1}^{d} \frac{\mu_{i,m}}{h_m} \right)^2 .
\end{aligned}$$

It remains to use Lemma 4.5.8 for the estimating of $\mathbf{I}_l$, $l = 2,3,4,5$ and Lemma A.4.3. □

Proof of Theorem 4.3.8. Let $h_m = a\sqrt{\lambda^0_{m,k}}$. Observe that due to assumptions we have

$$\lambda^0_{m,k} \geqslant \frac{3C_0 - 2}{2C_0 - 1} \sum_{i=1}^{n} \sum_{l=1}^{K} p^0_{i,m}(l), \qquad \beta^{(2)}_{i,m} = \sum_{l=1}^{K} l^2 p^0_{i,m}(l) \leqslant K^2 \sum_{l=1}^{K} p^0_{i,m}(l).$$

Therefore,

$$\sum_{i=1}^{n} \sum_{m=1}^{d} \frac{\beta^{(2)}_{i,m}}{\lambda^0_{m,k}} \leqslant K^2 \sum_{m=1}^{d} \frac{1}{\lambda^0_{m,k}} \sum_{i=1}^{n} \sum_{l=1}^{K} p^0_{i,m}(l) \leqslant \frac{K^2(2C_0 - 1)}{3C_0 - 2}.$$

Similarly,

$$\begin{aligned}
&\mu_{i,m} = \sum_{l=1}^{K} l p^0_{i,m}(l) \leqslant K \sum_{l=1}^{K} p^0_{i,m}(l), \qquad \mu^2_{i,m} \leqslant K^2 \sum_{l=1}^{K} p^0_{i,m}(l), \\
&B_i^2 \leqslant \frac{d^2}{a^2} \sum_{m=1}^{d} \frac{\mu^2_{i,m}}{\lambda^0_{m,k}} \leqslant \frac{d^2 K^2}{a^2} \sum_{m=1}^{d} \frac{1}{\lambda^0_{m,k}} \sum_{l=1}^{K} p^0_{i,m}(l),
\end{aligned}$$

$$\sum_{i=1}^{n} B_i^2 \leqslant \frac{d^3K^2(2C_0-1)}{a^2(3C_0-2)}.$$

Next observe that $\beta_{i,m}^{(2)} \leqslant K\mu_{i,m}$, $\lambda_{m,k}^0 \geqslant \sqrt{\lambda_{m,k}^0}$, $B_i \leqslant dK/a$ and

$$\sum_{m=1}^{d} \frac{\beta_{i,m}^{(2)}}{h_m^2} \leqslant \frac{KB_i}{a^2}, \sum_{i=1}^{n} B_{i,k}^{k+1} \leqslant \left(\frac{dK}{a}\right)^{k-1} \sum_{i=1}^{n} B_i^2 \leqslant \frac{d^{k+2}K^{k+1}(2C_0-1)}{a^{k+1}(3C_0-2)}.$$

Substituting these estimates into (4.18) and noting that $a^{k+3}, a^{2(k+1)} \geqslant a^{k+2}$ we get the estimate

$$\varepsilon_{d,k}(a) \geqslant \frac{C_1(d,k)}{a^{k+1}} \sum_{i=1}^{n} \left(\sum_{m=1}^{d} \frac{\mu_{i,m}}{\sqrt{\lambda_{m,k}^0}} \right)^{k+1} \left(1 - \frac{C_2(d,k,K)}{a}\right),$$

for some constant $C(d,k,K)$. To complete the proof of Corollary it suffices to take $a = 2C_2(d,k,K)$. □

Proof of Example 4.3.6. We have $\mu_{i,m} = p_{i,m} = p_{i,m}^0$, $\lambda_{m,1}^0 \geqslant C_0\lambda(m)$, $\lambda_{m,1}^0 \geqslant 1$. This allows to get the estimate

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1}\|_\infty \geqslant C(d,k)\, \omega_{d,1}^{-1/2} \sum_{i=1}^{n} \min\left(\left(\sum_{m=1}^{d} \frac{p_{i,m}^0}{\sqrt{\bar{\lambda}(m)}} \right)^2, \left(\sum_{m=1}^{d} p_{i,m}^0 \right)^2 \right).$$

It remains to note that

$$\left(\sum_{m=1}^{d} \frac{p_{i,m}^0}{\sqrt{\bar{\lambda}(m)}} \right)^2 \geqslant \sum_{m=1}^{d} \frac{(p_{i,m}^0)^2}{\bar{\lambda}(m)}.$$

□

Proof of Proposition 4.3.10. We recall that $u_j := \widehat{F}_j(t_j) - 1$, $\bar{u}_j := \widehat{F}_j(-t_j) - 1$. Note that $\bar{u}_j$ is conjugate to u_j and

$$Re\, u_j = Re\, \bar{u}_j = Re\, \widehat{F}_j(t_j) - 1 < 0,\ u_j + \bar{u}_j = 2(Re\, \widehat{F}_j(t_j) - 1), \quad (4.67)$$

$$u_j \bar{u}_j = |u_j|^2. \quad (4.68)$$

It is not difficult to check that

$$Re(\widehat{F}_{3\star}-1)\leqslant 0.$$

Thus by Lemma 4.5.1

$$\begin{aligned}|\widehat{H}_{3\star}| &\leqslant \left|\exp\left\{(\widehat{F}-1)-\frac{(\widehat{F}-1)^2}{2}\right\}\right| \\ &\leqslant \exp\left\{q\sum_{m=1}^{d}p_m(Re\,\widehat{F}_m(t_m)-1)\right\}. \qquad (4.69)\end{aligned}$$

Applying Lemma 4.5.3 and (4.69) we get

$$\begin{aligned}|\widehat{F}^n-\widehat{H}^n_{3\star}| &\leqslant Cn\exp\left\{nq\sum_{j=1}^{d}p_j(Re\,F_j(t_j)-1)\right\} \\ &\quad\times\left(|\widehat{F}-\widehat{H}_3|+|\widehat{H}_3-\widehat{H}_{3\star}|\right) \\ &\leqslant C(d)n\exp\left\{nq\sum_{j=1}^{d}p_j(Re\,\widehat{F}_j(t_j)-1)\right\} \\ &\quad\times\left(|\widehat{F}-1|^4+|(\widehat{F}-1)^3-(\widehat{F}_{3\star}-1)|\right). \qquad (4.70)\end{aligned}$$

Next observe, that

$$|\widehat{F}-1|^4\leqslant\left(\sum_{j=1}^{d}p_j|\widehat{F}_j-1|\right)^4\leqslant C(d)\sum_{j=1}^{d}p_j^4|\widehat{F}_j-1|^4\leqslant C(d)\sum_{j=1}^{d}p_j^4(Re\,\widehat{F}_j-1)^2$$

and by (4.67) and (4.68)

$$\begin{aligned}|u_m^3-\bar{u}_m|u_m|^2| &= |u_m||u_m-\bar{u}_m||u_m+\bar{u}_m| \\ &\leqslant 4|u_m|^2|Re\,\widehat{F}_j-1|\leqslant 8(Re\,\widehat{F}_j-1)^2, \\ |u_m^2u_j-|u_m|^2\bar{u}_j| &= |u_m||u_mu_j-\bar{u}_m\bar{u}_j| \\ &= |u_m||u_m(u_j+\bar{u}_j)-\bar{u}_j(u_m+\bar{u}_m)| \\ &\leqslant C(|Re\,\widehat{F}_m-1||Re\,\widehat{F}_j-1| \\ &\quad+|Re\,\widehat{F}_m-1|^{3/2}|Re\,\widehat{F}_j-1|^{1/2}).\end{aligned}$$

Similarly estimating other differences we arrive at

$$
\begin{aligned}
|(\widehat{F}-1)^3-(\widehat{F}_{3\star}-1)| &\leqslant C(d)\sum_{m=1}^{d} p_m^3|Re\,\widehat{F}_m-1|^2\\
&\quad +C(d)\sum_{j\neq m}^{d} p_j^2 p_m\Big(|Re\,\widehat{F}_j-1||Re\,\widehat{F}_m-1|\\
&\quad +|Re\,\widehat{F}_j-1|^{3/2}|Re\,\widehat{F}_m-1|^{1/2}\Big)\\
&= C(d)\sum_{m<j<l} p_m p_j p_l\Big(|Re\,\widehat{F}_m-1|^{1/2}|Re\,\widehat{F}_j-1|^{1/2}|Re\,\widehat{F}_l-1|\\
&\quad +|Re\,\widehat{F}_m-1|^{1/2}|Re\,\widehat{F}_j-1||Re\,\widehat{F}_l-1|^{1/2}\\
&\quad +|Re\,\widehat{F}_m-1||Re\,\widehat{F}_j-1|^{1/2}|Re\,\widehat{F}_l-1|^{1/2}\Big). \qquad (4.71)
\end{aligned}
$$

Substituting (4.71) into (4.70) and applying inequality $xe^{-x}\leqslant C$, $x>0$ we prove that

$$
|\widehat{F}^n-\widehat{H}^n_{3\star}|\leqslant C(nq)^{-2}(1-q)\exp\left\{\frac{nq}{2}\sum_{j=1}^{d} p_j(Re\,\widehat{F}_j(t_j)-1)\right\}.
$$

It remains to apply (A.36) and observe that by (A.6)

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\exp\left\{\frac{nq}{2}\sum_{j=1}^{d} p_j(Re\,\widehat{F}_j(t_j)-1)\right\}\mathrm{d}t_1\dots\mathrm{d}t_d\\
&\qquad\qquad \leqslant \frac{C(d)}{(nq)^{d/2}}\prod_{m=1}^{d}\frac{1}{\sqrt{p_m}}. \qquad (4.72)
\end{aligned}
$$

□

Proof of (4.23). We have

$$
e^{it_1+it_2}-1=(e^{it_1}-1)+(e^{it_2}-1)+(e^{it_1}-1)(e^{it_2}-1).
$$

Therefore

$$
\begin{aligned}
Re\widehat{F}_i-1 &\leqslant -2(p(1,0)+p(1,1))\sin^2(t_1/2)-2(p(0,1)\\
&\quad +p(1,1))\sin^2(t_2/2)\\
&\quad +4p(1,1)|\sin(t_1/2)\sin(t_2/2)|.
\end{aligned}
$$

Observing that

$$\begin{aligned} 4p(1,1)|\sin(t_1/2)\sin(t_2/2)| &= 2\Big(2\sqrt{p(1,1)}|\sin(t_1/2)|\Big) \\ &\quad \times\Big(\sqrt{p(1,1)}|\sin(t_2/2)|\Big) \\ &\leqslant 4p(1,1)\sin^2(t_1/2)+p(1,1)\sin^2(t_2/2) \end{aligned}$$

and $q = 1-p(1,0)-p(0,1)-p(1,1) = 0.7-1/n \geqslant 0.45$, we obtain the estimate

$$\begin{aligned} \widehat{\Psi}_2(t_1,t_2) &\leqslant \exp\big\{-0.45n\big(2(p(1,0)-p(1,1))\sin^2(t_1/2) \\ &\quad +2(p(0,1)+p(1,1)/2)\sin^2(t_2/2)\big)\big\} \\ &\leqslant \exp\big\{-0.09n\sin^2(t_1/2)-0.045n\sin^2(t_2/2)\big\}. \end{aligned}$$

Therefore, instead of Lemma 4.5.2 we can use estimate

$$\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\widehat{\Psi}_2(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 \leqslant Cn^{-1}$$

to prove (4.23). □

Proof of Example 4.4.2. To make the proof more explicit we use notation

$$f = q+n^{-1}\mathrm{e}^{\mathrm{i}t_1}+n^{-1}\mathrm{e}^{\mathrm{i}t_2}+0.2\mathrm{e}^{\mathrm{i}(t_1+t_2)}, \quad g = \exp\{(f-1)-(f-1)^2/2\}$$

instead of $\widehat{F}_i$ and $\widehat{H}_{i,2}$. Without loss of generality we assume further that $n \geqslant 40$. Indeed, if $n < 40$, then by (A.35) and(4.29)

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_\infty \leqslant \sup_{\vec{t}}|f^n(\vec{t})-g^n(\vec{t})| \leqslant 2 \leqslant 160n^{-1}.$$

We have $0.7 \leqslant q = 0.8-2/n \leqslant 0.8$. Therefore, taking into account Lemma 4.5.3, (4.28), (4.29) and inequality $x^3\mathrm{e}^{-x^2} \leqslant Cn^{-3/2}$, $x > 0$ we obtain

$$\begin{aligned} |f^m-g^m| &\leqslant Cm|f-g|\mathrm{e}^{0.7m(Ref-1)} \\ &\leqslant Cm(1-Ref)^{3/2}\mathrm{e}^{0.7m(Ref-1)} \leqslant Cm^{-1/2}. \end{aligned} \tag{4.73}$$

Similarly

$$|f-g||f|^m \leqslant Cm^{-3/2}, \quad |f-g||g|^m \leqslant Cm^{-3/2}. \tag{4.74}$$

Let $\lfloor n/2\rfloor$ denote the integer part of $n/2$. It is easier to apply (4.24) to initial

distribution than to CP or SCP approximation. Therefore we write

$$f^n - g^n = (f-g)\sum_{i=1}^{n} f^{i-1}g^{n-i} = (f-g)\sum_{i=\lfloor n/2\rfloor+1}^{n} f^{i-1}g^{n-i}$$
$$+(f-g)\sum_{i=1}^{\lfloor n/2\rfloor} f^{i-1}g^{(n-i)-\lfloor (n-i)/2\rfloor}\left(g^{\lfloor (n-i)/2\rfloor}-f^{\lfloor (n-i)/2\rfloor}\right)$$
$$+(f-g)\sum_{i=1}^{\lfloor n/2\rfloor} f^{i-1}g^{(n-i)-\lfloor (n-i)/2\rfloor}f^{[(n-i)/2]}. \tag{4.75}$$

Observe that $\lfloor n/2\rfloor \geqslant 2\lfloor n/5\rfloor$, $\lfloor n/5\rfloor \geqslant n/10$. Therefore applying (4.74) we get

$$|f-g|\sum_{i=\lfloor n/2\rfloor+1}^{n}|f|^{i-1}|g|^{n-i} \leqslant n|f-g||f|^{2\lfloor n/5\rfloor} \leqslant Cn^{-1/2}|f|^{\lfloor n/5\rfloor}. \tag{4.76}$$

Similarly, $(n-i)-\lfloor (n-i)/2\rfloor \geqslant (n-i)/2 \geqslant n/4$ and $\lfloor (n-i)/2\rfloor \geqslant n/2-i/2-1 \geqslant n/4-1 \geqslant n/5$ if $i \leqslant \lfloor n/2\rfloor$. Therefore from (4.73) and (4.74) it follows that

$$|f-g|\sum_{i=1}^{\lfloor n/2\rfloor}|f|^{i-1}|g|^{(n-i)-\lfloor (n-i)/2\rfloor}|g^{\lfloor (n-i)/2\rfloor}-f^{\lfloor (n-i)/2\rfloor}|$$
$$\leqslant \quad Cn|f-g||g|^{n/4}n^{-1/2} \leqslant Cn^{-1}. \tag{4.77}$$

Finally,

$$|f-g|\sum_{i=1}^{\lfloor n/2\rfloor}|f|^{i-1}|g|^{(n-i)-\lfloor (n-i)/2\rfloor}|f^{\lfloor (n-i)/2\rfloor}|$$
$$\leqslant \quad n|f-g||g|^{n/4}|f|^{\lfloor n/5\rfloor} \leqslant Cn^{-1/2}|f|^{\lfloor n/5\rfloor}. \tag{4.78}$$

Substituting (4.76), (4.77), (4.78) into (4.75) and applying (A.36) we obtain

$$\begin{aligned}\|\mathbf{F}-\mathbf{H}_{n,2}\|_\infty &\leqslant Cn^{-1}+Cn^{-1/2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(\vec{t})|^{\lfloor n/5\rfloor}\mathrm{d}t_1\mathrm{d}t_2\\ &\leqslant Cn^{-1}+Cn^{-1/2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(\vec{t})|^{2\lfloor n/20\rfloor}\mathrm{d}t_1\mathrm{d}t_2\\ &= Cn^{-1}+Cn^{-1/2}\mathbb{P}(\tilde{S}_{\lfloor n/20\rfloor}=\vec{0}),\end{aligned} \tag{4.79}$$

where $\tilde{S}_{\lfloor n/20\rfloor}=\tilde{X}_1+\tilde{X}_2+\ldots+\tilde{X}_{\lfloor n/20\rfloor}$ and $\tilde{X},\tilde{X}_1+\tilde{X}_2+\ldots+\tilde{X}_{\lfloor n/20\rfloor}$ are i.i.d.

r.v.s. We assumed $n \geqslant 40$, therefore

$$0.6 \leqslant \sup_{\vec{x} \in \mathbb{Z}^2} \mathbb{P}(\tilde{X} = \vec{x}) = \mathbb{P}(\tilde{X} = \vec{0}) = q^2 + 0.2^2 + 2/n \leqslant 0.7$$

and by (4.24) $\mathbb{P}(\tilde{S}_{\lfloor n/20 \rfloor} = \vec{0}) \leqslant Cn^{-1/2}$. Substituting this estimate into (4.79) we prove (4.25). $\square$

5 ESTIMATES IN TOTAL VARIATION FOR SYMMETRIC LATTICE DISTRIBUTIONS

5.1 Known results

Though some recent research in multidimensional approximations is related to discrete Normal distribution (see [11, 12]), the dominating approximations are still related to compound Poisson laws. Dependent vectors were considered in Theorem 6.8 [62]. General CP approximations for independent symmetric vectors in Kolmogorov and Prokhorov metrics were investigated in [43, 90] (see also references therein). CP approximations in total variation for non-symmetric vectors were considered by Roos [74, 77, 78], who used so-called Kerstan's method. In [56] Kerstan's method was applied for symmetric vectors concentrated on coordinate axes of $\mathbb{Z}^d$. Set $\bar{e}_1 = (1,0,\dots,0),\dots,\bar{e}_m = (0,\dots,0,1,0,\dots,0),\dots,\ \bar{e}_d = (0,\dots,0,1)$. Let $\vec{\bar{X}}_1^0, \vec{\bar{X}}_2^0, \dots, \vec{\bar{X}}_n^0$ be independent r.v.s. taking values in $\mathbb{Z}^d$ and let for $m = 1,2,\dots,d$; $i = 1,2,\dots,n$:

$$\mathscr{L}(\vec{\bar{X}}_i^0) = \bar{q}_i I + \sum_{m=1}^{d} \bar{p}_{i,m}\bar{F}_m,\ \bar{q}_i = \mathbb{P}(\vec{\bar{X}}_i^0 = \vec{0}),\ \bar{q}_i + \sum_{m=1}^{d} \bar{p}_{i,m} = 1, \tag{5.1}$$

$$\operatorname{supp}\bar{F}_m \subset \{j\bar{e}_m,\ j = \pm 1, \pm 2, \dots\}, \quad \bar{F}_m\{j\bar{e}_m\} = \bar{F}_m\{-j\bar{e}_m\}, \tag{5.2}$$

$$\bar{\lambda}_m = \sum_{i=1}^{n} \bar{p}_{i,m}, \quad \bar{\sigma}_m^2 = 2\sum_{j=1}^{\infty} j^2 \bar{F}_m\{j\bar{e}_m\}, \quad 0 \leqslant \bar{q}_i, \bar{p}_{i,m} \leqslant 1,$$

$$\bar{\alpha} = \sum_{i=1}^{n} g(2(1-\bar{q}_i)) \min\left\{ 2^{-3/2} \sum_{m=1}^{d} \frac{\bar{p}_{i,m}^2}{\bar{\lambda}_m}, \Big(\sum_{m=1}^{d} \bar{p}_{i,m} \Big)^2 \right\}, \tag{5.3}$$

where $g(\cdot)$ is defined by (3.7) (see p. 44). Observe that in above setting coordinates of $\vec{\bar{X}}_i^0$ are uncorrelated and, for example, $\mathbb{P}(\vec{\bar{X}}_i^0 = (1,1,1,\dots,1)) = 0$. In [56] it was proved that if $2\bar{\alpha}\mathrm{e} < 1$, $\bar{\sigma}_m^2 < \infty$, $(m = 1,2,\dots,d)$, then

$$\begin{aligned} &\left\| \prod_{i=1}^{*n} \mathscr{L}(\vec{\bar{X}}_i^0) - \prod_{i=1}^{*n} \exp\{\mathscr{L}(\vec{\bar{X}}_i^0) - I\} \right\|_{TV} \\ &\qquad \leqslant \frac{15.98}{(1-2\bar{\alpha}\mathrm{e})^{3/2}} \sum_{l=1}^{d} (1+\bar{\sigma}_l) \sum_{m=1}^{d} \frac{1}{\bar{\lambda}_m^2} \sum_{i=1}^{n} \bar{p}_{i,m}^2. \end{aligned} \tag{5.4}$$

If $d, \bar{\sigma}_l, \bar{p}_{i,m} \asymp C$, then the rate of accuracy in (5.4) is $O(n^{-1})$.

In the case of i.i.d. r.v.s., when $\bar{p}_{i,m} \equiv \bar{p}_m$, $\bar{q}_i = \bar{q}$ and $\mathbb{P}(\vec{\bar{X}}_i^0 = \vec{0}) \geqslant 4/5$, it

was proved that

$$\left\| \left(\bar{q}I + \sum_{m=1}^{d} \bar{p}_m \bar{F}_m \right)^{*n} - \exp\left\{ n \sum_{m=1}^{d} \bar{p}_m (\bar{F}_m - I) \right\} \right\|_{TV} \leqslant \frac{17.34}{n} \left(\sum_{m=1}^{d} \sqrt{1+\bar{\sigma}_m} \right)^2, \qquad (5.5)$$

see [56], Theorem 3.2. For (5.4) and (5.5) to hold, probabilities $\mathbb{P}(\vec{\bar{X}}_i^0 = \vec{0})$ must be large. Consider, for example, the case $d = 1$, $\bar{q}_i = \bar{p}_i = 1/2$. Then $2\bar{\alpha}\mathrm{e} > 1$. Moreover, $\bar{\sigma}_m$ can be large even if $\operatorname{supp} \vec{\bar{X}}_i^0$ is finite.

Note that any distribution with $\bar{\mathsf{q}} = \bar{\mathsf{F}}\{\vec{0}\} > 0$ can be written as $\bar{\mathsf{F}} = \bar{\mathsf{q}}I + (1-\bar{\mathsf{q}})\bar{\mathsf{V}}$. From one-dimensional Poisson approximation to the binomial law it follows then that

$$\|\bar{\mathsf{F}}^{*n} - \exp(n(\bar{\mathsf{F}} - I))\|_{TV} \leqslant \min(1-\bar{\mathsf{q}}, n(1-\bar{\mathsf{q}})^2), \qquad (5.6)$$

see [30], p.207 and (3.15) for details. However, (5.6) completely ignores the symmetry of $\bar{\mathsf{F}}$ and requires $\bar{\mathsf{q}}$ to be very close to 1, much closer than needed for (5.5).

5.2 Results

Hipp-type approximation of order-k ($k \in \{0, 1, 2, \dots\}$) is defined as

$$D_k = \exp\left\{ \sum_{m=1}^{k} \frac{(-1)^{m+1}(F-I)^{*m}}{m} \right\}. \qquad (5.7)$$

Note that $D_1 = \exp\{F - I\}$ is accompanying CP distribution to F. Meanwhile D_k, $k \geqslant 2$ are SCP measures. We also use notation

$$D_k^{*a} := \exp\left\{ a \sum_{m=1}^{k} \frac{(-1)^{m+1}(F-I)^{*m}}{m} \right\},$$

for any real (positive or negative) a.

5.2.1 The general case of vectors with possibly correlated coordinates

In (5.5) it was assumed that all lattice distributions are concentrated on coordinate axes, have finite variances and take value zero with probability larger

or equal to $4/5$. We demonstrate that these assumptions can be significantly weakened. Further we consider

$$\vec{X}_{S_1} \sim F_{S_1} \in \mathscr{F}_d^S; \qquad \operatorname{supp} F_{S_1} \subset \mathbb{Z}^d, \qquad q_{S_1} := F_{S_1}\{\vec{0}\} \in (0,1). \tag{5.8}$$

Assumptions (5.8) are more general than those used in (5.4), (5.5), since q_{S_1} can be reasonably small and non-zero probabilities are not restricted to coordinate axes. Further $p_{\vec{x}} = \mathbb{P}(\vec{X}_{S_1} = \vec{x}) = F_{S_1}\{\vec{x}\}$ and $\sum_{\vec{x}}$ denotes summation over all $\vec{x} \in \mathbb{Z}^d \setminus \{\vec{0}\}$. Let

$$\delta(y) = \sum_{\vec{x}} \min(1; 4y p_{\vec{x}}). \tag{5.9}$$

Note that $\delta(y) \leqslant 4y(1-q_{S_1})$, $\delta(y/j) \leqslant \delta(y)$ if $j \geqslant 1$ and $\delta(y) \leqslant 2N$ if $\operatorname{supp} F = \{\vec{0}, \pm\vec{x}_1, \ldots, \pm\vec{x}_N\}$.

As can be seen from cited above results, all emphasis is on the order of approximation, not on the magnitude of constants. The following theorem gives a general idea what constant can be expected for large n.

Theorem 5.2.1. *Let assumption (5.8) hold. Then, for any $n \in \mathbb{N}$*

$$\|F_{S_1}^{*n} - \exp\{n(F_{S_1} - I)\}\|_{TV} \leqslant \frac{\delta^2(n/2)}{2n}\left(1 + \frac{C(\delta(nq_{S_1})+1)}{q_{S_1}^4\sqrt{n}}\right) + \frac{C(1-q_{S_1})^{9/2}}{q_{S_1}^5 n\sqrt{n}}. \tag{5.10}$$

The following shorter version of Theorem 5.2.1 with the smaller power of q_{S_1} in denominator holds.

Theorem 5.2.1*. *Let assumption (5.8) hold. Then, for any $n \in \mathbb{N}$*

$$\|F_{S_1}^{*n} - \exp\{n(F_{S_1} - I)\}\|_{TV} \leqslant \frac{C(1+\delta^2(n))}{n q_{S_1}^{7/2}}. \tag{5.11}$$

Theorem 5.2.1 can be used even if (5.4) and (5.5) are not applicable.

The following example demonstrates that Theorem 5.2.1 is useful even if (5.4) and (5.5) can not be used.

Example 5.2.1. *Let $d = 2$ and $\vec{X}^{(S_1)}, \vec{X}_1^{(S_1)}, \ldots, \vec{X}_n^{(S_1)}$ be i.i.d. random vectors*

with

$$\mathbb{P}(\vec{X}^{(S_1)} = (0,0)) = \frac{1}{2},$$
$$\mathbb{P}(\vec{X}^{(S_1)} = (\pm k,0)) = \frac{1}{2k(k+1)(k+2)}, \qquad k = 1,2,\ldots$$
$$\mathbb{P}(\vec{X}^{(S_1)} = (0,\pm j)) = \frac{1}{2j(j+1)(j+2)}, \qquad j = 1,2,\ldots.$$

Observe that (5.5) is not applicable, since $q_{S_1} = 1/2 < 4/5$ *and* $\sigma_1, \sigma_2 = \infty$. *On the other hand,*

$$\delta(n) \leqslant 2\left(2\sum_{k=1}^{\lfloor n^{1/3}\rfloor+1} 1 + ne\sum_{k=\lfloor n^{1/3}\rfloor+2}^{\infty} \frac{1}{k(k+1)(k+2)}\right) < 10n^{1/3}$$

and from (5.11) follows estimate $\|F_{S_1}^{*n} - \exp(n(F_{S_1} - I))\|_{TV} \leqslant Cn^{-1/3}$.

Next, we use the SCP approximation of Hipp's type, see [46, 77].

Theorem 5.2.2. *Let assumption (5.8) hold. Then, for any* $n \in \mathbb{N}$,

$$\begin{aligned}\|F_{S_1}^{*n} - D_2^{*n}\|_{TV} &\leqslant \frac{31.5\,\delta^3(nq_{S_1}/6)}{q_{S_1}^{7/2} n^2}\left(1 + \frac{C}{q_{S_1}^{7/2}\sqrt{n}}\right) \\ &\quad + \frac{C(1-q_{S_1})^{15/2}}{q_{S_1}^{8} n^2\sqrt{n}}. \end{aligned} \tag{5.12}$$

SCP approximation D_2 can be viewed as a first-order asymptotic expansion in the exponent. Next, we consider a more usual first order asymptotic expansion to accompanying CP distribution.

Theorem 5.2.3. *Let assumption (5.8) hold. Then, for any* $n \in \mathbb{N}$

$$\begin{aligned}&\|F_{S_1}^{*n} - \exp\{n(F_{S_1} - I)\} * (I - n(F_{S_1} - I)^{*2}/2)\|_{TV} \\ &\qquad \leqslant \frac{C(\delta^3(nq_{S_1}) + \delta^4(nq_{S_1}))}{q_{S_1}^{11/2} n^2} + \frac{C(1-q_{S_1})^6}{q_{S_1}^{13/2} n^2}. \end{aligned} \tag{5.13}$$

It is easy to check that for the Example 5.2.1 the accuracy in (5.12) (resp. (5.13)) is of the order $O(n^{-1})$ (resp. $O(n^{-2/3})$).

The following result holds for subsequent convolutions of F_{S_1}.

Theorem 5.2.4. *Let assumption (5.8) hold. Then, for any $n \in \mathbb{N}$*

$$\|F_{S_1}^{*n} - F_{S_1}^{*(n+1)}\|_{TV} \leqslant \frac{C(\delta^2(n)+1)}{q_{S_1}^{7/2} n}. \tag{5.14}$$

Orders of accuracy in (5.10)–(5.14) are explicit, when $\vec{X}_{S_1}$ takes only finite number of values.

Corollary 5.2.1. *Let* $\operatorname{supp} F_{S_1} = \{\vec{0}, \pm\vec{x}_1, \pm\vec{x}_2, \ldots, \pm\vec{x}_N\}$, $N \geqslant d$, *and let (5.8) hold. Then*

$$\|F_{S_1}^{*n} - \exp\{n(F_{S_1} - I)\}\|_{TV} \leqslant \frac{2N^2}{n}\left(1 + \frac{CN}{q_{S_1}^5\sqrt{n}}\right);$$

$$\|F_{S_1}^{*n} - F_{S_1}^{*(n+1)}\|_{TV} \leqslant \frac{CN^2}{q_{S_1}^{7/2} n}, \qquad \|F_{S_1}^{*n} - D_2^{*n}\|_{TV} \leqslant \frac{CN^3}{q_{S_1}^{7/2} n^2},$$

$$\|F_{S_1}^{*n} - \exp\{n(F_{S_1} - I)\} * (I - n(F_{S_1} - I)^{*2}/2)\|_{TV} \leqslant \frac{CN^4}{q_{S_1}^{13/2} n^2}.$$

Note that only the number of points matters, not their values. The following example shows that even for r.v.s. with bounded supports, the estimate (5.10) can be more accurate than (5.5).

Example 5.2.2. *Let* $d = 2$, $\mathbb{P}(\vec{X}_{S_1} = (\pm 1, 0)) = \mathbb{P}(\vec{X}_{S_1} = (\pm n, 0)) = \mathbb{P}(\vec{X}_{S_1} = (0, \pm 1)) = \mathbb{P}(\vec{X}_{S_1} = (0, \pm n)) = 0.025$, $\mathbb{P}(\vec{X}_{S_1} = (0,0)) = 0.8$. *Then (5.5) gives trivial order of approximation* $O(1)$. *Meanwhile from Corollary 5.2.1 it follows that approximation by* $\exp(n(F_{S_1} - I))$ *is of the order* $O(n^{-1})$.

Observe also that assumptions for (5.4) and (5.5) to hold require $q_{S_1} > 1/2$. On the other hand, estimates in Corollary 5.2.1 remain meaningful, even if $q_{S_1} \to 0, N \to \infty$ slowly. For example, if $q_{S_1} = 1/\ln n, N = O(\ln n)$ the estimates still have quite good order of approximation.

Though absolute constant in Corollary 5.2.1 is very reasonable, the following result shows that, at least in the scheme of sequences, it might be much smaller.

Proposition 5.2.5. *Let* $\operatorname{supp} F_{S_1} = \{\vec{0}, \pm\vec{x}_1, \pm\vec{x}_2, \ldots, \pm\vec{x}_N\}$, $N \geqslant d$ *and let* $N, q, p_{\vec{x}}$ *and support of* F_{S_1} *do not depend on n. Then*

$$\lim_{n\to\infty} n\|F_{S_1}^{*n} - \exp(n(F_{S_1} - I))\|_{TV} \leqslant 0.3501N + 0.1171N(N-1). \tag{5.15}$$

Theorem 5.2.6. *Let (5.8) hold and* $\operatorname{supp} F_{S_1} = \{\vec{0}, \pm\vec{x}_1, \pm\vec{x}_2, \dots, \pm\vec{x}_N\}$, $N \geqslant d$. *Then, for any* $a, b > 0$, *the following estimate holds*

$$\|\exp\{a(F_{S_1} - I)\} - \exp\{b(F_{S_1} - I)\}\|_{TV} \leqslant \frac{|b-a|(N+2)}{\max(a,b)}.$$

If q_{S_1} is very large (F_{S_1} is almost degenerate), the following trivial estimate can be used

$$\|\exp\{a(F_{S_1} - I)\} - \exp\{b(F_{S_1} - I)\}\|_{TV} \leqslant 2|b-a|(1-q_{S_1}).$$

So far, we have considered short asymptotic expansions. Next, we demonstrate that, at least for D_2, it is possible to construct a long so-called Bergström-type [14] asymptotic expansion. Let

$$M_{S_1,k} := F_{S_1}^{*n} - \sum_{m=0}^{k} \binom{n}{m} D_2^{*(n-m)} * (F_{S_1} - D_2)^{*m}, \qquad k \leqslant n-1. \tag{5.16}$$

Theorem 5.2.7. *Let assumption (5.8) hold. Then, for any* $n \in \mathbb{N}$, $k < n$, *the following estimate holds*

$$\|M_{S_1,k}\|_{TV} \leqslant \frac{C(k)\delta^{3(k+1)}(nq_{S_1})}{q_{S_1}^{6k-3/2} n^{2(k+1)}} + \frac{C(k)(1-q_{S_1})^{6(k+1)}}{n^{2(k+1)} q_{S_1}^{6k+13/2}}.$$

Corollary 5.2.2. *Let* $\operatorname{supp} F = \{\vec{0}, \pm\vec{x}_1, \pm\vec{x}_2, \dots, \pm\vec{x}_N\}$, $N \geqslant d$, *and let assumption (5.8) hold. Then*

$$\|M_{S_1,k}\|_{TV} \leqslant \frac{C(k)N^{3(k+1)}}{q^{6k+13/2} n^{2(k+1)}}.$$

Theorem 5.2.7 can be used as an intermediate result for constructing longer expansions.

5.2.2 The case of r.v.s. concentrated on finite number of lines

We begin from simple example, which demonstrates that for some cases estimate (5.5) is more accurate than (5.10).

Example 5.2.3. *Let* $d = 2$ *and* $\vec{X}^{(S_1)}, \vec{X}_1^{(S_1)}, \dots, \vec{X}_n^{(S_1)}$ *be i.i.d. random vectors with*

$$\mathbb{P}(\vec{X}^{(S_1)} = (0,0)) = \frac{7}{9},$$

$$\mathbb{P}(\vec{X}^{(S_1)} = (\pm k, 0)) = \frac{1}{2k(k+1)(k+2)(k+3)}, \qquad k = 1,2,\dots$$
$$\mathbb{P}(\vec{X}^{(S_1)} = (0, \pm j)) = \frac{1}{2j(j+1)(j+2)(j+3)}, \qquad j = 1,2,\dots.$$

Then, arguing similarly to Example 5.2.1, we get

$$\delta(n) \leqslant 2\left(2\sum_{k=1}^{\lfloor n^{1/4}\rfloor+1} 1 + 2ne\sum_{k=\lfloor n^{1/4}\rfloor+2}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)}\right) \leqslant Cn^{1/4}.$$

From (5.11) it follows that $\|F_{S_1}^{*n} - \exp(n(F_{S_1} - I))\|_{TV} \leqslant Cn^{-1/2}$. *Meanwhile,* $\sigma_1^2, \sigma_2^2 \leqslant 2$ *and from (5.5) it follows that*

$$\|F_{S_1}^{*n} - \exp(n(F_{S_1} - I))\|_{TV} \leqslant Cn^{-1}.$$

Below we obtain some extensions of (5.5). We consider a more general case, assuming that random vectors are concentrated on a finite number of lines and satisfy certain moment conditions. As before we assume that $\vec{\mathsf{X}}^{(S_1)}, \vec{\mathsf{X}}_1^{(S_1)}, \dots, \vec{\mathsf{X}}_n^{(S_1)}$ are i.i.d. r.v.s. in $\mathbb{R}^d$, $\mathscr{L}(\vec{\mathsf{X}}^{(S_1)}) = \mathsf{F}_{S_1}$. We also assume that

1. For some natural $K \geqslant d$

$$\mathsf{F}_{S_1} = \mathsf{q}_{S_1} I + \sum_{m=1}^{K} \mathsf{p}_m \mathsf{F}_{S_1,m}, \quad \mathsf{q}_{S_1}, \mathsf{p}_1, \dots, \mathsf{p}_m \in (0,1), \tag{5.17}$$
$$\sum_{m=1}^{K} \mathsf{p}_m = 1 - \mathsf{q}_{S_1}.$$

2. There exist non-zero vectors $\vec{y}_1, \vec{y}_2, \dots, \vec{y}_K \in \mathbb{R}^d$ such that

$$\operatorname{supp} \mathsf{F}_{S_1,m} \subset \{j\vec{y}_m : j \in \mathbb{Z}\setminus\{0\}\},$$
$$\operatorname{supp} \mathsf{F}_{S_1,m} \cap \operatorname{supp} \mathsf{F}_{S_1,j} = \emptyset, \quad j \neq m.$$

3. All $\operatorname{supp} \mathsf{F}_{S_1,m} \neq \emptyset$ and $\mathsf{F}_{S_1,m}$ are symmetric:

$$\mathsf{F}_{S_1,m}\{j\vec{y}_m\} = \mathsf{F}_{S_1,m}\{-j\vec{y}_m\}, \quad j \in \{1,2,\dots\}; \quad m = 1,\dots,K.$$

4. There exist j_m^0 such that

$$\mathsf{F}_{S_1,m}\{j_m^0 \vec{y}_m\} > 0, \quad \mathsf{F}_{S_1,m}\{(j_m^0+1)\vec{y}_m\} > 0. \tag{5.18}$$

Distribution considered in (5.5) is a partial case of the above setting, since it suffices to take $K = d$ and $\vec{y}_m = \bar{e}_m$. We can define 'projection' of $\mathsf{F}_{S_1,m}$ to $\vec{y}_m$ by considering one-dimensional integer-valued random variable Y_{S_1} such that

$$\mathbb{P}(\mathsf{Y}_{S_1} = j) := \mathsf{F}_{S_1,m}\{j\vec{y}_m\}, \quad j \in \mathbb{N}. \tag{5.19}$$

Then (5.18) is an assumption that Y_{S_1} is concentrated on a lattice with the maximal span equal to one. Though not formulated explicitly, assumption (5.18) was used for the proof of (5.5) in [56]. We denote the variance of Y_{S_1} by

$$\sigma_m^2 = \sum_{j=-\infty}^{\infty} j^2 \mathsf{F}_{S_1,m}\{j\vec{y}_m\} = 2\sum_{j=1}^{\infty} j^2 \mathsf{F}_{S_1,m}\{j\vec{y}_m\}.$$

First, we show that assumption $\mathsf{q}_{S_1} \geqslant 4/5$ used in (5.5) can be dropped.

Theorem 5.2.8. *Let assumptions (5.17)–(5.18) hold. Then, for any $n \in \mathbb{N}$,*

$$\begin{aligned} &\|\mathsf{F}_{S_1}^{*n} - \exp\{n(\mathsf{F}_{S_1} - I)\}\|_{TV} \\ &\quad \leqslant \frac{3.51}{n}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^2\left(1+\frac{C}{\sqrt{n}\,\mathsf{q}_{S_1}^5}\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right). \end{aligned} \tag{5.20}$$

A shorter version with a non-explicit constant can also be proved.

Theorem 5.2.8*. *Let (5.17)–(5.18) hold. Then, for any $n \in \mathbb{N}$,*

$$\|\mathsf{F}_{S_1}^{*n} - \exp\{n(\mathsf{F}_{S_1} - I)\}\|_{TV} \leqslant \frac{C}{n\mathsf{q}_{S_1}^{7/2}}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^2. \tag{5.21}$$

Next we consider SCP measure D_2, which can be viewed as short asymptotic expansion in the exponent.

Theorem 5.2.9. *Let assumptions (5.17)–(5.18) hold. Then, for any $n \in \mathbb{N}$,*

$$\|\mathsf{F}_{S_1}^{*n} - D_2^{*n}\|_{TV} \leqslant \frac{C}{n^2\,\mathsf{q}_{S_1}^{13/2}}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^2. \tag{5.22}$$

We can formulate analogues of Theorems 5.2.3 and 5.2.4.

Theorem 5.2.10. *Let (5.17)–(5.18) hold. Then, for any $n \in \mathbb{N}$,*

$$\|\mathsf{F}_{S_1}^{*n} - \exp\{n(\mathsf{F}_{S_1} - I)\} * (I - n(\mathsf{F}_{S_1} - I)^{*2}/2)\|_{TV}$$

$$\leqslant \frac{C}{n^2\,\mathsf{q}_{S_1}^{13/2}}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^4. \tag{5.23}$$

Theorem 5.2.11. *Let (5.17)–(5.18) hold. Then, for any $n \in \mathbb{N}$,*

$$\|\mathsf{F}_{S_1}^{*n} - \mathsf{F}_{S_1}^{*(n+1)}\|_{TV} \leqslant \frac{C}{n\mathsf{q}_{S_1}^{7/2}}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^2. \tag{5.24}$$

Finally we formulate analogue of Theorem 5.2.6.

Theorem 5.2.12. *Let (5.17)–(5.18) hold. Then, for any $a, b > 0$,*

$$\|\exp\{a(\mathsf{F}_{S_1}-I)\} - \exp\{b(\mathsf{F}_{S_1}-I)\}\|_{TV} \leqslant \frac{3.4|b-a|}{\max(a,b)}\sum_{m=1}^{K}\sqrt{1+\sigma_m}. \tag{5.25}$$

If F_{S_1} is concentrated on finite number of points, then we get the same order of accuracy as in (5.2.6). On the other hand, (5.25) can be applied, when support of (F_{S_1} is infinite, but assumptions (5.17)–(5.18) hold.

5.3 Auxiliary results

First we show, that so called Bergström expansion allows for intermediate result. The following result is of special interest, since it shows that long asymptotic expansions can be constructed without assumptions of symmetry. Let, as in above,

$$D_2 = \exp\left\{(\mathsf{F}-I) - \frac{(\mathsf{F}-I)^{*2}}{2}\right\},$$
$$M_k := \mathsf{F}^{*n} - D_2^{*n} - \sum_{m=1}^{k}\binom{n}{m} D_2^{*(n-m)} * (\mathsf{F}-D_2)^{*m}.$$

First we prove some facts about D_2.

Lemma 5.3.1. *Let $\mathsf{F} = \mathsf{q}I + \mathsf{p}\mathsf{V} \in \mathscr{F}_d$, $\mathsf{q} = 1-\mathsf{p} \in (0,1)$, $m \in \mathbb{N}$. Then for any $m \in \mathbb{N}$, $a \in \mathbb{R}$*

$$D_2 - I = C(\mathsf{F}-I) * \Theta, \tag{5.26}$$
$$D_2^{*m} = \frac{3.5}{\sqrt{\mathsf{q}}}\mathrm{e}^{m\mathsf{q}(\mathsf{F}-I)/2} * \Theta_m,\ \left\|D_2^{*m}\right\|_{TV} \leqslant \frac{3.5}{\sqrt{\mathsf{q}}},\ \|D_2^{*a}\|_{TV} \leqslant \mathrm{e}^{4|a|}, \tag{5.27}$$
$$\mathsf{F} - D_2 = \frac{(\mathsf{F}-I)^{*3}}{3} + C_1(\mathsf{F}-I)^{*4} * \Theta$$

$$= C_2(\mathsf{F}-I)^{*3} * \Theta = C_2 p^3 (\mathsf{V}-I)^{*3} * \Theta. \tag{5.28}$$

Proof. The first estimate in (5.27) follows from (A.25). The second estimate in (5.27) follows from the fact that the total variation of any CP distribution is equal to 1 and, therefore,

$$\left\|D_2^{*m}\right\|_{TV} \leqslant \frac{3.5}{\sqrt{\mathsf{q}}} \left\|\exp\{m\mathsf{q}(\mathsf{F}-I)/2\}\right\|_{TV} \|\Theta_m\|_{TV} \leqslant \frac{3.5}{\sqrt{\mathsf{q}}}.$$

Observe that $\|\mathsf{F}-I\|_{TV} \leqslant \|\mathsf{F}\|_{TV} + \|I\|_{TV} = 2$ and, therefore, by (A.27)

$$\|D_2^{*a}\|_{TV} \leqslant \exp\{|a|\|\mathsf{F}-I\|_{TV} + |a|\|\mathsf{F}-I\|_{TV}^2/2\} \leqslant \mathrm{e}^{4|a|}.$$

Expressions (5.26) and (5.28) follow directly from the definition of exponential measure. □

Lemma 5.3.2. *Let* $\mathsf{F} = \mathsf{q}I + \mathsf{p}\mathsf{V} \in \mathscr{F}_d$, $\mathsf{q} = 1-\mathsf{p} \in (0,1)$, $k,n \in \mathbb{N}$, $k \leqslant n-1$. *Then*

$$\|M_k\|_{TV} \leqslant \frac{C(k)\mathsf{p}^{3(k+1)/2}}{n^{(k+1)/2}\mathsf{q}^{(3k+4)/2}}.$$

Proof. If $n \leqslant 12$, then

$$\begin{aligned} \|M_k\|_{TV} &\leqslant 1 + \sum_{m=0}^{k} \binom{n}{m} \mathrm{e}^{4(n-m)} (1+\mathrm{e}^4)^m \\ &\leqslant 1 + (1+2\mathrm{e}^4)^{12} = C = \frac{n^k C}{n^k} \leqslant \frac{12^k C}{n^k}. \end{aligned}$$

Therefore, without loss of generality, we further assume $n \geqslant 13$. Bergström identity (A.32) allows to write

$$M_k = \sum_{m=k+1}^{n} \binom{m-1}{k} \mathsf{F}^{*(n-m)} * (\mathsf{F}-D_2)^{*(k+1)} * D_2^{*(m-k-1)}.$$

Noting that

$$\sum_{m=k+1}^{n} \binom{m-1}{k} = \binom{n}{k+1} \tag{5.29}$$

and applying (5.28), (5.27), (A.18), (A.22), (A.33) we obtain

$$\|M_k\|_{TV} \leqslant \sum_{m=k+1}^{n} \binom{m-1}{k} \left\|\mathsf{F}^{*(n-m)} * (\mathsf{F}-D_2)^{*(k+1)} * D_2^{*(m-1)}\right\|_{TV}$$

$$
\begin{aligned}
&\leqslant C(k)\sum_{m\leqslant\lfloor n/2\rfloor}\binom{m-1}{k}\left\|(\mathsf{F}-I)^{*3(k+1)}*\mathsf{F}^{*(n-m)}\right\|_{TV}\left\|D_2^{*(m-1)}\right\|_{TV}\\
&\quad+C(k)\sum_{m>\lfloor n/2\rfloor}\binom{m-1}{k}\left\|\mathsf{F}^{*(n-m)}\right\|_{TV}\left\|(\mathsf{F}-I)^{*3(k+1)}*D_2^{*(m-1)}\right\|_{TV}\\
&\leqslant \frac{C(k)n^{k+1}}{\sqrt{\mathsf{q}}}\mathsf{p}^{3(k+1)}\left\|(\mathsf{V}-I)^{*3(k+1)}*(\mathsf{q}I+\mathsf{p}V)^{\lfloor n/2\rfloor}\right\|_{TV}\\
&\quad+C(k)n^{k+1}\mathsf{p}^{3(k+1)}\left\|(\mathsf{V}-I)^{*3(k+1)}*\mathsf{e}^{n\mathsf{pq}(\mathsf{V}-I)/6}\right\|_{TV}\\
&\leqslant \frac{C(k)\mathsf{p}^{3(k+1)/2}}{n^{(k+1)/2}\mathsf{q}^{(3k+4)/2}}.
\end{aligned}
$$

□

Next we estimate smoothing effect of accompanying distribution.

Lemma 5.3.3. *Let assumptions (5.8) hold. Then, for any $a>0$, $k\in\mathbb{N}$, the following estimate holds*

$$
\begin{aligned}
&\|(F_{S_1}-I)^{*k}*\exp\{a(F_{S_1}-I)\}\|_{TV}\\
&\qquad\leqslant \left(\frac{k}{2a}\right)^k\delta^k(a/k)\leqslant\frac{C(k)\delta^k(a)}{a^k},
\end{aligned}
$$

where $\delta(a)$ is defined by (5.9).

Proof. F_{S_1} is symmetric and we can write

$$
F_{S_1}-I=\sum_{\vec{x}}p_{\vec{x}}(I_{\vec{x}}-I)=\sum_{\vec{x}}p_{\vec{x}}(I_{-\vec{x}}-I)=\frac{1}{2}\sum_{\vec{x}}p_{\vec{x}}(I_{\vec{x}}+I_{-\vec{x}}-2I).
$$

Total variation of CP distribution equals 1. Therefore,

$$
\exp\{a(F_{S_1}-I)\}=\exp\left(p_{\vec{x}}(I_{\vec{x}}+I_{-\vec{x}}-2I)\right)*\Theta,
$$

for any $\vec{x}$ from F_{S_1} support and some $\Theta=\Theta(F_{S_1})$ such that $\|\Theta\|_{TV}\leqslant 1$.

Therefore, applying (A.19) we get

$$
\begin{aligned}
&\|(F_{S_1}-I)*\exp\{a(F_{S_1}-I)\}\|_{TV}\\
&\quad\leqslant\frac{1}{2}\sum_{\vec{x}}p_{\vec{x}}\|(I_{\vec{x}}+I_{-\vec{x}}-2I)*\exp\left(p_{\vec{x}}(I_{\vec{x}}+I_{-\vec{x}}-2I)\right)\|_{TV}\\
&\quad\leqslant\frac{1}{2}\sum_{\vec{x}}p_{\vec{x}}\min\left(4,\frac{1}{ap_{\vec{x}}}\right)=\frac{1}{2a}\delta(a).
\end{aligned}
$$

The required estimate for $k > 1$ follows from (A.29). □

Lemma 5.3.4. *Let assumptions (5.17)–(5.18) hold. Then, for any $a>0$, $k \in \mathbb{N}$, the following estimate holds*

$$\|(\mathsf{F}_{S_1} - I)^{*k} * \exp\{a(\mathsf{F}_{S_1} - I)\}\|_{TV} \leqslant \frac{(3.6k)^k}{\mathrm{e}^k a^k} \left(\sum_{m=1}^{K} \sqrt{1+\sigma_m} \right)^k. \qquad (5.30)$$

Proof. Observe that $\mathrm{e}^{a(F-I)}$ can be expressed as convolution of two CP distributions

$$\begin{aligned} \mathrm{e}^{a(\mathsf{F}_{S_1} - I)} &= \exp\{a\mathsf{p}_m(\mathsf{F}_{S_1,m} - I)\} * \Theta, \\ \|\Theta\|_{TV} &= \left\| \exp\left\{ a \sum_{j \neq m}^{K} \mathsf{p}_j (\mathsf{F}_{S_1,j} - I) \right\} \right\|_{TV} = 1, \end{aligned}$$

where the last equality follows from the fact that the total variation of any distribution equals 1. Noting that estimates for convolutions of $\mathsf{F}_{S_1,m}$ can be treated as estimates for one-dimensional distributions (see (5.19)) and applying Lemma A.1.1, we get

$$\begin{aligned} &\|(\mathsf{F}_{S_1} - I) * \exp\{a(\mathsf{F}_{S_1} - I)\}\|_{TV} \\ &\quad \leqslant \sum_{m=1}^{K} \mathsf{p}_m \|(\mathsf{F}_{S_1,m} - I) * \exp\{a\mathsf{p}_m(\mathsf{F}_{S_1,m} - I)\}\|_{TV} \|\Theta\|_{TV} \\ &\quad = \sum_{m=1}^{K} \mathsf{p}_m \|(\mathsf{F}_{S_1,m} - I) * \exp\{a\mathsf{p}_m(\mathsf{F}_{S_1,m} - I)\}\|_{TV} \\ &\quad \leqslant \frac{3.6}{a\mathrm{e}} \sum_{m=1}^{K} \sqrt{1+\sigma_m}. \end{aligned}$$

For the proof, when $k > 1$, we use (A.29). □

We recall that Θ is used for various measures, satisfying $\|\Theta\|_{TV} \leqslant 1$.

Lemma 5.3.5. *Let $\mathsf{F} = \mathsf{q}I + \mathsf{p}\mathsf{V} \in \mathscr{F}_d$, $\mathsf{q} \in (0,1)$, and let D_2 be defined by (5.7). Then for any $n \in \mathbb{N}$ the following equalities hold*

$$D_2^{*n} - \mathrm{e}^{n(\mathsf{F}-I)} = \frac{3.5n}{2\sqrt{\mathsf{q}}} (\mathsf{F}-I)^{*2} * \mathrm{e}^{n\mathsf{q}(\mathsf{F}-I)/2} * \Theta, \qquad (5.31)$$

$$
\begin{aligned}
D_2^{*n} - \mathrm{e}^{n(\mathsf{F}-I)} &= -\frac{n}{2}\mathrm{e}^{n(\mathsf{F}-I)} * (\mathsf{F}-I)^{*2} \\
&\quad + \frac{3.5n^2}{4\sqrt{\mathsf{q}}}\mathrm{e}^{n\mathsf{q}(\mathsf{F}-I)/2} * (\mathsf{F}-I)^{*4} * \Theta. \qquad (5.32)
\end{aligned}
$$

Proof. Observe that $\mathsf{F} - I = \mathsf{p}(\mathsf{V} - I)$. Then applying (A.30) and (A.25) we obtain

$$
\begin{aligned}
D_2^{*n} - \mathrm{e}^{n(\mathsf{F}-I)} &= \mathrm{e}^{n(\mathsf{F}-I)} * \left(\mathrm{e}^{-n(\mathsf{F}-I)^{*2}/2} - I\right) \\
&= -\frac{n}{2}\mathrm{e}^{n(\mathsf{F}-I)} * (\mathsf{F}-I)^{*2} * \int_0^1 \mathrm{e}^{-n\tau(\mathsf{F}-I)^{*2}/2}\mathrm{d}\tau \\
&= \frac{n}{2}\mathrm{e}^{n\mathsf{q}(\mathsf{F}-I)/2} * (\mathsf{F}-I)^{*2} * \int_0^1 (-1)\mathrm{e}^{n\mathsf{p}(1+\mathsf{p})(\mathsf{V}-I)/2 - \tau n\mathsf{p}^2(\mathsf{V}-I)^{*2}/2}\mathrm{d}\tau \\
&= \frac{3.5n}{2\sqrt{\mathsf{q}}}\mathrm{e}^{n\mathsf{q}(\mathsf{F}-I)/2} * (\mathsf{F}-I)^{*2} * \Theta,
\end{aligned}
$$

which proves (5.31). The proof of (5.32) is similar:

$$
\begin{aligned}
D_2^{*n} - \mathrm{e}^{n(\mathsf{F}-I)} &= \mathrm{e}^{n(\mathsf{F}-I)} * \left(\mathrm{e}^{-n(\mathsf{F}-I)^{*2}/2} - I + n(\mathsf{F}-I)^{*2}/2\right) \\
&\quad -\frac{n}{2}\mathrm{e}^{n(\mathsf{F}-I)} * (\mathsf{F}-I)^{*2} \\
&= \mathrm{e}^{n\mathsf{q}(\mathsf{F}-I)/2} * \frac{n^2}{4}(\mathsf{F}-I)^{*4} * \int_0^1 (-1)\mathrm{e}^{n\mathsf{p}(1+\mathsf{p})(\mathsf{V}-I)/2 - \tau n\mathsf{p}^2(\mathsf{V}-I)^{*2}/2}\mathrm{d}\tau \\
&\quad -\frac{n}{2}\mathrm{e}^{n(\mathsf{F}-I)} * (\mathsf{F}-I)^{*2} \\
&= \frac{3.5n^2}{4\sqrt{q}}\mathrm{e}^{nq(\mathsf{F}-I)/2} * (\mathsf{F}-I)^{*4} * \Theta - \frac{n}{2}\mathrm{e}^{n(\mathsf{F}-I)} * (\mathsf{F}-I)^{*2}.
\end{aligned}
$$

□

5.4 Proofs

Proof of Theorem 5.2.1 and Theorem 5.2.8. In the proof of theorems we use F instead of F_{S_1} or F_{S_1}. Let M_2 be defined by (5.16). Then

$$
\begin{aligned}
\left\| F^{*n} - \mathrm{e}^{n(F-I)} \right\|_{TV} &\leqslant \|M_2\|_{TV} + \left\| D_2^{*n} - \mathrm{e}^{n(F-I)} \right\|_{TV} \\
&\quad + \sum_{m=1}^{2} \binom{n}{m} \left\| D_2^{*(n-m)} * (F - D_2)^{*m} \right\|_{TV}. \qquad (5.33)
\end{aligned}
$$

From Lemma 5.3.2 it follows that

$$\|M_2\|_{TV} \leqslant \frac{C(1-q)^{9/2}}{n\sqrt{n}\,q^5}.$$

Taking into account (5.27), (5.28), (A.18) and observing that $\|F-I\|_{TV} \leqslant 2$ we obtain

$$\begin{aligned}
&\sum_{m=1}^{2} \binom{n}{m} \left\| D_2^{*(n-m)} * (F-D_2)^{*m} \right\|_{TV} \\
&\qquad \leqslant C \sum_{m=1}^{2} n^m \|D_2^{-m}\|_{TV} \left\| D_2^{*n} * (F-I)^{*3m} \right\|_{TV} \\
&\qquad \leqslant \frac{C}{\sqrt{q}} \sum_{m=1}^{2} n^m \left\| \mathrm{e}^{nq(F-I)/2} * (F-I)^{*3m} \right\|_{TV} \\
&\qquad \leqslant \frac{C\sqrt{n}}{\sqrt{q}} \left\| \mathrm{e}^{nq(F-I)/4} * (F-I)^{*2} \right\|_{TV} \\
&\qquad\quad \times \sum_{m=1}^{2} n^{m-1/2} \left\| \mathrm{e}^{nq(F-I)/4} * (F-I)^{*(2m-1)} \right\|_{TV} \|F-I\|_{TV}^{m-1} \\
&\qquad \leqslant \frac{C\sqrt{n}}{q^2} \left\| \mathrm{e}^{nq(F-I)/4} * (F-I)^{*2} \right\|_{TV}.
\end{aligned}$$

Applying (5.30) we get

$$\begin{aligned}
&\sum_{m=1}^{2} \binom{n}{m} \left\| D_2^{*(n-m)} * (F-D_2)^{*m} \right\|_{TV} \\
&\qquad\qquad \leqslant \frac{C\delta^2(nq/4)}{n\sqrt{n}\,q^4} \leqslant \frac{C\delta^2(n/2)}{n\sqrt{n}\,q^4}.
\end{aligned} \tag{5.34}$$

Applying (5.30) we get

$$\begin{aligned}
&\sum_{m=1}^{2} \binom{n}{m} \left\| D_2^{*(n-m)} * (F-D_2)^{*m} \right\|_{TV} \\
&\qquad\qquad \leqslant \frac{C}{n\sqrt{n}\,q^4} \left(\sum_{m=1}^{K} \sqrt{1+\sigma_m} \right)^2.
\end{aligned} \tag{5.35}$$

From (5.32) and (A.18) it follows that

$$\|D_2^{*n} - \mathrm{e}^{n(F-I)}\|_{TV} \leqslant \frac{n}{2} \left\| (F-I)^{*2} * \mathrm{e}^{n(F-I)} \right\|_{TV}$$

$$
\begin{aligned}
&+\frac{3.5n^2}{4\sqrt{q}}\left\|(F-I)^{*4}*\mathrm{e}^{nq(F-I)/2}\right\|_{TV}\\
\leqslant\ & \frac{n}{2}\left\|(F-I)^{*2}*\mathrm{e}^{n(F-I)}\right\|_{TV}+\frac{3.5n^2}{4\sqrt{q}}\left\|(F-I)^{*3}*\mathrm{e}^{nq(F-I)/4}\right\|_{TV}\\
&\times\left\|(F-I)*\mathrm{e}^{nq(F-I)/4}\right\|_{TV}\\
\leqslant\ & \frac{n}{2}\left\|(F-I)^{*2}*\mathrm{e}^{n(F-I)}\right\|_{TV}+\frac{Cn\sqrt{n}}{q}\left\|(F-I)^{*3}*\mathrm{e}^{nq(F-I)/4}\right\|_{TV}.
\end{aligned}
$$

Applying (5.30) we get

$$
\begin{aligned}
\|D_2^{*n}-\mathrm{e}^{n(F-I)}\|_{TV} &\leqslant \frac{\delta^2(n/2)}{2n}+\frac{C\delta^3(nq/4)}{n\sqrt{n}q^4}\\
&\leqslant \frac{\delta^2(n/2)}{2n}+\frac{C\delta^2(n/2)\delta(nq)}{n\sqrt{n}q^4}. \qquad (5.36)
\end{aligned}
$$

Applying (5.30) we get

$$
\begin{aligned}
\|D_2^{*n}-\mathrm{e}^{n(F-I)}\|_{TV} &\leqslant \frac{25.92}{\mathrm{e}^2 n}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^2\\
&\quad+\frac{C}{n\sqrt{n}q^4}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^3. \qquad (5.37)
\end{aligned}
$$

Substituting (5.34) and (5.36) into (5.33) we complete the proof of (5.10). Substituting (5.35) and (5.37) into (5.33) we complete the proof of (5.20). □

Proof of Theorem 5.2.1* and Theorem 5.2.8*. In the proof of theorems we use F instead of F_{S_1} or F_{S_1}. Let M_1 be defined by (5.16). Then applying Lemma 5.3.2, (5.31), (5.27), (5.28) we get

$$
\begin{aligned}
&\left\|F^{*n}-\mathrm{e}^{n(F-I)}\right\|_{TV}\\
&\quad\leqslant \|M_1\|_{TV}+\left\|D_2^{*n}-\mathrm{e}^{n(F-I)}\right\|_{TV}+n\left\|D_2^{*(n-1)}*(F-D_2)\right\|_{TV}\\
&\quad\leqslant \frac{C(1-q)}{nq^{7/2}}+\frac{Cn}{\sqrt{q}}\left\|(F-I)^{*2}*\mathrm{e}^{nq(F-I)/4}\right\|_{TV}\left\|\mathrm{e}^{nq(F-I)/4}\right\|_{TV}\\
&\qquad+\frac{Cn}{\sqrt{q}}\|D_2^{*(-1)}\|_{TV}\left\|(F-I)^{*2}*\mathrm{e}^{nq(F-I)/4}\right\|_{TV}\\
&\qquad\times\|F-I\|_{TV}\left\|\mathrm{e}^{nq(F-I)/4}\right\|_{TV}
\end{aligned}
$$

$$\leqslant \frac{C(1-q)}{nq^{7/2}} + \frac{Cn}{\sqrt{q}} \left\| (F-I)^{*2} * \mathrm{e}^{nq(F-I)/4} \right\|_{TV}.$$

It remains to apply (5.30) or (5.30). □

Proof of Theorem 5.2.2. From Lemma 5.3.2 it follows that

$$\begin{aligned} \|F_{S_1}^{*n} - D_2^{*n}\|_{TV} &\leqslant \frac{C(1-q_{S_1})^{15/2}}{q_{S_1}^8 n^2\sqrt{n}} \\ &\quad + \sum_{m=1}^{4} \binom{n}{m} \left\| D_2^{*(n-m)} * (F_{S_1} - D_2)^{*m} \right\|_{TV}. \end{aligned} \tag{5.38}$$

Therefore, applying (5.27), (5.28), (A.18) and (5.30) we prove that

$$\begin{aligned} &\sum_{m=2}^{4} \binom{n}{m} \left\| D_2^{*(n-m)} * (F_{S_1} - D_2)^{*m} \right\|_{TV} \\ &\quad \leqslant \sum_{m=2}^{4} \binom{n}{m} \|D_2^{*n} * (F_{S_1} - D_2)^{*m}\|_{TV} \, \mathrm{e}^{4m} \\ &\quad \leqslant \frac{C}{\sqrt{q_{S_1}}} \sum_{m=2}^{4} \binom{n}{m} \left\| \exp\left\{ \frac{nq_{S_1}(F_{S_1}-I)}{2} \right\} * (F_{S_1} - D_2)^{*m} \right\|_{TV} \\ &\quad \leqslant \frac{C}{\sqrt{q_{S_1}}} \sum_{m=2}^{4} n^m \left\| \exp\left\{ \frac{nq_{S_1}(F_{S_1}-I)}{2} \right\} * (F_{S_1} - I)^{*3m} \right\|_{TV} \\ &\quad \leqslant \frac{C}{\sqrt{q_{S_1}}} \left\| \exp\left\{ \frac{nq_{S_1}(F_{S_1}-I)}{4} \right\} * (F_{S_1} - I)^{*3} \right\|_{TV} \\ &\qquad \times \sum_{m=2}^{4} n^m \left\| \exp\left\{ \frac{nq_{S_1}(F_{S_1}-I)}{4} \right\} * (F_{S_1} - I)^{*(2m-1)} \right\|_{TV} \\ &\qquad \times \|F_{S_1} - I\|_{TV}^{m-2} \\ &\quad \leqslant \frac{C}{\sqrt{q_{S_1}}} \left\| \exp\left\{ \frac{nq_{S_1}(F_{S_1}-I)}{12} \right\} * (F_{S_1} - I) \right\|_{TV}^{3} \sum_{m=2}^{4} \frac{n^m 2^{m-2}}{(nq_{S_1})^{m-1/2}} \\ &\quad \leqslant \frac{C\delta^3(nq_{S_1}/12)}{q_{S_1}^7 n^2\sqrt{n}} \leqslant \frac{C\delta^3(nq_{S_1}/6)}{q_{S_1}^7 n^2\sqrt{n}}. \end{aligned} \tag{5.39}$$

Similarly, applying (5.28), (5.26), (5.27), (A.18) and (5.30), we get

$$\begin{aligned} &n\|D_2^{*(n-1)} * (F_{S_1} - D_2)\|_{TV} \\ &\quad \leqslant \frac{n}{3}\|D_2^{*(n-1)} * (F_{S_1} - I)^{*3}\|_{TV} + Cn\|D_2^{*(n-1)} * (F_{S_1} - I)^{*4}\|_{TV} \\ &\quad \leqslant \frac{n}{3}\|D_2^{*n} * (F_{S_1} - I)^{*3}\|_{TV} \end{aligned}$$

$$
\begin{aligned}
&+\frac{n}{3}\|D_2^{*(-1)}\|_{TV}\|D_2^{*n}*(I-D_2)*(F_{S_1}-I)^{*3}\|_{TV}\\
&+Cn\|D_2^{*(-1)}\|_{TV}\|D_2^{*n}*(F_{S_1}-I)^{*4}\|_{TV}\\
\leqslant\ & \frac{3.5n}{3\sqrt{q}}\left\|\exp\left\{\frac{nq_{S_1}}{2}(F_{S_1}-I)\right\}*(F_{S_1}-I)^{*3}\right\|_{TV}\\
&+\frac{Cn}{\sqrt{qs_1}}\left\|\exp\left\{\frac{nq_{S_1}}{2}(F_{S_1}-I)\right\}*(F_{S_1}-I)^{*4}\right\|_{TV}\\
\leqslant\ & \frac{31.5\,\delta^3(nq_{S_1}/6)}{q_{S_1}^{7/2}n^2}+\frac{Cn}{\sqrt{qs_1}}\left\|\exp\left\{\frac{nq_{S_1}}{4}(F_{S_1}-I)\right\}*(F_{S_1}-I)^{*3}\right\|_{TV}\\
&\times\left\|\exp\left\{\frac{nq_{S_1}}{4}(F_{S_1}-I)\right\}*(F_{S_1}-I)\right\|_{TV}\\
\leqslant\ & \frac{31.5\,\delta^3(nq_{S_1}/6)}{q_{S_1}^{7/2}n^2}+\frac{C\sqrt{n}}{qs_1}\left\|\exp\left\{\frac{nq_{S_1}}{4}(F_{S_1}-I)\right\}*(F_{S_1}-I)^{*3}\right\|_{TV}\\
\leqslant\ & \frac{31.5\,\delta^3(nq_{S_1}/6)}{q_{S_1}^{7/2}n^2}+\frac{C\delta^3(nq_{S_1})}{q_{S_1}^4n^2\sqrt{n}}. \qquad (5.40)
\end{aligned}
$$

Collecting estimates (5.38), (5.39) and (5.40) we complete the proof of Theorem. □

Proof of Theorem 5.2.3 and Theorem 5.2.10. In the proof of theorems we use F instead of F_{S_1} or F_{S_1}.

We have

$$
\begin{aligned}
&\left\|F^{*n}-\mathrm{e}^{n(F-I)}*\left(I-\frac{n}{2}(F-I)^{*2}\right)\right\|_{TV}\\
&\qquad\leqslant\ \|M_3\|_{TV}+\sum_{m=1}^{3}\binom{n}{m}\left\|D_2^{*(n-m)}*(F-D_2)^{*m}\right\|_{TV}\\
&\qquad\quad+\left\|D_2^{*n}-\mathrm{e}^{n(F-I)}*\left(I-\frac{n}{2}(F-I)^{*2}\right)\right\|_{TV} \qquad (5.41)
\end{aligned}
$$

From Lemma 5.3.2 it follows that

$$
\|M_3\|_{TV}\leqslant\frac{C(1-q)^6}{n^2q^{13/2}}. \qquad (5.42)
$$

Taking into account (5.27), (5.28) and (A.18) we obtain

$$
\sum_{m=1}^{3}\binom{n}{m}\left\|D_2^{*(n-m)}*(F-D_2)^{*m}\right\|_{TV}
$$

$$
\begin{aligned}
&\leqslant C\sum_{m=1}^{3}\binom{n}{m}\|D_2^{*(-m)}\|_{TV}\left\|D_2^{*n}*(F-I)^{*3m}\right\|_{TV}\\
&\leqslant \frac{C}{\sqrt{q}}\sum_{m=1}^{3}\binom{n}{m}\mathrm{e}^{4m}\left\|\exp\left\{\frac{nq(F-I)}{2}\right\}*(F-I)^{*3m}\right\|_{TV}\\
&\leqslant \frac{Cn}{\sqrt{q}}\left\|\exp\left\{\frac{nq(F-I)}{4}\right\}*(F-I)^{*3}\right\|_{TV}\\
&\quad\times\sum_{m=1}^{3}n^{m-1}\left\|\exp\left\{\frac{nq(F-I)}{4}\right\}*(F-I)^{*2(m-1)}\right\|_{TV}\|F-I\|_{TV}^{m-1}\\
&\leqslant \frac{Cn}{\sqrt{q}}\left\|\exp\left\{\frac{nq(F-I)}{4}\right\}*(F-I)^{*3}\right\|_{TV}\sum_{m=2}^{3}\frac{n^{m-1}2^{m-1}}{n^{m-1}q^{m-1}}\\
&\leqslant \frac{Cn}{q^{5/2}}\left\|\exp\left\{\frac{nq(F-I)}{4}\right\}*(F-I)^{*3}\right\|_{TV}. \qquad (5.43)
\end{aligned}
$$

From (5.32) it follows that

$$
\begin{aligned}
&\left\|D_2^{*n}-\mathrm{e}^{n(F-I)}*\left(I-\frac{n}{2}(F-I)^{*2}\right)\right\|_{TV}\\
&\qquad\leqslant \frac{Cn^2}{\sqrt{q}}\left\|\mathrm{e}^{nq(F-I)/2}*(F-I)^{*4}\right\|_{TV}. \qquad (5.44)
\end{aligned}
$$

To proof (5.13) it suffices to use (5.30) in (5.43)–(5.44) and substitute resulting estimates and (5.42) into (5.41). To proof (5.23) it suffices to use (5.30) in (5.43)–(5.44) and substitute resulting estimates into (5.41). □

Proof of Theorem 5.2.4 and Theorem 5.2.11. In the proof of theorems we use F instead of F_{S_1} or F_{S_1}.

By the triangle inequality

$$
\begin{aligned}
\left\|F^{*n}-F^{*(n+1)}\right\|_{TV} \leqslant{}& \left\|F^{*n}-\mathrm{e}^{n(F-I)}\right\|_{TV}+\left\|\mathrm{e}^{n(F-I)}-\mathrm{e}^{(n+1)(F-I)}\right\|_{TV}\\
&+\left\|F^{*(n+1)}-\mathrm{e}^{(n+1)(F-I)}\right\|_{TV}.
\end{aligned}
$$

From definition of exponential measure it follows that

$$
\begin{aligned}
\left\|\mathrm{e}^{n(F-I)}-\mathrm{e}^{(n+1)(F-I)}\right\|_{TV} &= \left\|\mathrm{e}^{n(F-I)}*\left(I-\mathrm{e}^{F-I}\right)\right\|_{TV}\\
&\leqslant C\left\|\mathrm{e}^{n(F-I)}*(F-I)\right\|_{TV}.
\end{aligned}
$$

It remains to apply (5.11) and (5.30) or (5.21) and (5.30). □

Proof of Theorem 5.2.6. Without loss of generality let $a < b$. From definition of exponential measure, the fact that total variation of any distribution equals 1 and Lemma 5.3.3 we obtain

$$\begin{aligned}
\left\| \mathrm{e}^{b(F_{S_1}-I)} - \mathrm{e}^{a(F_{S_1}-I)} \right\|_{TV} &= \left\| \mathrm{e}^{a(F_{S_1}-I)} * \left(\mathrm{e}^{(b-a)(F_{S_1}-I)} - I \right) \right\|_{TV} \\
&= \left\| (b-a)\mathrm{e}^{a(F_{S_1}-I)} * (F_{S_1}-I) * \int_0^1 \mathrm{e}^{\tau(b-a)(F_{S_1}-I)} \mathrm{d}\tau \right\|_{TV} \\
&\leqslant (b-a) \left\| \mathrm{e}^{a(F_{S_1}-I)} * (F_{S_1}-I) \right\|_{TV} \int_0^1 \left\| \mathrm{e}^{\tau(b-a)(F_{S_1}-I)} \right\|_{TV} \mathrm{d}\tau \\
&\leqslant (b-a) \left\| \mathrm{e}^{a(F_{S_1}-I)} * (F_{S_1}-I) \right\|_{TV} \\
&\leqslant \frac{(b-a)\delta(a)}{2a} \leqslant \frac{(b-a)N}{a}. \qquad (5.45)
\end{aligned}$$

If $b \leqslant a(1+2/N)$, then

$$\frac{(b-a)N}{a} \leqslant \frac{(b-a)N}{b}\left(1+\frac{2}{N}\right)$$

and theorem's statement follows from (5.45). If $b > a(1+2/N)$, then directly

$$\begin{aligned}
\left\| \mathrm{e}^{b(F_{S_1}-I)} - \mathrm{e}^{a(F_{S_1}-I)} \right\|_{TV} &\leqslant \left\| \mathrm{e}^{b(F_{S_1}-I)} \right\|_{TV} + \left\| \mathrm{e}^{a(F_{S_1}-I)} \right\|_{TV} = 2 \\
&\leqslant \frac{(b-a)(N+2)}{b}.
\end{aligned}$$

□

Proof of Theorem 5.2.7. Applying Lemma 5.3.2, (5.27), (A.18) and (5.30) we obtain

$$\begin{aligned}
\|M_k\|_{TV} &\leqslant \|M_{4k+3}\|_{TV} + \sum_{m=k+1}^{4k+3} \binom{n}{m} \left\| D_2^{*(n-m)} * (F_{S_1} - D_2)^{*m} \right\|_{TV} \\
&\leqslant \frac{C(k)(1-q_{S_1})^{6(k+1)}}{n^{2(k+1)} q_{S_1}^{6k+13/2}} \\
&\quad + C \sum_{m=k+1}^{4k+3} n^m \|D_2^{*(-m)}\|_{TV} \left\| D_2^{*n} * (F_{S_1} - I)^{*3m} \right\|_{TV} \\
&\leqslant \frac{C(k)(1-q_{S_1})^{6(k+1)}}{n^{2(k+1)} q_{S_1}^{6k+13/2}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C(k)}{\sqrt{q_{S_1}}}\sum_{m=k+1}^{4k+3} n^m \left\| \mathrm{e}^{nq_{S_1}(F_{S_1}-I)/2} * (F_{S_1}-I)^{*3m} \right\|_{TV} \\
\leqslant\ & \frac{C(k)(1-q_{S_1})^{6(k+1)}}{n^{2(k+1)} q_{S_1}^{6k+13/2}} \\
&+\frac{C(k)n^{k+1}}{\sqrt{q_{S_1}}} \left\| \mathrm{e}^{nq_{S_1}(F_{S_1}-I)/4} * (F_{S_1}-I)^{*3(k+1)} \right\|_{TV} \\
&\times \left(1+\sum_{m=1}^{3k-2} n^m \left\| \mathrm{e}^{nq_{S_1}(F_{S_1}-I)/4} * (F_{S_1}-I)^{*2m} \right\|_{TV} \|F_{S_1}-I\|_{TV}^m \right) \\
\leqslant\ & \frac{C(k)(1-q_{S_1})^{6(k+1)}}{n^{2(k+1)} q_{S_1}^{6k+13/2}} + \frac{C(k)\delta^{3(k+1)}(nq_{S_1})}{q_{S_1}^{6k-3/2} n^{2(k+1)}}.
\end{aligned}
$$

□

Proof of Theorem 5.2.9. Applying Lemma 5.3.2, (5.27), (5.28), (A.16) and (5.30) we get

$$
\begin{aligned}
&\|\mathsf{F}_{S_1}^{*n} - D_2^{*n}\|_{TV} \\
&\quad\leqslant\ \|M_3\|_{TV} + C\sum_{m=1}^{3}\binom{n}{m}\|D_2^{*(-m)}\|_{TV}\|D_2^{*n} * (\mathsf{F}_{S_1}-I)^{*3m}\|_{TV} \\
&\quad\leqslant\ \frac{C(1-\mathsf{q}_{S_1})^6}{n^2\mathsf{q}_{S_1}^{13/2}} + \frac{C}{\sqrt{q}}\sum_{m=1}^{3}\mathrm{e}^{4m}n^m \left\| \mathrm{e}^{n\mathsf{q}_{S_1}(\mathsf{F}_{S_1}-I)/2} * (\mathsf{F}_{S_1}-I)^{*3m} \right\|_{TV} \\
&\quad\leqslant\ \frac{C(1-\mathsf{q}_{S_1})^6}{n^2\mathsf{q}_{S_1}^{13/2}} + \frac{Cn}{\sqrt{\mathsf{q}_{S_1}}} \left\| \mathrm{e}^{n\mathsf{q}_{S_1}(\mathsf{F}_{S_1}-I)/4} * (\mathsf{F}_{S_1}-I)^{*3} \right\|_{TV} \\
&\qquad\times \sum_{m=1}^{3} n^{m-1} \left\| \mathrm{e}^{n\mathsf{q}_{S_1}(\mathsf{F}_{S_1}-I)/4} * (\mathsf{F}_{S_1}-I)^{*2(m-1)} \right\|_{TV} 2^{m-1} \\
&\quad\leqslant\ \frac{C(1-\mathsf{q}_{S_1})^6}{n^2\mathsf{q}_{S_1}^{13/2}} + \frac{C}{n^2\mathsf{q}_{S_1}^{7/2}}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^3 \sum_{m=2}^{3}\frac{n^{m-1}2^{m-1}}{n^{m-1}\mathsf{q}_{S_1}^{m-1}} \\
&\quad\leqslant\ \frac{C}{n^2\,\mathsf{q}_{S_1}^{13/2}}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^3.
\end{aligned}
$$

□

Proof of Theorem 5.2.12. Without loss of generality let $a < b$. Repeating the

proof of (5.45) but using (5.30) we prove that

$$\begin{aligned}\left\|\mathrm{e}^{b(\mathsf{F}_{S_1}-I)}-\mathrm{e}^{a(\mathsf{F}_{S_1}-I)}\right\|_{TV} &\leqslant (b-a)\left\|\mathrm{e}^{a(\mathsf{F}_{S_1}-I)}*(\mathsf{F}_{S_1}-I)\right\|_{TV}\\ &\leqslant \frac{3.6(b-a)}{a\mathrm{e}}\sum_{m=1}^{K}\sqrt{1+\sigma_m}.\end{aligned}$$

If $b\leqslant 2.5a$, then

$$\frac{3.6(b-a)}{a\mathrm{e}}\sum_{m=1}^{K}\sqrt{1+\sigma_m}\leqslant\frac{3.4(b-a)}{b}\sum_{m=1}^{K}\sqrt{1+\sigma_m}.$$

and theorem's statement follows from (5.45). If $b>2.5a$, then directly

$$\left\|\mathrm{e}^{b(\mathsf{F}_{S_1}-I)}-\mathrm{e}^{a(\mathsf{F}_{S_1}-I)}\right\|_{TV}\leqslant 2\leqslant\frac{3.4(b-a)}{b}\leqslant\frac{3.4(b-a)}{b}\sum_{m=1}^{K}\sqrt{1+\sigma_m}.$$

□

Proof of Proposition 5.2.5. From the proof of Theorem 5.2.1 we see that

$$n\|\mathsf{F}_{S_1}^{*n}-\exp(n(\mathsf{F}_{S_1}-I))\|\leqslant\frac{n^2}{2}\left\|(\mathsf{F}_{S_1}-I)^{*2}*\mathrm{e}^{n(\mathsf{F}_{S_1}-I)}\right\|_{TV}+\frac{C}{\sqrt{n}}.$$

By assumptions

$$\begin{aligned}\mathsf{F}_{S_1}-I &= \sum_{j=1}^{N}p_{\vec{x}_j}\left(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I\right),\\ (\mathsf{F}_{S_1}-I)^{*2} &= \sum_{j=1}^{N}p_{\vec{x}_j}^2\left(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I\right)^{*2}\\ &\quad+\sum_{j\neq k}^{N}p_{\vec{x}_j}\left(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I\right)p_{\vec{x}_k}\left(I_{-\vec{x}_k}+I_{\vec{x}_k}-2I\right)\end{aligned}$$

and

$$\begin{aligned}&\exp\left(n(\mathsf{F}_{S_1}-I)\right)\\ &\quad=\exp\left(np_{\vec{x}_j}(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I)\right)*\exp\left(np_{\vec{x}_k}(I_{-\vec{x}_k}+I_{\vec{x}_k}-2I)\right)*\Theta\\ &\quad=\exp\left(n(\mathsf{F}_{S_1}-I)\right)=\exp\left(np_{\vec{x}_j}(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I)\right)*\Theta.\end{aligned}$$

Therefore, taking into account (A.21), we get

$$\begin{aligned}
&\frac{n^2}{2}\left\|(\mathsf{F}_{S_1}-I)^{*2}*\mathrm{e}^{n(F-I)}\right\|_{TV}\\
&\quad\leqslant\ \frac{n^2}{2}\sum_{j=1}^{N}p_{\vec{x}_j}^2\left\|\left(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I\right)^{*2}*\exp\left(np_{\vec{x}_j}\left(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I\right)\right)\right\|_{TV}\\
&\qquad+\frac{n^2}{2}\sum_{j\neq k}^{N}p_{\vec{x}_j}\left\|\left(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I\right)*\exp\left(np_{\vec{x}_j}\left(I_{-\vec{x}_j}+I_{\vec{x}_j}-2I\right)\right)\right\|_{TV}\\
&\qquad\times p_{\vec{x}_k}\left\|\left(I_{-\vec{x}_k}+I_{\vec{x}_k}-2I\right)*\exp\left(np_{\vec{x}_k}\left(I_{-\vec{x}_k}+I_{\vec{x}_k}-2I\right)\right)\right\|_{TV}\\
&\quad\leqslant\ \frac{C^*N}{2}+\frac{N(N-1)}{\pi\,\mathrm{e}}+\frac{C}{\sqrt{n}}.
\end{aligned}$$

To complete the proof it suffices to calculate limits for estimates as $n\to\infty$.$\square$

6 LOCAL ESTIMATES AND ESTIMATES IN ℓ_r METRIC FOR SYMMETRIC LATTICE DISTRIBUTIONS

In this Chapter a distribution of the sum of independent lattice vectors having symmetric distributions is approximated by the Hipp-type signed Compound Poisson measures. The upper and lower bounds are established for the point metric and ℓ_r, $r \geqslant 1$ metrics. Marginal non-uniform local estimates are obtained.

6.1 Known results

The Hipp-type [46] one-dimensional lattice approximations were investigated in [18] and [30], Section 7.3. Particularly, from Theorem 1 [18] it follows that

$$\left\| F^{*n} - \exp\left\{ n \sum_{j=1}^{k} (-1)^{j+1} (F-I)^{*j}/j \right\} \right\|_{\infty} \leqslant \frac{C}{n^{k+1/2}\sqrt{1-F\{0\}}},$$

if $F\{0\} \geqslant 1/2$.

Not many CP approximation results are known for symmetric multidimensional r.v.s. The most general results are in Kolmogorov metric, see [42, 87]. In [87] it is proved that if for some $0 \leqslant \alpha \leqslant 2$, $\widehat{F}(\vec{t}) \geqslant -1+\alpha$, for all $\vec{t} \in \mathbb{R}^d$, then

$$\begin{aligned} \sup_{\vec{x}\in\mathbb{R}^d} & |F^{*n}(\vec{x}) - \exp\{n(F-I)\}(\vec{x})| \\ & \leqslant \quad C(d)(n^{-1} + \exp\{-n\alpha + Cd\ln^3 n\}). \end{aligned} \tag{6.1}$$

Here $F(\vec{x}) = F\{(-\infty, x_1] \times (-\infty, x_2] \times \ldots \times (-\infty, x_d]\}$ denotes a cumulative distribution function. Though assumption $\widehat{F}(\vec{t}) \geqslant -1+\alpha$ seems very technical it is always satisfied with $\alpha = 2F\{\vec{0}\}$. Therefore (6.1) can be applied to any symmetric distribution with some probabilistic mass at zero. In [42], the estimate (6.1) was reformulated for infinite random vectors.

In [56], Kerstan's method from [74] was adapted for estimating the accuracy of CP approximation in the total variation metric.

Local estimates for CP approximations to sums of symmetric lattice r.v.s. are poorly analyzed. We are only aware of some results proved in [56].

Let $\vec{X}_1, \ldots, \vec{X}_n$ be independent r.v.s. taking values in $\mathbb{Z}^d$ and let H_m be a distribution concentrated on integers of mth coordinate-axis except zero, that is on $\{k\bar{e}_m, \quad k = \pm 1, \pm 2, \ldots\}$, where $\bar{e}_m = (0, \ldots, 0, 1, 0, \ldots)$ is mth coordinate

vector. Let for $m = 1,2,\dots,d$; $i = 1,2,\dots,n$:

$$\mathscr{L}(\vec{X}_i) = q_i I + \sum_{m=1}^{d} p_{i,m} H_m,$$

$$q_i = \mathbb{P}(\vec{X}_i = \vec{0}), \quad q_i + \sum_{m=1}^{d} p_{i,m} = 1, \quad 0 \leqslant q_i, p_{i,m} \leqslant 1,$$

$$M_i = \mathscr{L}(\vec{X}_i) - I = \sum_{m=1}^{d} p_{i,m}(H_m - I), \qquad a_r = \sum_{j=1}^{n} q_j p_{j,r}.$$

In [56] it was proved that if H_i, $i = 1,\dots,n$ are symmetric distributions, then

$$\left\| \prod_{i=1}^{*n}(I + M_i) - \exp\left\{ \sum_{i=1}^{n} M_i \right\} \right\|_{\infty} \leqslant 3.12 \left(\prod_{l=1}^{d} \sqrt{\frac{2}{a_l \mathrm{e}}} \right) \sum_{j=1}^{n} \left(\sum_{r=1}^{d} \frac{p_{j,r}}{a_r} \right)^2.$$

Note that if $d, p_{j,r} \asymp C$, then the rate of accuracy in (6.2) is $O(n^{-(1+d/2)})$. A more general result was proved for identically distributed r.v.s. Let F be symmetric distribution concentrated on $\mathbb{Z}^k$. If $F\{\vec{0}\} \geqslant 1/2$, then

$$\|F^{*n} - \exp\{n(F-I)\}\|_{\infty} \leqslant \frac{C(d)}{n} \prod_{m=1}^{d} \min\left(1, \frac{1}{\sqrt{nF\{\bar{e}_m\}}}\right). \tag{6.2}$$

One of our goals is to obtain estimates similar to (6.2) for the more general Hipp-type approximations.

6.2 Results

Suppose, $\vec{X}_{S_2,i} = (X_{i1}^{S_2}, X_{i2}^{S_2}, \dots, X_{id}^{S_2})$, $(i = 1,\dots,n)$ are d-dimensional independent r.v.s. taking values in $\mathbb{Z}^d$ and having symmetric distributions $F_{S_2,1}, F_{S_2,2}, \dots, F_{S_2,n} \in \mathscr{F}_d^S$, i.e. $F_{S_2,i}\{\vec{x}\} = F_{S_2,i}\{-\vec{x}\}$, $\vec{x} \in \mathbb{Z}^d$, $i = 1,2,\dots,n$. Our aim is to approximate distribution of $\vec{S}_n^{\star} = \sum_{i=1}^{n} \vec{X}_{S_2,i}$ by Hipp-type SCP measures.

We recall some notation from Chapter 4. Let, for $k \in \mathbb{N}$, $i = 1,2,\dots,n$,

$$\mathbf{F} := \mathscr{L}(\vec{S}_n^{\star}) = \prod_{i=1}^{*n} F_{S_2,i}, \qquad \mathbf{H}_{\mathrm{n,k}} := \prod_{i=1}^{*n} H_{i,k},$$

$$H_{i,k} := \exp\left\{\sum_{j=1}^{k}(-1)^{j+1}(F_{S_2,i}-I)^{*j}/j\right\}.$$

Note that $\mathbf{H}_{\mathrm{n},1} = \exp\{\sum_{i=1}^{n}(F_{S_2,i}-I)\}$ is convolution of accompanying CP distributions. Let $i,j,m,k = 1,2,\ldots,n$. Set

$$\sum_{\vec{x}} := \sum_{\vec{x}\in\mathbb{Z}^d\setminus\{\vec{0}\}}, \quad p_{S_2,i}(\vec{x}) = \mathbb{P}(\vec{X}_{S_2,i} = \vec{x}),$$
$$q_{S_2,i} := \mathbb{P}(\vec{X}_{S_2,i} = \vec{0}) = p_{S_2,i}(\vec{0}),$$
$$\bar{\beta}_{i,j} = \sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x}), \quad \bar{\mu}_{i,jm} = \sum_{\vec{x}}x_j x_m p_{S_2,i}(\vec{x}),$$
$$\bar{\beta}_{i,jm} = \sum_{\vec{x}}|x_j||x_m|p_{S_2,i}(\vec{x}), \quad \bar{\beta}^{(2)}_{i,j} = \sum_{\vec{x}}x_j^2 p_{S_2,i}(\vec{x}),$$
$$\bar{\beta}^{(4)}_{i,j} = \sum_{\vec{x}}x_j^4 p_{S_2,i}(\vec{x}),$$
$$\lambda_1^{S_2}(\vec{x}) = \max\left(1; \sum_{i=1}^{n}q_{S_2,i}p_{S_2,i}(\vec{x})\right), \quad \bar{L}_{i,1} = \sum_{\vec{x}}\frac{p_{S_2,i}(\vec{x})}{\lambda_1^{S_2}(\vec{x})},$$
$$\bar{p}^0_{i,1}(j) = p_{S_2,i}(j,0,\ldots,0), \quad \bar{p}^0_{i,m}(j) = p_{S_2,i}(0,\ldots,0,j,0,\ldots,0),$$
$$\bar{p}^0_{i,d}(j) = p_{S_2,i}(0,\ldots,0,j),$$
$$\bar{\lambda}^0_{m,1} = \max\left(1; \sum_{i=1}^{n}q_{S_2,i}\sum_{j\in\mathbb{Z}\setminus\{0\}}\bar{p}^0_{i,m}(j)\right); \quad \bar{\omega}_{d,1} = \prod_{m=1}^{d}\bar{\lambda}^0_{m,1},$$
$$\widehat{\Psi}_1 = \exp\left\{\frac{1}{2}\sum_{i=1}^{n}q_{S_2,i}(\widehat{F}_{S_2,i}(\vec{t})-1)\right\}.$$

We begin from the estimates in point metric and analog of Theorem 1 in [48].

Theorem 6.2.1. *Let $k\in\mathbb{N}$ and let $F_{S_2,i}\in\mathscr{F}_d^{\mathbb{Z}}$ be symmetric, $0<q_{S_2,i}<1$, $i=1,\ldots,n$. Then*

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n,k}}\|_{\infty} \leqslant 6^d \mathrm{e}^{1-k}(2(k+1))^{k+1}\,\bar{\omega}_{d,1}^{-1/2}\sum_{i=1}^{n}\bar{L}_{i,1}^{k+1}. \tag{6.3}$$

If, in addition, $F_{S_2,i}\equiv F_{S_2}$, then

$$\|F_{S_2}^{*n}-H_{1,k}^{*n}\|_{\infty} \leqslant \frac{(2(k+1))^{k+1}(3\sqrt{2})^d C(q_{S_2})}{n^k\,\mathrm{e}^{k-1}} \times\prod_{m=1}^{d}\min\left(1,\frac{1}{\sqrt{nF_{S_2}\{j\bar{e}_m,\,j\in\mathbb{N}\}}}\right), \tag{6.4}$$

where

$$C(q_{S_2}) = \begin{cases} 1, & \text{if } q_{S_2} \geqslant 1/2, \\ q_{S_2}^{-(k+1+d/2)}, & \text{if } q_{S_2} < 1/2. \end{cases}$$

The estimate (6.3) holds under more general conditions than (6.2), since $k \geqslant 1$ and there are no additional specific structural assumptions on the lattice support of $F_{S_2,i}$. Observe that in (6.2) distributions of summands are various mixtures of the same set of distributions concentrated on coordinate axes. The difference in supports makes it difficult to compare estimates. However, if $k = 1$ and $\vec{X}_{S_2,i}$ are concentrated on zero and $\bar{e}_m$, then the only difference between (6.3) and (6.2) is that the former has a larger constant. Comparison of estimates is easier for identically distributed r.v.s. Obviously, (6.6) is an improvement over (6.2), since $F_{S_2}\{j\bar{e}_m,\, j \in \mathbb{N}\} \geqslant F_{S_2}\{\bar{e}_m\}$. Moreover, (6.6) contains explicit constants and holds for $k \geqslant 1$.

To better understand the accuracy, we provide the following example.

Example 6.2.1. *Let $\vec{X}_{S_2,i}$, $(i = 1,\dots,n)$ be i.i.d. two-dimensional vectors concentrated at seven points; let $\mathscr{L}(\vec{X}_{S_2}) = F_{S_2} \in \mathscr{F}_2$ be symmetric,*

$$p_{S_2}(1,0) = p_{S_2}(-1,0), \qquad p_{S_2}(0,1) = p_{S_2}(0,-1),$$
$$p_{S_2}(1,1) = p_{S_2}(-1,-1), \qquad q_{S_2} = p(0,0) \in (0,1).$$

Then $\bar{L}_{1,1} \leqslant 7/nq_{S_2}$; $\bar{\lambda}^0_{1,1} \geqslant nq_{S_2}p_{S_2}(1,0)$, $\bar{\lambda}^0_{2,1} \geqslant nq_{S_2}p_{S_2}(0,1)$ and

$$\|F^{*n}_{S_2} - \exp\{n(F_{S_2} - I)\}\|_\infty \leqslant \frac{C}{n^2} \cdot \frac{1}{q^3_{S_2}\sqrt{p_{S_2}(1,0)p_{S_2}(0,1)}},$$

$$\|F^{*n}_{S_2} - H^{*n}_{1,2}\|_\infty \leqslant \frac{C}{n^3} \cdot \frac{1}{q^4_{S_2}\sqrt{p_{S_2}(1,0)p_{S_2}(0,1)}}.$$

If $p_{S_2}(1,0) = p_{S_2}(-1,0) = p_{S_2}(0,1) = p_{S_2}(0,-1) = p_{S_2}(1,1) = p_{S_2}(-1,-1) = p_{S_2}$, then

$$\|F^{*n}_{S_2} - \exp\{n(F_{S_2} - I)\}\|_\infty \leqslant \frac{C}{n^2 q^3_{S_2} p_{S_2}}, \qquad \|F^{*n}_{S_2} - H^{*n}_{1,2}\|_\infty \leqslant \frac{C}{n^3 q^4_{S_2} p_{S_2}}.$$

If $p_{S_2} = const$, the approximation accuracy is C/n^2 and C/n^3, respectively.

Theorem's 6.2.1 estimates can be reformulated for ℓ_r, $r \geqslant 2$ metric.

Theorem 6.2.2. *Let $k \in \mathbb{N}$ and let $F_{S_2,i} \in \mathscr{F}_d^{\mathbb{Z}}$ be symmetric, $0 < q_{S_2,i} < 1$, $i = 1, \dots, n$, $r \geqslant 2$. Then*

$$\begin{aligned} \|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_r &\leqslant 3^{d(1-1/r)} 2^{d(1-3/2r)} \bar{\omega}_{d,1}^{(1-r)/2r} \\ &\quad \times \mathrm{e}^{1-k} \left(2(k+1)\right)^{k+1} \sum_{i=1}^{n} \bar{L}_{i,1}^{k+1}. \end{aligned} \tag{6.5}$$

If, in addition, $F_{S_2,i} \equiv F_{S_2,i}$, then

$$\begin{aligned} \|F_{S_2,i}^{*n} - H_{1,k}^{*n}\|_r &\leqslant \frac{(2(k+1))^{k+1} 3^{d(r-1)/r} 2^{d(r-2)/2r} \tilde{C}(q_{S_2}, r)}{n^{k+d(r-1)/2r}\, \mathrm{e}^{k-1}} \\ &\quad \times \prod_{m=1}^{d} \left(F_{S_2}\{j\bar{e}_m,\, j \in \mathbb{N}\}\right)^{(1-r)/2r}. \end{aligned}$$

Here

$$\tilde{C}(q_{S_2}, r) = \begin{cases} 1, & \text{if } q_{S_2} \geqslant 1/2, \\ q_{S_2}^{-(k+1+d(r-1)/2r)}, & \text{if } q_{S_2} < 1/2. \end{cases}$$

The Hipp-type approximation can be viewed as a special case of asymptotic expansion in the exponent. It is also possible to construct a non-exponential asymptotic expansion. Let

$$\mathbf{A}_1 = -\frac{1}{2} \sum_{i=1}^{n} (F_{S_2,i} - I)^{*2},$$

Theorem 6.2.3. *Let $0 < q_{S_2,i} < 1$, $(i = 1, \dots, n)$. Then*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1} * (I + \mathbf{A_1})\|_\infty \leqslant 6^d C \bar{\omega}_{d,1}^{-1/2} \left(\sum_{i=1}^{n} \bar{L}_{i,1}^3 + \left(\sum_{i=1}^{n} \bar{L}_{i,1}^2 \right)^2 \right).$$

Example 6.2.2. *Let assumptions of Example 6.2.1 hold and*

$$\begin{aligned} &p_{S_2}(1,0) = p_{S_2}(-1,0) = p_{S_2}(0,1) = p_{S_2}(0,-1) \\ &= p_{S_2}(1,1) = p_{S_2}(-1,-1) = p_{S_2} = const. \end{aligned}$$

Then

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1} * (I + \mathbf{A_1})\|_\infty \leqslant C n^{-3}.$$

We got the same accuracy as holds for $\mathbf{H}_{\mathrm{n},2}$.

Next, we consider nonuniform estimates for separate coordinates. As before, $k, n \in \mathbb{N}$.

Theorem 6.2.4. *Let $j \in \{1,2,\dots,d\}$, $k \in \mathbb{N}$, and let $\bar{\beta}^{(2)}_{i,j} < \infty$ for $i = 1,2,\dots,n$. Then for any $\vec{x} \in \mathbb{Z}^d$,*

$$\begin{aligned}\left|x_j\right| \left|\mathbf{F}\{\vec{x}\} - \mathbf{H}_{\mathrm{n,k}}\{\vec{x}\}\right| \;\leqslant\; & C(d,k)\, \bar{\omega}^{-1/2}_{d,1} \left(\sum_{m=1}^{n} \bar{L}^{k+1}_{m,1} \sum_{i=1}^{n} \sqrt{\bar{L}_{i,1} \bar{\beta}^{(2)}_{i,j}} \right. \\ & \left. + \sum_{i=1}^{n} \bar{L}^{k+1/2}_{i,1} \sqrt{\bar{\beta}^{(2)}_{i,j}} \right).\end{aligned}$$

Theorem 6.2.5. *Let $j \in \{1,2,\dots,d\}$, $k \in \mathbb{N}$, and let $\bar{\beta}^{(2)}_{i,j} < \infty$ for $i = 1,2,\dots,n$. Then for any $\vec{x} \in \mathbb{Z}^d$,*

$$\begin{aligned}(x_j)^2 \left|\mathbf{F}\{\vec{x}\} - \mathbf{H}_{\mathrm{n,k}}\{\vec{x}\}\right| \leqslant\; & C(d,k)\, \bar{\omega}^{-1/2}_{d,1} \left(\sum_{i=1}^{n} \bar{\beta}^{(2)}_{i,j} \sum_{l=1}^{n} \bar{L}^{k+1}_{l,1} \right. \\ & + \sum_{i=1}^{n} \bar{\beta}^{(2)}_{i,j}\, \bar{L}^{k}_{i,1} + \left(\sum_{i=1}^{n} \sqrt{\bar{L}_{i,1} \bar{\beta}^{(2)}_{i,j}} \right) \left(\sum_{l=1}^{n} \bar{L}^{k+1/2}_{l,1} \sqrt{\bar{\beta}^{(2)}_{l,j}} \right) \\ & \left. + \sum_{l=1}^{n} \bar{L}^{k+1}_{l,1} \left(\sum_{i=1}^{n} \sqrt{\bar{L}_{i,1} \bar{\beta}^{(2)}_{i,j}} \right)^2 \right).\end{aligned}$$

Theorem 6.2.6. *Let $j,m \in \{1,2,\dots,d\}$, $k \in \mathbb{N}$, and $\bar{\beta}^{(2)}_{i,j} < \infty$, $\bar{\beta}_{i,jm} < \infty$ for $i = 1,2,\dots,n$. Then for any $\vec{x} \in \mathbb{Z}^d$,*

$$\begin{aligned}\left|x_j x_m\right| \left|\mathbf{F}\{\vec{x}\} - \mathbf{H}_{\mathrm{n,k}}\{\vec{x}\}\right| \leqslant\; & C(d,k)\, \bar{\omega}^{-1/2}_{d,1} \left(\sum_{i=1}^{n} \bar{\beta}_{i,jm} \sum_{l=1}^{n} L^{k+1}_{l,1} \right. \\ & + \sum_{l=1}^{n} \bar{L}^{k+1}_{l,1} \left(\sum_{i=1}^{n} \sqrt{\bar{L}_{i,1} \bar{\beta}^{(2)}_{i,j}} \right)^2 \\ & \left. + \sum_{i=1}^{n} \sqrt{\bar{L}_{i,1} \bar{\beta}^{(2)}_{i,j}} \sum_{k=1}^{n} \bar{L}^{k+1/2}_{l,1} \sqrt{\bar{\beta}^{(2)}_{l,j}} + \sum_{i=1}^{n} \bar{L}^{k}_{i,1} \sqrt{\bar{\beta}^{(2)}_{i,j}\, \bar{\beta}^{(2)}_{i,m}} \right).\end{aligned}$$

Observe, that we left x_j, x_m on the left-hand sides of estimates. It is obvious, that if $x_j, x_m \neq 0$, then all estimates can be rewritten with factors $|x_j|^{-1}$, x_j^{-2}, $|x_m x_j|^{-1}$ on the right-hand sides of inequalities.

Example 6.2.3. *Let assumptions of Example 6.2.1 hold and*

$$\begin{aligned}& p_{S_2}(1,0) = p_{S_2}(-1,0) = p_{S_2}(0,1) = p_{S_2}(0,-1) \\ & = p_{S_2}(1,1) = p_{S_2}(-1,-1) = p_{S_2} = const.\end{aligned}$$

Then

$$|x_1|\,|\mathbf{F}\{\vec{x}\}-\mathbf{H}_{\mathrm{n},2}\{\vec{x}\}| \leqslant \frac{C}{n^{5/2}}, \quad x_1^2\,|\mathbf{F}\{\vec{x}\}-\mathbf{H}_{\mathrm{n},2}\{\vec{x}\}| \leqslant \frac{C}{n^2},$$
$$|x_1x_2|\,|\mathbf{F}\{\vec{x}\}-\mathbf{H}_{\mathrm{n},\mathrm{k}}\{\vec{x}\}| \leqslant \frac{C}{n^2}.$$

To prove that in some cases, estimates (6.3) and (6.5) are optimal, we need lower estimates. Let

$$\begin{aligned}
B_{i,h} &= \sum_{j=1}^{d} \frac{\bar{\beta}_{i,j}^{(2)}}{h_j^2},\\
R_h &= \frac{1}{4}\sum_{i=1}^{n} B_{i,h}^2 - d^6 \sum_{i=1}^{n} B_{i,h} \sum_{j=1}^{d} \frac{\bar{\beta}_{i,j}^{(4)}}{h_j^4} - d^6 \sum_{i=1}^{n} B_{i,h}^2 \sum_{l=1}^{n} B_{l,h}\\
&\quad -2d^6 \sum_{i=1}^{n} B_{i,h}^3 - d^8 \left(\sum_{i=1}^{n} B_{i,h}^2\right)^2;\\
R_{h,k} &= C_1(k)\sum_{i=1}^{n} B_{i,h}^{k+1} - C_2(k)\left(\sum_{i=1}^{n} B_{i,h}^{k} \sum_{j=1}^{d} \frac{\bar{\beta}_{i,j}^{(4)}}{h_j^4}\right.\\
&\quad \left. + \sum_{i=1}^{n} B_{i,h}^{k+1} \sum_{l=1}^{n} B_{l,h} + \sum_{i=1}^{n} B_{i,h}^{k+2} + \left(\sum_{i=1}^{n} B_{i,h}^{k+1}\right)^2\right).
\end{aligned}$$

We say that $F_{S_2,i}$ is concentrated on coordinate axes if

$$\operatorname{supp} F_{S_2,i} \subset \{\pm k\bar{e}_1, \pm k\bar{e}_2, \ldots, \pm k\bar{e}_d;\ k \in \mathbb{N}\}.$$

Theorem 6.2.7. *Let $F_{S_2,i} \in \mathscr{F}_d^Z$ be symmetric, $r \geqslant 1$, $0 < q_{S_2,i} < 1$, $\bar{\beta}_{i,j}^{(4)} < \infty$, $h_j > 0$, $(i = 1, \ldots, n)$; $(j = 1, \ldots, d)$. Then*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1}\|_\infty \geqslant \frac{R_h}{4^d} \prod_{j=1}^{d} \frac{1}{h_j}, \quad \|\mathbf{F} - \mathbf{H}_{\mathrm{n},1}\|_r \geqslant \frac{V}{5^{d(r-1)/r}} \prod_{j=1}^{d} \frac{1}{h_j^{1-1/r}}. \tag{6.6}$$

If, in addition, all F_i are concentrated on coordinate axes, then, for any $k \in \mathbb{N}$, there exist $C_1(k)$ and $C_2(k)$ such that

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},\mathrm{k}}\|_\infty \geqslant \frac{R_{h,k}}{4^d} \prod_{j=1}^{d} \frac{1}{h_j}, \quad \|\mathbf{F} - \mathbf{H}_{\mathrm{n},\mathrm{k}}\|_r \geqslant \frac{R_{h,k}}{5^{d(r-1)/r}} \prod_{j=1}^{d} \frac{1}{h_j^{1-1/r}}.$$

Observe that the ℓ_1 norm is equal to the total variation norm. Therefore, Theorem 6.2.7 yields the following corollary.

Corollary 6.2.1. *Let all assumptions of Theorem 6.2.7 hold. Then*

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}\|_{TV} = \sum_{\vec{m}\in\mathbb{Z}^d} |\mathbf{F}\{\vec{m}\}-\mathbf{H}_{\mathrm{n},1}\{\vec{m}\}| \geqslant R_h,$$
$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n,k}}\|_{TV} = \sum_{\vec{m}\in\mathbb{Z}^d} |\mathbf{F}\{\vec{m}\}-\mathbf{H}_{\mathrm{n,k}}\{\vec{m}\}| \geqslant R_{h,k}.$$

Theorem 6.2.7 is abstract in the sense that it holds for any set of $h_1,\dots,h_d>0$ and the RHS of (6.6) can be negative. The general idea is to choose $h_1,\dots,h_d$ in such a way, that $R_h \approx \sum_{i=1}^n B_{i,1}^2$. If $F_{S_2,i}\equiv F_{S_2}$, then one can take $h_j^2=\alpha n\bar{\beta}_{1,j}^{(2)}$, $\alpha>0$, to get the following corollary. Note that, in this case, $\mathbf{H}_{\mathrm{n},1}=\exp\{n(F_{S_2}-I)\}$.

Corollary 6.2.2. *Let assumptions of Theorem 6.2.7 hold and $F_{S_2,i}\equiv F_{S_2}$, for $i=1,\dots,n$. Then, for any $\alpha>0$,*

$$\begin{aligned}
\|F_{S_2}^{*n}-\exp\{n(F_{S_2}-I)\}\|_{\infty} &\geqslant \frac{\tilde{R}}{n^{d/2}\alpha^{d/2}4^d}\prod_{j=1}^{d}\left(\bar{\beta}_{1,j}^{(2)}\right)^{-1/2},\\
\|F_{S_2}^{*n}-\exp\{n(F_{S_2}-I)\}\|_{r} &\geqslant \frac{\tilde{R}}{n^{d(r-1)/2r}\alpha^{d(r-1)/2r}5^{d(r-1)/r}}\\
&\quad\times\prod_{j=1}^{d}\left(\bar{\beta}_{1,j}^{(2)}\right)^{(1-r)/2r},
\end{aligned}$$

where

$$\tilde{R}=\frac{d^2}{4\alpha^2 n}\left(1-\frac{4d^5}{\alpha n}\sum_{j=1}^{d}\frac{\bar{\beta}_{1,j}^{(4)}}{\left(\bar{\beta}_{1,j}^{(2)}\right)^2}-\frac{4d^7}{\alpha}-\frac{8d^7}{\alpha n}-\frac{4d^{10}}{\alpha^2 n}\right). \tag{6.7}$$

If, for some C_0,

$$\frac{1}{n}\sum_{j=1}^{d}\frac{\bar{\beta}_{1,j}^{(4)}}{\left(\bar{\beta}_{1,j}^{(2)}\right)^2}\leqslant C_0, \tag{6.8}$$

then by choosing sufficiently large $\alpha=C(d)$ we can get the estimate $\tilde{R}\geqslant d^2/4C^2(d)n$.

Example 6.2.4. *Let assumptions of Example 6.2.1 hold, $q_{S_2}\geqslant C_3$, and $np_{S_2}(1,0)>C_4$, $np_{S_2}(0,1)>C_5$, for some C_3,C_4,C_5. We have $\bar{\beta}_{1,1}^{(4)}=\bar{\beta}_{1,1}^{(2)}=2p_{S_2}(1,0)+2p_{S_2}(1,1)$, $\bar{\beta}_{1,2}^{(4)}=\bar{\beta}_{1,2}^{(2)}=2p_{S_2}(0,1)+2p_{S_2}(1,1)$ and, therefore,*

(6.8) is satisfied. By choosing a sufficiently large constant α in (6.7), we get

$$\|F_{S_2}^{*n} - \exp\{n(F_{S_2} - I)\}\|_\infty \geqslant \frac{C}{n^2\sqrt{(p_{S_2}(1,0)+p_{S_2}(1,1))(p_{S_2}(0,1)+p_{S_2}(1,1))}}. \quad (6.9)$$

Comparing (6.9) with the upper estimate in Example 6.2.1, we see that the accuracy is of the same order if $p_{S_2}(1,1) \leqslant \min(p_{S_2}(1,0); p_{S_2}(0,1))$. For example, if

$$\begin{aligned} p_{S_2} &= p_{S_2}(1,0) = p_{S_2}(-1,0) = p_{S_2}(0,1) \\ &= p_{S_2}(0,-1) = p_{S_2}(1,1) = p_{S_2}(-1,-1), \end{aligned}$$

then

$$\frac{C_4}{n^2 p_{S_2}} \leqslant \|F_{S_2}^{*n} - \exp\{n(F_{S_2} - I)\}\|_\infty \leqslant \frac{C_5}{n^2 p_{S_2}},$$

for some absolute constants C_4, C_5. Another obvious example, when upper and lower bounds are of the same order, is when $p_{S_2}(1,1) = 0$, that is, when F_{S_2} is concentrated on coordinate axes.

Finally, we formulate an analog of Theorem 6.2.8 for the sequences of random vectors when all probabilities do not depend on the number of vectors n.

Theorem 6.2.8. *Let $n \in \mathbb{N}$ and let $F_{S_2} \in \mathscr{F}_d^Z$ be symmetric, $F_{S_2}\{j\bar{e}_m; j \in \mathbb{N}\} > 0$, $\bar{\beta}_{1,m}^{(4)} < \infty$, $m = 1, \ldots, d$; $F_{S_2}\{\vec{0}\} > 0$, $r \geqslant 2$. Then there exist constants $C_6(d, F_{S_2})$, $C_7(d, F)$, $C_8(d, F, r)$, $C_9(d, F, r)$, such that*

$$\begin{aligned} C_6(d, F_{S_2})\, n^{-1-d/2} &\leqslant \|F_{S_2}^{*n} - \exp\{n(F_{S_2} - I)\}\|_\infty \\ &\leqslant C_7(d, F_{S_2})\, n^{-1-d/2}, \\ C_8(d, F_{S_2}, r)\, n^{-1-d(r-1)/2r} &\leqslant \|F_{S_2}^{*n} - \exp\{n(F_{S_2} - I)\}\|_r \\ &\leqslant C_9(d, F_{S_2}, r)\, n^{-1-d(r-1)/2r}. \end{aligned}$$

If, in addition, F_{S_2} is concentrated on coordinate axes, then, for any $k \in \mathbb{N}$, there exist $C_{10}(k, d, F_{S_2}, r)$ and $C_{11}(k, d, F_{S_2}, r)$ such that

$$\begin{aligned} C_{10}(k, d, F_{S_2}, r)\, n^{-k-d(r-1)/2r} &\leqslant \|F_{S_2}^{*n} - H_{1,k}^{*n}\}\|_r \\ &\leqslant C_{11}(k, d, F_{S_2}, r)\, n^{-k-d(r-1)/2r}. \end{aligned}$$

The statement of Theorem 6.2.8 follows directly from Theorems 6.2.1,

6.2.2 and 6.2.7.

6.3 Auxiliary results

Lemma 6.3.1. *Let $k \in \mathbb{N}$ and let $F_{S_2,i} \in \mathscr{F}_d^Z$ be symmetric, $0 < q_{S_2,i} < 1$, $i = 1, \ldots, n$. Then*

$$|\widehat{F}_{S_2,i}(\vec{t})|, \widehat{H}_{i,k} \leqslant \Psi_1^2 = \exp\left\{-2q_{S_2,i}\sum_{\vec{x}} p_{S_2,i}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right\}.$$

Proof. Recall that $F_{S_2,i}$ is symmetric. Observe that $\widehat{F}_{S_2,i}(\vec{t}) = \widehat{F}_{S_2,i}(-\vec{t})$ and

$$\begin{aligned}\widehat{F}_{S_2,i}(\vec{t}) &= Re\,\widehat{F}_{S_2,i}(\vec{t}) = \frac{1}{2}(\widehat{F}_{S_2,i}(\vec{t}) + \widehat{F}_{S_2,i}(-\vec{t})) \\ &= \frac{1}{2}\sum_{\vec{x}\in\mathbb{Z}^d} p_{S_2,i}(\vec{x})\left(\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} + \mathrm{e}^{-\mathrm{i}(\vec{x},\vec{t})}\right) = \sum_{\vec{x}\in\mathbb{Z}^d} p_{S_2,i}(\vec{x})\cos(\vec{x},\vec{t})\end{aligned}$$

and, therefore,

$$\widehat{F}_{S_2,i}(\vec{t}) - 1 = -2\sum_{\vec{x}} p_{S_2,i}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}. \tag{6.10}$$

By Eq. (4.10) from [48]

$$\begin{aligned}|\widehat{F}_{S_2,i}(\vec{t})| &\leqslant \exp\{q_{S_2,i}(\widehat{F}_{S_2,i}(\vec{t}) - 1) \\ &= \exp\left\{-2q_{S_2,i}\sum_{\vec{x}} p_{S_2,i}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right\}. \end{aligned}\tag{6.11}$$

By the properties of characteristic functions $\widehat{F}_{S_2,i} \in [-1,1]$. Therefore $\widehat{F}_{S_2,i}(\vec{t}) - 1 \leqslant 0$ and, obviously,

$$\begin{aligned}|\widehat{H}_{i,k}(\vec{t})| &= \widehat{H}_{i,k}(\vec{t}) \leqslant \exp\{\widehat{F}_{S_2,i}(\vec{t}) - 1\} \\ &= \exp\left\{-2\sum_{\vec{x}} p_{S_2,i}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right\}. \end{aligned}\tag{6.12}$$

□

Lemma 6.3.2. *Let $\bar{\beta}_{i,j}^{(4)} < \infty$, $j = 1, \ldots, d$. Then*

$$|\widehat{F}_{S_2,i}(\vec{t}) - 1| \leqslant \frac{d^2}{2}\sum_{j=1}^{d} \bar{\beta}_{i,j}^{(2)} t_j^2, \tag{6.13}$$

$$|\widehat{F}_{S_2,i}(\vec{t}) - 1 - U_i(\vec{t})| \;\leqslant\; \frac{d^4}{24}\sum_{j=1}^{d}\bar{\beta}_{i,j}^{(4)}t_j^4, \tag{6.14}$$

where

$$U_{S_2,i}(\vec{t}) = -\frac{1}{2}\sum_{\vec{x}}(\vec{x},\vec{t})^2 p_{S_2,i}(\vec{x}), \quad (\vec{x},\vec{t}) = \sum_{j=1}^{d} x_j t_j. \tag{6.15}$$

Proof. Observe that $(\vec{x},\vec{t})^2 \leqslant d^2(\max(|x_1t_1|,\ldots,|x_dt_d|)^2$. From (6.10) it follows that

$$\begin{aligned}
|\widehat{F}_{S_2,i}(\vec{t}) - 1| &\leqslant \frac{1}{2}\sum_{\vec{x}} p_{S_2,i}(\vec{x})(\vec{x},\vec{t})^2 \\
&\leqslant \frac{d^2}{2}\sum_{\vec{x}} p_{S_2,i}(\vec{x})(\max(|x_1t_1|,\ldots,|x_dt_d|)^2 \\
&\leqslant \frac{d^2}{2}\sum_{j=1}^{d}\beta_{i,j}^{(2)}t_j^2.
\end{aligned}$$

Next, observe that by (A.48)

$$\begin{aligned}
\widehat{F}_{S_2,i}(\vec{t}) - 1 &= \frac{1}{2}\sum_{\vec{x}} p_{S_2,i}(\vec{x})(\mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} + \mathrm{e}^{-\mathrm{i}(\vec{x},\vec{t})} - 2) \\
&= U_{S_2,i}(\vec{t}) + \frac{\theta}{24}\sum_{\vec{x}} p_{S_2,i}(\vec{x})(\vec{x},\vec{t})^4, \quad |\theta| \leqslant 1.
\end{aligned}$$

To complete the proof, note that $(\vec{x},\vec{t})^4 \leqslant d^4(\max(|x_1t_1|,\ldots,|x_dt_d|)^4$. □

6.4 Proofs

Proof of Theorem 6.2.1. Assuming that $\prod_{j=k}^{m} = 1$, when $k > m$, and taking into account (6.12) and (6.11) we get

$$\begin{aligned}
\left|\prod_{i=1}^{n}\widehat{F}_{S_2,i} - \prod_{i=1}^{n}\widehat{H}_{i,k}\right| &\leqslant \sum_{i=1}^{n}|\widehat{F}_{S_2,i} - \widehat{H}_{i,k}|\prod_{j=1}^{i-1}|\widehat{F}_{S_2,j}|\prod_{m=i+1}^{n}\widehat{H}_{m,k} \\
&\leqslant \widehat{\bar{\Psi}}_1^2\sum_{i=1}^{n}\exp\{q_{S_2,i}(1-\widehat{F}_{S_2,i})\}|\widehat{F}_{S_2,i} - \widehat{H}_{i,k}| \\
&\leqslant \mathrm{e}^2\widehat{\bar{\Psi}}_1^2\sum_{i=1}^{n}|\widehat{F}_{S_2,i} - \widehat{H}_{i,k}|.
\end{aligned} \tag{6.16}$$

Here, we used an obvious inequality $q_{S_2,i}(1-\widehat{F}_{S_2,i}) \leqslant 1+|\widehat{F}_{S_2,i}| \leqslant 2$. Observe that by equation (4.5) from [48]

$$\begin{aligned}
|\widehat{F}_{S_2,i}-\widehat{H}_{i,k}| &\leqslant \widehat{H}_{i,k}|\widehat{F}_{S_2,i}-1|^{k+1}\int_0^1 \tau^k \exp\Big\{\sum_{j=1}^{k}\frac{(-1)^j}{j}(\widehat{F}_{S_2,i}-1)^j\tau^j\Big\}\mathrm{d}\tau \\
&= |\widehat{F}_{S_2,i}-1|^{k+1}\int_0^1 \tau^k \exp\Big\{\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}(\widehat{F}_{S_2,i}-1)^j(1-\tau^j)\Big\}\mathrm{d}\tau \\
&\leqslant |\widehat{F}_{S_2,i}-1|^{k+1}. \qquad (6.17)
\end{aligned}$$

Next, applying estimate $x\mathrm{e}^{-x} \leqslant \mathrm{e}^{-1}$, we get

$$\begin{aligned}
\widehat{\Psi}_1|\widehat{F}_{S_2,i}-1|^{k+1} &= \widehat{\Psi}_1\left(2\sum_{\vec{x}} p_{S_2,i}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right)^{k+1} \\
&\leqslant 2^{k+1}\Bigg(\sum_{\vec{x}} p_{S_2,i}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2} \\
&\quad \times\exp\Big\{-\frac{1}{k+1}\sum_{j=1}^{n} q_{S_2,j}\,p_{S_2,j}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\Big\}\Bigg)^{k+1} \\
&\leqslant \left(\frac{2(k+1)}{\mathrm{e}}\right)^{k+1}\left(\sum_{\vec{x}\neq\vec{0}}\frac{p_{S_2,i}(\vec{x})}{\bar{\lambda}_1(\vec{x})}\right)^{k+1} = \left(\frac{2(k+1)}{\mathrm{e}}\right)^{k+1}\bar{L}_{i,1}^{k+1}. \qquad (6.18)
\end{aligned}$$

Substituting estimates (6.17), (6.18) into (6.16) we obtain

$$\left|\prod_{i=1}^{n}\widehat{F}_{S_2,i}-\prod_{i=1}^{n}\widehat{H}_{i,k}\right| \leqslant \widehat{\bar{\Psi}}_1\mathrm{e}^2\left(\frac{2(k+1)}{\mathrm{e}}\right)^{k+1}\sum_{i=1}^{n}\bar{L}_{i,1}^{k+1}. \qquad (6.19)$$

Application of (A.36) and Lemma 4.5.2 completes the proof of (6.3). For the proof of (6.6) observe that

$$\begin{aligned}
\left|\prod_{i=1}^{n}\widehat{F}_{S_2,i}-\prod_{i=1}^{n}\widehat{H}_{i,k}\right| &\leqslant \mathrm{e}^2 n|\widehat{F}_{S_2}-1|^{k+1}\mathrm{e}^{nq_{S_2}(\widehat{F}_{S_2}-1)} \\
&\leqslant \frac{(2(k+1))^{k+1}}{q_{S_2}^{k+1}\mathrm{e}^{k-1}n^k}\bar{\Psi}_1,
\end{aligned}$$

$$\bar{\omega}_{d,1} = \prod_{m=1}^{n}\max(1, n\,q_{S_2}F_{S_2}\{j\bar{e}_m, j\in\mathbb{N}\}).$$

If $q_{S_2} \geqslant 1/2$, then $\widehat{F}_{S_2} \geqslant 0$ and

$$|\widehat{F}_{S_2}| = \widehat{F}_{S_2} \leqslant \exp\{\widehat{F}_{S_2}-1\},$$

$$\left|\prod_{i=1}^{n}\widehat{F}_{S_2,i}-\prod_{i=1}^{n}\widehat{H}_{i,k}\right| \leqslant \frac{(2(k+1))^{k+1}}{\mathrm{e}^{k-1}n^k}\exp\left\{\frac{n}{2}(\widehat{F}_{S_2}-1)\right\}.$$

Therefore, Lemma 4.5.2 can be applied with $q_{S_2,i}=1$. □

Proof of Theorem 6.2.2. From (6.19), (A.35) and Lemma 4.5.2 it follows that

$$\begin{aligned}\|\mathbf{F}-\mathbf{H}_{\mathrm{n,k}}\|_2^2 &\leqslant \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\left(\prod_{i=1}^{n}\widehat{F}_{S_2,i}-\prod_{i=1}^{n}\widehat{H}_{i,k}\right)^2 \mathrm{d}t_1\dots\mathrm{d}t_d \\ &\leqslant (3\sqrt{2})^d\bar{\omega}_{d,1}^{-1/2}\left(\mathrm{e}^{1-k}(2(k+1))^{k+1}\sum_{i=1}^{n}\bar{L}_{i,1}^{k+1}\right)^2.\end{aligned}$$

Combining the last estimate with Theorem 6.2.1 and estimate

$$\|M\|_r \leqslant (\|M\|_\infty)^{(r-2)/r}(\|M\|_2)^{2/r}, \tag{6.20}$$

we prove (6.5). The proof for $F_{S_2,i}\equiv F_{S_2}$ is very similar and, therefore, omitted. □

Proof of Theorem 6.2.3. By the triangle inequality

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},1}*(I+\mathbf{A_1})\|_\infty \leqslant \|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_\infty + \|\mathbf{H}_{\mathrm{n},2}-\mathbf{H}_{\mathrm{n},1}*(I+\mathbf{A_1})\|_\infty. \tag{6.21}$$

We use Theorem 6.2.1 to estimate

$$\|\mathbf{F}-\mathbf{H}_{\mathrm{n},2}\|_\infty \leqslant 6^d C\bar{\omega}_{d,1}^{-1/2}\sum_{i=1}^{n}\bar{L}_{i,1}^3.$$

Next observe that

$$\begin{aligned}|\widehat{F}_{S_2,i}-1|\widehat{\bar{\Psi}}_1^{1/4} &\leqslant 2\sum_{\vec{x}}p_{S_2,i}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2} \\ &\quad\times\exp\left\{-\frac{1}{4}\sum_{j=1}^{n}q_{S_2,j}p_{S_2,j}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right\} \\ &\leqslant C\sum_{\vec{x}\neq\vec{0}}\frac{p_{S_2,i}(\vec{x})}{\bar{\lambda}_1(\vec{x})}=C\bar{L}_{i,1}.\end{aligned}$$

Applying (A.48) we get similarly

$$\left|\widehat{\mathbf{H}}_{\mathrm{n},2}-\widehat{\mathbf{H}}_{\mathrm{n},1}(1+\widehat{\mathbf{A}}_{\mathbf{1}})\right|$$

$$\begin{aligned}
&\leqslant \frac{1}{4}\widehat{\mathbf{H}}_{\mathrm{n},1}\left(\sum_{i=1}^{n}|\widehat{F}_{S_2,i}-1|^2\right)^2\left|\int_0^1\exp\left\{\frac{-\tau}{2}\sum_{i=1}^{n}(\widehat{F}_{S_2,i}-1)^2\right\}(1-\tau)\mathrm{d}\tau\right| \\
&\leqslant C\widehat{\bar{\Psi}}_1^2\left(\sum_{i=1}^{n}|\widehat{F}_{S_2,i}-1|^2\right)^2\leqslant C\widehat{\bar{\Psi}}_1\left(\sum_{i=1}^{n}\bar{L}_{i,1}^2\right)^2. \qquad (6.22)
\end{aligned}$$

Applying Lemma 4.5.2 and (6.22) we prove that

$$\|\mathbf{H}_{\mathrm{n},2}-\mathbf{H}_{\mathrm{n},1}*(I+\mathbf{A_1})\|_\infty\leqslant 6^d C\bar{\omega}_{d,1}^{-1/2}\left(\sum_{i=1}^{n}\bar{L}_{i,1}^2\right)^2.$$

Substituting above estimates into (6.21) we complete the proof of the theorem.

□

Proof of Theorem 6.2.4. We have

$$\begin{aligned}
\left|\frac{\partial}{\partial t_j}\left(\prod_{i=1}^{n}\widehat{F}_{S_2,i}-\prod_{i=1}^{n}\widehat{H}_{i,k}\right)\right|&\leqslant\sum_{i=1}^{n}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\prod_{l\neq i}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i}^{n}\widehat{H}_{l,k}\right| \\
&\quad+\sum_{i=1}^{n}\left|\frac{\partial}{\partial t_j}(\widehat{F}_{S_2,i}-\widehat{H}_{i,k})\right|\prod_{l\neq i}^{n}\widehat{H}_{l,k}. \qquad (6.23)
\end{aligned}$$

Observe that

$$\begin{aligned}
\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right| &= \left|\sum_{\vec{x}}p_{S_2,i}(\vec{x})\frac{\partial}{\partial t_j}\cos(\vec{x},\vec{t})\right|\leqslant\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})|\sin(\vec{x},\vec{t})| \\
&\leqslant 2\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|. \qquad (6.24)
\end{aligned}$$

Then similarly to the proof of Theorem 6.2.1:

$$\begin{aligned}
\left|\prod_{l\neq i}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i}^{n}\widehat{H}_{l,s}\right| &\leqslant \sum_{l\neq i}^{n}|\widehat{F}_{S_2,l}-\widehat{H}_{l,k}|\prod_{m\neq i,l}^{n}\exp\{q_{S_2,m}(\widehat{F}_{S_2,m}-1)\} \\
&\leqslant \mathrm{e}^4\widehat{\bar{\Psi}}_1^2\sum_{l\neq i}^{n}|\widehat{F}_{S_2,l}-\widehat{H}_{l,k}|\leqslant\mathrm{e}^4\widehat{\bar{\Psi}}_1^2\sum_{l=1}^{n}|\widehat{F}_{S_2,l}-1|^{k+1} \\
&\leqslant C(k)\widehat{\bar{\Psi}}_1^{3/2}\sum_{l=1}^{n}\bar{L}_{l,1}^{k+1}. \qquad (6.25)
\end{aligned}$$

The same simple inequality $x\mathrm{e}^{-x^2}\leqslant 1/\sqrt{2\mathrm{e}}$ and Hölder's inequality allow to

prove

$$\begin{aligned}\widehat{\bar{\Psi}}_1^{1/2}\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right| &\leqslant 2\widehat{\bar{\Psi}}_1^{1/2}\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \\ &\leqslant \sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \\ &\quad\times\exp\left\{-\frac{1}{2}\sum_{l=1}^{n}q_{S_2,l}p_{S_2,i}(\vec{x})\sin^2\frac{(\vec{x},\vec{t})}{2}\right\} \\ &\leqslant C\sum_{\vec{x}\neq\vec{0}}|x_j|\frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}}\leqslant C\sqrt{\bar{L}_{i,1}\bar{\beta}_{i,j}^{(2)}}. \end{aligned} \tag{6.26}$$

Collecting above estimates a we get

$$\begin{aligned}&\sum_{i=1}^{n}\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\prod_{l\neq i}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i}^{n}\widehat{H}_{l,k}\right| \\ &\qquad\leqslant C(k)\,\widehat{\bar{\Psi}}_1\sum_{l=1}^{n}\bar{L}_{l,1}^{k+1}\sum_{i=1}^{n}\sqrt{\bar{L}_{i,1}\bar{\beta}_{i,j}^{(2)}}. \end{aligned} \tag{6.27}$$

By (4.5) in [48]

$$\begin{aligned}&\widehat{F}_{S_2,i}-\widehat{H}_{i,k}=(\widehat{F}_{S_2,i}-1)^{k+1}\int_0^1(-1)^k\tau^k\exp\{A_\tau\}\mathrm{d}\tau, \\ &A_\tau:=\sum_{m=1}^{k}\frac{(-1)^{m+1}}{m}(\widehat{F}_{S_2,i}-1)^m(1-\tau^m).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t_j}\left(\widehat{F}_{S_2,i}-\widehat{H}_{i,k}\right)&=(k+1)(\widehat{F}_{S_2,i}-1)^k\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j} \\ &\quad\times\int_0^1(-\tau)^k\mathrm{e}^{A_\tau}\,\mathrm{d}\tau+(\widehat{F}_{S_2,i}-1)^{k+1}\int_0^1(-\tau)^k\mathrm{e}^{A_\tau}\frac{\partial A_\tau}{\partial t_j}\,\mathrm{d}\tau.\end{aligned}$$

Taking into account that $0\leqslant\tau\leqslant 1$, $|\widehat{F}_{S_2,i}-1|\leqslant 2$ we can write

$$\left|\frac{\partial A_\tau}{\partial t_j}\right| \leqslant \sum_{m=1}^{k}|1-\tau^m||\widehat{F}_{S_2,i}-1|\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|\leqslant\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|\sum_{m=1}^{k}2^{m-1}$$

$$= \quad (2^k-1)\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|. \tag{6.28}$$

Note that $A_\tau \leqslant 0$, $\exp\{A_\tau\} \leqslant 1$. Thus,

$$\begin{aligned}\left|\frac{\partial}{\partial t_j}\left(\widehat{F}_{S_2,i}-\widehat{H}_{i,k}\right)\right| &\leqslant (2^{k+1}+k-1)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right| \\ &\leqslant C(k)|\widehat{F}_{S_2,i}-1|^k\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|. \end{aligned} \tag{6.29}$$

Obviously,

$$\prod_{l\neq i}^{n}\widehat{H}_{l,k} = \widehat{\bar{\Psi}}_1^2\prod_{l\neq i}\exp\{q_{S_2,l}(\widehat{F}_{S_2,l}-1)\} \leqslant \mathrm{e}^2\widehat{\bar{\Psi}}_1^2. \tag{6.30}$$

Taking into account (6.18), (6.26), (6.29), Hölder's inequality and the above estimate we can write

$$\begin{aligned}\sum_{i=1}^{n}\left|\frac{\partial}{\partial t_j}(\widehat{F}_{S_2,i}-\widehat{H}_{i,k})\right|\prod_{l\neq i}^{n}\widehat{H}_{l,k} &\leqslant C(k)\widehat{\bar{\Psi}}_1^2\sum_{i=1}^{n}|\widehat{F}_{S_2,i}-1|^k \\ &\quad\times\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \\ &\leqslant C(k)\,\widehat{\bar{\Psi}}_1\sum_{i=1}^{n}L_{i,1}^k\sum_{\vec{x}\neq\vec{0}}|x_j|\frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}} \\ &\leqslant C(k)\,\widehat{\bar{\Psi}}_1\sum_{i=1}^{n}\bar{L}_{i,1}^{k+1/2}\sqrt{\bar{\beta}_{i,j}^{(2)}}. \end{aligned} \tag{6.31}$$

Substituting estimates (6.27), (6.31) into (6.23) and applying (A.37), Lemma 4.5.2 we complete the proof of Theorem 6.2.4. □

Proof of Theorem 6.2.5. Observe that

$$\begin{aligned}\left|\frac{\partial^2}{\partial t_j^2}\left(\prod_{i=1}^{n}\widehat{F}_{S_2,i}-\prod_{i=1}^{n}\widehat{H}_{i,k}\right)\right| &\leqslant \sum_{i=1}^{n}\left|\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j^2}\right|\left|\prod_{l\neq i}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i}^{n}\widehat{H}_{l,k}\right| \\ &\quad+\sum_{i=1}^{n}\left|\frac{\partial^2}{\partial t_j^2}(\widehat{F}_{S_2,i}-\widehat{H}_{i,k})\right|\prod_{l\neq i}^{n}\widehat{H}_{l,k}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|\sum_{m\neq i}^{n}\left|\frac{\partial}{\partial t_j}(\widehat{F}_{S_2,i}-\widehat{H}_{i,k})\right|\prod_{l\neq i,m}^{n}\widehat{H}_{l,k}\\
&+\sum_{i=1}^{n}\left|\frac{\partial}{\partial t_j}(\widehat{F}_{S_2,i}-\widehat{H}_{i,k})\right|\sum_{m\neq i}\left|\frac{\partial\widehat{H}_{m,k}}{\partial t_j}\right|\prod_{l\neq i,m}^{n}\widehat{H}_{l,k}\\
&+\sum_{i=1}^{n}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|\sum_{m\neq i}\left|\frac{\partial\widehat{F}_{S_2,m}}{\partial t_j}\right|\left|\prod_{l\neq i,m}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i,m}^{n}\widehat{H}_{l,k}\right|.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j^2}=\frac{\partial^2}{\partial t_j^2}\sum_{\vec{x}}p_{S_2,i}(\vec{x})\cos(\vec{x},\vec{t})=-\sum_{\vec{x}}x_j^2p_{S_2,i}(\vec{x})\cos(\vec{x},\vec{t}),\\
&\left|\frac{\partial^2}{\partial t_j^2}\widehat{F}_i\right|\leqslant\sum_{\vec{x}}x_j^2p_{S_2,i}(\vec{x})=\bar{\beta}_{i,j}^{(2)}.
\end{aligned}
\tag{6.32}
$$

Similarly, as in the proof of (6.25), we get

$$
\begin{aligned}
\left|\prod_{l\neq i}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i}^{n}\widehat{H}_{l,k}\right| &\leqslant C\widehat{\bar{\Psi}}_1^2\sum_{m=1}^{n}|\widehat{F}_{S_2,m}-1|^{k+1}\\
&\leqslant C(k)\widehat{\bar{\Psi}}_1\sum_{m=1}^{n}\bar{L}_{m,k}^{k+1}.
\end{aligned}
\tag{6.33}
$$

Thus

$$
\sum_{i=1}^{n}\left|\frac{\partial^2}{\partial t_j^2}\widehat{F}_{S_2,i}\right|\left|\prod_{l\neq i}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i}^{n}\widehat{H}_{l,k}\right|\leqslant C(k)\widehat{\bar{\Psi}}_1\sum_{i=1}^{n}\bar{\beta}_{i,j}^{(2)}\sum_{m=1}^{n}\bar{L}_{m,k}^{k+1}.
\tag{6.34}
$$

Observe that

$$
\begin{aligned}
\frac{\partial^2}{\partial t_j^2}A_\tau &= (1-\tau)\frac{\partial^2}{\partial t_j^2}\widehat{F}_{S_2,i}\\
&\quad+\sum_{m=2}^{k}(-1)^{m+1}(1-\tau^m)(m-1)(\widehat{F}_{S_2,i}-1)^{m-2}\left(\frac{\partial}{\partial t_j}\widehat{F}_{S_2,i}\right)^2\\
&\quad+\sum_{m=1}^{k}(-1)^{m+1}(1-\tau^m)(\widehat{F}_{S_2,i}-1)^{m-1}\frac{\partial^2}{\partial t_j^2}\widehat{F}_{S_2,i}
\end{aligned}
$$

and

$$
\left|\frac{\partial^2A_\tau}{\partial t_j^2}\right| \leqslant \left|\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j^2}\right|+\sum_{m=1}^{k}(m-1)|\widehat{F}_{S_2,i}-1|^{m-2}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|^2
$$

$$
\begin{aligned}
&+\sum_{m=1}^{k}|\widehat{F}_{S_2,i}-1|^{m-1}\left|\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j^2}\right| \\
\leqslant\;& C(k)\left|\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j^2}\right|+C(k)\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|^2. \qquad (6.35)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{\partial^2}{\partial t_j^2}(\widehat{F}_{S_2,i}-\widehat{H}_{i,k}) \\
&\quad= (k+1)k(\widehat{F}_{S_2,i}-1)^{k-1}\left(\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right)^2\int_0^1(-1)^k\tau^s\mathrm{e}^{A_\tau}\mathrm{d}\tau \\
&\quad\quad+(k+1)(\widehat{F}_{S_2,i}-1)^k\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j^2}\widehat{F}_{S_2,i}\int_0^1(-\tau)^k\mathrm{e}^{A_\tau}\mathrm{d}\tau \\
&\quad\quad+2(k+1)(\widehat{F}_i-1)^k\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\int_0^1(-\tau)^k\mathrm{e}^{A_\tau}\frac{\partial A_\tau}{\partial t_j}\,\mathrm{d}\tau \\
&\quad\quad+(\widehat{F}_{S_2,i}-1)^{k+1}\int_0^1(-\tau)^k\mathrm{e}^{A_\tau}\left(\frac{\partial A_\tau}{\partial t_j}\right)^2\mathrm{d}\tau \\
&\quad\quad+(\widehat{F}_{S_2,i}-1)^{k+1}\int_0^1(-\tau)^k\mathrm{e}^{A_\tau}\frac{\partial^2 A_\tau}{\partial t_j^2}\,\mathrm{d}\tau.
\end{aligned}
$$

Recalling that $A_\tau\leqslant 0$ and, taking into account (6.28), (6.35), (6.24) and (6.32), we obtain

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial t_j^2}(\widehat{F}_{S_2,i}-\widehat{H}_{i,k})\right| &\leqslant C(k)|\widehat{F}_{S_2,i}-1|^{k-1}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|^2 \\
&\quad+C(k)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|^2+|\widehat{F}_{S_2,i}-1|^{k+1}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|^2 \\
&\quad+C(k)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j^2}\right|+|\widehat{F}_{S_2,i}-1|^{k+1}\left|\frac{\partial^2 A_\tau}{\partial t_j^2}\right| \\
&\leqslant C(k)|\widehat{F}_{S_2,i}-1|^{k-1}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|^2+C(k)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j^2}\right|.
\end{aligned}
$$

At the last step we applied (6.24) and (6.32). Similarly, as in proofs of (6.18) and (6.26), we get

$$
\widehat{\Psi}_1^{1/2}|\widehat{F}_{S_2,i}-1|^{k-1}\;\leqslant\;C(k)\bar{L}_{i,1}^{k-1},
$$

$$
\begin{aligned}
\widehat{\bar{\Psi}}_1 |\widehat{F}_{S_2,i}-1|^k &\leqslant C(k)\bar{L}_{i,1}^k, \\
\widehat{\bar{\Psi}}_1^{1/4} \sum_{\vec{x}} |x_j| p_{S_2,i}(\vec{x}) \left|\sin\frac{(\vec{x},\vec{t})}{2}\right| &\leqslant C \sum_{\vec{x}\neq\vec{0}} |x_j| \frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}}.
\end{aligned} \tag{6.36}
$$

Collecting above estimates, (6.30) and applying (A.52) we get

$$
\begin{aligned}
&\sum_{i=1}^n \left|\frac{\partial^2}{\partial t_j^2}\left(\widehat{F}_{S_2,i}-\widehat{H}_{i,k}\right)\right| \prod_{m\neq i}^n \widehat{H}_{m,k} \\
&\quad\leqslant C(k)\widehat{\bar{\Psi}}_1^2 \sum_{i=1}^n \left(|\widehat{F}_{S_2,i}-1|^{k-1} \left(\sum_{\vec{x}} |x_j| p_{S_2,i}(\vec{x}) \left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \right)^2 \right. \\
&\qquad \left. + |\widehat{F}_{S_2,i}-1|^k \bar{\beta}_{i,j}^{(2)} \right) \\
&\quad\leqslant C(k)\widehat{\bar{\Psi}}_1 \left(\sum_{i=1}^n \left(\sum_{\vec{x}\neq\vec{0}} |x_j| \frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}} \right)^2 \bar{L}_{i,1}^{k-1} + \sum_{i=1}^n \bar{\beta}_{i,j}^{(2)} \bar{L}_{i,1}^k \right) \\
&\quad\leqslant C(k)\widehat{\bar{\Psi}}_1 \sum_{i=1}^n \bar{\beta}_{i,j}^{(2)} \bar{L}_{i,1}^k.
\end{aligned} \tag{6.37}
$$

Collecting the estimates (6.24), (6.29), (6.30), and applying

$$
\widehat{\bar{\Psi}}_1^{1/3} |\widehat{F}_{S_2,i}-1|^k \leqslant C(k)\bar{L}_{i,1}^k, \tag{6.38}
$$

$$
\widehat{\bar{\Psi}}_1^{1/3} \sum_{\vec{x}} |x_j| p_{S_2,i}(\vec{x}) \left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \leqslant C \sum_{\vec{x}\neq\vec{0}} |x_j| \frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}}, \tag{6.39}
$$

we get

$$
\begin{aligned}
&\sum_{i=1}^n \left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right| \sum_{m\neq i}^n \left|\frac{\partial}{\partial t_j}\left(\widehat{F}_{S_2,i}-\widehat{H}_{m,k}\right)\right| \prod_{l\neq i,m}^n \widehat{H}_{l,k} \\
&\quad\leqslant C(k)\widehat{\bar{\Psi}}_1^2 \sum_{i=1}^n \sum_{\vec{x}} |x_j| p_{S_2,i}(\vec{x}) \left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \sum_{m=1}^n |\widehat{F}_{S_2,i}-1|^k \\
&\qquad \times \sum_{\vec{y}} |y_j| p_{S_2,m}(\vec{y}) \left|\sin\frac{(\vec{y},\vec{t})}{2}\right| \\
&\quad\leqslant C(k)\widehat{\bar{\Psi}}_1 \left(\sum_{i=1}^n \sum_{\vec{x}\neq\vec{0}} |x_j| \frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}} \right) \left(\sum_{m=1}^n \bar{L}_{m,1}^k \sum_{\vec{y}\neq\vec{0}} |y_j| \frac{p_{S_2,m}(\vec{y})}{\sqrt{\bar{\lambda}_1(\vec{y})}} \right) \\
&\quad\leqslant C(k)\widehat{\bar{\Psi}}_1 \left(\sum_{i=1}^n \sqrt{\bar{L}_{i,1}\bar{\beta}_{i,j}^{(2)}} \right) \left(\sum_{m=1}^n \bar{L}_{m,1}^{k+1/2} \sqrt{\bar{\beta}_{m,j}^{(2)}} \right).
\end{aligned} \tag{6.40}
$$

At the last step we applied (A.52).

Observe that

$$\begin{aligned}\left|\frac{\partial \widehat{H}_{m,k}}{\partial t_j}\right| &= \left|\exp\left\{\sum_{i=1}^{k}\frac{(-1)^{i+1}}{i}(\widehat{F}_{S_2,m}-1)^i\right\}\right. \\ &\quad \left.\times\sum_{l=1}^{k}(-1)^{l+1}(\widehat{F}_{S_2,m}-1)^{l-1}\frac{\partial \widehat{F}_{S_2,m}}{\partial t_j}\right| \\ &\leqslant \widehat{H}_{m,k}\sum_{l=1}^{k}\left|\widehat{F}_{S_2,m}-1\right|^{l-1}\left|\frac{\partial \widehat{F}_{S_2,m}}{\partial t_j}\right| \\ &\leqslant C(k)\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|.\end{aligned}$$

At the last step we used $|H_{m,k}|\leqslant C$, $|\widehat{F}_{S_2,i}-1|\leqslant 2$ and (6.24). Collecting above estimate, (6.29), (6.30) and applying (6.39), (6.38), (A.52) we get

$$\begin{aligned}&\sum_{i=1}^{n}\left|\frac{\partial}{\partial t_j}\left(\widehat{F}_{S_2,i}-\widehat{H}_{i,k}\right)\right|\sum_{m\neq i}^{n}\left|\frac{\partial \widehat{H}_{m,k}}{\partial t_j}\right|\prod_{l\neq i,m}^{n}\widehat{H}_{l,k} \\ &\quad\leqslant C\widehat{\bar{\Psi}}_1^2\sum_{i=1}^{n}\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right||\widehat{F}_{S_2,i}-1|^k \\ &\qquad\times\sum_{m=1}^{n}\sum_{\vec{y}}|y_j|p_{S_2,m}(\vec{y})\left|\sin\frac{(\vec{y},\vec{t})}{2}\right| \\ &\quad\leqslant C(k)\widehat{\bar{\Psi}}_1\left(\sum_{i=1}^{n}\bar{L}_{i,1}^{k}\sum_{\vec{x}\neq\vec{0}}|x_j|\frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}}\right)\left(\sum_{m=1}^{n}\sum_{\vec{y}\neq\vec{0}}|y_j|\frac{p_{S_2,m}(\vec{y})}{\sqrt{\bar{\lambda}_1(\vec{y})}}\right) \\ &\quad\leqslant C(k)\widehat{\bar{\Psi}}_1\left(\sum_{i=1}^{n}\bar{L}_{i,1}^{k+1/2}\sqrt{\bar{\beta}_{i,j}^{(2)}}\right)\left(\sum_{m=1}^{n}\sqrt{\bar{L}_{m,1}\bar{\beta}_{m,j}^{(2)}}\right).\end{aligned}\tag{6.41}$$

Collecting estimates (6.24), (6.25) and applying (A.52) we get

$$\begin{aligned}&\sum_{i=1}^{n}\left|\frac{\partial}{\partial t_j}\widehat{F}_{S_2,i}\right|\sum_{m\neq i}^{n}\left|\frac{\partial}{\partial t_j}\widehat{F}_{S_2,m}\right|\left|\prod_{l\neq i,m}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i,m}^{n}\widehat{H}_{l,k}\right| \\ &\qquad\leqslant C(k)\widehat{\bar{\Psi}}_1^{3/2}\sum_{l=1}^{n}\bar{L}_{l,1}^{k+1}\left(\sum_{i=1}^{n}\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|\right)^2 \\ &\qquad\leqslant C(k)\widehat{\bar{\Psi}}_1\sum_{l=1}^{n}\bar{L}_{l,1}^{k+1}\left(\sum_{i=1}^{n}\sum_{\vec{x}}|x_j|\frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}}\right)^2\end{aligned}$$

$$\leqslant \quad C(k)\widehat{\bar{\Psi}}_1 \sum_{l=1}^{n} \bar{L}_{l,1}^{k+1} \left(\sum_{i=1}^{n} \sqrt{\bar{L}_{i,1}\, \bar{\beta}_{i,j}^{(2)}} \right)^2. \tag{6.42}$$

Substituting estimates (6.34), (6.37), (6.40), (6.41), (6.42) into (A.55) and applying (A.38) and Lemma 4.5.2 we complete the proof of Theorem 6.2.5. □

Proof of Theorem 6.2.6. According to (A.56), we have

$$\begin{aligned}
\left| \frac{\partial^2}{\partial t_j \partial t_m} \left(\prod_{l=1}^{n} \widehat{F}_{S_2,l} - \prod_{l=1}^{n} \widehat{H}_{l,k} \right) \right| \leqslant & \sum_{i=1}^{n} \left| \frac{\partial^2 \widehat{F}_{S_2,i}}{\partial t_j \partial t_m} \right| \left| \prod_{l\neq i}^{n} \widehat{F}_{S_2,l} - \prod_{l\neq i}^{n} \widehat{H}_{l,k} \right| \\
& + \sum_{i=1}^{n} \left| \frac{\partial \widehat{F}_{S_2,i}}{\partial t_j} \right| \left| \frac{\partial}{\partial t_m} \left(\prod_{l\neq i}^{n} \widehat{F}_{S_2,l} - \prod_{l\neq i}^{n} \widehat{H}_{l,k} \right) \right| \\
& + \sum_{i=1}^{n} \left| \frac{\partial^2}{\partial t_j \partial t_m} \left(\widehat{F}_{S_2,i} - \widehat{H}_{i,k} \right) \right| \prod_{l\neq i}^{n} \widehat{H}_{l,k} \\
& + \sum_{i=1}^{n} \left| \frac{\partial}{\partial t_j} \left(\widehat{F}_{S_2,i} - \widehat{H}_{i,k} \right) \right| \sum_{s\neq i}^{n} \left| \frac{\partial \widehat{H}_{s,k}}{\partial t_m} \right| \prod_{l\neq i,s}^{n} \widehat{H}_{l,k}.
\end{aligned}$$

Observe that

$$\left| \frac{\partial^2 \widehat{F}_{S_2,i}}{\partial t_j \partial t_m} \right| \leqslant \sum_{\vec{x}} |x_j||x_m| p_{S_2,i}(\vec{x}) = \bar{\beta}_{i,jm}. \tag{6.43}$$

Then from above estiamte and (6.33) follows that

$$\sum_{i=1}^{n} \left| \frac{\partial^2 \widehat{F}_{S_2,i}}{\partial t_j \partial t_m} \right| \left| \prod_{l\neq i}^{n} \widehat{F}_{S_2,l} - \prod_{l\neq i}^{n} \widehat{H}_{l,k} \right| \leqslant C(k)\, \widehat{\bar{\Psi}}_1 \sum_{i=1}^{n} \bar{\beta}_{i,jm} \sum_{l=1}^{n} \bar{L}_{l,k}^{k+1}. \tag{6.44}$$

Substituting estimates (6.24), (6.25), (6.29) and (6.30) into (6.23), we get

$$\begin{aligned}
& \left| \frac{\partial}{\partial t_m} \left(\prod_{l\neq i}^{n} \widehat{F}_{S_2,l} - \prod_{l\neq i}^{n} \widehat{H}_{l,k} \right) \right| \\
& \qquad \leqslant \quad \widehat{\bar{\Psi}}_1^{3/2} \sum_{i=1}^{n} \sum_{\vec{x}} |x_j| p_{S_2,i}(\vec{x}) \left| \sin \frac{(\vec{x},\vec{t})}{2} \right| \sum_{l=1}^{n} \bar{L}_{l,1}^{k+1} \\
& \qquad\quad + \widehat{\bar{\Psi}}_1^{2} \sum_{i=1}^{n} |\widehat{F}_{S_2,i} - 1|^k \sum_{\vec{x}} |x_j| p_{S_2,i}(\vec{x}) \left| \sin \frac{(\vec{x},\vec{t})}{2} \right|.
\end{aligned}$$

Taking into account (6.24), the estimate above, (6.36), (6.38), (6.39) and ap-

plying (A.52) we get

$$\begin{aligned}
&\sum_{i=1}^{n}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\frac{\partial}{\partial t_m}\left(\prod_{l\neq i}^{n}\widehat{F}_{S_2,l}-\prod_{l\neq i}^{n}\widehat{H}_{l,k}\right)\right| \\
&\quad\leqslant\ C(k)\widehat{\bar{\Psi}}_1\Bigg(\widehat{\bar{\Psi}}_1^{1/2}\Bigg(\sum_{i=1}^{n}\sum_{\vec{x}}|x_j|p\widehat{F}_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|\Bigg)^2\sum_{l=1}^{n}\bar{L}_{l,1}^{k+1} \\
&\qquad+\widehat{\bar{\Psi}}_1\sum_{i=1}^{n}\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right| \\
&\qquad\times\sum_{l=1}^{n}|\widehat{F}_{S_2,l}-1|^k\sum_{\vec{y}}|x_j|p_l(\vec{y})\left|\sin\frac{(\vec{y},\vec{t})}{2}\right|\Bigg) \\
&\quad\leqslant\ C(k)\widehat{\Psi}_1\Bigg(\sum_{l=1}^{n}\bar{L}_{l,1}^{k+1}\Bigg(\sum_{i=1}^{n}\sum_{\vec{x}}|x_j|\frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}}\Bigg)^2 \\
&\qquad+\Bigg(\sum_{i=1}^{n}\sum_{\vec{x}}|x_j|\frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}}\Bigg)\Bigg(\sum_{l=1}^{n}\bar{L}_{l,1}^{k}\sum_{\vec{y}}|y_j|\frac{p_{S_2,l}(\vec{y})}{\sqrt{\bar{\lambda}_1(\vec{y})}}\Bigg)\Bigg) \\
&\quad\leqslant\ C(k)\widehat{\bar{\Psi}}_1\Bigg(\sum_{l=1}^{n}\bar{L}_{l,1}^{k+1}\left(\sum_{i=1}^{n}\sqrt{\bar{L}_{i,1}\bar{\beta}_{i,j}^{(2)}}\right)^2 \\
&\qquad+\sum_{i=1}^{n}\sqrt{\bar{L}_{i,1}\bar{\beta}_{i,j}^{(2)}}\sum_{l=1}^{n}\bar{L}_{l,1}^{k+1/2}\sqrt{\bar{\beta}_{l,j}^{(2)}}\Bigg). \qquad (6.45)
\end{aligned}$$

Observe that

$$\begin{aligned}
\frac{\partial^2 A_\tau}{\partial t_j\partial t_m} &= \frac{\partial}{\partial t_m}\left(\sum_{l=1}^{k}(-1)^{l+1}(1-\tau^l)(\widehat{F}_{S_2,i}-1)^{l-1}\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right) \\
&= \sum_{l=2}^{k}(-1)^{l+1}(1-\tau^l)(l-1)(\widehat{F}_{S_2,i}-1)^{l-2}\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\frac{\partial\widehat{F}_{S_2,i}}{\partial t_m} \\
&\quad+\sum_{l=1}^{k}(-1)^{l+1}(1-\tau^l)(\widehat{F}_{S_2,i}-1)^{l-1}\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j\partial t_m}
\end{aligned}$$

Then

$$\begin{aligned}
\left|\frac{\partial^2 A_\tau}{\partial t_j\partial t_m}\right| &\leqslant \sum_{l=2}^{k}(l-1)|\widehat{F}_{S_2,i}-1|^{l-2}\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\frac{\partial\widehat{F}_{S_2,i}}{\partial t_m}\right| \\
&\quad+\sum_{l=1}^{k}|\widehat{F}_{S_2,i}-1|^{l-1}\left|\frac{\partial^2\widehat{F}_{S_2,i}}{\partial t_j\partial t_m}\right|
\end{aligned}$$

$$\leqslant \quad C(k)\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_m}\right|+C(k)\left|\frac{\partial^2 \widehat{F}_{S_2,i}}{\partial t_j \partial t_m}\right|.$$

It allows to write

$$\begin{aligned}
\left|\frac{\partial^2}{\partial t_j \partial t_m}\left(\widehat{F}_{S_2,i}-\widehat{H}_{i,k}\right)\right| &= \left|\frac{\partial}{\partial t_m}\left((k+1)(\widehat{F}_{S_2,i}-1)^k\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right.\right.\\
&\quad \times\int_0^1(-1)^k\tau^k \mathrm{e}^{A_\tau}\,\mathrm{d}\tau\\
&\quad \left.\left.+(\widehat{F}_{S_2,i}-1)^{k+1}\int_0^1(-1)^k\tau^k\mathrm{e}^{A_\tau}\frac{\partial A_\tau}{\partial t_j}\,\mathrm{d}\tau\right)\right|\\
&\leqslant \quad C(k)|\widehat{F}_{S_2,i}-1|^{k-1}\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_m}\right|+C(k)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial^2 \widehat{F}_{S_2,i}}{\partial t_j\partial t_m}\right|\\
&\quad +C(k)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\frac{\partial A_\tau}{\partial t_m}\right|\\
&\quad +C(k)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_m}\right|\left|\frac{\partial A_\tau}{\partial t_j}\right|\\
&\quad +|\widehat{F}_{S_2,i}-1|^{k+1}\left|\frac{\partial A_\tau}{\partial t_j}\right|\left|\frac{\partial A_\tau}{\partial t_m}\right|+|\widehat{F}_{S_2,i}-1|^{k+1}\left|\frac{\partial^2 A_\tau}{\partial t_j\partial t_m}\right|\\
&\leqslant \quad C(k)|\widehat{F}_{S_2,i}-1|^{k-1}\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_m}\right|(1+2|\widehat{F}_{S_2,i}-1|+2|\widehat{F}_{S_2,i}-1|^2)\\
&\quad +C(k)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial^2 \widehat{F}_{S_2,i}}{\partial t_j\partial t_m}\right|(1+|\widehat{F}_{S_2,i}-1|)\\
&\leqslant \quad C(k)|\widehat{F}_{S_2,i}-1|^{k-1}\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_j}\right|\left|\frac{\partial \widehat{F}_{S_2,i}}{\partial t_m}\right|+C(k)|\widehat{F}_{S_2,i}-1|^k\left|\frac{\partial^2 \widehat{F}_{S_2,i}}{\partial t_j\partial t_m}\right|.
\end{aligned}$$

Collecting estimate above, estimates (6.30), (6.43), (6.24), (6.38), (6.39) and applying (A.52) we get

$$\begin{aligned}
&\sum_{i=1}^n\left|\frac{\partial^2}{\partial t_j\partial t_m}\left(\widehat{F}_{S_2,i}-\widehat{H}_{i,k}\right)\right|\prod_{l\neq i}^n\widehat{H}_{l,k}\\
&\qquad\leqslant \quad C(k)\widehat{\bar{\Psi}}_1^2\sum_{i=1}^n|\widehat{F}_{S_2,i}-1|^{k-1}\sum_{\vec{x}}|x_j|p_{S_2,i}(\vec{x})\left|\sin\frac{(\vec{x},\vec{t})}{2}\right|\\
&\qquad\quad \times\sum_{\vec{y}}|y_m|p_{S_2,i}(\vec{y})\left|\sin\frac{(\vec{y},\vec{t})}{2}\right|+C(k)\widehat{\bar{\Psi}}_1^2\sum_{i=1}^n|\widehat{F}_{S_2,i}-1|^k\bar{\beta}_{i,jm}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C(k)\widehat{\bar{\Psi}}_1 \sum_{i=1}^{n} L_{i,1}^{k-1} \sum_{\vec{x}\neq\vec{0}} |x_j| \frac{p_{S_2,i}(\vec{x})}{\sqrt{\bar{\lambda}_1(\vec{x})}} \sum_{\vec{y}\neq\vec{0}} |y_m| \frac{p_{S_2,i}(\vec{y})}{\sqrt{\bar{\lambda}_1(\vec{y})}} \\
&\quad + C(k)\widehat{\bar{\Psi}}_1 \sum_{i=1}^{n} \bar{L}_{i,1}^{k} \bar{\beta}_{i,jm} \\
&\leqslant C(k)\widehat{\bar{\Psi}}_1 \sum_{i=1}^{n} \bar{L}_{i,1}^{k-1} \sqrt{\bar{L}_{i,1}\beta_{i,j}^{(2)}} \sqrt{\bar{L}_{i,1}\beta_{i,m}^{(2)}} + C(k)\widehat{\bar{\Psi}}_1 \sum_{i=1}^{n} \bar{L}_{i,1}^{k} \bar{\beta}_{i,jm} \\
&\leqslant C(k)\widehat{\bar{\Psi}}_1 \left(\sum_{i=1}^{n} \bar{L}_{i,1}^{k} \sqrt{\bar{\beta}_{i,j}^{(2)} \bar{\beta}_{i,m}^{(2)}} + \sum_{i=1}^{n} \bar{L}_{i,1}^{k} \bar{\beta}_{i,jm} \right). \qquad (6.46)
\end{aligned}
$$

Substituting estimates (6.41), (6.44), (6.45), (6.46) into (6.43) and applying (A.39) and Lemma 4.5.2 we complete the proof of Theorem 6.2.6. □

Proof of Theorem 6.2.7. Let $U = (-1/2)\sum_{i=1}^{n} U_i^2$, where U_i is defined by (6.15). Then

$$
\widehat{\mathbf{F}} - \widehat{\mathbf{H}}_{\mathrm{n},1} = U + (\widehat{\mathbf{A}}_{\mathbf{1}} - U) + (\widehat{\mathbf{H}}_{\mathrm{n},1}\widehat{\mathbf{A}}_{\mathbf{1}} - \widehat{\mathbf{A}}_{\mathbf{1}}) + (\widehat{\mathbf{F}} - \widehat{\mathbf{H}}_{\mathrm{n},1}(1+\widehat{\mathbf{A}}_{\mathbf{1}}))
$$

and $|V(\vec{0},\vec{h})| \geqslant I_1(\vec{t}_{\mathbf{h}}) - I_2(\vec{t}_{\mathbf{h}}) - I_3(\vec{t}_{\mathbf{h}}) - I_4(\vec{t}_{\mathbf{h}})$, where $\mathrm{d}\vec{t} := \mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_d$,

$$
\begin{aligned}
I_1(\vec{t}_{\mathbf{h}}) &= \left| \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \tilde{\Psi} U(\vec{t}_{\mathbf{h}})\, \mathrm{d}\vec{t} \right|, \\
I_2(\vec{t}_{\mathbf{h}}) &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \tilde{\Psi} \left| \widehat{\mathbf{A}}_{\mathbf{1}}(\vec{t}_{\mathbf{h}}) - U(\vec{t}_{\mathbf{h}}) \right| \mathrm{d}\vec{t}\,, \\
I_3(\vec{t}_{\mathbf{h}}) &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \tilde{\Psi} \left| (\widehat{\mathbf{H}}_{\mathrm{n},1}(\vec{t}_{\mathbf{h}}) - 1)\widehat{\mathbf{A}}_{\mathbf{1}}(\vec{t}_{\mathbf{h}}) \right| \mathrm{d}\vec{t}\,, \\
I_4(\vec{t}_{\mathbf{h}}) &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \tilde{\Psi} \left| \widehat{\mathbf{F}}(\vec{t}_{\mathbf{h}}) - \widehat{\mathbf{H}}_{\mathrm{n},1}(\vec{t}_{\mathbf{h}}) \left(1+\widehat{\mathbf{A}}_{\mathbf{1}}(\vec{t}_{\mathbf{h}})\right) \right| \mathrm{d}\vec{t}\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\sum_{\vec{x}} p_{S_2,i}(\vec{x})(\vec{x},\vec{t})^2 = \sum_{j=1}^{d} \bar{\beta}_{i,j}^{(2)} t_j^2 + 2\sum_{m=1}^{d}\sum_{l=1}^{m-1} \bar{\mu}_{i,ml} t_m t_l, \\
&\int_{-\infty}^{\infty} t^{2s+1} \mathrm{e}^{-t^2/2} \mathrm{d}t = 0, \quad s = 0,1,2\ldots
\end{aligned}
$$

Therefore,

$$\begin{aligned} I_1(\vec{t}_{\mathbf{h}}) &\geqslant \frac{3(2\pi)^{d/2}}{4}\sum_{i=1}^{n}\sum_{j=1}^{d}\left(\frac{\bar{\beta}_{i,j}^{(2)}}{h_j^2}\right)^2 + \frac{2(2\pi)^{d/2}}{4}\sum_{i=1}^{n}\sum_{m=1}^{d}\sum_{l=1}^{m-1}\frac{\bar{\beta}_{i,m}^{(2)}\bar{\beta}_{i,l}^{(2)}}{h_m h_l} \\ &\geqslant \frac{(2\pi)^{d/2}}{4}\sum_{i=1}^{n}B_{i,h}^2. \end{aligned} \tag{6.47}$$

It is easy to check that

$$\begin{aligned} \left|\widehat{\mathbf{A}}_{\mathbf{1}}(\vec{t}_{\mathbf{h}}) - U(\vec{t}_{\mathbf{h}})\right| &\leqslant \frac{1}{2}\sum_{i=1}^{n}\left|\widehat{F}_{S_2,i}(\vec{t}_{\mathbf{h}}) - 1 - U_i(\vec{t}_{\mathbf{h}})\right| \\ &\quad \times\left(\left|\widehat{F}_{S_2,i}(\vec{t}_{\mathbf{h}}) - 1\right| + \left|U_i(\vec{t}_{\mathbf{h}})\right|\right). \end{aligned}$$

Taking into account (6.13) and (6.14), we prove

$$I_2(\vec{t}_{\mathbf{h}}) \leqslant d^6(2\pi)^{d/2}\sum_{i=1}^{n}B_{i,h}\sum_{j=1}^{d}\frac{\bar{\beta}_{i,j}^{(4)}}{h_j^4}. \tag{6.48}$$

Similarly to the proof of (6.16) and (6.22) we obtain

$$\begin{aligned} &\left|(\widehat{\mathbf{H}}_{\mathrm{n},1} - 1)\widehat{\mathbf{A}}_{\mathbf{1}}\right| \leqslant \frac{1}{2}\sum_{i=1}^{n}(\widehat{F}_{S_2,i} - 1)^2\sum_{l=1}^{n}|\widehat{F}_{S_2,l} - 1|, \\ &\left|\widehat{\mathbf{F}} - \widehat{\mathbf{H}}_{\mathrm{n},1}(1+\widehat{\mathbf{A}}_{\mathbf{1}})\right| \leqslant \sum_{i=1}^{n}|\widehat{F}_{S_2,i} - 1|^3 + \frac{1}{8}\left(\sum_{i=1}^{n}(\widehat{F}_{S_2,i} - 1)^2\right)^2. \end{aligned}$$

Combining these estimates with (6.13) we get

$$\begin{aligned} I_3(\vec{t}_{\mathbf{h}}) + I_4(\vec{t}_{\mathbf{h}}) &\leqslant (2\pi)^{d/2}\left(d^6\sum_{i=1}^{n}B_{i,h}^2\sum_{l=1}^{n}B_{l,h} + 2d^6\sum_{i=1}^{n}B_{i,h}^3\right. \\ &\quad \left. + 2d^6\sum_{i=1}^{n}B_{i,h}^3 + d^8\left(\sum_{i=1}^{n}B_{i,h}^2\right)^2\right). \end{aligned} \tag{6.49}$$

Collecting estimates (6.47), (6.48) and (6.49) we get $|V(\vec{0},\vec{h})| \geqslant (2\pi)^{d/2}R_h$. It remains to apply Lemma A.4.3.

The proof when all r.v.s. are concentrated on coordinate axes is very similar. Observe that, in this case, $p_{S_2,i}(\vec{x})x_m x_j = 0$ $(m \neq j)$. Therefore

$$|\widehat{F}_{S_2,i} - 1| \leqslant \frac{1}{2}\sum_{\vec{x}}p_{S_2,i}(\vec{x})(\vec{x},\vec{t})^2$$

$$= \frac{1}{2}\sum_{\vec{x}} p_{S_2,i}(\vec{x})x_j^2 t_j^2 = \frac{1}{2}\sum_{j=1}^{d}\bar{\beta}_{i,j}^{(2)}t_j^2 \tag{6.50}$$

and

$$U_i = -\frac{1}{2}\sum_{j=1}^{d}\bar{\beta}_{i,j}^{(2)}t_j^2, \tag{6.51}$$

$$|\widehat{F}_{S_2,i} - 1 - U_i| \leqslant \frac{1}{24}\sum_{\vec{x}} p_{S_2,i}(\vec{x})(\vec{x},\vec{t})^4 = \frac{1}{24}\sum_{j=1}^{d}\bar{\beta}_{i,j}^{(4)}t_j^4. \tag{6.52}$$

We have

$$\widehat{\mathbf{F}}(\vec{t}) - \widehat{\mathbf{H}}_{\mathrm{n},1}(\vec{t}) = J_1(\vec{t}) + J_2(\vec{t}) + J_3(\vec{t}) + J_4(\vec{t}) + J_5(\vec{t}),$$

where,

$$\begin{aligned}
J_1(\vec{t}) &= \frac{(-1)^{k+2}}{k+1}\sum_{i=1}^{n}U_i^{k+1},\\
J_2(\vec{t}) &= \left(\widehat{\mathbf{H}}_{\mathrm{n,k}} - 1\right)\frac{(-1)^{k+2}}{k+1}\sum_{i=1}^{n}U_i^{k+1},\\
J_3(\vec{t}) &= \widehat{\mathbf{H}}_{\mathrm{n,k}}\frac{(-1)^{k+2}}{k+1}\sum_{i=1}^{n}\left((\widehat{F}_{S_2,i}-1)^{k+1} - U_i^{k+1}\right);\\
J_4(\vec{t}) &= \widehat{\mathbf{H}}_{\mathrm{n,k+1}} - \widehat{\mathbf{H}}_{\mathrm{n,k}}\left(1 + \frac{(-1)^{k+2}}{k+1}\sum_{i=1}^{n}(\widehat{F}_{S_2,i}-1)^{k+1}\right),\\
J_5(\vec{t}) &= \widehat{\mathbf{F}} - \widehat{\mathbf{H}}_{\mathrm{n,k+1}}.
\end{aligned}$$

Taking into account (A.48), (6.50), (6.51) and (6.52) we prove that

$$\begin{aligned}
\left|\widehat{\mathbf{H}}_{\mathrm{n,k}} - 1\right| &\leqslant \sum_{m=1}^{n}\sum_{j=1}^{k}\frac{|\widehat{F}_{S_2,m}-1|^j}{j} \leqslant \sum_{m=1}^{n}|\widehat{F}_{S_2,m}-1|\sum_{j=1}^{k}\frac{2^{j-1}}{j}\\
&\leqslant \frac{1}{2}\sum_{m=1}^{n}|\widehat{F}_{S_2,m}-1|,\\
|J_2(\vec{t})| &\leqslant C(k)\sum_{i=1}^{n}|U_i|^{k+1}\sum_{m=1}^{n}|\widehat{F}_{S_2,m}-1|\\
&\leqslant C(k)\sum_{i=1}^{n}\left(\sum_{j=1}^{d}\bar{\beta}_{i,j}^{(2)}t_j^2\right)^{k+1}\sum_{m=1}^{n}\sum_{l=1}^{d}\bar{\beta}_{m,l}^{(2)}t_l^2,
\end{aligned}$$

$$
\begin{aligned}
|J_3(\vec{t})| &\leqslant C(k)\sum_{i=1}^{n}|\widehat{F}_{S_2,i}-1-U_i|(|\widehat{F}_{S_2,i}-1|^k+|U_i|^k)\\
&\leqslant C(k)\sum_{i=1}^{n}\sum_{j=1}^{d}\bar{\beta}_{i,j}^{(4)}t_j^4\left(\sum_{m=1}^{d}\bar{\beta}_{i,m}^{(2)}t_m^2\right)^k,\\
|J_4(\vec{t})| &\leqslant \frac{1}{2}\widehat{\mathbf{H}}_{\mathrm{n,k}}\left(\frac{(-1)^{k+2}}{k+1}\sum_{i=1}^{n}(\widehat{F}_{S_2,i}-1)^{k+1}\right)^2\\
&\leqslant C(k)\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{d}\bar{\beta}_{i,j}^{(2)}\right)^{k+1}\right)^2,\\
|J_5(\vec{t})| &\leqslant \sum_{i=1}^{n}|\widehat{F}_{S_2,i}-1|^{k+2}\leqslant\sum_{i=1}^{n}\left(\frac{1}{2}\sum_{j=1}^{d}\bar{\beta}_{i,j}^{(2)}t_j^2\right)^{k+2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{\Psi}J_1(\vec{t}_{\mathbf{h}})\,\mathrm{d}\vec{t}\right|\geqslant C(k)\sum_{i=1}^{n}B_{i,h}^{k+1},\\
&\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{\Psi}|J_2(\vec{t}_{\mathbf{h}})|\,\mathrm{d}\vec{t}\leqslant C(k)\sum_{i=1}^{n}B_{i,h}^{k+1}\sum_{l=1}^{n}B_{l,h},\\
&\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{\Psi}|J_3(\vec{t}_{\mathbf{h}})|\,\mathrm{d}\vec{t}\leqslant C(k)\sum_{i=1}^{n}B_{i,h}^{k}\sum_{j=1}^{d}\frac{\bar{\beta}_{i,j}^{(4)}}{h_j^4},\\
&\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{\Psi}|J_4(\vec{t}_{\mathbf{h}})|\,\mathrm{d}\vec{t}\leqslant C(k)\left(\sum_{i=1}^{n}B_{i,h}^{k+1}\right)^2,\\
&\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\tilde{\Psi}|J_5(\vec{t}_{\mathbf{h}})|\,\mathrm{d}\vec{t}\leqslant C(k)\sum_{i=1}^{n}B_{i,h}^{k+2}.
\end{aligned}
$$

Applying Lemma A.4.3 we complete the proof of theorem. □

7 SKELLAM APPROXIMATIONS FOR SUMS OF SYMMETRIC LATTICE VECTORS

7.1 Preliminary remarks

From the practical point of view, $\exp\{F-I\}$ is not very useful approximation if support of F is the whole $\mathbb{Z}^d$, since then we deal with infinite number of Poisson distributions. In this chapter we consider the possibility to simplify the structure of accompanying CP approximation. For symmetric lattice r.v.s. Skellam distribution seems the natural choice. Skellam distribution was introduced in [79]. Its one-dimensional applications in medicine, sport, and finances are discussed in [3, 17, 47, 49, 58, 67], see also references therein. One-dimensional Skellam approximations are applied in [37, 66]. The bivariate Skellam distribution, its properties, applications, and relation to various copulas are investigated in [1, 38].

The idea of one-dimensional Skellam approximation can be described in the following way: to approximate symmetric integer-valued random variable we use Poisson r.v. π_a for non-negative semi-line and another Poisson variable $-\tilde{\pi}_b$ for non-positive semi-line. Assuming their independence and ensuring matching of (zero) mean and variance to approximated variable, we arrive at Skellam approximation $\pi_{\sigma^2/2}-\tilde{\pi}_{\sigma^2/2}$.

At first glance, multivariate analogue of Skellam approximation can be derived by replacing the main part of characteristic function

$$\exp\left\{-\frac{1}{2}(t_1,t_2,\dots,t_d)\Sigma(t_1,t_2,\dots,t_d)^T+\dots\right\},$$

where Σ denotes covariance matrix, with

$$\exp\left\{-\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}t_1}-1,\dots,\mathrm{e}^{\mathrm{i}t_d}-1\right)\Sigma\left(\mathrm{e}^{-\mathrm{i}t_1}-1,\dots,\mathrm{e}^{-\mathrm{i}t_d}-1\right)^T+\dots\right\}.$$

However such approach runs into serious difficulties, when the accuracy of approximation is estimated, since the main advantage of Poisson and discrete CP approximation over the Normal one lies in the usage of expansions in factorial moments. It is not clear, how to construct such expansion for multivariate characteristic function of symmetric lattice vector. Therefore, we slightly generalize approach of [56] by considering only mixtures of distributions concentrated on different lines in $\mathbb{R}^d$. Note that multidimensional Skellam distribution (even two-dimensional one) can be defined in various ways, see [2, 17, 38].

7.2 Known results

Estimates in total variation. Arguably the first one-dimensional estimate in total variation for Skellam approximation (though the name Skellam was not used) was obtained by Presman [66], who used it to approximate the sum of symmetrized Bernoulli variables. Presman's result was extended to sums of integer-valued random variables in [30] Theorem 7.7. Skellam approximation was used for special subset of random variables in [37]. Multidimensional version of Skellam approximation was considered in [56]. Let $\vec{X}_1,\dots,\vec{X}$ be independent vectors taking values in $\mathbb{Z}^d$ and let F_m be distribution concentrated on integers of mth coordinate-axis except zero, that is on $\{k\bar{e}_m,\quad k=\pm1,\pm2,\dots\}$, $\widehat{F}_m(\vec{t})=\widehat{F}_m(t_m)$. Let for $(m=1,2,\dots,d;$ $i=1,2,\dots,n)$:

$$\begin{aligned}
\mathscr{L}(\vec{X}_i) &= q_iI+\sum_{m=1}^{d}p_{i,m}F_m,\quad F_m=\sum_{k=1}^{\infty}q_{k,m}(I_{k\bar{e}_m}+I_{-k\bar{e}_m}),\\
&\quad \sum_{k=1}^{\infty}q_{k,m}=\frac{1}{2},\quad q_i+\sum_{m=1}^{d}p_{i,m}=1,\quad 0\leqslant q_i,p_{i,m}\leqslant 1,\\
H &= \prod_{i=1}^{*n}\mathscr{L}(\vec{X}_i),\quad \lambda_m:=\sum_{i=1}^{n}p_{i,m},\\
\alpha &:= \sum_{i=1}^{n}g(2(1-q_i))\min\left\{2^{-3/2}\sum_{m=1}^{d}\frac{p_{i,m}^2}{\lambda_m},\left(\sum_{m=1}^{d}p_{i,m}\right)^2\right\},\\
\nu_{k,m} &:= 2\sum_{l=1}^{\infty}l(l-1)\cdot\ldots\cdot(l-k+1)q_{l,m},\quad k=1,2,3,4;\\
\tilde{G} &= \exp\left\{\sum_{m=1}^{d}\frac{\sigma_m^2\lambda_m}{2}(I_{\bar{e}_m}+I_{-\bar{e}_m}-2I)\right\}.
\end{aligned}$$

Here $g(\cdot)$ is defined by (3.7).

If symmetric F_m distribution has finite fourth moment $\nu_{4,m}<\infty$ and

$$\nu_{2,m}\leqslant\frac{\nu_{1,m}}{2},\quad m=1,2,\dots, \tag{7.1}$$

and $2\alpha\mathrm{e}<1$, then

$$\|H-\tilde{G}\|_{TV}\quad\leqslant\quad\frac{C}{(1-2\alpha\mathrm{e})^{3/2}}\sum_{l=1}^{d}(1+\sigma_l)\sum_{m=1}^{d}\frac{1}{\lambda_m^2}\sum_{i=1}^{n}p_{i,rm}^2$$

$$+C_5(d)\sum_{m=1}^{d}\frac{\nu_{2,m}+\nu_{3,m}+\nu_{4,m}}{\lambda_m\nu_{1,m}^2}. \tag{7.2}$$

Once more we observe that, if all probabilities are of constant order, the order of accuracy in (7.2) is $O(n^{-1})$. On the other hand, assumptions (7.1) and $2\alpha\mathrm{e}<1$ are very restrictive.

Local estimates. In [56] it was also proved that if $\nu_{4,m}<\infty$ and $\nu_{2,m}\leqslant\nu_{1,m}$, $(m=1,2,\dots,d)$, then

$$\begin{aligned}\|H-\tilde{G}\|_\infty \;\leqslant\; & 2.3(0.651)^d\left(\prod_{l=1}^{d}\frac{1}{\sqrt{b_l}}\right)\sum_{j=1}^{n}\left\{\left(\sum_{m=1}^{d}\frac{p_{j,m}\sigma_m^2}{b_m}\right)^2\right.\\ &\left.+\sum_{m=1}^{k}\frac{p_{j,m}(\nu_{2,m}+\nu_{3,m}+0.2\nu_{4,m})}{b_m^2}\right\}.\end{aligned}$$

Here $b_m=(\nu_{1,m}-\nu_{2,m})\sum_{j=1}^{n}p_{j,0}p_{j,m}$. If all probabilities are of constant order, the local estimate is of $O(n^{-1-d/2})$ order of accuracy. Assumption $\nu_{2,m}\leqslant\nu_{1,m}$ means that $F_m\{[3,\infty)\}$ is very small.

7.3 Setting

We consider the case already investigated in subsection 5.2.2. We recall assumptions that were used. Let $\vec{X},\vec{X}_1,\dots,\vec{X}_n$ be i.i.d. r.v.s. in $\mathbb{R}^d$, $\mathscr{L}(\vec{X})=F$. We also assume that

1. For some natural $K\geqslant d$

$$F=qI+\sum_{m=1}^{K}p_mF_m,\ q,p_1,\dots,p_m\in(0,1),\ \sum_{m=1}^{K}p_m=1-q. \tag{7.3}$$

2. There exist non-zero vectors $\vec{y}_1,\vec{y}_2,\dots,\vec{y}_K\in\mathbb{R}^d$ such that

$$\begin{aligned}&\operatorname{supp}F_m\subset\{j\vec{y}_m:j\in\mathbb{Z}\setminus\{0\}\},\\ &\operatorname{supp}F_m\cap\operatorname{supp}F_k=\emptyset,\quad k\neq m.\end{aligned}$$

3. All $\operatorname{supp}F_m\neq\emptyset$ and F_m are symmetric:

$$F_m\{j\vec{y}_m\}=F_m\{-j\vec{y}_m\},\quad j\in\{1,2,\dots\};\quad m=1,\dots,K.$$

4. There exist j_m^0 such that

$$F_m\{j_m^0\vec{y}_m\} > 0, \quad F_m\{(j_m^0+1)\vec{y}_m\} > 0. \tag{7.4}$$

To make notations simpler we set

$$q_{j,m} = F\{j\vec{y}_m\}, \quad U_m = I_{\vec{y}_m} - I, \quad \bar{U}_m = I_{-\vec{y}_m} - I.$$

Then

$$\begin{aligned} F_m - I &= \sum_{k=1}^{\infty} q_{k,m}(I_{k\vec{y}_m} + I_{-k\vec{y}_m} - 2I) \\ &= \sum_{k=1}^{\infty} q_{k,m}\left((U_m+I)^{*k} + (\bar{U}_m+I)^{*k} - 2I\right). \end{aligned}$$

Accompanying distribution to F is

$$\exp\{F-I\} = \exp\left\{\sum_{m=1}^{K} p_m \sum_{k=1}^{\infty} q_{k,m}\left((U_m+I)^{*k} - I + (\bar{U}_m+I)^{*k} - I\right)\right\}.$$

Our goal is to replace accompanying distribution by some CP distribution with simpler structure. We will replace $(U_m+I)^{*k} - I$ and $(\bar{U}_m+I)^{*k} - I$ by k^2U_m and $k^2\bar{U}_m$ respectively. Set

$$\sigma_m^2 = \sum_{k=-\infty}^{\infty} k^2 F_m\{k\vec{y}_m\} = 2\sum_{k=1}^{\infty} k^2 q_{k,m}; \quad \mu_{j,m} = 2\sum_{k=1}^{\infty} k^j q_{k,m}.$$

We recall that σ_m^2 and $\mu_{k,m}$ are variance and kth moment of integer-valued one-dimensional 'projection' of F_m to $\vec{y}_m$, see (5.19).

Similar to one-dimensional Skellam approximation, we choose parameters for Poisson distributions to be match weighted variances of F_m:

$$\begin{aligned} H &= \prod_{m=1}^{*K} H_m, \\ H_m &= \exp\left\{\frac{1}{2}p_m\sigma_m^2(U_m + \bar{U}_m)\right\} = \exp\left\{-\frac{1}{2}p_m\sigma_m^2(U_m * \bar{U}_m)\right\}. \end{aligned}$$

We denote the variance and j'th absolute moment of 'projection' of F_m (see (5.19))

Example 7.3.1. *Distributions concentrated on coordinate axes are partial*

case of above setting, since it suffices to take $\vec{y}_m = (0,0,\dots,1,\dots)$. In this case $K = d$ and measures introduced above have the following Fourier transforms:

$$\widehat{F}(\vec{t}) = q + \sum_{m=1}^{d} p_m \sum_{k=1}^{\infty} q_{k,m} \left(\mathrm{e}^{\mathrm{i}kt_m} + \mathrm{e}^{-\mathrm{i}kt_m} \right)$$

$$\widehat{H}(\vec{t}) = \prod_{m=1}^{d} \widehat{H}_m(t_m),\ \widehat{H}_m(t_m) = \exp\left\{ \frac{p_m \sigma_m^2}{2} \left(\mathrm{e}^{\mathrm{i}t_m} - 1 + \mathrm{e}^{-\mathrm{i}t_m} - 1 \right) \right\}.$$

Obviously, H is convolution of Skellam distributions concentrated on different coordinate axes.

7.4 Results

7.4.1 Estimates in total variation

First, we estimate, how close are accompanying distribution and corresponding Skellam distribution.

Theorem 7.4.1. *Let F satisfy (7.3)–(7.4) and let $q_{1,m} > 0$, $\mu_{4,m} < \infty$, $(m = 1,\dots,K)$. Then, for any $n \in \mathbb{N}$*

$$\| \exp\{n(F-I)\} - H^{*n} \|_{TV} \leqslant \frac{4}{3} \sum_{m=1}^{K} \left(\mu_{4,m} - \sigma_m^2 \right) \min\left(np_m, \frac{1}{np_m q_{1,m}^2 \mathrm{e}^2} \right).$$

If $p_m = O(1)$, $q_{1,m} = O(1)$, then accuracy of approximation in Theorem 7.4.1 is $O(Kn^{-1})$.

In Theorem 7.4.1, estimate contains $q_{1,m}^2$ in denominator. If at least one $q_{1,m}$ is very small, then estimates can become trivial. This problem can be partially solved if we use special quantities:

$$v_m := 1 - \frac{1}{2} \sum_{j=1}^{\infty} |F_m\{j\} - F_m\{j-1\}| = 1 - \frac{1}{2} \sum_{j=1}^{\infty} |q_{j,m} - q_{j-1,m}|. \qquad (7.5)$$

For estimation of total variation distance between lattice distribution and it's shifted version v_m was used in [7]. In Probability Theory $1 - v_m$ is known as smoothing function, see [36] and references therein.

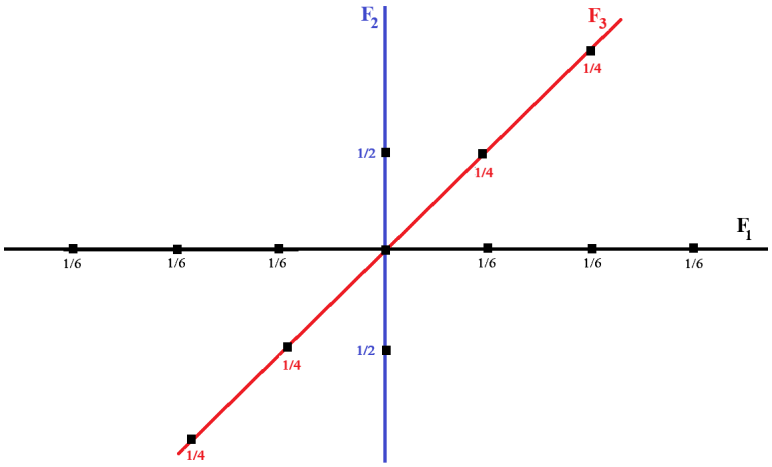

Figure 1: Example 7.4.1. Distributions F_1, F_2, F_3.

Let

$$\tilde{\Delta} := \frac{1}{q^{13/2}}\left(\sum_{m=1}^{K}\sqrt{\sigma_m}\right)^3 + \frac{1}{\sqrt{q}}\left(\sum_{m=1}^{K}\sigma_m^2 \min\left(np_m, \frac{1}{v_m}, \frac{1}{qq_{1,m}}\right)\right)^2 + \frac{1}{\sqrt{q}}\sum_{m=1}^{K}\frac{\mu_{4,m}-\sigma_m^2}{p_m}\left(\min\left(np_m, \frac{1}{v_m}, \frac{1}{qq_{1,m}}\right)\right)^2.$$

The following result is new even in one-dimensional case, $d = 1$.

Theorem 7.4.2. *Let F satisfy (7.3)–(7.4) and let $v_m > 0$, $\mu_{4,m} < \infty$, $(m = 1, \dots, N)$. Then, for any $n \in \mathbb{N}$*

$$\|F^{*n} - H^{*n}\|_{TV} \leqslant \frac{C\tilde{\Delta}}{n}. \tag{7.6}$$

If p_m are not very small, we can use simplified version of Theorem 7.4.2.

Corollary 7.4.1. *Let F satisfy (7.3)–(7.4) and let $v_m > 0$, $\mu_{4,m} < \infty$, $(m = 1, \dots, N)$. Then, for any $n \in \mathbb{N}$*

$$\|F^{*n} - H^{*n}\|_{TV} \leqslant \frac{C_2 K^2}{nq^{13/2}}\sum_{m=1}^{K}\frac{\mu_{4,m}}{p_m}\left(\min\left(\frac{1}{v_m}; \frac{1}{q_{1,m}}\right)\right)^2. \tag{7.7}$$

Example 7.4.1. *Let F be distribution of vector (X_1, X_2) concentrated on lines, going through $(0,0)$ and $\vec{y}_1 = (1,0)$, $\vec{y}_2 = (0,1)$, $\vec{y}_3 = (1,1)$. More precisely,*

let

$$\begin{aligned} F &= \frac{1}{2}I+\frac{1}{6}(F_1+F_2+F_3), \\ F_1\{(1,0)\} &= F_1\{(-1,0)\}=F_1\{(2,0)\}=F\{(-2,0)\}=F\{(4,0)\} \\ &= F\{-(4,0)\}=1/6, \\ F_2\{(0,1)\} &= F_2\{(0,-1)\}=1/2, \\ F_3\{(1,1)\} &= F\{(-1,-1)\}=F\{(2,2)\}=F\{(-2,-2)\}=1/4. \end{aligned}$$

Then,

$$p_1=p_2=p_3=\frac{1}{6}, \quad q_{1,1}=\frac{1}{6}; \quad q_{1,2}=\frac{1}{2}, \quad q_{1,3}=\frac{1}{4}$$
$$\sigma_1^2=7, \quad \sigma_2^2=1, \quad \sigma_3^2=\frac{5}{2}, \quad \mu_{4,1}=91, \quad \mu_{4,2}=1, \quad \mu_{4,3}=\frac{17}{2}$$

and

$$\begin{aligned} H &= \exp\left\{\frac{7}{6}\left(I_{(1,0)}-I+I_{(-1,0)}-I\right)\right\} \\ &\quad *\exp\left\{\frac{1}{6}\left(I_{(0,1)}-I+I_{(0,-1)}-I\right)\right\} \\ &\quad *\exp\left\{\frac{5}{12}\left(I_{(1,1)}-I+I_{(-1,-1)}-I\right)\right\}. \end{aligned}$$

From (7.7) it follows that

$$\|F^{*n}-H^{*n}\|_{TV}\leqslant\frac{C}{n}.$$

Note, that (X_1,X_2) have dependent coordinate variables, $Corr(X_1,X_2)=1/\sqrt{133}$. Moreover, $\nu_{11}<\nu_{2,1}$. Therefore, (7.2) is not applicable.

Next we demonstrate that v_m in (7.6) can not dropped.

Example 7.4.2. *Let $d=1$, $n\geqslant 10$ and*

$$q=p_1=1/2, \quad F_1\{\pm1\}=\frac{1}{n}, \quad F_1\{\pm2\}=\frac{1}{6}, \quad F_1\{\pm3\}=\frac{1}{3}-\frac{1}{n}.$$

Then

$$q_{1,m}=\frac{1}{n}, \quad \sigma_1^2=\frac{22}{3}-\frac{16}{n}, \quad \mu_{4,1}=\frac{178}{3}-\frac{160}{n}, \quad v_1=\frac{1}{3}+\frac{2}{n}.$$

Therefore,

$$\begin{aligned}\min\left(np_m;\frac{1}{v_m};\frac{1}{q_{1,m}}\right) &= \min\left(\frac{n}{2};\frac{1}{1/3+2/n};n\right)\\ &= \frac{1}{1/3+2/n}=\frac{1}{v_m}\leqslant 3\end{aligned}$$

and

$$d_{TV}((qI+p_1F_1)^{*n},H^{*n})\leqslant\frac{C}{n}.$$

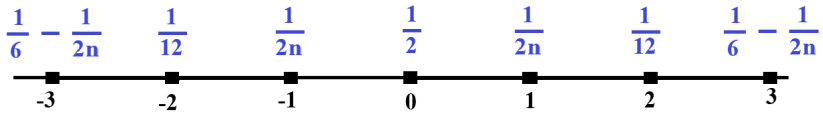

Figure 2: Example 7.4.2. Distribution of F.

Similarly, there exist situations, when $q_{1,m}$ is preferable to v_m.

Example 7.4.3. *Let $d=1$, $n\geqslant 10$ and*

$$q=p_1=1/2,\quad F_1\{\pm1\}=\frac{1}{2}-\frac{1}{n},\quad F_1\{\pm2\}=\frac{1}{n}.$$

Then

$$v_1=\frac{2}{n},\quad \sigma_1^2=2\left(\frac{1}{2}-\frac{1}{n}\right)^2+\frac{8}{n}\asymp C;\quad \mu_{4,1}-\sigma_1^2=\frac{8}{n}$$

and

$$\min\left(np_m;\frac{1}{v_m};\frac{1}{q_{1,m}}\right)=\min\left(\frac{n}{2};n;\frac{1}{1/4-1/2n}\right)=\frac{1}{q_{1,m}}\leqslant 5.$$

Therefore,

$$\|(qI+p_1F_1)^{*n}-H^{*n}\|_{TV}\leqslant\frac{C}{n}.$$

Finally, there are situations, when Skellam approximation is much worse than approximation by accompanying law.

Example 7.4.4. *Let $d=1$, $n\geqslant 10$ and*

$$q=p_1=\frac{1}{2},\quad F_1\{\pm1\}=\frac{1}{n};\quad F_1\{\pm2\}=\frac{1}{2}-\frac{1}{n}.$$

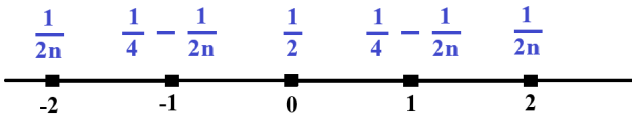

Figure 3: Example 7.4.3. Distribution of F.

Then $\mu_{4,1}, \sigma_1^2 \asymp C$ *and*

$$v_1 = \frac{2}{n}, \quad q_{1,1} = \frac{1}{n},$$

so that estimate (7.6) is of the order $O(n)$. *Note that trivial order for the difference of two distributions is* $O(1)$, *therefore we have very bad estimate. On the other hand, from corollary 5.2.1 it follows that*

$$d_{TV}\left(\left(\frac{1}{2}I + \frac{1}{2}F_1\right)^{*n}\right) \leqslant \frac{C}{n}.$$

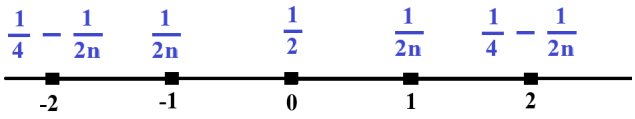

Figure 4: Example 7.4.4. Distribution of F.

Thus, we see that Skellam approximation is not always effective and additional research for structurally more advanced CP approximation is needed.

Next we construct short asymptotic expansion to H^{*n}. Let for brevity

$$\begin{aligned} B &:= \frac{1}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m} - \sigma_m^2)(U_m * \bar{U}_m)^{*2}, \\ V &:= \frac{1}{2}\sum_{m=1}^{K} p_m\sigma_m^2(U_m + \bar{U}_m) = -\frac{1}{2}\sum_{m=1}^{K} p_m\sigma_m^2(U_m * \bar{U}_m). \end{aligned}$$

and

$$R = \frac{1}{q^{13/2}}\left(\sum_{m=1}^{K}\sqrt{\sigma_m}\right)^4 + \left(\sum_{m=1}^{K}(\mu_{4,m} - \sigma_m^2)\min\left(n^2 p_m; \frac{1}{p_m q_{1,m}^2}\right)\right)^2,$$

$$r_m = \frac{2}{6!}\sum_{k=1}^{\infty} k^2(k^2-1)(k^2-4)q_{k,m} = \frac{\mu_{6,m}-5\mu_{4,m}+4\sigma_m^2}{6!}, \tag{7.8}$$

$$\Delta = \left(\sum_{m=1}^{K}\frac{\sigma_m^2}{v_m}\right)^2 + \sum_{m=1}^{K}\frac{\mu_{4,m}-\sigma_m^2}{p_m v_m^2}. \tag{7.9}$$

Theorem 7.4.3. *Let F satisfy (7.3)–(7.4) and let $q_{1,m} > 0$, $\mu_{6,m} < \infty$, $(m = 1,\dots,N)$. Then, for any $n \in \mathbb{N}$*

$$\left\| F^{*n} - H^{*n} * \left(I + nB - \frac{nV^{*2}}{2}\right)\right\|_{TV} \leqslant \frac{C}{n^2}\left(\sum_{m=1}^{K}\frac{r_m}{p_m^2\sigma_m^6} + \min(\Delta^2, R)\right).$$

Observe that $r_m = 0$ if F_m takes only values $\{\pm\vec{y}_m; \pm 2\vec{y}_m\}$. However, in all cases the following Corollary holds.

Corollary 7.4.2. *Let F satisfy (7.3)–(7.4) and let $q_{1,m} > 0$, $\mu_{6,m} < \infty$, $(m = 1,\dots,N)$. Then, for any $n \in \mathbb{N}$*

$$\begin{aligned}&\left\| F^{*n} - H^{*n} * \left(I + nB - \frac{nV^{*2}}{2}\right)\right\|_{TV} \\ &\quad\leqslant \frac{CK^6}{n^2 q^{13/2}}\left(\sum_{m=1}^{K}\frac{r_m}{p_m^2\sigma_m^6} + \min\left(\max_{1\leqslant m\leqslant K}\frac{\mu_{4,m}^2}{p_m^2 q_{1,m}^2}; \max_{1\leqslant k\leqslant K}\frac{\mu_{4,k}^2}{p_k^2 v_k^2}\right)\right).\end{aligned}$$

We see that, in many cases, the order of accuracy in Theorem 7.4.3 is $O(n^{-2})$.

Example 7.4.5. *Let $F \in \mathscr{F}_2$ be distribution defined in Example 7.4.1. Then*

$$\begin{aligned}B &= \frac{21}{36}(I_{(1,0)}-I)^{*2} * (I_{(-1,0)}-I)^{*2} + \frac{1}{24}(I_{(1,1)}-I)^{*2} * (I_{(-1,-1)}-I)^{*2}, \\ V &= -\frac{7}{2}(I_{(1,0)}-I) * (I_{(-1,0)}-I) - \frac{1}{6}(I_{(0,1)}-I) * (I_{(0,-1)}-I) \\ &\quad -\frac{5}{12}(I_{(1,1)}-I) * (I_{(-1,-1)}-I)\end{aligned}$$

and

$$\left\| F^{*n} - H^{*n} * \left(I + nB - \frac{nV^{*2}}{2}\right)\right\|_{TV} \leqslant \frac{C}{n^2}.$$

The proofs do not allow to get small absolute constants. However, from Theorem 7.4.3 we can get the following form of Theorem 7.4.2, which allows to get some idea about the magnitude of constants.

Theorem 7.4.2*. *Let F satisfy (7.3)–(7.4) and let $q_{1,m} > 0$, $\mu_{6,m} < \infty$, $(m = 1, \dots, N)$. Then, for any $n \in \mathbb{N}$*

$$\begin{aligned} \|F^{*n} - H^{*n}\|_{TV} &\leqslant \frac{32K^2}{n\mathrm{e}^2} + \frac{8}{3\mathrm{e}^2 n} \sum_{m=1}^{K} \frac{\mu_{4,m} - \sigma_m^2}{p_m \sigma_m^4} \\ &\quad + \frac{C}{n^2} \left(\sum_{m=1}^{K} \frac{r_m}{p_m^2 \sigma_m^6} + \min(\Delta^2, R) \right). \end{aligned}$$

Remarkably, the first summand on the RHS of the above estimate depends solely on K and n.

7.4.2 Local estimates and estimates in ℓ_r metric

We consider the same setting as used for local estimates in [56]. Our aim is to get rid of the restrictive assumption $\nu_{2,m} \leqslant \nu_{1,m}$. In addition to the assumptions (7.3)–(7.4) we assume that $F_1, \dots, F_d$ distributions are concentrated on coordinate axes:

$$\vec{y}_m = \bar{e}_m = (0, \dots, 0, 1, 0, \dots), \quad m = 1, \dots, d; \tag{7.10}$$

Let

$$\begin{aligned} \Delta_0 &:= \left(\sum_{m=1}^{K} \sigma_m^2 \min\left(np_m, \frac{1}{\nu_m}, \frac{1}{q q_{1,m}} \right) \right)^2 \\ &\quad + \sum_{m=1}^{K} \frac{\mu_{4,m} - \sigma_m^2}{p_m} \min\left(n^2 p_m^2, \frac{1}{\nu_m^2}; \frac{1}{q^2 q_{1,m}^2} \right). \end{aligned}$$

Theorem 7.4.4. *Let F satisfy (7.3)–(7.4) and (7.10). Let $\nu_m > 0$, $\mu_{4,m} < \infty$, $(m = 1, \dots, N)$. Then, for any $n \in \mathbb{N}$*

$$\|F^{*n} - H^{*n}\|_{\infty} \leqslant \frac{C(d)\Delta_0}{n} \prod_{j=1}^{d} (\max(1, nqp_j))^{-1/2}. \tag{7.11}$$

Corollary 7.4.3. *Let F satisfy (7.3)–(7.4) and (7.10). Let $\nu_m > 0$, $\mu_{4,m} < \infty$, $(m = 1, \dots, N)$. Then, for any $n \in \mathbb{N}$*

$$\|F^{*n} - H^{*n}\|_{\infty} \leqslant \frac{C(d)K^2}{nq^2} \prod_{j=1}^{d} (\max(1, nqp_j))^{-1/2}$$

$$\times \sum_{m=1}^{K} \frac{\mu_{4,m}}{p_m} \left(\min\left(\frac{1}{v_m}; \frac{1}{q_{1,m}} \right) \right)^2. \tag{7.12}$$

Combining (7.7) and (7.13) we get estimate for $\ell_r$metric.

Theorem 7.4.5. *Let F satisfy (7.3)–(7.4) and (7.10). Let $r \in \mathbb{N}$, $v_m > 0$, $\mu_{4,m} < \infty$, $(m = 1, \ldots, N)$. Then, for any $n \in \mathbb{N}$*

$$\begin{aligned} \|F^{*n} - H^{*n}\|_{\infty} \leqslant & \frac{C_3(d)K^2}{nq^{2+9/2r}} \prod_{j=1}^{d} (\max(1, nqp_j))^{-(1-1/r)/2} \\ & \times \sum_{m=1}^{K} \frac{\mu_{4,m}}{p_m} \left(\min\left(\frac{1}{v_m}; \frac{1}{q_{1,m}} \right) \right)^2. \end{aligned} \tag{7.13}$$

7.5 Auxiliary results

Essential to the proofs is expansion of F_m in analogues of factorial moments. We recall that $U_m := I_{\vec{y}_m} - I$, $\bar{U}_m := I_{-\vec{y}_m} - I$. and introduce new notation

$$W := \exp\left\{ \sum_{m=1}^{K} p_m q_{1,m} (U_m + \bar{U}_m) \right\}. \tag{7.14}$$

For exponential measures H and W real powers H^{*a}, W^{*a} are understood in the following sense

$$H^{*a} := \mathrm{e}^{aV}, \quad W^{*a} := \exp\left\{ a \sum_{m=1}^{K} p_m q_{1,m} (U_m + \bar{U}_m) \right\}.$$

Observe that $H = \mathrm{e}^{V}$ and

$$U_m + \bar{U}_m = -U_m * \bar{U}_m, \quad \bar{U}_m = -U_m * I_{-\vec{y}_m}, \tag{7.15}$$

$$\|U_m + I\|_{TV} = \|\bar{U}_m + I\|_{TV} = 1. \tag{7.16}$$

The last equality follows from the fact that the total variation norm of any distribution is equal to 1.

Lemma 7.5.1. *Let F satisfy (7.3)–(7.4) and let $q_{1,m} > 0$, $\sigma_m^2 < \infty$, $(m = 1, \ldots, K)$. Then,*

$$F_m - I = \frac{\sigma_m^2}{2} (U_m * \bar{U}_m) * \Theta_m. \tag{7.17}$$

If $\mu_{4,m} < \infty$, then

$$F_m - I = -\frac{\sigma_m^2}{2}(U_m * \bar{U}_m) + \frac{\mu_{4,m} - \sigma_m^2}{4!}(U_m * \bar{U}_m)^{*2} * \Theta_m. \tag{7.18}$$

If $\mu_{8,m} < \infty$, then

$$\begin{aligned} F_m - I &= -\frac{\sigma_m^2}{2}(U_m * \bar{U}_m) + \frac{\mu_{4,m} - \sigma_m^2}{4!}(U_m * \bar{U}_m)^{*2} \\ &\quad + r_m (U_m * \bar{U}_m)^{*3} * \Theta_m. \end{aligned} \tag{7.19}$$

Here r_m is defined by (7.8).

It is obvious, that $r_m \geqslant 0$. Observe also that

$$\mu_{4,m} - \sigma_m^2 = 2\sum_{k=1}^{\infty} k^2(k^2-1)q_{k,m} \geqslant 0.$$

Proof. For completeness we prove all expressions, though estimate (7.18) was already proved in [56]. For $k = 1$ expansion (7.17) follows from (7.15), (7.16). If $k > 1$, then by Bergström identity (A.32)

$$\begin{aligned} (U_m + I)^{*k} - I &= kU_m + U_m^{*2} * \sum_{j=1}^{k} (j-1)(U_m + I)^{*(k-j)} \\ &= kU_m + U_m * \bar{U}_m * (-I_{\vec{y}_m}) * \sum_{j=1}^{k} (j-1)(U_m + I)^{*(k-j)}. \end{aligned}$$

By (A.33) and (7.16)

$$\left\| (-I_{\vec{y}_m}) * \sum_{j=1}^{k} (j-1)(U_m + I)^{*(k-j)} \right\|_{TV} \leqslant \sum_{j=1}^{k} (j-1) = \binom{k}{2}.$$

Therefore we can write

$$(U_m + I)^{*k} - I = kU_m + \binom{k}{2}(U_m * \bar{U}_m) * \Theta_{m,1}, \quad \|\Theta_{m,1}\|_{TV} \leqslant 1,$$

and, similarly,

$$(\bar{U}_m + I)^{*k} - I = k\bar{U}_m + \binom{k}{2}(U_m * \bar{U}_m) * \Theta_{m,2}, \quad \|\Theta_{m,2}\|_{TV} \leqslant 1.$$

Combining both expressions and applying (7.15) we get

$$\begin{aligned}(U_m+I)^{*k}+(\bar{U}_m+I)^{*k}-2I&\\ &= -k(U_m*\bar{U}_m)+\binom{k}{2}(U_m*\bar{U}_m)*(\Theta_{m,1}+\Theta_{m,2})\\ &= (U_m*\bar{U}_m)*\left(-kI+\binom{k}{2}(\Theta_{m,1}+\Theta_{m,2})\right)\\ &= k^2(U_m*\bar{U}_m)*\Theta_{m,3}, \qquad \|\Theta_{m,3}\|_{TV}\leqslant 1,\end{aligned}$$

since

$$k+2\binom{k}{2}=k^2.$$

Expression (7.17) now follows from (7.5). To make the proof explicit, in above, we added additional indices to $\Theta_{i,m}$. To make notation shorter below this practice is abandoned. For the proof of (7.18) we need longer expansions. Further we assume that $\binom{k}{m}=0$, if $m>k$. Applying (A.32) and taking into account (A.33) and (7.17) we can write

$$\begin{aligned}(U_m+I)^{*k}-I &= kU_m+\binom{k}{2}U_m^{*2}+\binom{k}{3}U_m^{*3}+\binom{k}{4}(U_m*\bar{U}_m)^{*2}*\Theta_m,\\ (\bar{U}_m+I)^{*k}-I &= k\bar{U}_m+\binom{k}{2}\bar{U}_m^{*2}+\binom{k}{3}\bar{U}_m^{*3}+\binom{k}{4}(U_m*\bar{U}_m)^{*2}*\Theta_m,\end{aligned}$$

From (7.15) it follows that

$$\begin{aligned}k(U_m+\bar{U}_m)+\binom{k}{2}(U_m^{*2}+\bar{U}_m^{*2})&\\ =-k(U_m*\bar{U}_m)+\binom{k}{2}\left(U_m^{*2}+2(U_m*\bar{U}_m)+\bar{U}_m^{*2}\right)-2\binom{k}{2}U_m*\bar{U}_m&\\ =-k^2(U_m*\bar{U}_m)+\binom{k}{2}(U_m+\bar{U}_m)^{*2}&\\ =-k^2(U_m*\bar{U}_m)+\binom{k}{2}(U_m*\bar{U}_m)^{*2}.&\end{aligned} \tag{7.20}$$

Similarly

$$\begin{aligned}U_m^{*3}+\bar{U}_m^{*3} &= (U_m+\bar{U}_m)*(U_m^{*2}-(U_m*\bar{U}_m)+\bar{U}_m^{*2})\\ &= (U_m*\bar{U}_m)^{*2}*(I_{-\vec{y}_m}+I+I_{\vec{y}_m})\\ &= 3(U_m*\bar{U}_m)^{*2}*\Theta_m.\end{aligned} \tag{7.21}$$

Taking into account (7.20) and (7.21) we get

$$\begin{aligned}
&(U_m+I)^{*k}+(\bar{U}_m+I)^{*k}-2I \\
&\quad = -k^2(U_m*\bar{U}_m)+\left(\binom{k}{2}+3\binom{k}{3}+2\binom{k}{4}\right)(U_m*\bar{U}_m)^{*2}*\Theta_m \\
&\quad = -k^2(U_m*\bar{U}_m)+\frac{k^2(k^2-1)}{12}(U_m*\bar{U}_m)^{*2}*\Theta_m.
\end{aligned}$$

Substituting the last expression into (7.5) we prove (7.18).

For the proof of (7.19) we use (A.32) to write

$$\begin{aligned}
&(U_m+I)^{*k}+(\bar{U}_m+I)^{*k}-2I \\
&\qquad = \sum_{j=1}^{5}\binom{k}{j}\left(U_m^{*j}+\bar{U}_m^{*j}\right)+2\binom{k}{6}(U_m*\bar{U}_m)^{*3}*\Theta_m. \qquad (7.22)
\end{aligned}$$

We have

$$\begin{aligned}
U_m^{*3}+\bar{U}_m^{*3} &= (U_m+\bar{U}_m)^{*3}-3U_m*\bar{U}_m*(U_m+\bar{U}_m) \\
&= -(U_m*\bar{U}_m)^{*3}+3(U_m*\bar{U}_m)^{*2}. \qquad (7.23)
\end{aligned}$$

Similarly

$$\begin{aligned}
U_m^{*4}+\bar{U}_m^{*4} &= (U_m+\bar{U}_m)^{*4}-4(U_m*\bar{U}_m)*(U_m^{*2}+\bar{U}_m^{*2})-6(U_m*\bar{U}_m)^{*2} \\
&= -(U_m*\bar{U}_m)^{*3}*(U_m+\bar{U}_m)-4(U_m*\bar{U}_m)(U_m+\bar{U}_m)^{*2} \\
&\quad +2(U_m*\bar{U}_m)^{*2} \\
&= -(U_m*\bar{U}_m)^{*3}*(U_m+\bar{U}_m+4I)+2(U_m*\bar{U}_m)^{*2} \\
&= 2(U_m*\bar{U}_m)^{*2}+4(U_m*\bar{U}_m)^{*3}*\Theta_m.
\end{aligned}$$

Here we took into account that by (7.16)

$$\|U_m+\bar{U}_m+4I\|_{TV} \leqslant \|U_m+I\|_{TV}+\|\bar{U}_m+I\|_{TV}+2\|I\|_{TV} \leqslant 4.$$

Simple identity and (7.15) allow to write

$$\begin{aligned}
U_m^{*5}+\bar{U}_m^{*5} &= (U_m+\bar{U}_m)*(U_m^{*4}-U_m^{*3}*\bar{U}_m+U_m^{*2}*\bar{U}_m^{*2} \\
&\quad -U_m*\bar{U}_m^{*3m}+\bar{U}_m^{*4}) \\
&= 5(U_m*\bar{U}_m)^{*3}*\Theta_m. \qquad (7.24)
\end{aligned}$$

Substituting (7.20), (7.23)–(7.24) into (7.22) we obtain

$$\begin{aligned}
&(U_m+I)^{*k}+(\bar{U}_m+I)^{*k}-2I \\
&\quad = -k^2(U_m*\bar{U}_m)+(U_m*\bar{U}_m)^{*2}\left(\binom{k}{2}+3\binom{k}{3}+2\binom{k}{4}\right) \\
&\qquad +\left(\binom{k}{3}+4\binom{k}{4}+5\binom{k}{5}+2\binom{k}{6}\right)(U_m*\bar{U}_m)^{*3}*\Theta_m \\
&\quad = -k^2(U_m*\bar{U}_m)+\frac{k^2(k^2-1)}{12}(U_m*\bar{U}_m)^{*2} \\
&\qquad +\frac{k^2(k^2-1)(k^2-4)}{360}(U_m*\bar{U}_m)^{*3}*\Theta_m.
\end{aligned}$$

To complete the proof of (7.19) suffices to substitute the above expression into (7.5). □

Similar result holds for Fourier transforms of U_m, $\bar{U}_m$. Let

$$u_m := \exp\left\{\mathrm{i}(\vec{t},\vec{y}_m)\right\}-1, \quad \bar{u}_m := \exp\left\{-\mathrm{i}(\vec{t},\vec{y}_m)\right\}-1; \quad m=1,\dots,K.$$

Note that $\bar{u}_m$ is conjugate complex number to u_m. Moreover, the following identities hold:

$$\begin{aligned}
&u_m+\bar{u}_m=-u_m\bar{u}_m; \quad u_m\bar{u}_m=|u_m|^2\geqslant 0, \\
&|u_m|=|\bar{u}_m|, |u_m+1|=|\bar{u}_m+1|=1.
\end{aligned}$$

Below we use symbol θ_m, for all (different and probably dependent on m) complex or real quantities, satisfying $|\theta_m|\leqslant 1$.

Lemma 7.5.2. *Let F satisfy (7.3)–(7.4) and let $q_{1,m}>0$, $\sigma_m^2<\infty$, $(m=1,\dots,K)$. Then,*

$$\widehat{F}_m(\vec{t})-1=\frac{\sigma_m^2}{2}(u_m\bar{u}_m)\,\theta.$$

If $\mu_{4,m}<\infty$, then

$$\widehat{F}_m(\vec{t})-1=-\frac{\sigma_m^2}{2}(u_m\bar{u}_m)+\frac{\mu_{4,m}-\sigma_m^2}{4!}(u_m\bar{u}_m)^2\,\theta_m.$$

If $\mu_{8,m}<\infty$, then

$$\widehat{F}_m(\vec{t})-1=-\frac{\sigma_m^2}{2}(u_m\bar{u}_m)+\frac{\mu_{4,m}-\sigma_m^2}{4!}(u_m\bar{u}_m)^2+r_m(u_m\bar{u}_m)^6\,\theta_m,$$

where r_m is defined by (7.8).

Proof. It suffices to replace in the proof of Lemma 7.5.1 convolutions $U_m * \bar{U}_m$ by products $u_m \bar{u}_m$. □

The next two lemmas allows to shorten the proofs. We recall that W is defined by (7.14) and Θ_a is used for different measures, dependent on a, and having total variation bounded by 1.

Lemma 7.5.3. *Let F satisfy (7.3)–(7.4) and let $q_{1,m} > 0$, $\sigma_m^2 < \infty$, $(m = 1, \dots, K)$. Then, for any $a > 0$*

$$H^{*a} = W^{*a} * \Theta_a, \quad \mathrm{e}^{a(F-I)} = W^{*a} * \Theta_a. \tag{7.25}$$

Proof. We have

$$\frac{\sigma_m^2}{2} - q_{1,m} = \sum_{k=2}^{\infty} k^2 q_{k,m} \geqslant 0.$$

Therefore in expression

$$H^{*a} = W^{*a} * \exp\left\{ a \sum_{m=1}^{N} p_m \left(\frac{\sigma_m^2}{2} - q_{1,m} \right) (U_m + \bar{U}_m) \right\}$$

the second exponent is CP *distribution* and total variation of any distribution equals 1. It remains to denote it by Θ_a.

Similarly

$$\mathrm{e}^{a(F-I)} = W^{*a} * \exp\left\{ a \sum_{m=1}^{N} p_m \sum_{k=2}^{\infty} q_{k,m} (I_{k\vec{y}_m} - I + I_{-k\vec{y}_m} - I) \right\}.$$

and the second CP distribution is denoted by Θ_a. We recall that the same symbol Θ is used for different measures in the proofs. □

Lemma 7.5.4. *Let $p_m q_{1,m} > 0$, $a > 0$. Then*

$$\| W^{*a} * U_m * \bar{U}_m \|_{TV} \leqslant \frac{2}{\mathrm{e} a p_m q_{1,m}}. \tag{7.26}$$

Proof. Observe that W^{*a} is convolution of N CP distributions. Taking into account that any distribution has total variation equal to 1, we can write

$$W_m^{*a} = \exp\{a p_m q_{1,m} U_m\} * \exp\{a p_m q_{1,m} \bar{U}_m\} * \Theta_m,$$

where

$$\|\Theta_m\|_{TV} = \left\| \exp\left\{ \sum_{k\neq m}^{N} p_k q_{1,k}(U_k + \bar{U}_k) \right\} \right\|_{TV} = 1.$$

Therefore

$$\|W^{*a} * U_m * \bar{U}_m\|_{TV} \leqslant \left\|U_m * \mathrm{e}^{a p_m q_{1,m} U_m}\right\|_{TV} \left\|\bar{U}_m * \mathrm{e}^{a p_m q_{1,m} \bar{U}_m}\right\|_{TV}$$

and it remains to use (A.16). □

Lemma 7.5.5. *Let $a, p_m > 0$, $m = 1, \ldots, K$. Then*

$$\|H^{*a} * U_m * \bar{U}_m\|_{TV} \leqslant \frac{4}{a p_m \sigma_m^2 \mathrm{e}}, \tag{7.27}$$

$$\|H^{*a} * (U_m * \bar{U}_m)^{*2}\|_{TV} \leqslant \frac{64}{a^2 p_m^2 \sigma_m^4 \mathrm{e}^2}, \tag{7.28}$$

$$\|H^{*a} * (F - I)\|_{TV} \leqslant \frac{2K}{a\mathrm{e}}. \tag{7.29}$$

Proof. Repeating the proof of Lemma 7.5.4 and applying (A.16) we get (7.27) and (7.28).Then by (7.17) and (7.27)

$$\begin{aligned} \|H^{*a} * (F - I)\|_{TV} &\leqslant \sum_{m=1}^{K} p_m \|H^{*a} * (F_m - I)\|_{TV} \\ &\leqslant \frac{1}{2} \sum_{m=1}^{K} p_m \sigma_m^2 \|H^{*a} * U_m * \bar{U}_m\|_{TV} \\ &\leqslant \frac{1}{2} \sum_{m=1}^{K} \frac{p_m \sigma_m^2 4}{a p_m \sigma_m^2 \mathrm{e}} = \frac{2K}{a\mathrm{e}}. \end{aligned}$$

□

Lemma 7.5.6. *Let F satisfy (7.3)–(7.4) and let $\mu_{4,m} < \infty$, $(m = 1, \ldots, K)$. Then*

$$\mathrm{e}^{F-I} - H = W * \sum_{m=1}^{K} p_m \frac{\mu_{4,m} - \sigma_m^2}{4!} (U_m * \bar{U}_m)^{*2} * \Theta_m, \tag{7.30}$$

$$\begin{aligned} F - H &= \frac{1}{2}(F - I)^{*2} * \Theta + \frac{1}{4!} \sum_{m=1}^{K} p_m (\mu_{4,m} - \sigma_m^2)(U_m * \bar{U}_m)^{*2} * \Theta_m \\ &= \frac{1}{8} \left(\sum_{m=1}^{K} p_m \sigma_m^2 U_m * \bar{U}_m * \Theta_m \right)^{*2} * \Theta \end{aligned}$$

$$+\frac{1}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)(U_m*\bar{U}_m)^{*2}*\Theta_m. \tag{7.31}$$

Proof. From (A.30), (7.25) and (A.31) it follows that

$$\begin{aligned}
\mathrm{e}^{F-I}-H &= \mathrm{e}^{F-I}*\left(I-H*\mathrm{e}^{I-F}\right)\\
&= \left(F-I-\sum_{m=1}^{K}\frac{p_m\sigma_m^2}{2}(U_m+\bar{U}_m)\right)*\int_0^1 \mathrm{e}^{(1-\tau)(F-I)}*H^{*\tau}\mathrm{d}\tau\\
&= W*\left(\sum_{m=1}^{K} p_m(F_m-I)+\sum_{m=1}^{K}\frac{p_m\sigma_m^2}{2}(U_m*\bar{U}_m)\right)*\Theta\\
&= W*\sum_{m=1}^{K} p_m\frac{\mu_{4,m}-\sigma_m^2}{4!}(U_m*\bar{U}_m)^{*2}*\Theta_m.
\end{aligned}$$

For the proof of (7.31) observe that

$$F-H=(F-\mathrm{e}^{F-I})+(\mathrm{e}^{F-I}-H)$$

and apply (A.30), (7.17) and (7.30). □

Lemma 7.5.7. *Let F satisfy (7.3)–(7.4) and let $v_m>0$, $\mu_{4,m}<\infty$, for $m=1,\dots,K$. Then, for any $k\in\mathbb{N}\cup\{0\}$,*

$$\begin{aligned}
\|F_m^{*k}*U_m*\bar{U}_m\|_{TV} &\leqslant \frac{C}{(k+1)v_m}, &(7.32)\\
\|F_m^{*k}*(U_m*\bar{U}_m)^{*2}\|_{TV} &\leqslant \frac{C}{v_m^2(k+1)^2}, &(7.33)\\
\|F^{*k}*U_m*\bar{U}_m\|_{TV} &\leqslant \frac{C}{(k+1)p_mv_m}, &(7.34)\\
\|F^{*k}*(U_m*\bar{U}_m)^{*2}\|_{TV} &\leqslant \frac{C}{p_m^2v_m^2(k+1)^2}, &(7.35)\\
\|F^{*k}*(F-H)\|_{TV} &\leqslant \frac{C\Delta}{(k+1)^2}. &(7.36)
\end{aligned}$$

Here Δ is defined by (7.9).

Proof. Let $k\leqslant 15$. Observe that $F^{*k}\in\mathscr{F}_d$ and by (A.28) $\|F_m^{*k}\|_{TV}=1$. Then due to $\|U_m\|_{TV}=\|\bar{U}_m\|_{TV}\leqslant 2$ and $v_m\in(0,1)$, we get

$$\|F_m^{*k}*U_m*\bar{U}_m\|_{TV} \leqslant 4\leqslant\frac{64}{k+1}\leqslant\frac{64}{(k+1)v_m},$$

$$\|F_m^{*k} * (U_m * \bar{U}_m)^{*2}\|_{TV} \quad \leqslant \quad 16 \leqslant \frac{C}{v_m^2 (k+1)^2}.$$

Therefore, we can assume that $k \geqslant 16$. Then $\lfloor k/2 \rfloor \geqslant k/2 - 1 \geqslant k/4$, $\lfloor k/4 \rfloor \geqslant k/8$ and by (A.5)

$$\begin{aligned}
\|F_m^{*k} * U_m * \bar{U}_m\|_{TV} &\leqslant \|F_m^{*\lfloor k/2 \rfloor} * U_m\|_{TV} \|F_m^{\lfloor k/2 \rfloor} * \bar{U}_m\|_{TV} \\
&\leqslant \frac{C}{v_m (k+1)}, \\
\|F_m^{*k} * (U_m * \bar{U}_m)^2\|_{TV} &\leqslant \|F_m^{\lfloor k/2 \rfloor} * \bar{U}_m\|_{TV}^2 \leqslant \frac{C}{v_m^2 (k+1)^2}.
\end{aligned}$$

Thus, (7.32) and (7.33) are proved.

Next observe that

$$F = p_m F_m + (1 - p_m) \Psi_m, \quad \Psi_m = (1 - p_m)^{-1} \left(qI + \sum_{j \neq m}^{K} p_j F_j \right) \in \mathscr{F}_d.$$

Therefore, taking into account $\|\Psi_m\|_{TV} = 1$ and (7.32) we obtain

$$\begin{aligned}
&\|F^{*k} * U_m * \bar{U}_m\|_{TV} \\
&\quad \leqslant \sum_{j=0}^{k} \binom{k}{j} p_m^j (1 - p_m)^{k-j} \|F_m^{*j} * U_m * \bar{U}_m\|_{TV} \|\Psi_m\|_{TV}^{k-j} \\
&\quad \leqslant \frac{C}{v_m} \sum_{j=0}^{k} \binom{k}{j} p_m^j (1 - p_m)^{k-j} \frac{1}{(j+1)} \leqslant \frac{C}{p_m v_m (k+1)}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\|F^{*k} * (U_m * \bar{U}_m)^{*2}\|_{TV} &\leqslant \sum_{j=0}^{k} \binom{k}{j} p_m^j (1-p_m)^{k-j} \|F_m^{*j} * (U_m * \bar{U}_m)^{*2}\|_{TV} \\
&\leqslant \frac{C}{v_m^2} \sum_{j=0}^{k} \binom{k}{j} p_m^j (1-p_m)^{k-j} \frac{1}{(j+1)^2} \\
&\leqslant \frac{C}{p_m^2 v_m^2 (k+1)^2}.
\end{aligned}$$

Estimates (7.34) and (7.35) are proved.

From (7.31) it follows that

$$\|F - H\|_{TV} \leqslant C \left\{ \left(\sum_{m=1}^{K} p_m \sigma_m^2 \right)^2 + \sum_{m=1}^{K} p_m (\mu_{4,m} - \sigma_m^2) \right\} \leqslant C\Delta. \tag{7.37}$$

Let $k \leqslant 15$. Then (7.36) follows from (7.37):

$$\|F^{*k} * (F-H)\|_{TV} \leqslant \|F-H\|_{TV} \leqslant C\Delta \leqslant \frac{16^2 C\Delta}{(k+1)^2}.$$

Let $k \geqslant 16$. Then $\lfloor k/2 \rfloor \geqslant k/4$ and, therefore, applying (7.31) and (7.34) we get

$$\|F^{*k} * (F-H)\|_{TV} \leqslant \frac{1}{8}\left(\sum_{m=1}^{K} p_m \sigma_m^2 \|F^{*\lfloor k/2 \rfloor} * U_m * \bar{U}_m\|_{TV}\right)^2$$
$$+\frac{1}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\|F^{*k} * (U_m * \bar{U}_m)^{*2}\|_{TV} \leqslant \frac{C\Delta}{(k+1)^2}.$$

□

7.6 Proofs

In the proofs we frequently use the following relations. If $F \in \mathscr{F}_d$, $j \in \mathbb{N}$, then $F^{*j} \in \mathscr{F}_d$, and

$$\|F^{*j}\|_{TV} = 1, \ \sigma_m^2 = 2\sum_{k=1}^{\infty} k^2 q_{k,m} \geqslant 2\sum_{k=1}^{\infty} q_{k,m} = 1, \ \sigma_m^4 \leqslant \mu_{4,m}, \tag{7.38}$$

$$\|U_m\|_{TV} = \|\bar{U}_m\|_{TV} \leqslant 2. \tag{7.39}$$

Similarly total variation of any CP distribution equals 1.

Proof of Theorem 7.4.1. Therefore, taking account (7.25) and (7.30) we obtain

$$\left\|\mathrm{e}^{n(F-I)} - H^{*n}\right\|_{TV} \leqslant \sum_{j=1}^{n} \left\|\mathrm{e}^{(j-1)(F-I)} * H^{*(n-j)} * (\mathrm{e}^{F-I} - H)\right\|_{TV}$$
$$\leqslant \frac{n}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\left\|(U_m * \bar{U}_m)^{*2} * W^{*n}\right\|_{TV}$$
$$\leqslant \frac{n}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\left\|(U_m * \bar{U}_m) * W^{*n/2}\right\|_{TV}^2.$$

It remains to apply (7.26) or trivial estimate (7.39)

$$\left\|(U_m * \bar{U}_m) * W^{*n/2}\right\|_{TV}^2 \leqslant \|U_m\|_{TV}^2 \|\bar{U}_m\|_{TV}^2 \leqslant 16. \tag{7.40}$$

Theorem is proved. □

Proof of Theorem 7.4.2. Let $n \leqslant 15$. Observe that by (7.38)

$$p_m \leqslant 1 \leqslant \frac{1}{v_m}; \quad p_m \leqslant \frac{1}{qq_{1,m}}, \quad p_m \leqslant np_m.$$

Then by (7.31) and (A.27)

$$\begin{aligned}
\|F^{*n}-H^{*n}\|_{TV} &\leqslant n\|F-H\|_{TV} \leqslant \frac{15}{8}\left(\sum_{m=1}^{K} p_m\sigma_m^2\|U_m\|_{TV}\|\bar{U}_m\|_{TV}\right)^2 \\
&\quad +\frac{15}{24}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\|U_m\|_{TV}^2\|\bar{U}_m\|_{TV}^2 \\
&\leqslant \frac{15\cdot 16}{8}\left(\sum_{m=1}^{K} p_m\sigma_m^2\right)^2 + \frac{15\cdot 16}{24}\sum_{m=1}^{K}\frac{\mu_{4,m}-\sigma_m^2}{p_m}p_m^2 \\
&\leqslant C\tilde{\Delta} \leqslant \frac{15C\tilde{\Delta}}{n}.
\end{aligned}$$

Combining the last two estimates we get (7.6) for $n \leqslant 15$. Let $n \geqslant 16$. Then

$$n-1-\lfloor n/2\rfloor \geqslant n/3; \quad \lfloor n/2\rfloor+1 \geqslant n/2 \geqslant 2\lfloor n/4\rfloor, \quad \lfloor n/4\rfloor \geqslant n/5.$$

Therefore

$$\begin{aligned}
\|F^{*n}-H^{*n}\|_{TV} &\leqslant \sum_{j=0}^{n-1}\left\|F^{*j}*(F-H)*H^{*(n-1-j)}\right\|_{TV} \\
&\leqslant \sum_{j\leqslant\lfloor n/2\rfloor}\|F^{*j}\|_{TV}\left\|(F-H)*H^{*(n-1-j)}\right\|_{TV} \\
&\quad +\sum_{j\geqslant\lfloor n/2\rfloor+1}\left\|F^{*j}*(F-H)\right\|_{TV}\|H^{*(n-1-j)}\|_{TV} \\
&\leqslant n\left\|F^{*(\lfloor n/2\rfloor+1)}*(F-H)\right\|_{TV} + n\left\|(F-H)*H^{*n/3}\right\|_{TV}.
\end{aligned}$$

Taking account of (5.22), (5.27) and (7.25) we get

$$\begin{aligned}
n\left\|F^{*\lfloor n/2\rfloor+1}*(F-H)\right\|_{TV} &\leqslant n\left\|F^{*\lfloor n/4\rfloor}*F^{*\lfloor n/4\rfloor}*(F-H)\right\|_{TV} \\
&\leqslant n\left\|F^{*\lfloor n/4\rfloor}-D^{*\lfloor n/4\rfloor}\right\|_{TV}\|F^{\lfloor n/4\rfloor}\|_{TV}\left(\|F\|_{TV}+\|D\|_{TV}\right) \\
&\quad +n\left\|D^{*\lfloor n/4\rfloor}*F^{*\lfloor n/4\rfloor}*(F-H)\right\|_{TV} \leqslant \frac{C}{nq^{13/2}}\left(\sum_{m=1}^{K}\sqrt{1+\sigma_m}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{Cn}{\sqrt{q}}\left\|\mathrm{e}^{q\lfloor n/4\rfloor(F-I)} * F^{*\lfloor n/4\rfloor} * (F-H)\right\|_{TV} \\
\leqslant\ & \frac{C}{nq^{13/2}}\left(\sum_{m=1}^{K}\sqrt{\sigma_m}\right)^2 + \frac{Cn}{\sqrt{q}}\left\|W^{*q\lfloor n/4\rfloor} * F^{*\lfloor n/4\rfloor} * (F-H)\right\|_{TV}.
\end{aligned}
$$

From (7.31) we get

$$
\begin{aligned}
&\frac{Cn}{\sqrt{q}}\left\|W^{*q\lfloor n/4\rfloor} * F^{*\lfloor n/4\rfloor} * (F-H)\right\|_{TV} \\
&\quad\leqslant \frac{Cn}{\sqrt{q}}\left(\sum_{m=1}^{K} p_m\sigma_m^2\left\|W^{*qn/15} * F^{*\lfloor n/20\rfloor} * U_m * \bar{U}_m\right\|_{TV}\right)^2 \\
&\qquad + \frac{Cn}{\sqrt{q}}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\left\|W^{*qn/5} * F^{*\lfloor n/4\rfloor} * (U_m * \bar{U}_m)^{*2}\right\|_{TV}
\end{aligned}
$$

From (7.39) it follows that

$$
\begin{aligned}
&\left\|W^{*n/15} * F^{*\lfloor n/20\rfloor} * U_m * \bar{U}_m\right\|_{TV} \\
&\qquad\leqslant \|W^{*qn/15}\|_{TV}\|F^{*\lfloor n/20\rfloor}\|_{TV}\|U_m * \bar{U}_m\|_{TV} \leqslant 4
\end{aligned}
$$

and

$$
\left\|W^{*qn/5} * F^{*\lfloor n/4\rfloor} * (U_m * \bar{U}_m)^{*2}\right\|_{TV} \leqslant 16.
$$

From (7.34) it follows that

$$
\begin{aligned}
&\left\|W^{*qn/15} * F^{*\lfloor n/20\rfloor} * U_m * \bar{U}_m\right\|_{TV} \\
&\qquad\leqslant \|W^{*qn/15}\|_{TV}\left\|F^{*\lfloor n/20\rfloor} * U_m * \bar{U}_m\right\|_{TV} \\
&\qquad= \left\|F^{*\lfloor n/20\rfloor} * U_m * \bar{U}_m\right\|_{TV} \leqslant \frac{C}{np_m v_m}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\left\|W^{*qn/5} * F^{*\lfloor n/4\rfloor} * (U_m * \bar{U}_m)^{*2}\right\|_{TV} &\leqslant \left\|F^{*\lfloor n/4\rfloor} * (U_m * \bar{U}_m)^{*2}\right\|_{TV} \\
&\leqslant \frac{C}{n^2 p_m^2 v_m^2}.
\end{aligned}
$$

On the other hand, from (7.26) we get

$$
\begin{aligned}
&\left\|W^{*qn/15} * F^{*\lfloor n/20\rfloor} * U_m * \bar{U}_m\right\|_{TV} \\
&\qquad\leqslant \|F^{*\lfloor n/20\rfloor}\|_{TV}\left\|W^{*qn/15} * U_m * \bar{U}_m\right\|_{TV}
\end{aligned}
$$

$$= \quad \left\|W^{*qn/15} * U_m * \bar{U}_m\right\|_{TV} \leqslant \frac{C}{nqp_m q_{1,m}}.$$

Similarly,

$$\begin{aligned}\left\|W^{*qn/5} * F^{*\lfloor n/4\rfloor} * (U_m * \bar{U}_m)^{*2}\right\|_{TV} &\leqslant \left\|W^{*qn/10} * U_m * \bar{U}_m\right\|_{TV}^2 \\ &\leqslant \frac{C}{n^2 q^2 p_m^2 q_{1,m}^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{Cn}{\sqrt{q}}\left\|W^{*q\lfloor n/4\rfloor} * F^{*\lfloor n/4\rfloor} * (F-H)\right\|_{TV} \\ &\qquad\leqslant \frac{C}{n\sqrt{q}}\left(\sum_{m=1}^{K}\sigma_m^2 \min\left(np_m; \frac{1}{v_m}; \frac{1}{qq_{1,m}}\right)\right)^2 \\ &\qquad\quad + \frac{C}{n\sqrt{q}}\sum_{m=1}^{K}\frac{(\mu_{4,m}-\sigma_m^2)}{p_m}\min\left(n^2p_m^2\frac{1}{v_m^2}; \frac{1}{q^2q_{1,m}^2}\right)\end{aligned}$$

and

$$n\left\|F^{*\lfloor n/2\rfloor+1} * (F-H)\right\|_{TV} \leqslant \frac{C\tilde{\Delta}}{n}. \tag{7.41}$$

Applying (7.31) and (7.28) we get

$$\begin{aligned}n\left\|(F-H) * H^{*n/3}\right\|_{TV} &\leqslant Cn\left(\sum_{m=1}^{K} p_m\sigma_m^2\|H^{*n/6} * U_m * \bar{U}_m\|_{TV}\right)^2 \\ &\quad + Cn\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\|H^{*n/3} * (U_m * \bar{U}_m)^{*2}\|_{TV} \\ &\leqslant Cn\left(\sum_{m=1}^{K} p_m\sigma_m^2 \min\left(1, \frac{1}{np_m\sigma_m^2}\right)\right)^2 \\ &\quad + Cn\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\min\left(1; \frac{1}{n^2p_m^2\sigma_m^4}\right) \\ &= \frac{C}{n}\left(\sum_{m=1}^{K}\sigma^2\min\left(np_m; \frac{1}{\sigma_m^2}\right)\right)^2 \\ &\quad + \frac{C}{n}\sum_{m=1}^{K}\frac{\mu_{4,m}-\sigma_m^2}{p_m}\min\left(n^2p_m^2; \frac{1}{\sigma_m^4}\right) \leqslant \frac{C\tilde{\Delta}}{n}.\end{aligned} \tag{7.42}$$

For the last estimate we used (7.38) so that

$$\frac{1}{\sigma_m^2} \leqslant 1 \leqslant \min\left(\frac{1}{v_m}; \frac{1}{qq_{1,m}}\right).$$

Substituting (7.41) and (7.42) into (7.41) we complete the proof of Theorem. □

Proof of Corollary 7.4.1. We recall that $v_m, q, q_{1,m} \in (0,1)$. We have

$$\left(\sum_{m=1}^K p_m\sigma_m^2\right)^2 \leqslant K^2 \max_m p_m^2\sigma_m^4 \leqslant K^2 \max_m p_m^2\mu_{4,m} \leqslant K^2 \sum_{m=1}^K p_m^2\mu_{4,m}.$$

Similarly

$$\begin{aligned}\left(\sum_{m=1}^K \sqrt{\sigma_m}\right)^3 &\leqslant \left(K\max_m \sqrt{\sigma_m}\right)^3 \leqslant K^3 \max_m(\sigma_m)^{3/2} \\ &\leqslant K^3 \max_m \sigma_m^2 \leqslant K^3 \sum_{m=1}^K \sigma_m^2 \leqslant K^3 \sum_{m=1}^K \mu_{4,m} \\ &\leqslant K^3 \sum_{m=1}^K \frac{\mu_{4,m}}{p_m \max(q^2q_{1,m}^2, v_m^2)}. \end{aligned} \tag{7.43}$$

Applying these estimates to $\tilde{\Delta}$ we prove (7.7). □

Proof of Theorem 7.4.3. Recall that $H = \mathrm{e}^V$. Moreover $H \in \mathscr{F}_d$ is a distribution. Therefore $\|H^{*a}\|_{TV} = 1$, for any $a > 0$. Similarly $\|U_m\|_{TV} \leqslant 2$. First we assume that $n \geqslant 16$.

We have

$$\begin{aligned}&\left\|F^{*n} - H^{*n} * \left(I + nB - \frac{nV^{*2}}{2}\right)\right\|_{TV} \\ &\qquad \leqslant \|\mathscr{K}_0\|_{TV} + \|\mathscr{K}_1\|_{TV} + \ldots + \|\mathscr{K}_6\|_{TV},\end{aligned}$$

where

$$\begin{aligned}\mathscr{K}_0 &= F^{*n} - \mathrm{e}^{n(F-I)} * \left(I - \frac{n}{2}(F-I)^{*2}\right), \\ \mathscr{K}_1 &= nH^{*n} * (F - I - V - B), \\ \mathscr{K}_2 &= nH^{*n} * \left(\mathrm{e}^{F-I} - I - (F-I) - \frac{(F-I)^{*2}}{2}\right),\end{aligned}$$

$$
\begin{aligned}
\mathscr{K}_3 &= -nH^{*n} * \left(\mathrm{e}^V - I - V - \frac{V^{*2}}{2} \right), \\
\mathscr{K}_4 &= \frac{n(F-I)^{*2}}{2} * \left(H^{*n} - \mathrm{e}^{n(F-I)} \right), \\
\mathscr{K}_5 &= \mathrm{e}^{n(F-I)} - H^{*n} - nH^{*(n-1)} * (\mathrm{e}^{F-I} - H), \\
\mathscr{K}_6 &= nH^{*(n-1)} * (I-H) * (\mathrm{e}^{F-I} - H).
\end{aligned}
$$

From Theorem 5.2.10 it follows that

$$
\|\mathscr{K}_0\|_{TV} \leqslant \frac{C}{n^2 q^{13/2}} \left(\sum_{m=1}^{K} \sqrt{1+\sigma_m} \right)^4 \leqslant \frac{C}{n^2 q^{13/2}} \left(\sum_{m=1}^{K} \sqrt{\sigma_m} \right)^4. \tag{7.44}
$$

Applying (7.19) and (7.27) we prove that

$$
\begin{aligned}
\|\mathscr{K}_1\|_{TV} &\leqslant n \sum_{m=1}^{K} p_m r_m \left\| H^{*n} * (U_m * \bar{U}_m)^{*3} \right\|_{TV} \\
&\leqslant n \sum_{m=1}^{K} p_m r_m \left\| H^{*n/3} * U_m * \bar{U}_m \right\|_{TV}^3 \leqslant \frac{C}{n^2} \sum_{m=1}^{K} \frac{r_m}{p_m^2 \sigma_m^6}.
\end{aligned} \tag{7.45}
$$

Applying (A.30), (7.17) and (7.27) we obtain

$$
\begin{aligned}
\|\mathscr{K}_2\|_{TV} &\leqslant \frac{n}{6} \|H^{*n} * (F-I)^{*3}\|_{TV} \leqslant \frac{n}{3} \|H^{*n/3} * (F-I)\|_{TV}^3 \\
&\leqslant Cn \left(\sum_{m=1}^{K} p_m \sigma_m^2 \left\| H^{*n/3} * U_m * \bar{U}_m \right\|_{TV} \right)^3 \leqslant \frac{CK^3}{n^2}.
\end{aligned} \tag{7.46}
$$

Similarly applying (A.30), (7.17) and (7.27) we obtain

$$
\begin{aligned}
\|\mathscr{K}_3\|_{TV} &\leqslant Cn \|H^{*n} * V^{*3}\|_{TV} \leqslant Cn \|H^{*n/3} * V\|_{TV}^3 \\
&\leqslant Cn \left(\sum_{m=1}^{K} p_m \sigma_m^2 \left\| H^{*n/3} * U_m * \bar{U}_m \right\|_{TV} \right)^3 \leqslant \frac{CK^3}{n^2}.
\end{aligned} \tag{7.47}
$$

Taking into account that total variation of any distribution equals 1 and applying (7.25) and (7.30) we get

$$
\begin{aligned}
\|\mathscr{K}_4\|_{TV} &\leqslant \frac{n}{2} \left\| (F-I)^{*2} * \sum_{j=1}^{n} H^{*j} * \mathrm{e}^{(n-j-1)(F-I)} * (H - \mathrm{e}^{F-I}) \right\|_{TV} \\
&\leqslant \frac{n}{2} \left\| (F-I)^{*2} * \sum_{j \geqslant \lfloor n/2 \rfloor + 1} (F-I)^{*2} * \sum_{j < \lfloor n/2 \rfloor + 1} \right\|_{TV}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{n}{2}\left\|(F-I)^{*2}*H^{*n/2}*(H-\mathrm{e}^{F-I})\right\|_{TV}\\
&\quad\times\sum_{j\geqslant\lfloor n/2\rfloor+1}\|H^{*(j-n/2)}\|_{TV}\|\mathrm{e}^{(n-j-1)(F-I)}\|_{TV}\\
&\quad+\frac{n}{2}\left\|(F-I)^{*2}*\mathrm{e}^{n(F-I)/2}*(H-\mathrm{e}^{F-I})\right\|_{TV}\\
&\quad\times\sum_{j<\lfloor n/2\rfloor+1}\|F^{*j}\|_{TV}\|\mathrm{e}^{(n/2-j-1)(F-I)}\|_{TV}\\
&\leqslant \frac{n^2}{4}\left\|(F-I)^{*2}*H^{*n/4}\right\|_{TV}\left\|H^{*n/4}*(H-\mathrm{e}^{F-I})\right\|_{TV}\\
&\quad+\frac{n}{2}\left(\frac{n}{2}+1\right)\left\|(F-I)^{*2}*\mathrm{e}^{n(F-I)/4}\right\|_{TV}\\
&\quad\times\left\|\mathrm{e}^{n(F-I)/4}*(H-\mathrm{e}^{F-I})\right\|_{TV}\mathrm{e}^{3\|F-I\|_{TV}}\\
&\leqslant Cn^2\sum_{m=1}^{K}p_m(\mu_{4,m}-\sigma_m^2)\left\|W^{*n/4}*(U_m*\bar{U}_m)^{*2}\right\|_{TV}\\
&\quad\times\left(\|(F-I)*H^{*n/8}\|_{TV}^2+\|(F-I)*\mathrm{e}^{n(F-I)/8}\|_{TV}^2\right).
\end{aligned}
\tag{7.48}
$$

From (7.29) and (5.30) it follows that

$$
\begin{aligned}
n^2\|(F-I)*H^{*n/8}\|_{TV}^2 &\leqslant CK^2,\\
n^2\|(F-I)*\mathrm{e}^{n(F-I)/8}\|_{TV}^2 &\leqslant C\left(\sum_{m=1}^{K}\sqrt{\sigma_m^2}\right)^2.
\end{aligned}
\tag{7.49}
$$

Similarly to the proof of Theorem 7.4.1 taking into account (7.26) and analogue of (7.40) we prove that

$$
\left\|W^{*n/4}*(U_m*\bar{U}_m)^{*2}\right\|_{TV}\leqslant\min\left(16,\frac{4\cdot 8^2}{\mathrm{e}^2n^2p_m^2q_{1,m}^2}\right).
\tag{7.50}
$$

Substituting (7.49) and (7.50) into (7.48) we obtain

$$
\|\mathscr{K}_4\|_{TV}\leqslant\frac{C}{n^2}\left(\sum_{m=1}^{K}\sqrt{\sigma_m}\right)^2\sum_{j=1}^{K}(\mu_{4,j}-\sigma_j^2)\min\left(n^2p_j;\frac{1}{p_jq_{1,j}^2}\right).
\tag{7.51}
$$

From (A.32) and (7.25) we get

$$
\|\mathscr{K}_5\|_{TV}\leqslant\sum_{j=2}^{n}(j-1)\left\|\mathrm{e}^{(n-j)(F-I)}*H^{*(j-1)}*(\mathrm{e}^{F-I}-H)^{*2}\right\|_{TV}
$$

$$
\begin{aligned}
&\leqslant C\sum_{j=1}^{n}(j-1)\left\|W^{*(n-1)}*(\mathrm{e}^{F-I}-H)^{*2}\right\|_{TV}\\
&\leqslant Cn^2\left\|W^{*(n-1)/2}*(\mathrm{e}^{F-I}-H)\right\|_{TV}^2\\
&\leqslant Cn^2\left(\sum_{m=1}^{K}p_m(\mu_{4,m}-\sigma_m^2)\|W^{*n/4}*(U_m*\bar{U}_m)^2\|_{TV}\right)^2.
\end{aligned}
$$

On one hand,

$$
\|W^{*n/4}*(U_m*\bar{U}_m)^2\|_{TV}\leqslant\|W^{*n/4}\|_{TV}^2\|U_m\|_{TV}^2\|\bar{U}_m\|_{TV}^2\leqslant 16.
$$

On the other hand, by (7.26)

$$
\|W^{*n/4}*(U_m*\bar{U}_m)^2\|_{TV}\leqslant\|W^{*n/8}*U_m*\bar{U}_m\|_{TV}^2\leqslant\frac{16^2}{n^2p_m^2q_{1,m}^2\mathrm{e}^2}.
$$

Therefore,

$$
\|\mathscr{K}_5\|_{TV}\leqslant\frac{C}{n^2}\left(\sum_{m=1}^{K}(\mu_{4,m}-\sigma_m^2)\min\left(n^2p_m,\frac{1}{p_mq_{1,m}^2}\right)\right)^2. \tag{7.52}
$$

From (A.30) and (7.27) it follows that

$$
\begin{aligned}
n\left\|H^{*(n-1)/2}*(I-\mathrm{e}^V)\right\|_{TV}&\leqslant Cn\|H^{*(n-1)}*V\|_{TV}\\
&\leqslant Cn\sum_{m=1}^{K}p_m\sigma_m^2\|H^{*(n-1)}*U_m*\bar{U}_m\|_{TV}\\
&\leqslant Cn\sum_{m=1}^{K}\frac{p_m\sigma_m^2}{(n-1)p_m\sigma_m^2}\leqslant\frac{CKn}{n-1}\leqslant CK,
\end{aligned}
$$

since by assumptions $n\geqslant 5$. Similarly from (7.30) and (7.27) we get

$$
\begin{aligned}
&\left\|H^{*(n-1)/2}*(\mathrm{e}^{F-I}-H)\right\|_{TV}\\
&\qquad\leqslant C\sum_{m=1}^{K}p_m(\mu_{4,m}-\sigma_m^2)\|H^{*(n-1)/4}*U_m*\bar{U}_m\|_{TV}^2\\
&\qquad\leqslant\frac{C}{n^2}\sum_{m=1}^{K}\frac{\mu_{4,m}-\sigma_m^2}{p_m\sigma^4}.
\end{aligned}
$$

On the other hand,

$$\|H^{*(n-1)/4} * U_m * \bar{U}_m\|_{TV}^2 \leqslant \|H^{*(n-1)}\|_{TV}^2 \|U_m\|_{TV}^2 \|\bar{U}_m\|_{TV}^2 \leqslant 1 \cdot 2^2 \cdot 2^2 = 16.$$

Collecting above estimates we get

$$\begin{aligned} \|\mathscr{K}_6\|_{TV} &\leqslant n \left\| H^{*(n-1)/2} * (I - \mathrm{e}^V) \right\|_{TV} \left\| H^{*(n-1)/2} * (\mathrm{e}^{F-I} - H) \right\|_{TV} \\ &\leqslant \frac{CK}{n^2} \sum_{m=1}^{K} (\mu_{4,m} - \sigma_m^2) \min\left(n^2 p_m; \frac{1}{p_m \sigma_m^4} \right). \end{aligned} \tag{7.53}$$

Substituting (7.44), (7.45), (7.46), (7.47), (7.51), (7.52), (7.53) into (7.44) and observing that

$$\|\mathscr{K}_2\|_{TV} + \|\mathscr{K}_3\|_{TV} \leqslant \frac{CK^3}{n^2} \leqslant \frac{CK^4}{n^2 q^{13/2}} \leqslant \frac{C}{n^2} \left(\sum_{m=1}^{K} \sqrt{1+\sigma_m} \right)^4,$$

$$\sigma_m^2 \geqslant 2q_{1,m}; \quad \frac{1}{p_m \sigma_m^4} \leqslant \frac{1}{p_m q_{1,m}^2},$$

and, therefore,

$$\begin{aligned} &\|\mathscr{K}_4\|_{TV} + \|\mathscr{K}_6\|_{TV} \\ &\quad\leqslant \frac{C}{n^2} \left(\sum_{m=1}^{K} \sqrt{1+\sigma_m} \right)^2 \sum_{m=1}^{K} (\mu_{4,m} - \sigma_m^2) \min\left(n^2 p_m; \frac{1}{p_m q_{1,m}^2} \right) \\ &\quad\leqslant \frac{C}{n^2} \left(\sum_{m=1}^{K} \sqrt{1+\sigma_m} \right)^4 \\ &\qquad + \frac{C}{n^2} \left(\sum_{m=1}^{K} (\mu_{4,m} - \sigma_m^2) \min\left(n^2 p_m; \frac{1}{p_m q_{1,m}^2} \right) \right)^2 \end{aligned}$$

we prove that

$$\left\| F^{*n} - H^{*n} * \left(I + nB - \frac{nV^{*2}}{2} \right) \right\|_{TV} \leqslant \frac{C}{n^2} \left(\sum_{m=1}^{K} \frac{r_m}{p_m^2 \sigma_m^6} + R \right). \tag{7.54}$$

Next we use the following identity:

$$F^{*n} - H^{*n} * \left(I + nB - \frac{nV^{*2}}{2} \right) = \mathscr{K}_1 + \mathscr{K}_3 + \mathscr{K}_7 + \mathscr{K}_8,$$

where

$$\mathscr{K}_7 = F^{*n} - H^{*n} - nH^{*(n-1)} * (F-H), \quad \mathscr{K}_8 = -nH^{*(n-1)} * (H-I) * (F-H)$$

and $\mathscr{K}_1, \mathscr{K}_3$ are the same as defined in above. From (7.47) and (7.38) it follows that

$$\|\mathscr{K}_3\|_{TV} \leqslant \frac{CK^3}{n^2} \leqslant \frac{C}{n^2} K^4 \leqslant \frac{C}{n^2} \left(\sum_{m=1}^{K} \frac{\sigma_m^2}{v_m} \right)^4 \leqslant \frac{C\Delta^2}{n^2}. \tag{7.55}$$

By Bergström identity (A.32) and (7.36)

$$\begin{aligned}
\|\mathscr{K}_7\|_{TV} &\leqslant \sum_{j=2}^{n} (j-1) \left\| F^{*(n-j)} * (F-H)^{*2} * H^{*(j-2)} \right\|_{TV} \\
&\leqslant \sum_{j=2}^{\lfloor n/2 \rfloor} (j-1) \left\| F^{*(n-j)} * (F-H)^{*2} \right\|_{TV} \|H^{*(j-2)}\|_{TV} \\
&\quad + \sum_{j=\lfloor n/2 \rfloor + 1}^{n} (j-1) \|F^{*(n-j)}\|_{TV} \left\| (F-H)^{*2} * H^{*(j-2)} \right\|_{TV} \\
&\leqslant n^2 \left\| F^{*\lfloor n/2 \rfloor} * (F-H)^{*2} \right\|_{TV} + n^2 \left\| H^{*n/3} * (F-H)^{*2} \right\|_{TV} \\
&\leqslant n^2 \left\| F^{*\lfloor n/5 \rfloor} * (F-H) \right\|_{TV}^2 + n^2 \left\| H^{*n/6} * (F-H) \right\|_{TV}^2 \\
&\leqslant \frac{C\Delta^2}{n^2} + n^2 \left\| H^{*n/6} * (F-H) \right\|_{TV}^2.
\end{aligned}$$

Repeating the proof of (7.42) we get

$$\begin{aligned}
n^2 \left\| H^{*n/6} * (F-H) \right\|_{TV}^2 &\leqslant \frac{C}{n^2} \left(K^2 + \sum_{m=1}^{K} \frac{\mu_{4,m} - \sigma_m^2}{p_m v_m^2} \right)^2 \\
&\leqslant \frac{C}{n^2} \left(\left(\sum_{m=1}^{K} \frac{\sigma_m^2}{v_m} \right)^2 + \sum_{m=1}^{K} \frac{\mu_{4,m} - \sigma_m^2}{p_m v_m^2} \right)^2 = \frac{C\Delta^2}{n^2}.
\end{aligned}$$

Thus,

$$\|\mathscr{K}_7\|_{TV} \leqslant \frac{C\Delta^2}{n^2}. \tag{7.56}$$

Next observe that by (A.27) and (7.42)

$$\begin{aligned}
\|\mathscr{K}_8\|_{TV} &\leqslant n \left\| H^{*n/3} * (H-I) \right\|_{TV} \left\| H^{*n/3} * (F-H) \right\|_{TV} \\
&\leqslant \frac{C\Delta}{n} \left\| H^{*n/3} * (H-I) \right\|_{TV}.
\end{aligned} \tag{7.57}$$

Recalling that $H = \mathrm{e}^V$ and applying (A.30) and (7.27) we get

$$\begin{aligned}\left\|H^{*n/3} * (H-I)\right\|_{TV} &= \left\|H^{*n/3} * (\mathrm{e}^V - I)\right\|_{TV} \leqslant \left\|H^{*n/3} * V\right\|_{TV} \\ &\leqslant \frac{1}{2}\sum_{m=1}^{K} p_m \sigma_m^2 \|H^{*n/3} * U_m * \bar{U}_m\|_{TV} \leqslant \frac{CK}{n} \leqslant \frac{C\Delta}{n}. \end{aligned} \tag{7.58}$$

Substituting (7.58) into (7.57) we obtain

$$\|\mathscr{K}_8\|_{TV} \leqslant \frac{C\Delta^2}{n^2}. \tag{7.59}$$

Collecting estimates (7.45), (7.55), (7.56) and (7.59) we get

$$\left\|F^{*n} - H^{*n} * \left(I + nB - \frac{nV^{*2}}{2}\right)\right\|_{TV} \leqslant \frac{C}{n^2}\left(\sum_{m=1}^{K} \frac{r_m}{p_m^2 \sigma_m^6} + \Delta^2\right). \tag{7.60}$$

For $n \geqslant 16$ Theorem's statement now follows from (7.54) and (7.60).

Now let us assume that $n \leqslant 15$. Then

$$\begin{aligned}&\|F^{*n} - H^{*n} * (I + nB - nV^{*2}/2)\|_{TV} \\ &\quad\leqslant \|F^{*n}\|_{TV} + \|H^{*n}\|_{TV} + 5\|H^{*n} * B\|_{TV} + 3\|H^{*n/2} * V\|_{TV}^2. \end{aligned} \tag{7.61}$$

Obviously,

$$\begin{aligned}\|F^{*n}\|_{TV} + \|H^{*n}\|_{TV} &= 2 \leqslant \frac{450}{n^2} \\ &\leqslant \min\left(\frac{450}{n^2 q^{13/2}}\left(\sum_{m=1}^{n}\sqrt{1+\sigma_m}\right)^4, \frac{450\Delta^2}{n^2}\right).\end{aligned}$$

From (7.27) it follows that

$$\begin{aligned}\|H^{*n/2} * V\|_{TV}^2 &\leqslant \left(\sum_{m=1}^{K} p_m \sigma^2 \|H^{*n/2} * U_m * \bar{U}_m\|_{TV}\right)^2 \leqslant \frac{CK^2}{n^2} \\ &\leqslant \frac{C}{n^2}\min\left(\left(\sum_{m=1}^{n}\sqrt{1+\sigma_m}\right)^4, \Delta^2\right).\end{aligned}$$

Similarly from (7.27) it follows that

$$\begin{aligned}
\|H^{*n} * B\|_{TV} &\leqslant \frac{1}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma^2)\|H^{*n/2} * U_m * \bar{U}_m\|_{TV}^2 \\
&\leqslant \frac{1}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma^2)\min\left(16, \frac{16}{\mathrm{e}^2 n^2 p_m^2 q_{1,m}^2}\right) \\
&\leqslant \frac{2}{3n^2}\sum_{m=1}^{n}(\mu_{4,m}-\sigma_m^2)\min\left(n^2 p_m; \frac{1}{p_m q_{1,m}^2}\right) \\
&\leqslant 1+\left(\frac{2}{3n^2}\sum_{m=1}^{n}(\mu_{4,m}-\sigma_m^2)\min\left(n^2 p_m; \frac{1}{p_m q_{1,m}^2}\right)\right)^2 \\
&\leqslant \frac{25}{n^2}\left(\sum_{m=1}^{n}\sqrt{1+\sigma_m}\right)^4 \\
&\quad +\left(\frac{2}{3n^2}\sum_{m=1}^{n}(\mu_{4,m}-\sigma_m^2)\min\left(n^2 p_m; \frac{1}{p_m q_{1,m}^2}\right)\right)^2 \\
&\leqslant CR. \qquad (7.62)
\end{aligned}$$

On the other hand, applying (7.34) and taking into account that $\Delta \geqslant 1$ we obtain

$$\begin{aligned}
\|H^{*n} * B\|_{TV} &\leqslant \frac{1}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma^2)\|H^{*n/2} * U_m * \bar{U}_m\|_{TV}^2 \\
&\leqslant C\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2) \leqslant C\Delta \leqslant C\Delta^2 \leqslant \frac{225C\Delta^2}{n^2}.
\end{aligned}$$

Substituting above estimates into (7.61) we prove that (7.54) and (7.60) hold for $n \leqslant 15$. Theorem is proved. □

Proof of Corollary 7.4.2. Recalling that $1 \leqslant \sigma_m^2 \leqslant \sigma_m^4 \leqslant \mu_{4,m}$, we get

$$\left(\sum_{m=1}\sqrt{\sigma_m}\right)^4 \leqslant K^4 \max_m \sigma_m^2 \leqslant K^4 \max_m \mu_{4,m} \leqslant K^4 \max_m \frac{\mu_{4,m}^2}{p_m q_{1,m}^4}$$

and

$$\left(\sum_{m=1}^{K}\frac{\sigma_m^2}{v_m}\right)^2 \leqslant K^2 \max_m \frac{\sigma_m^4}{v_m^2} \leqslant K^2 \max_m \frac{\mu_{4,m}}{p_m v_m^2}.$$

Now the proof is obvious. □

Proof of Theorem 7.4.2*. We use triangle inequality, Theorem 7.4.2, (7.27) and (7.28), so that

$$\begin{aligned} n\left\|H^{*n}*B\right\|_{TV} &\leqslant \frac{n}{24}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\left\|H^{*n}*(U_m*\bar{U}_m)^{*2}\right\|_{TV} \\ &\leqslant \frac{64}{24\mathrm{e}^2 n}\sum_{m=1}^{K}\frac{\mu_{4,m}-\sigma_m^2}{p_m\sigma_m^4}. \end{aligned}$$

and

$$\frac{n}{2}\left\|H^{*n/2}*V\right\|_{TV}^2 \leqslant \frac{n}{2}\left(\sum_{m=1}^{K} p_m\sigma_m^2\left\|H^{*n/2}*U_m*\bar{U}_m\right\|_{TV}\right)^2 \leqslant \frac{32K^2}{\mathrm{e}^2 n}.$$

□

Proof of Theorems 7.4.4, 7.4.5. Observe that

$$\begin{aligned} |\widehat{F}^n-\widehat{H}^n| &\leqslant \sum_{j=0}^{n-1}\left|\widehat{F}^j(\widehat{F}-\widehat{H})\widehat{H}^{n-1-j}\right| \\ &\leqslant n|\widehat{F}-\widehat{H}|\max(|\widehat{F}|^{n-1},|\widehat{H}|^{n-1}) \\ &\leqslant n|\widehat{F}|^{n-1}|\widehat{F}-\widehat{H}|+n|\widehat{H}|^{n-1}|\widehat{F}-\widehat{H}|. \end{aligned}$$

From (4.29) it follows that

$$|\widehat{F}| \leqslant \exp\left\{q(\widehat{F}-1)\right\} = \exp\left\{q\sum_{m=1}^{K} p_m(\widehat{F}_m-1)\right\}.$$

Let $n \geqslant 20$. It is easy to check that for any finite measure $|\widehat{M}(t)| \leqslant \|M\|_{TV}$. Therefore, taking into account (7.10) and (7.25), we obtain

$$\begin{aligned} n|\widehat{F}|^{n-1}|\widehat{F}-\widehat{H}| &\leqslant |\widehat{F}|^{\lfloor n/4\rfloor}\exp\left\{\frac{nq(\widehat{F}-1)}{2}\right\}|\widehat{F}-\widehat{H}| \\ &\leqslant \exp\left\{\frac{nq(\widehat{F}-1)}{4}\right\}\left\|F^{*\lfloor n/4\rfloor}*\exp\left\{\frac{nq(F-I)}{4}\right\}*(F-H)\right\|_{TV} \\ &\leqslant \prod_{m=1}^{d}\exp\left\{\frac{nqp_m(\widehat{F}_m(t_m)-1)}{4}\right\} \\ &\quad\times\left\|F^{*\lfloor n/4\rfloor}*W^{*nq/4}*(F-H)\right\|_{TV}. \end{aligned} \tag{7.63}$$

Applying (7.31) we get

$$\begin{aligned}
&\|W^{*(nq/4)} * F^{*\lfloor n/4\rfloor} * (F-H)\|_{TV} \\
&\quad\leqslant \left\| W^{*(nq/4)} * F^{*\lfloor n/4\rfloor} * \frac{1}{8}\left(\sum_{m=1}^{K} p_m\sigma_m^2 U_m * \bar{U}_m\right)^{*2}\right\|_{TV} \\
&\qquad + \left\| W^{*(nq/4)} * F^{*\lfloor n/4\rfloor} * \frac{1}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)(U_m * \bar{U}_m)^{*2}\right\|_{TV} \\
&\quad\leqslant \frac{1}{8}\left(\sum_{m=1}^{K} p_m\sigma_m^2 \|W^{*(nq/8)} * F^{*\lfloor n/16\rfloor} * U_m * \bar{U}_m\|_{TV}\right)^2 \\
&\qquad + \frac{1}{4!}\sum_{m=1}^{K} p_m(\mu_{4,m}-\sigma_m^2)\|W^{*(nq/4)} * F^{*\lfloor n/4\rfloor} * (U_m * \bar{U}_m)^{*2}\|_{TV}.
\end{aligned}$$

From (A.28), Lemma 7.5.4 and (7.34) it follows that

$$\begin{aligned}
\|W^{*(nq/8)} * F^{*\lfloor n/16\rfloor} * U_m * \bar{U}_m\|_{TV} &\leqslant \|W^{*(nq/8)}\|_{TV}\|F^{*\lfloor n/16\rfloor}\|_{TV} \\
&\quad \times \|U_m\|_{TV}\|\bar{U}_m\|_{TV} \leqslant 4, \\
\|W^{*(nq/8)} * F^{*\lfloor n/16\rfloor} * U_m * \bar{U}_m\|_{TV} &\leqslant \|F^{*\lfloor n/16\rfloor} * U_m * \bar{U}_m\|_{TV} \leqslant \frac{C}{np_m v_m}, \\
\|W^{*(nq/8)} * F^{*\lfloor n/16\rfloor} * U_m * \bar{U}_m\|_{TV} &\leqslant \|W^{*(nq/8)}\|_{TV}\|U_m\|_{TV}\|\bar{U}_m\|_{TV} \\
&\leqslant \frac{C}{nq\,q_{1,m}p_m}.
\end{aligned}$$

Similarly estimating expressions with U^{*2}, we prove that

$$\begin{aligned}
&\|W^{*(nq/4)} * F^{*\lfloor n/4\rfloor} * (F-H)\|_{TV} \\
&\quad\leqslant C\left(\sum_{m=1}^{K}\frac{p_m\sigma_m^2}{np_m}\min\left(np_m, \frac{1}{qq_{1,m}}, \frac{1}{v_m}\right)\right)^2 \\
&\qquad + C\sum_{m=1}^{K} p_m\frac{(\mu_{4,m}-\sigma_m^2)}{n^2p_m^2}\min\left(n^2p_m^2, \frac{1}{q^2q_{1,m}^2}, \frac{1}{v_m^2}\right) \\
&\quad\leqslant \frac{C\Delta_0}{n^2}.
\end{aligned} \tag{7.64}$$

We have

$$n|\widehat{H}|^{n-1}|\widehat{F}-\widehat{H}| \leqslant n|\widehat{H}|^{n/3}\left\|H^{*(n/2)} * (F-H)\right\|_{TV}.$$

From Lemma 7.5.5 and (7.31) it follows that

$$\begin{aligned}\left\|H^{*(n/2)} * (F-H)\right\|_{TV} &\leqslant C\left(\sum_{m=1}^{K} p_m\sigma_m^2\|H^{*(n/4)} * U_m{*}\bar{U}_m\|_{TV}\right)^2 \\ &\quad + C\sum_{m=1}^{K}(\mu_{4,m}-\sigma_m^2)p_m\|H^{*(n/3)} * (U_m{*}\bar{U}_m)^{*2}\|_{TV} \\ &\leqslant \frac{C}{n^2}\left(\sum_{m=1}^{K}\sigma_m^2\min\left(np_m;\frac{1}{\sigma_m^2}\right)\right)^2 \\ &\leqslant \frac{C}{n^2}\sum_{m=1}^{K}\frac{\mu_{4,m}-\sigma_m^2}{p_m}\min\left(n^2p_m^2,\frac{1}{\sigma_m^4}\right).\end{aligned}$$

By definition

$$\sigma^2 m \geqslant q_{1,m} \geqslant q\,q_{1,m}; \quad \sigma^2 \geqslant 1 \geqslant v_m.$$

Therefore

$$\frac{1}{\sigma_m^2} \leqslant \min\left(\frac{1}{v_m};\frac{1}{q\,q_{1,m}}\right)$$

and

$$\left\|H^{*(n/2)} * (F-H)\right\|_{TV} \leqslant \frac{C\Delta_0}{n^2}. \tag{7.65}$$

We recall that support of F_m does not contain zero. Therefore, applying (A.7) and (A.6) we get

$$\int_{-\pi}^{\pi}\exp\left\{\frac{nqp_m(\widehat{F}_m(t_m)-1)}{4}\right\}\mathrm{d}t_m \leqslant \frac{C}{\sqrt{nqp_m}}.$$

Direct integration allows to prove that

$$\int_{-\pi}^{\pi}\exp\left\{-\frac{2}{3}np_m\sigma_m^2\sin^2(t_m/2)\right\}\mathrm{d}t_m \leqslant \frac{C}{\sqrt{n\sigma^2}} \leqslant \frac{C}{\sqrt{n}} \leqslant \frac{C}{\sqrt{nqp_m}}.$$

Formula of inversion and estimates (7.63), (7.64) and (7.65) allows to write

$$\begin{aligned}\|F^{*n}-H^{*n}\|_\infty &\leqslant \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}|\widehat{F}^n-\widehat{H}^n|\,\mathrm{d}t_1\dots\mathrm{d}t_d \\ &\leqslant \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}n|\widehat{F}|^{n-1}|\widehat{F}-\widehat{H}|\,\mathrm{d}t_1\dots\mathrm{d}t_d \\ &\quad + \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}n|\widehat{H}|^{n-1}|\widehat{F}-\widehat{H}|\,\mathrm{d}t_1\dots\mathrm{d}t_d\end{aligned}$$

$$\leqslant \frac{C\Delta_0}{n}\left(\prod_{m=1}^{d}\int_{-\pi}^{\pi}\exp\left\{\frac{nqp_m(\widehat{F}_m(t_m)-1)}{4}\right\}\mathrm{d}t_m + \prod_{m=1}^{d}\int_{-\pi}^{\pi}\exp\left\{-\frac{2np_m\sigma_m^2\sin^2(t_m/2)}{3}\right\}\mathrm{d}t_m\right)$$

$$\leqslant \frac{C\Delta_0}{n}\prod_{m=1}^{d}\left(\max(1;\sqrt{nqp_m})\right)^{-1}. \tag{7.66}$$

Theorem is proved for $n \geqslant 20$. If $n \leqslant 20$, then by (A.28) and (7.31)

$$\begin{aligned}\|F^{*n}-H^{*n}\|_{\infty} &\leqslant \|F^{*n}-H^{*n}\|_{TV} \leqslant 20\|F-H\|_{TV}\\ &\leqslant \frac{C(d\|F-H\|_{TV})}{n^{d/2+1}} \leqslant \frac{C(d)}{n}\prod_{m=1}^{d}\frac{1}{\sqrt{nqp_m}}\|F-H\|_{TV}\\ &\leqslant \frac{C(d)}{n}\prod_{m=1}^{d}\frac{1}{\sqrt{nqp_m}}\left(\left(\sum_{m=1}^{K}p_m\sigma_m^2\right)^2+\sum_{m=1}^{K}(\mu_{4,m}-\sigma_m^2)p_m\right). \end{aligned}\tag{7.67}$$

Observe that $v_m \leqslant 1$ and $qq_{1,m} \leqslant 1$. Therefore

$$p_m \leqslant \min\left(np_m;\frac{1}{v_m};\frac{1}{qq_{1,m}}\right).$$

Substituting the last estimate into (7.67) we prove (7.11) for $n \leqslant 20$. Theorem 7.4.4 is proved.

Proof of Theorem 7.4.5 follows from (7.13), (7.7) and estimate

$$\sum_{\vec{x}}|F^{*n}\{\vec{x}\}-H^{*n}\{\vec{x}\}|^r \leqslant \|F^{*n}-H^{*n}\|_{\infty}^{r-1}\|F^{*n}-H^{*n}\|_{TV}.$$

□

8 COMPOUND POISSON APPROXIMATION FOR A SEQUENCE OF TWO-DIMENSIONAL LATTICE VECTORS

In Chapter 2, it was noted that the best general infinitely divisible approximation proved by Presman [64] is of the order $O(n^{-1/3})$, see (2.2). Meanwhile, the best one-dimensional infinitely divisible approximation is of the order $O(n^{-2/3})$, see (2.1). Presman used accompanying distribution and Kolmogorov metric. As proved by Zaitsev [88], in general, uniform metric can not be replaced by stronger total variation metric. However, for lattice one-dimensional distributions having finite 4'th moments total variation metric can be used, see [9], Theorem 4.3, or [30] Theorem 7.5.

Accompanying CP distributions proved to be sufficiently accurate approximations for symmetric vectors, see [87]. Accompanying distributions also play a crucial role in the second uniform Kolmogorov theorem, when convolutions of distributions mostly concentrated on finite intervals are estimated in Lévy and Lévy-Prokhorov metrics, see [43, 86] and the references therein. In total variation accompanying CP approximations applied to non-symmetric distribution can be effective only if we consider a scheme of triangular arrays with growing probability mass at zero, see [10, 70, 78]. Particularly, estimates are of the trivial order of accuracy $O(1)$ for the sequence of r.v.s. Therefore, to ensure better accuracy, a more structurally complex CP approximation is needed.

In this chapter, for a sequence of two-dimensional non-symmetric lattice r.v.s. with finite $4th$ moments, we construct CP approximation matching the first three moments of the initial distribution. The accuracy of approximation in total variation is of the order $O(n^{-1})$.

8.1 Known results

If we consider not the general case of triangular arrays, but just a sequence of random variables, then some improved orders of approximation are available. In short, for triangular arrays all probabilities of X can depend on n, meanwhile in a scheme of sequences, n is just the number of summands and probabilities of X do not depend on n. This allows to simplify all proofs.

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. integer valued random variables, having finite fourth moments. Let $\mathscr{L}(X) = F \in \mathscr{F}_1$ and let there exist

$m<k\in\mathbb{Z}_+$ such that $F\{m\}F\{k\}>0$ Let $\gamma=\mathbb{E}X+\upsilon\in\mathbb{Z}$, $\upsilon\in[0,1)$ and

$$\widehat{G}_0(t)\,=\,\exp\Biggl\{\mathrm{i}t\frac{\gamma}{n}-\frac{\upsilon}{n(k-m)}(\mathrm{e}^{\mathrm{i}t(k-m)}-1)+\frac{\upsilon}{6n(k-m)}(\mathrm{e}^{2\mathrm{i}t(k-m)}-1)$$
$$+\frac{\upsilon}{3n(k-m)}(\mathrm{e}^{-\mathrm{i}t(k-m)}-1)+\frac{1}{6}(\widehat{F}(2t)\widehat{F}^2(-t)-1)\Biggr\}. \qquad (8.1)$$

If $n\geqslant 6(F\{k\})^{-2}(F\{m\})^{-1}$, then

$$\|F^{*n}-G_0^{*n}\|_{TV}\leqslant\frac{C(F)}{n};\quad \|F^{*n}-G_0^{*n}\|_{\infty}\leqslant\frac{C(F)}{n\sqrt{n}}, \qquad (8.2)$$

see [25], Theorem 7 and [30] Theorem 11.4, respectively.

It is obvious, that (8.2) gives us the following version of Kolmogorov's theorem for sequences of random variables: if assumptions of (8.2) are met, then there exists CP distribution D such that $\|F^{*n}-D\|_{TV}\leqslant C(F)n^{-1}$. Indeed, if $n\geqslant 6(F\{k\})^{-2}(F\{m\})^{-1}$, then we can take $D=G_0^{*n}$ and use (8.2). If $n<6(F\{k\})^{-2}(F\{m\})^{-1}$, then we can use any CP distribution D. Indeed, taking into account that total variation of any distribution equals 1, then we write

$$\|F^{*n}-D\|_{TV}\leqslant\|F^{*n}\|_{TV}+\|D\|_{TV}=2\leqslant\frac{12}{n(F\{k\})^2F\{m\}}=\frac{C(F)}{n}.$$

The aim of this chapter is to prove analogue of (8.2) for two-dimensional vectors.

8.2 Results

Let $\vec{X},\vec{X}_1,\vec{X}_2,\ldots$ be a sequence of i.i.d. two-dimensional random vectors concentrated on a subset of $\mathbb{Z}^2$. Set $\vec{X}=(X_1,X_2)$ and $\mathscr{L}(\vec{X})=F\in\mathscr{F}_2$, $F\{(j,l)\}=\mathbb{P}(X_1=j,X_2=l)$. Let

$$F\{(0,0)\}>0; \qquad (8.3)$$
$$F\{(1,0)\}+F\{(-1,0)\}>0,\quad F\{(0,1)\}+F\{(0,-1)\}>0; \qquad (8.4)$$
$$\mathbb{E}X_1^4+\mathbb{E}X_2^4<\infty. \qquad (8.5)$$

Conditions (8.3) and (8.5) are essential for the proofs. Condition (8.4) can be replaced by assumption that there exist k and m such that $F\{(k,0)\}>0$, $F\{(k+1,0)\}>0$, $F\{(0,m)\}>0$, $F\{(0,m+1)\}>0$, but then the proofs be-

come significantly longer.

Let $\vec{S}_n := \vec{X}_1 + \ldots + \vec{X}_n$, so that $\mathscr{L}(\vec{S}_n) = F^{*n}$. Observe that $\vec{S}_n$ takes values from $\mathbb{Z}^2$, therefore, CP approximation also must be concentrated on $\mathbb{Z}^2$. Let

$$\delta_1 := n\mathbb{E}X_1 - \lfloor n\mathbb{E}X_1 \rfloor \in [0,1), \quad \delta_2 := n\mathbb{E}X_2 - \lfloor n\mathbb{E}X_2 \rfloor \in [0,1).$$

We use CP approximation with the characteristic function:

$$\begin{aligned} \widehat{D}_n(\vec{t}) &= \exp\Big\{ \mathrm{i}t_1 \lfloor \mathbb{E}X_1 \rfloor + \mathrm{i}t_2 \lfloor \mathbb{E}X_2 \rfloor + \delta_1(\mathrm{e}^{\mathrm{i}t_1} - 1) + \delta_2(\mathrm{e}^{\mathrm{i}t_2} - 1) \\ &\quad + \frac{n}{6}(\widehat{F}(2\vec{t}\,)\widehat{F}^2(-\vec{t}\,) - 1)\Big\}. \end{aligned} \tag{8.6}$$

Observe that (8.6) has slightly simpler structure than (8.1), though both approximations contain the same expression $\widehat{F}(2\vec{t}\,)\widehat{F}^2(-\vec{t}\,) - 1$, the usage of which is directly inspired by Arak's solution to the first Kolmogorov problem. In terms of random vectors, $\widehat{F}(2\vec{t}\,)\widehat{F}^2(-\vec{t}\,)$ is the characteristic function of $2\vec{X}_1 - \vec{X}_2 - \vec{X}_3$ where, as before, r.v.s. $\vec{X}_1, \vec{X}_2, \vec{X}_3$ are independent and distributed as $\vec{X}$. In terms of exponential measures we can write

$$\begin{aligned} D_n &= I_{(1,0)}^{*\lfloor n\mathbb{E}X_1 \rfloor} * \exp\{\delta_1(I_{(1,0)} - I)\} * I_{(0,1)}^{*\lfloor n\mathbb{E}X_2 \rfloor} * \exp\{\delta_2(I_{(0,1)} - I)\} \\ &\quad * \exp\Big\{\frac{n}{6}(\mathscr{L}(2\vec{X}_1 - \vec{X}_2 - \vec{X}_3) - I)\Big\}. \end{aligned}$$

Theorem 8.2.1. *Let assumptions (8.3)–(8.5) hold, $n \in \mathbb{N}$. Then there exists $C_1(F)$ such that*

$$\|F^{*n} - D_n\|_{TV} \leqslant \frac{C_1(F)}{n}. \tag{8.7}$$

Local estimates typically have additional factor $n^{-d/2}$, where d is the dimension of lattice, see [48]. The same rule applies for D_n.

Theorem 8.2.2. *Let assumptions (8.3)–(8.5) hold, $n \in \mathbb{N}$. Then there exists $C_2(F)$ such that*

$$\|F^{*n} - D_n\|_{\infty} \leqslant \frac{C_2(F)}{n^2}. \tag{8.8}$$

Theorem 8.2.2 can be reformulated to include nonuniform estimate.

Theorem 8.2.3. *Let assumptions (8.3)–(8.5) hold, $n, m, k \in \mathbb{N}$. Then there exists $C_3(F)$ such that*

$$|F^{*n}\{(k,m)\} - D_n\{(k,m)\}| \leqslant \frac{C_3(F)}{n^2}\left(1 + \frac{|k - \mathbb{E}X_1||m - \mathbb{E}X_2|}{n}\right)^{-1}.$$

We can combine Theorems 8.2.1 and 8.2.2 for estimate in ℓ_r metric.

Theorem 8.2.4. *Let assumptions (8.3)–(8.5) hold, $n \in \mathbb{N}$, $r \geqslant 1$. Then there exists $C(F)$ such that*

$$\|F^{*n} - D_n\|_r \leqslant \frac{C_4(F)}{n^{2-1/r}}. \tag{8.9}$$

For the proofs, we use the characteristic function method. Our conjecture is that (8.9) can be extended to the d - dimensional case; however, some other methods of proof might be needed.

8.3 Proofs

To make all proofs shorter, we introduce auxiliary notation. Let $\vec{t} = (t_1, t_2)$, $\vec{X} = (X_1, X_2)$, $a := X_1 t_1 + X_2 t_2$. Then $\mathbb{E}a = t_1 \mathbb{E}X_1 + t_2 \mathbb{E}X_2$ and $\widehat{F}(\vec{t}) = \mathbb{E}e^{ia}$. Set

$$\begin{aligned}
f &:= e^{-i\mathbb{E}a}\widehat{F}(\vec{t}), \quad \delta := \delta_1(e^{it_1} - 1 - it_1) + \delta_2(e^{it_2} - 1 - it_2),\\
b &:= \frac{1}{6}(\widehat{F}(2\vec{t}\,)\widehat{F}^2(-\vec{t}\,) - 1),\\
v &:= (\widehat{F}(\vec{t}) - 1 - i\mathbb{E}a) - \frac{(\widehat{F}(\vec{t}) - 1)^2}{2} + \frac{(\widehat{F}(\vec{t}) - 1)^3}{3},\\
\widehat{M} &:= \widehat{F}(\vec{t}) - \exp\left\{\sum_{j=1}^{3} \frac{(-1)^{j+1}(\widehat{F}(\vec{t}) - 1)^j}{j}\right\},\\
\beta(\vec{t}) &:= \exp\{-C_5(F)t_1^2 - C_6(F)t_2^2\},\\
C_5(F) &= 2F\{(0,0)\}(F\{(1,0)\} + F\{(-1,0)\})/9\pi^2,\\
C_6(F) &= 2F\{(0,0)\}(F\{(0,1)\} + F\{(0,-1)\})/9\pi^2.
\end{aligned}$$

Our goal is to estimate

$$\Delta(\vec{t}) := e^{-in\mathbb{E}a}\widehat{F}^n(\vec{t}) - e^{-in\mathbb{E}a}\widehat{D}_n(\vec{t}) = f^n - e^{nb+\delta}.$$

First, we prove some auxiliary results. We recall that θ is used for all quantities satisfying $|\theta| \leqslant 1$. Note also that $|e^{i\varkappa}| = 1$, for any real $\varkappa$, and from the general properties of characteristic functions it follows that $|\widehat{F}(\vec{t})| \leqslant 1$.

Lemma 8.3.1. *Let assumptions (8.3)–(8.5) hold and let $|t_1|, |t_2| \leqslant \pi$. Then*

$$\begin{aligned}
\widehat{F}(\vec{t}) &= 1 + i\mathbb{E}a + \frac{i^2\mathbb{E}a^2}{2} + \frac{i^3\mathbb{E}a^3}{6} + \theta C(F)(t_1^4 + t_2^4),\\
|\widehat{F}(\vec{t}) - 1| &\leqslant C(F)(|t_1| + |t_2|),
\end{aligned}$$

$$
\begin{aligned}
\widehat{F}(-\vec{t}) &= 1-\mathrm{i}\mathbb{E}a+\frac{\mathrm{i}^2\mathbb{E}a^2}{2}-\frac{\mathrm{i}^3\mathbb{E}a^3}{6}+\theta C(F)(t_1^4+t_2^4),\\
|\widehat{F}(-\vec{t})-1| &\leqslant C(F)(|t_1|+|t_2|),\\
\widehat{F}(2\vec{t}) &= 1+2\mathrm{i}\mathbb{E}a+2\mathrm{i}^2\mathbb{E}a^2+\frac{4\mathrm{i}^3\mathbb{E}a^3}{3}+\theta C(F)(t_1^4+t_2^4),\\
|\widehat{F}(2\vec{t})-1| &\leqslant C(F)(|t_1|+|t_2|).
\end{aligned}
$$

For $j=1,2$

$$
\begin{aligned}
\frac{\partial\widehat{F}(\vec{t})}{\partial t_j} &= \mathrm{i}\mathbb{E}X_j+\mathrm{i}^2\mathbb{E}X_ja+\frac{\mathrm{i}^3\mathbb{E}X_ja^2}{2}+\theta C(F)(|t_1|^3+|t_2|^3),\\
\frac{\partial\widehat{F}(-\vec{t})}{\partial t_j} &= -\mathrm{i}\mathbb{E}X_j+\mathrm{i}^2\mathbb{E}X_ja-\frac{\mathrm{i}^3\mathbb{E}X_ja^2}{2}+\theta C(F)(|t_1|^3+|t_2|^3),\\
\frac{\partial\widehat{F}(2\vec{t})}{\partial t_j} &= 2\mathrm{i}\mathbb{E}X_j+4\mathrm{i}^2\mathbb{E}X_ja+4\mathrm{i}^3\mathbb{E}X_ja^2+\theta C(F)(|t_1|^3+|t_2|^3)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2\widehat{F}(\vec{t})}{\partial t_1\partial t_2} &= \mathrm{i}^2\mathbb{E}X_1X_2+\mathrm{i}^3\mathbb{E}X_1X_2a+\theta C(F)(t_1^2+t_2^2),\\
\frac{\partial^2\widehat{F}(-\vec{t})}{\partial t_1\partial t_2} &= \mathrm{i}^2\mathbb{E}X_1X_2-\mathrm{i}^3\mathbb{E}X_1X_2a+\theta C(F)(t_1^2+t_2^2),\\
\frac{\partial^2\widehat{F}(2\vec{t})}{\partial t_1\partial t_2} &= 4\mathrm{i}^2\mathbb{E}X_1X_2+8\mathrm{i}^3\mathbb{E}X_1X_2a+\theta C(F)(t_1^2+t_2^2).
\end{aligned}
$$

Proof. It is well known that for any real number $\varkappa$

$$
\mathrm{e}^{\mathrm{i}\varkappa}=1+\mathrm{i}\varkappa+\frac{\mathrm{i}^2\varkappa^2}{2}+\frac{\mathrm{i}^3\varkappa^3}{6}+\theta\frac{|\varkappa|}{24}. \tag{8.10}
$$

We have

$$
\widehat{F}(\vec{t})=\mathbb{E}\mathrm{e}^{\mathrm{i}a},\quad \frac{\partial\widehat{F}(\vec{t})}{\partial t_j}=\mathrm{i}\mathbb{E}X_j\mathrm{e}^{\mathrm{i}a},\quad \frac{\partial^2\widehat{F}(\vec{t})}{\partial t_1\partial t_2}=\mathrm{i}^2\mathbb{E}X_1X_2\mathrm{e}^{\mathrm{i}a},
$$

etc. It remains to use (8.10) and observe that

$$
|a|^2\leqslant(X_1^2+X_2^2)(t_1^2+t_2^2),\quad (t_1^2+t_2^2)^2\leqslant 4\max(t_1^4,t_2^4)\leqslant 4(t_1^4+t_2^4).
$$

Similarly we can estimate other quantities containing t_1 and t_2. For example,

$$
(|t_1|+|t_2|)(t_1^2+t_2^2)\leqslant 4\max(|t_1|^3,|t_2|^3)\leqslant 4(|t_1|^3+|t_2|^3).
$$

Observe also that $|t_1|^4 \leqslant \pi |t_1|^3$, etc. $\square$

Similar estimates hold for v and b, δ and f.

Lemma 8.3.2. *Let assumptions (8.3)–(8.5) hold, $j = 1,2$ and $|t_1|, |t_2| \leqslant \pi$. Then*

$$
\begin{aligned}
&|v| \leqslant C(F)(t_1^2 + t_2^2), \quad \Big|\frac{\partial v}{\partial t_j}\Big| \leqslant C(F)(|t_1| + |t_2|), \quad \Big|\frac{\partial^2 v}{\partial t_1 \partial_2}\Big| \leqslant C(F),\\
&|b| \leqslant C(F)(t_1^2 + t_2^2), \quad \Big|\frac{\partial b}{\partial t_j}\Big| \leqslant C(F)(|t_1| + |t_2|), \quad \Big|\frac{\partial^2 b}{\partial t_1 \partial_2}\Big| \leqslant C(F),\\
&|v-b| \leqslant C(F)(t_1^4 + t_2^4), \quad \Big|\frac{\partial (v-b)}{\partial t_j}\Big| \leqslant C(F)(|t_1|^3 + |t_2|^3),\\
&\Big|\frac{\partial^2 (v-b)}{\partial t_1 \partial t_2}\Big| \leqslant C(F)(t_1^2 + t_2^2),\\
&|\delta| \leqslant t_1^2 + t_2^2, \quad \Big|\frac{\partial \delta}{\partial t_j}\Big| \leqslant |t_1| + |t_2|, \quad \Big|\frac{\partial^2 \delta}{\partial t_1 \partial t_2}\Big| = 0,\\
&|f| = |\widehat{F}(\vec{t})| \leqslant 1, \quad \Big|\frac{\partial f}{\partial t_j}\Big| \leqslant C(F)(|t_1| + |t_2|), \quad \Big|\frac{\partial^2 f}{\partial t_1 \partial t_2}\Big| \leqslant C(F).
\end{aligned}
$$

Proof. Relations for δ follow from (8.10). For other relations it suffices to use Lemma 8.3.1 and collect remainder terms. $\square$

Next we estimate $\widehat{M}$.

Lemma 8.3.3. *Let assumptions (8.3)–(8.5) hold, $j = 1,2$ and $|t_1|, |t_2| \leqslant \pi$. Then*

$$
\begin{aligned}
&|\widehat{M}| \leqslant C(F)(t_1^4 + t_2^4),\\
&\Big|\frac{\partial \widehat{M}}{\partial t_j}\Big| \leqslant C(F)(|t_1|^3 + |t_2|^3), \quad \Big|\frac{\partial^2 \widehat{M}}{\partial t_1 \partial t_2}\Big| \leqslant C(F)(t_1^2 + t_2^2).
\end{aligned}
$$

Proof. By Lemma 4 [48]

$$
\widehat{M} = -(\widehat{F}(\vec{t}) - 1)^4 \int_0^1 \exp\Big\{ \sum_{j=1}^{3} (\widehat{F}(\vec{t}) - 1)^j (-1)^{j+1} (1 - \tau^j)/j \Big\} \tau^3 \mathrm{d}\tau.
$$

It remains to use Lemma 8.3.1. $\square$

Lemma 8.3.4. *Let assumptions (8.3)–(8.5) hold, $j=1,2$ and $|t_1|,|t_2|\leqslant\pi$. Then*

$$|f-\mathrm{e}^b|\leqslant C(F)(t_1^4+t_2^4),$$
$$\left|\frac{\partial(f-\mathrm{e}^b)}{\partial t_j}\right|\leqslant C(F)(|t_1|^3+|t_2|^3),\quad \left|\frac{\partial^2(f-\mathrm{e}^b)}{\partial t_1\partial t_2}\right|\leqslant C(F)(t_1^2+t_2^2).$$

Proof. Observe that

$$f-\mathrm{e}^b=\mathrm{e}^{\mathrm{i}\mathbb{E}a}\widehat{M}+(\mathrm{e}^v-\mathrm{e}^b).$$

To estimate $\mathrm{e}^{\mathrm{i}\mathbb{E}a}\widehat{M}$ and its derivatives we use Lemma 8.3.3. For instance,

$$\begin{aligned}\left|\frac{\partial \mathrm{e}^{\mathrm{i}\mathbb{E}a}\widehat{M}}{\partial t_1}\right| &\leqslant \left|\frac{\partial \mathrm{e}^{\mathrm{i}\mathbb{E}a}}{\partial t_1}\right||\widehat{M}|+\left|\mathrm{e}^{\mathrm{i}\mathbb{E}a}\frac{\partial\widehat{M}}{\partial t_1}\right|=|\mathbb{E}X_1||\widehat{M}|+\left|\frac{\partial\widehat{M}}{\partial t_1}\right|\\ &\leqslant C(F)(|t_1|^3+|t_2|^3).\end{aligned}$$

To estimate $\mathrm{e}^v-\mathrm{e}^b$ we use expression

$$\mathrm{e}^v-\mathrm{e}^b=\mathrm{e}^b(\mathrm{e}^{v-b}-1)=\mathrm{e}^b\int_0^1\left(\mathrm{e}^{\tau(v-b)}\right)'_\tau\mathrm{d}\tau=(v-b)\int_0^1\mathrm{e}^{\tau v+(1-\tau)b}\mathrm{d}\tau.$$

Then we can use Lemma 8.3.2. For instance,

$$\begin{aligned}\left|\frac{\partial(\mathrm{e}^v-\mathrm{e}^b)}{\partial t_1}\right| &\leqslant \left|\frac{\partial(v-b)}{\partial t_1}\right|\int_0^1\mathrm{e}^{\tau|v|+(1-\tau)|b|}\mathrm{d}\tau\\ &\quad+|v-b|\int_0^1\mathrm{e}^{\tau|v|+(1-\tau)|b|}\left(\tau\left|\frac{\partial v}{\partial t_1}\right|+(1-\tau)\left|\frac{\partial b}{\partial t_1}\right|\right)\mathrm{d}\tau\\ &\leqslant C(F)(|t_1|^3+|t_2|^3).\end{aligned}$$

Here we also used estimates $|v|\leqslant C(F)(t_1^2+t_2^2)\leqslant 2C(F)\pi^2$, $|b|\leqslant C(F)$. All other proofs are very similar and, therefore, omitted. □

Lemma 8.3.5. *Let assumptions (8.3)–(8.4) hold and $|t_1|,|t_2|\leqslant\pi$. Then*

$$|f|\leqslant\beta^3(\vec{t});\quad |\mathrm{e}^b|\leqslant\beta^3(\vec{t}).$$

Proof. Essentially the proof repeats the proof of Lemma 6 from [48], see also (D.6) in [30]. We have $\widehat{F}(\vec{t})=q+(1-q)\widehat{V}(\vec{t})$, where $q=F\{(0,0)\}$ and

$$(1-q)\widehat{V}(\vec{t})=\sum_{(m,j)\neq(0,0)}F\{(m,j)\}\mathrm{e}^{\mathrm{i}(mt_1+jt_2)}.$$

Let $Re\widehat{V}(\vec{t})$ and $Im\widehat{V}(\vec{t})$ denote the real and imaginary parts of $\widehat{V}(\vec{t})$. Then $(Re\widehat{V}(\vec{t}))^2+(Im\widehat{V}(\vec{t}))^2=|\widehat{V}(\vec{t})|^2\leqslant 1$ and

$$\begin{aligned}|\widehat{F}(\vec{t})|^2 &= \left(q+(1-q)Re\widehat{V}(\vec{t})\right)^2+(1-q)^2\left(Im\widehat{V}(\vec{t})\right)^2\\ &= q^2+2q(1-q)Re\widehat{V}(\vec{t})+(1-q)^2\left((Re\widehat{V}(\vec{t}))^2+(Im\widehat{V}(\vec{t}))^2\right)\\ &\leqslant q^2+2q(1-q)Re\widehat{V}(\vec{t})+(1-q)^2\\ &= 1+2q(1-q)(Re\widehat{V}(\vec{t})-1)=1+2q(Re\widehat{F}(\vec{t})-1).\end{aligned}$$

Observe that

$$\begin{aligned}Re\widehat{F}(\vec{t})-1 &= -2\sum_{(m,j)\neq(0,0)}F\{(m,j)\}\sin^2\frac{t_1m+t_2j}{2}\\ &\leqslant -2(F\{(1,0)\}+F\{(-1,0)\})\sin^2\frac{t_1^2}{2}\\ &\quad -2(F\{(0,1)\}+F\{(0,-1)\})\sin^2\frac{t_2}{2}\\ &\leqslant -\frac{2}{\pi^2}(F\{(1,0)\}+F\{(-1,0)\})t_1^2\\ &\quad -\frac{2}{\pi^2}(F\{(0,1)\}+F\{(0,-1)\})t_2^2.\end{aligned}$$

Here we used inequality $|\sin(t/2)|\geqslant|t|/\pi$ valid for $|t|\leqslant\pi$. Therefore,

$$\begin{aligned}|\widehat{F}(\vec{t})|^2 &\leqslant 1-\frac{4}{\pi^2}F\{(0,0)\}(F\{(1,0)\}+F\{(-1,0)\})t_1^2\\ &\quad -\frac{4}{\pi^2}F\{(0,0)\}(F\{(0,1)\}+F\{(0,-1)\})t_2^2\\ &\leqslant \exp\{-18C_5(F)t_1^2-18C_6(F)t_2^2\}.\end{aligned}\tag{8.11}$$

Therefore

$$|\widehat{F}(\vec{t})|\leqslant\beta^9(\vec{t})\leqslant\beta^3(\vec{t}).$$

From (8.11) it follows that

$$\begin{aligned}Reb &\leqslant \frac{1}{6}\left(|\widehat{F}(2\vec{t})||\widehat{F}(-\vec{t})|^2-1\right)\leqslant\frac{1}{6}\left(|\widehat{F}(-\vec{t})|^2-1\right)\\ &\leqslant -3C_5(F)t_1^2-3C_6(F)t_2^2\end{aligned}$$

and, therefore,

$$|\mathrm{e}^b| = \mathrm{e}^{Re\, b} \leqslant \beta^3(\vec{t}).$$

The last estimate completes the proof of theorem. □

Lemma 8.3.6. *Let assumptions (8.3)–(8.5) hold, $j = 1,2$ and $|t_1|, |t_2| \leqslant \pi$. Then*

$$|f^n - \mathrm{e}^{nb}| \leqslant \frac{C(F)\beta(\vec{t})}{n},$$
$$\left|\frac{\partial(f^n - \mathrm{e}^{nb})}{\partial t_j}\right| \leqslant \frac{C(F)\beta(\vec{t})}{\sqrt{n}}, \quad \left|\frac{\partial^2(f^n - \mathrm{e}^{nb})}{\partial t_1 \partial t_2}\right| \leqslant C(F)\beta(\vec{t}).$$

Proof. Observe that

$$\beta^{n-1}(\vec{t}) \leqslant \mathrm{e}^{4t_1^2/9\pi^2 + 4t_2^2/9\pi^2}\beta^n(\vec{t}) \leqslant \mathrm{e}^{8/9}\beta^n(\vec{t}). \tag{8.12}$$

Trivial estimate $x\mathrm{e}^{-x} \leqslant 1$, $x > 0$ allows to get estimates for $j = 1,2,3,4$

$$|t_1|^j \beta^n(\vec{t}) \leqslant \frac{j^{j/2}}{(C_5(F)n)^{j/2}}; \quad |t_2|^j \beta^n(\vec{t}) \leqslant \frac{j^{j/2}}{(C_6(F)n)^{j/2}}. \tag{8.13}$$

The proof of Lemma 8.3.6 now follows from Lemmas 8.3.2, 8.3.4, 8.3.5. Indeed,

$$\begin{aligned} |f^n - \mathrm{e}^{nb}| &\leqslant n\beta^{3(n-1)}(\vec{t})|f - \mathrm{e}^b| \leqslant nC(F)\beta^{3n}(\vec{t})(t_1^4 + t_2^4) \\ &\leqslant C(F)\beta^2(\vec{t})n^{-1} \leqslant C(F)\beta(\vec{t})n^{-1}. \end{aligned}$$

Similarly,

$$\begin{aligned} \left|\frac{\partial(f^n - \mathrm{e}^{nb})}{\partial t_j}\right| &\leqslant n\left|f^{n-1}\frac{\partial f}{\partial t_j} - \mathrm{e}^{(n-1)b}\frac{\partial \mathrm{e}^b}{\partial t_j}\right| \\ &\leqslant n\left|f^{n-1} - \mathrm{e}^{(n-1)b}\right|\left|\frac{\partial f}{\partial t_j}\right| + n\left|\mathrm{e}^{(n-1)v}\right|\left|\frac{\partial(f - \mathrm{e}^v)}{\partial t_j}\right| \\ &\leqslant C(F)\beta^{2n}(\vec{t})(|t_1| + |t_2|) + nC(F)\beta^{3n}(\vec{t})(|t_1|^3 + |t_2|^3) \\ &\leqslant C(F)\beta(\vec{t})n^{-1/2}. \end{aligned}$$

The proof of the estimate for the second-order derivatives is very similar and, therefore, omitted. □

Lemma 8.3.7. *Let assumptions (8.3)–(8.5) hold, $j=1,2$ and $|t_1|,|t_2|\leqslant\pi$. Then*

$$|\mathrm{e}^{nb+\delta}-\mathrm{e}^{nb}|\leqslant\frac{C(F)\beta^n(\vec{t})}{n},\qquad\left|\frac{\partial(\mathrm{e}^{nb+\delta}-\mathrm{e}^{nb})}{\partial t_j}\right|\leqslant\frac{C(F)\beta^n(\vec{t})}{\sqrt{n}},$$
$$\left|\frac{\partial^2(\mathrm{e}^{nb+\delta}-\mathrm{e}^{nb})}{\partial t_1\partial t_2}\right|\leqslant C(F)\beta^n(\vec{t}).$$

Proof. Taking into account Lemma 8.3.2, (8.12), (8.13) and Lemma 8.3.5, we get

$$\begin{aligned}\left|\mathrm{e}^{nb+\delta}-\mathrm{e}^{nb}\right| &\leqslant \left|\mathrm{e}^{nb}\right|\left|\mathrm{e}^{b}-1\right|\leqslant C(F)\beta^{3n}(\vec{t})|\delta|\leqslant C(F)\beta^{3n}(\vec{t})(t_1^2+t_2^2)\\ &\leqslant C(F)\beta^n n^{-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\frac{\partial\mathrm{e}^{nb}(\mathrm{e}^{\delta}-1)}{\partial t_j}\right| &\leqslant n\left|\mathrm{e}^{nb}\right|\left|\frac{\partial b}{\partial t_j}\right|\left|\mathrm{e}^{\delta}-1\right|+\left|\mathrm{e}^{nb}\right|\left|\mathrm{e}^{\delta}\right|\left|\frac{\partial\delta}{t_j}\right|\\ &\leqslant nC(F)\beta^{3n}(\vec{t})(|t_1|+|t_2|)(t_1^2+t_2^2)+C(F)\beta^{3n}(\vec{t})(|t_1|+|t_2|)\\ &\leqslant C(F)\beta^n(\vec{t})n^{-1/2}\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial^2\mathrm{e}^{nb}(\mathrm{e}^{\delta}-1)}{\partial t_1\partial t_2}\right| &\leqslant n^2\left|\mathrm{e}^{nb}\right|\left|\frac{\partial b}{\partial t_1}\right|\left|\frac{\partial b}{\partial t_2}\right|\left|\mathrm{e}^{\delta}-1\right|+n\left|\mathrm{e}^{nb}\right|\left|\frac{\partial^2 b}{\partial t_1\partial t_2}\right|\left|\mathrm{e}^{\delta}-1\right|\\ &\quad+n\left|\mathrm{e}^{nb}\right|\left|\frac{\partial b}{\partial t_1}\right|\left|\mathrm{e}^{\delta}\right|\left|\frac{\partial\delta}{\partial t_2}\right|+n\left|\mathrm{e}^{nb}\right|\left|\frac{\partial b}{\partial t_2}\right|\left|\mathrm{e}^{\delta}\right|\left|\frac{\partial\delta}{\partial t_1}\right|\\ &\quad+\left|\mathrm{e}^{nb}\right|\left|\mathrm{e}^{\delta}\right|\left|\frac{\partial\delta}{\partial t_1}\right|\left|\frac{\partial\delta}{\partial t_2}\right|+\left|\mathrm{e}^{nb}\right|\left|\mathrm{e}^{\delta}\right|\left|\frac{\partial^2\delta}{\partial t_1\partial t_2}\right|\\ &\leqslant C(F)\beta^n(\vec{t}).\end{aligned}$$

Lemma is proved. □

The triangle inequality and Lemmas 8.3.6, 8.3.7 allow to get estimates for $\Delta(\vec{t}\,)$.

Lemma 8.3.8. *Let assumptions (8.3)–(8.5) hold, $j=1,2$ and $|t_1|,|t_2|\leqslant\pi$.*

Then

$$\begin{aligned} |\Delta(\vec{t}\,)| &\leqslant \frac{C(F)\beta^n(\vec{t}\,)}{n}, \\ \left|\frac{\partial\Delta(\vec{t}\,)}{\partial t_j}\right| &\leqslant \frac{C(F)\beta^n(\vec{t}\,)}{\sqrt{n}}, \quad \left|\frac{\partial^2\Delta(\vec{t}\,)}{\partial t_1\partial t_2}\right| \leqslant C(F)\beta^n(\vec{t}\,). \end{aligned}$$

Next, we formulate formulas of inversion in two-dimensional case. We write $W \in \mathscr{M}_2^Z$, if $\|W\|_{TV} < \infty$, $\operatorname{supp} W \subset \mathbb{Z}^2$ and maximal span of the lattice is equal to 1; that is, there are no $j,k \in \mathbb{N}$ such that $j+k > 1$ and $\operatorname{supp} W \subset \{(j\mathbb{Z}, k\mathbb{Z})\}$. We also assume that W can not be concentrated on some one-dimensional subset of $\mathbb{Z}^2$.

Lemma 8.3.9. *Let $W \in \mathscr{M}_2^{\mathbb{Z}}$. Then*

$$\|W\|_\infty \leqslant \frac{1}{(2\pi)^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} |\widehat{W}(\vec{t}\,)|\,\mathrm{d}t_1\mathrm{d}t_2. \tag{8.14}$$

If, in addition, $\sum_{(k,m)\in\mathbb{Z}^2}(k^2+m^2)|W\{(k,m)\}| < \infty$, then for any $\alpha,\beta \in \mathbb{R}$

$$\begin{aligned} &|k-\alpha||m-\beta|\,|W\{(k,m)\}| \\ &\qquad \leqslant \frac{1}{(2\pi)^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}\left|\frac{\partial^2}{\partial t_1\partial t_2}\left(\widehat{W}(\vec{t}\,)\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|\mathrm{d}t_1\mathrm{d}t_2 \end{aligned} \tag{8.15}$$

and, for any $\gamma_1,\gamma_2 > 0$,

$$\begin{aligned} \|W\|_{TV}^2 &\leqslant \frac{(1+\sqrt{\gamma_1}\pi)(1+\sqrt{\gamma_2}\pi)}{4\pi^2} \\ &\quad \times\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\left|\widehat{W}(\vec{t}\,)\right|^2 + \frac{1}{\gamma_1}\left|\frac{\partial}{\partial t_1}\left(\widehat{W}(\vec{t}\,)\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2\right. \\ &\quad + \frac{1}{\gamma_2}\left|\frac{\partial}{\partial t_2}\left(\widehat{W}(\vec{t}\,)\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2\frac{1}{\gamma_1\gamma_2} \\ &\quad \left. + \left|\frac{\partial^2}{\partial t_1\partial t_2}\left(\widehat{W}(\vec{t}\,)\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2\right)\mathrm{d}t_1\mathrm{d}t_2 \end{aligned} \tag{8.16}$$

Proof. Lemma follows from Lemmas A.4.1–A.4.2.

□

Now we are in position to prove all theorems.

Proof of Theorems 8.2.1, 8.2.2, 8.2.3. Observe that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\beta(\vec{t}\,)\mathrm{d}t_1\mathrm{d}t_2 \leqslant \frac{C(F)}{n}.$$

Therefore, it suffices to apply Lemma A.4.1 with $\gamma_1 = \gamma_2 = n$, $\alpha = \mathbb{E}X_1$, $\beta = \mathbb{E}X_2$ and to use estimates from Lemma 8.3.8. □

Proof of Theorem 8.2.4. Obviously,

$$\sum_{\vec{m}\in\mathbb{Z}^2} |M\{\vec{m}\}|^r \leqslant \|M\|_{\infty}^{r-1}\|M\|_{TV}.$$

Therefore, Theorems statement follows from (8.9) and (8.8) with $C_4(F) = \max(C_1(F), C_2(F))$. □

9 BIVARIATE APPROXIMATIONS AND OPEN PROBLEMS

In this chapter we consider the case $d = 2$ and provide some insight for possible solutions to open problems from previous chapters. We deal with the following problems

1. Possibility to weaken assumptions for the Hipp-type CP approximation of lattice vectors in total variation.

2. Approximation for non-identically distributed symmetric r.v.s.

3. Practical applicability of Hipp-type approximations.

The main tool used for the proofs is the characteristic function method, which does not guarantees small constants. Therefore, a question about the magnitude of absolute constants remains an open problem, but is not considered in this chapter.

9.1 Hipp-type approximation for bivariate lattice vectors

Let us assume in (3.5) $d = 2$, $q_i = 1/5$ and

$$F_i = F = \frac{1}{5} + \frac{4}{5}\left(\frac{1}{2}I_{(1,0)} + \frac{1}{2}I_{(0,1)}\right).$$

Then, for large n,

$$\beta_1(x) = \frac{4x}{5}, \quad c_3(2;0.8) \approx 62,04.$$

and assumption (3.10)

$$c_3(2,0.8)\beta_1(2^{-3/2}A_2^{-1}) < 1$$

is not satisfied for any $A_1 \in [0,1]$.

In this Section we demonstrate that for $\mathbf{H}_{\mathrm{n},2}$ assumptions that q_i must be not small can be dropped. We use the same notation as in Chapter 4. For the convenience we repeat it here. We consider $F_1, \ldots, F_n \in \mathscr{F}_2^Z$. Let

$$\mathbf{F} := \mathscr{L}(S_n) = \prod_{i=1}^{*n} F_i, \quad \mathbf{H}_{\mathrm{n},2} := \prod_{i=1}^{*n} H_{i,2} = \prod_{i=1}^{*n} \exp\left\{(F_i - I) - \frac{(F_i - I)^{*2}}{2}\right\}$$

and, for $i = 1,2,\ldots,n$, $j = 1,2$

$$\sum_{\vec{x}} := \sum_{\vec{x}\in\mathbb{Z}^d\setminus\{\vec{0}\}}, \quad p_i(\vec{x}) = \mathbb{P}(\vec{X}_i = \vec{x}), \quad q_i = p_i(\vec{0}),$$

$$\widehat{\Psi}_s = \exp\left\{\frac{1}{2}\sum_{i=1}^{n} q_i(Re\,\widehat{F}_i - 1)\right\},$$

$$p^0_{i,1}(k) = p_i(k,0), \quad p^0_{i,2}(k) = p_i(0,k),$$

$$\lambda^0_{j,2} = \max\left(1;\ \sum_{i=1}^{n} q_i \sum_{k\in\mathbb{Z}\setminus\{0\}} p^0_{i,j}(k)\right),$$

$$\mu_{i,j} = \sum_{\vec{x}} x_j p_i(\vec{x}); \quad \mu_{i,jm} = \sum_{\vec{x}} x_j x_m p_i(\vec{x}),$$

$$\mu_j = \sum_{i=1}^{n} \mu_{i,j}, \quad \omega_{2,2} = \lambda^0_{1,2}\lambda^0_{2,2},$$

$$\beta_{i,j} = \sum_{\vec{x}} |x_j| p_i(\vec{x}), \quad \beta_{i,jm} = \sum_{\vec{x}} |x_j||x_m| p_i(\vec{x}),$$

$$\beta^{(2)}_{i,j} = \sum_{\vec{x}} x_j^2 p_i(\vec{x}),$$

$$\lambda_2(\vec{x}) = \max\left(1;\ \sum_{i=1}^{n} q_i p_i(\vec{x})\right), \quad \Lambda_{i,2} = \sum_{\vec{x}} \frac{p_i(\vec{x})}{\sqrt{\lambda_2(\vec{x})}},$$

$$L_{i,2} = \sum_{\vec{x}} \frac{p_i(\vec{x})}{\lambda_2(\vec{x})}.$$

Let

$$\begin{aligned}
A_0 &= \sum_{i=1}^{n} \Lambda^3_{i,2}, \quad A_1 = \sum_{i=1}^{n} \Lambda^3_{i,2} \sum_{j=1}^{n} \beta^{(2)}_{j,1} L_{i,2} + \sum_{i=1}^{n} \beta_{i,1}\Lambda^2_{i,2},\\
A_2 &= \sum_{i=1}^{n} \Lambda^3_{i,2} \sum_{j=1}^{n} \beta^{(2)}_{j,2} L_{i,2} + \sum_{i=1}^{n} \beta_{i,2}\Lambda^2_{i,2},\\
A_3 &= \sum_{i=1}^{n} \Lambda^3_{i,2} \sum_{k=1}^{n} (\beta_{k,12} + \beta_{k,1}\beta_{k,2})\\
&\quad + \sum_{i=1}^{n} \Lambda^3_{i,2} \sum_{k=1}^{n} \sqrt{\beta^{(2)}_{k,1} L_{k,2}} \sum_{m=1}^{n} \sqrt{\beta^{(2)}_{m,2} L_{m,2}}\\
&\quad + \sum_{k=1}^{n} \sqrt{\beta^{(2)}_{k,2} L_{k,2}} \sum_{i=1}^{n} \beta_{i,1}\Lambda^2_{i,2}\\
&\quad + \sum_{k=1}^{n} \sqrt{\beta^{(2)}_{k,1} L_{k,2}} \sum_{i=1}^{n} \beta_{i,2}\Lambda^2_{i,2} + \sum_{i=1}^{n} (\beta_{i,12}\Lambda_{i,2} + \beta_{i,1}\beta_{i,2}\Lambda^2_{i,2}).
\end{aligned}$$

Theorem 9.1.1. *Let $F_i \in \mathscr{F}_2$ be concentrated on $\mathbb{Z}_+^2$, $q_i = F_i\{\vec{0}\} \in (0,1)$ and $\beta_{i,1}^{(2)}, \beta_{i,2}^{(2)} < \infty$, $(i = 1,\dots,n)$. Then*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_{TV} \leqslant C\left(A_0 + \frac{A_1}{\sqrt{\lambda_{1,2}^0}} + \frac{A_2}{\sqrt{\lambda_{2,2}^0}} + \frac{A_3}{\sqrt{\lambda_{1,2}^0\lambda_{2,2}^0}}\right).$$

On one hand, unlike (3.11) estimate holds without any assumption on the magnitude of q_i. On the other hand, (3.11) has much simpler structure. To get the better idea about the accuracy of approximation in Theorem 9.1.1 we reformulate it for the scheme of sequences.

Corollary 9.1.1. *Let $p_i(1,0) > 0$, $p_i(0,1) > 0$. Then*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_{TV} \leqslant \frac{C(F)}{\sqrt{n}}.$$

Next we consider two dimensional analog of Bernoulli variables.

Corollary 9.1.2. *Let $F_i = q_i I_{(0,0)} + p_i(1,0)I_{(1,0)} + p_i(0,1)I_{(0,1)}$, $i = 1,\dots,n$. Then*

$$\begin{aligned}\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2}\|_{TV} \\ \leqslant \quad \frac{C}{\min_{1\leqslant k\leqslant n} q_k}\sum_{i=1}^{n}\left(\frac{p_i(1,0)}{\lambda_2(1,0)\sqrt{\lambda_2(1,0)}} + \frac{p_i(0,1)}{\lambda_2(0,1)\sqrt{\lambda_2(0,1)}}\right).\end{aligned}$$

Here

$$\lambda_2(1,0) = \sum_{j=1}^{n} q_j p_j(1,0), \quad \lambda_2(0,1) = \sum_{j=1}^{n} q_j p_j(0,1).$$

Proof of Theorem 9.1.1. Let

$$\widehat{V}(t_1,t_2) = \prod_{j=1}^{n}\widehat{F}_j(t_1,t_2)\mathrm{e}^{-\mathrm{i}t_1\mu_{j,1}-\mathrm{i}t_2\mu_{j,2}} - \prod_{j=1}^{n}\widehat{H}_{j,2}(t_1,t_2)\mathrm{e}^{-\mathrm{i}t_1\mu_{j,1}-\mathrm{i}t_2\mu_{j,2}}.$$

Collecting estimates proof the proofs of Theorems 4.3.1 and 4.3.3 we get

$$|\widehat{V}(t_1,t_2)| \leqslant C\widehat{\Psi}_2 A_0, \quad \left|\frac{\partial\widehat{V}(t_1,t_2)}{\partial t_1}\right| \leqslant C\widehat{\Psi}_2 A_1, \quad \left|\frac{\partial\widehat{V}(t_1,t_2)}{\partial t_2}\right| \leqslant C\widehat{\Psi}_2 A_2,$$

$$\left|\frac{\partial^2\widehat{V}(t_1,t_2)}{\partial t_1\partial t_2}\right| \leqslant C\widehat{\Psi}_2 A_3.$$

It remains to apply Lemma A.4.2 with $\alpha=\mu_1$, $\beta=\mu_2$, $\gamma_1=\lambda^0_{1,2}$, $\gamma_2=\lambda^0_{2,2}$ and Lemma 4.5.2. $\square$

Is it possible to simplify estimate in Theorem 9.2.1 for all $q_i\in(0,1)$ remains an open problem.

9.2 Hipp-type approximation for non-identically distributed symmetric lattice random vectors

In chapter 5, Hipp-type approximations were applied to identically distributed sums of symmetric r.v.s. Lemma 5.3.1 played crucial role in the proofs. There is no analogue of Lemma 5.3.1 for non-identically distributed r.v.s. Therefore different approach must be used. In this Section we prove that, at least for bivariate vectors, some analogues of Theorem 5.2.1 is possible. Assumptions will be much weaker than used for (5.4).

Assuming that $d=2$, we use the same notation as in Chapter 4. For the convenience we repeat it. Let

$$\sum_{\vec{x}}:=\sum_{\vec{x}\in\mathbb{Z}^d\setminus\{\vec{0}\}},\quad q_i:=\mathbb{P}(\vec{X}_i=\vec{0}),$$
$$\beta_{i,j}=\sum_{\vec{x}}|x_j|p_i(\vec{x}),\quad \mu_{i,jm}=\sum_{\vec{x}}x_jx_mp_i(\vec{x}),$$
$$\beta_{i,jm}=\sum_{\vec{x}}|x_j||x_m|p_i(\vec{x}),\quad \beta^{(2)}_{i,j}=\sum_{\vec{x}}x_j^2p_i(\vec{x}),\quad \beta^{(4)}_{i,j}=\sum_{\vec{x}}x_j^4p_i(\vec{x}),$$
$$\lambda_1(\vec{x})=\max\left(1;\sum_{i=1}^n q_ip_i(\vec{x})\right),\quad L_{i,1}=\sum_{\vec{x}}\frac{p_i(\vec{x})}{\lambda_1(\vec{x})},$$
$$p^0_{i,1}(k)=p_i(k,0),\quad p^0_{i,2}(k)=p_i(0,k),$$
$$\lambda^0_{m,1}=\max\left(1;\sum_{i=1}^n q_i\sum_{j\in\mathbb{Z}\setminus\{0\}}p^0_{i,m}(j)\right);\quad \omega_{d,1}=\prod_{m=1}^d\lambda^0_{m,1},$$
$$\widehat{\Psi}_1=\exp\left\{\frac{1}{2}\sum_{i=1}^n q_i(\widehat{F}_i(\vec{t})-1)\right\}.$$

As in above, we assume that

$$\mathbf{F}:=\mathscr{L}(S_n)=\prod_{i=1}^{*n}F_i,\mathbf{H}_{\mathrm{n,k}}:=\prod_{i=1}^{*n}H_{i,k}=\prod_{i=1}^{*n}\exp\left\{\sum_{m=1}^k\frac{(-1)^{m+1}(F_i-I)^{*m}}{m}\right\}$$

and

$$
\begin{aligned}
B_0 &= \sum_{i=1}^{n} L_{i,1}^{k+1}, \quad B_1 = \sum_{i=1}^{n} L_{i,1}^{k} \left(\sqrt{L_{i,1}\beta_{i,1}^{(2)}} + L_{i,1} \sum_{m=1}^{n} \sqrt{L_{m,1}\beta_{i,1}^{(2)}} \right), \\
B_2 &= \sum_{i=1}^{n} L_{i,1}^{k} \left(\sqrt{L_{i,1}\beta_{i,2}^{(2)}} + L_{i,1} \sum_{m=1}^{n} \sqrt{L_{m,1}\beta_{i,2}^{(2)}} \right), \\
B_3 &= \sum_{i=1}^{n} L_{i,1}^{k} \left(\sqrt{\beta_{i,1}^{(2)}\beta_{i,2}^{(2)}} + L_{i,1} \sum_{m=1}^{n} \sqrt{\beta_{i,1}^{(2)}\beta_{i,2}^{(2)}} \right) \\
&\quad + \sum_{i=1}^{n} L_{i,1}^{k+1} \sum_{j=1}^{n} \sqrt{L_{i,1}\beta_{i,1}^{(2)}} \sum_{m=1}^{n} \sqrt{L_{m,1}\beta_{m,2}^{(2)}} \\
&\quad + \sum_{i=1}^{n} \sqrt{L_{i,1}\beta_{i,1}^{(2)}} \sum_{l=1}^{n} L_{l,1} \sqrt{L_{l,1}\beta_{l,2}^{(2)}} \\
&\quad + \sum_{i=1}^{n} \sqrt{L_{i,1}\beta_{i,2}^{(2)}} \sum_{l=1}^{n} L_{l,1} \sqrt{L_{l,1}\beta_{l,1}^{(2)}}.
\end{aligned}
$$

Theorem 9.2.1. *Let $F_i \in \mathscr{F}_2^S$ be concentrated on $\mathbb{Z}_+^2$, $q_i = F_i\{\vec{0}\} \in (0,1)$ and $\beta_{i,1}^{(2)}, \beta_{i,2}^{(2)} < \infty$, $(i = 1, \ldots, n)$. Then, for any $k \in \mathbb{N}$*

$$
\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_{TV} \leqslant C(k) \left(B_0 + \frac{B_1}{\sqrt{\lambda_{1,2}^0}} + \frac{B_2}{\sqrt{\lambda_{2,2}^0}} + \frac{B_3}{\sqrt{\lambda_{1,2}^0 \lambda_{2,2}^0}} \right).
$$

Theorem 9.2.1 improves (5.4) in the following way: it holds for $k > 1$ and for all $q_i \in (0,1)$. On the other hand, if $2\alpha\mathrm{e} < 1$, then (5.4) has a simpler structure.

Corollary 9.2.1. *Let $p_i(1,0) > 0$, $p_i(0,1) > 0$. Then*

$$
\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_{TV} \leqslant \frac{C(F,k)}{n^k}.
$$

Proof of Theorem 9.2.1. Let

$$
\widehat{V}_s(t_1,t_2) = \prod_{j=1}^{n} \widehat{F}_j(t_1,t_2) - \prod_{j=1}^{n} \widehat{H}_{j,k}(t_1,t_2).
$$

Collecting estimates we get

$$
|\widehat{V}_s(t_1,t_2)| \leqslant C(k)\widehat{\Psi}_1 B_0, \quad \left| \frac{\partial \widehat{V}_s(t_1,t_2)}{\partial t_1} \right| \leqslant C(k)\widehat{\Psi}_1^{1/2} B_1,
$$

$$\left|\frac{\partial \widehat{V}_s(t_1,t_2)}{\partial t_2}\right| \leqslant C(k)\widehat{\Psi}_1^{1/2}B_2, \quad \left|\frac{\partial^2 \widehat{V}_s(t_1,t_2)}{\partial t_1 \partial t_2}\right| \leqslant C(k)\widehat{\Psi}_1^{1/2}B_3.$$

It remains to apply Lemma A.4.2 with $\alpha = \mu_1$, $\beta = \mu_2$, $\gamma_1 = \lambda_{1,2}^0$, $\gamma_2 = \lambda_{2,2}^0$ and Lemma 4.5.2. □

Is it possible to simplify estimate in Theorem 9.2.1 for all $q_i \in (0,1)$ remains an open problem.

9.3 Recursive formulas for CP approximations

One-dimensional recursions for CP and SCP approximations. Panjer-type recursive algorithms were defined in [81], chapter 10.7 for one-dimensional Hipp's approximation. In [83] recursion was obtained for a one-dimensional Hipp-type approximation, Let

$$\widehat{G}_{cp}(t) = \exp\left\{\sum_{j=1}^{\infty} \lambda_j (\mathrm{e}^{\mathrm{i}tj} - 1)\right\} \tag{9.1}$$

and $\sum_{j=1}^{\infty} j|\lambda_j| < \infty$. Then, for all $k = 0,1,2,\ldots,$

$$(k+1)G_{cp}\{k+1\} = \sum_{m=0}^{k} G_{cp}\{m\}(k-m+1)\lambda_{k-m+1}. \tag{9.2}$$

Particularly, if

$$\widehat{G}(t) = \exp\left\{\lambda_1(\mathrm{e}^{\mathrm{i}t} - 1) + \lambda_2(\mathrm{e}^{2\mathrm{i}t} - 1)\right\},$$

then

$$kG\{k\} = \lambda_1 G\{k-1\} + 2\lambda_2 G\{k-2\}, \quad k = 2,3,4,\ldots \tag{9.3}$$

By expanding Fourier transform in powers of $\mathrm{e}^{\mathrm{i}t}$ it is not difficult to verify that

$$G\{0\} = \exp\{-\lambda_1 - \lambda_2\}, \quad G\{1\} = \lambda_1 \exp\{-\lambda_1 - \lambda_2\}. \tag{9.4}$$

Two-dimensional analogue of (9.3). Let

$$\begin{aligned} H &= \exp\Big\{\lambda_1(I_{(1,0)} - I) + \lambda_2(I_{(2,0)} - I) + \mu_1(I_{(0,1)} - I) \\ &\quad + \mu_2(I_{(0,2)} - I) + \varkappa(I_{(1,1)} - I)\Big\} \end{aligned}$$

Here $\lambda_1, \lambda_2, \mu_1, \mu_2, \varkappa \in \mathbb{R}$.

Set

$$\Lambda = -\lambda_1 - \lambda_2 - \mu_1 - \mu_2 - \varkappa.$$

Expanding Fourier transform in powers of e^{it_1} and e^{it_2} we establish that

$$\begin{aligned}
H\{0,0\} &= e^{\Lambda}, \quad H\{1,0\} = \lambda_1 e^{\Lambda}, \quad H\{0,1\} = \mu_1 e^{\Lambda}, \\
H\{1,1\} &= (\varkappa + \lambda_1\mu_1)e^{\Lambda},\ H\{2,0\} = \lambda_2 + \frac{\lambda_1^2}{2},\ H\{0,2\} = \mu_2 + \frac{\mu_1^2}{2}, \\
2H\{2,1\} &= \lambda_1 H\{1,1\} + 2\lambda_2 H\{0,1\} + \varkappa H\{1,0\}, \\
2H\{1,2\} &= \mu_1 H\{1,1\} + \mu_2 H\{1,0\} + \varkappa H\{0,1\}.
\end{aligned}$$

For the convenience we assume that $H\{k,m\} = 0$, if $k < 0$ or $m < 0$ and

Proposition 9.3.1. *For $k,m \in \mathbb{N}$ the following recursive relations hold:*

$$\begin{aligned}
(k+1)(m+1)H\{k+1,m+1\} &= (\lambda_1\mu_1 + \varkappa)H\{k,m\} \\
&+(4\lambda_2\mu_2 + \varkappa^2)H\{k-1,m-1\} \\
&+(\lambda_1\varkappa + 2\lambda_2\mu_1)H\{k-1,m\} + (\mu_1\varkappa + 2\lambda_1\mu_2)H\{k,m-1\} \\
&+2\lambda_2\varkappa H\{k-2,m\} + 2\mu_2\varkappa H\{k,m-2\}
\end{aligned}$$

and

$$\begin{aligned}
(k+1)H\{k+1,m\} &= \lambda_1 H\{k,m\} + 2\lambda_2 H\{k-1,m\} + \varkappa H\{k,m-1\}, \\
(m+1)H\{k,m+1\} &= \mu_1 H\{k,m\} + 2\mu_2 H\{k,m-1\} + \varkappa H\{k-1,m\}.
\end{aligned}$$

Proof. We will use the probability generating function:

$$\begin{aligned}
h(z_2,z_2) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} H\{k,m\}z_1^k z_2^m \\
&= \exp\Big\{\lambda_1(z_1-1) + \lambda_2(z_1^2-1) + \mu_1(z_2-1) + \mu_2(z_2^2-1) \\
&\quad +\varkappa(z_1z_2-1)\Big\}.
\end{aligned}$$

Then, on one hand,

$$\begin{aligned}
\frac{\partial h(z_1,z_2)}{\partial z_1} &= \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} kH\{k,m\}z_1^{k-1}z_2^m \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(k+1)H\{k+1,m\}z_1^k z_2^m,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial h(z_1,z_2)}{\partial z_2} &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(m+1)H\{k,m+1\}z_1^k z_2^m,\\
\frac{\partial^2 h(z_1,z_2)}{\partial z_1 \partial z_2} &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(k+1)(m+1)H\{k+1,m+1\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial h(z_1,z_2)}{\partial z_1} &= h(z_1,z_2)(\lambda_1+2\lambda_2 z_1+\varkappa z_2)\\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}H\{k,m\}z_1^k z_2^m(\lambda_1+2\lambda_2 z_1+\varkappa z_2)\\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Big(\lambda_1 H\{k,m\}+2\lambda_2 H\{k-1,m\}\\
&\quad +\varkappa H\{k,m-1\}\Big)z_1^k z_2^m,\\
\frac{\partial h(z_1,z_2)}{\partial z_2} &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Big(\mu_1 H\{k,m\}+2\mu_2 H\{k,m-1\}\\
&\quad +\varkappa H\{k-1,m\}\Big)z_1^k z_2^m,\\
\frac{\partial^2 h(z_1,z_2)}{\partial z_1 \partial z_2} &= h(z_1,z_2)((\lambda_1\mu_1+\varkappa)+(\lambda_1\varkappa+2\lambda_2\mu_1)z_1\\
&\quad +(\mu_1\varkappa+2\lambda_1\mu_2)z_2+2\lambda_2\varkappa z_1^2+2\mu_2\varkappa z_2^2\\
&\quad +(4\lambda_2\mu_2+\varkappa^2)z_1 z_2)\\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\big((\lambda_1\mu_1+\varkappa)H\{k,m\}\\
&\quad +(\lambda_1\varkappa+2\lambda_2\mu_1)H\{k-1,m\}\\
&\quad +(\mu_1\varkappa+2\lambda_1\mu_2)H\{k,m-1\}\\
&\quad +2\lambda_2\varkappa H\{k-2,m\}+2\mu_2\varkappa H\{k,m-2\}\\
&\quad +(4\lambda_2\mu_2+\varkappa^2)H\{k-1,m-1\}.
\end{aligned}$$

It remains to equate coefficients to powers of z_1 and z_2. □

In principle, it is not difficult to extend the above approach to more general SCP approximations. However, the general formulas might be very cumbersome. It would be interesting also to establish a Panjer-type recursive algorithms at least in two-dimensional case.

One-dimensional Skellam distribution. We will consider a symmetric

version of Skellam distribution, that is the difference of two identically distributed Poisson r.v.s. Let

$$\widehat{G}_{sk}(t) = \exp\left\{\lambda(e^{it}-1)+\lambda(e^{-it}-1)\right\}.$$

Then, see [79],

$$G_{sk}\{k\} = e^{-2\lambda} I_{|k|}(2\lambda). \tag{9.5}$$

Here $I_{|k|}(2\lambda)$ is the modified Bessel function of the first kind. Recursive formulas for Bessel functions allow to write

$$kG_{sk}\{k\} = \lambda G_{sk}\{k-1\} - \lambda G_{sk}\{k+1\} \quad k = \{2,3,\dots\}. \tag{9.6}$$

Two-dimensional Skellam distribution. We are using Skellam distribution with independent coordinate vectors, that is

$$\widehat{D}(t_1,t_2) = \exp\left\{\lambda_1(e^{it_1}-1)+\lambda_1(e^{-it_1}-1)\right\}\exp\left\{\lambda_2(e^{it_2}-1)+\lambda_2(e^{-it_2}-1)\right\}.$$

Therefore

$$D\{m,k\} = D_1\{m\}D_2\{k\}, \tag{9.7}$$

where D_i is one-dimensional Skellam distribution with parameter λ_i. Therefore, we can use (9.6) twice

$$\begin{aligned} mD_1\{m\} &= \lambda_1 D_1\{m-1\} - \lambda_1 D_1\{m+1\}, & (9.8)\\ kD_2\{k\} &= \lambda_2 D_2\{k-1\} - \lambda_2 D_2\{k+1\}, \quad k,m \in \{2,3,\dots\} & (9.9) \end{aligned}$$

and apply (9.7).

$$\begin{aligned} mkD\{m,k\} &= \lambda_1\lambda_2\big(D_1\{m+1\}D_2\{k+1\} + D_1\{m-1\}D_2\{k-1\} \\ &\quad - D_1\{m-1\}D_2\{k+1\} - D_1\{m+1\}D_2\{k-1\}\big). \end{aligned} \tag{9.10}$$

Note that D_1, D_2 are symmetric distributions, that is $D_1\{-m\} = D_1\{m\}$, $D_2\{-k\} = D_2\{k\}$.

9.4 Simulations

Hipp-type approximation of the second order. Let

$$\begin{aligned}\widehat{F}_i(t_i) &= q_i + p_i \mathrm{e}^{\mathrm{i}t_i}, \quad q_i = 1 - p_i, \quad i = 1,2,\\ \widehat{H}_i(t_i) &= \exp\left\{(p_i + p_i^2)(\mathrm{e}^{\mathrm{i}t_i} - 1) - \frac{p_i^2}{2}(\mathrm{e}^{2\mathrm{i}t_i} - 1)\right\}\end{aligned}$$

and

$$\begin{aligned}\widehat{F}(t_1,t_2) &= \widehat{F}_1(t_1)\widehat{F}_2(t_2) = q_1q_2 + p_1q_2\mathrm{e}^{\mathrm{i}t_1} + q_1p_2\mathrm{e}^{\mathrm{i}t_2} + p_1p_2\mathrm{e}^{\mathrm{i}(t_1+t_2)},\\ \widehat{H}(t_1,t_2) &= \widehat{H}_1(t_1)\widehat{H}_2(t_2) = \exp\left\{(p_1 + p_1^2)(\mathrm{e}^{\mathrm{i}t_1} - 1) - \frac{p_1^2}{2}(\mathrm{e}^{2\mathrm{i}t_1} - 1)\right\}\\ &\quad \times \exp\left\{(p_2 + p_2^2)(\mathrm{e}^{\mathrm{i}t_2} - 1) - \frac{p_2^2}{2}(\mathrm{e}^{2\mathrm{i}t_2} - 1)\right\}.\end{aligned}$$

We have a version of two-dimensional distribution of a vector with independent coordinate vectors. Therefore

$$F^{*n}\{k,m\} = F_1^{*n}\{k\}F_2^{*n}\{m\} = \binom{n}{k} p_1^k q_1^{n-k} \binom{n}{m} p_2^m q_2^{n-m}.$$

Noting that $F_i\{k\} = 0$, $k > n$, we can write

$$\begin{aligned}\|F^{*n} - H^{*n}\|_{TV} = d &+ \sum_{k=0}^{n} |H_1^{*n}\{k\}| \sum_{m=n+1}^{\infty} |H_2^{*n}\{m\}|\\ &+ \sum_{k=n+1}^{\infty} |H_1^{*n}\{k\}| \sum_{m=0}^{n} |H_2^{*n}\{m\}|\\ &+ \sum_{k=n+1}^{\infty} |H_1^{*n}\{k\}| \sum_{m=n+1}^{\infty} |H_2^{*n}\{m\}|. \qquad (9.11)\end{aligned}$$

Here

$$d := \sum_{k=0}^{n}\sum_{m=0}^{n} |F^{*n}\{k,m\} - H^{*n}\{k,m\}|.$$

Simulations show that the difference between d and $\|F^{*n} - H^{*n}\|_{TV}$ is usually quite small. Here we outline an approach, which can be used to estimate that difference. To deal with infinite sums we apply Lemma A.1.9, Lemma A.1.10 and Lemma A.1.11. Let for $i = 1,2$ $S_{ni} = Y_{1i} + \ldots + Y_{ni}$, where Y_{ki} are

independent and

$$\mathbb{E}\,\mathrm{e}^{\mathrm{i}t_iY_{ki}} = \exp\left\{p_i(1+p_i)(\mathrm{e}^{\mathrm{i}t_i}-1)+\frac{p_i^2}{2}(\mathrm{e}^{2\mathrm{i}t_i}-1)\right\}.$$

By Lemma A.1.11 $Y_{ki}-p_i(1+2p_i)$ satisfies Bernstein's inequality (A.12) with $\tau=2$. Let $T_i \geqslant 2$ be an integer satisfying

$$T_i \geqslant \frac{p_i(3+7p_i)}{2}. \tag{9.12}$$

Then $T_in - np_i(1+2p_i) \geqslant np_i(1+3p_i)/2 = \operatorname{Var} S_{ni}/2$ and by Lemma A.1.10

$$\begin{aligned}\mathbb{P}(S_{ni} \geqslant nT_i) &= \mathbb{P}\big(S_{ni}-np_i(1+2p_i) \geqslant T_in-np_i(1+2p_i)\big)\\ &\leqslant \exp\left\{-\frac{T_in-np_i(1+2p_i)}{8}\right\}.\end{aligned}$$

Therefore, for any T_i satisfying (9.12), from Lemma A.1.9 it follows that

$$\begin{aligned}\sum_{k=nT_i+1}^{\infty}|H_i^{*n}\{k\}| &\leqslant \exp\{np_i^2\}\mathbb{P}(S_{ni} \geqslant nT_i)\\ &\leqslant \exp\left\{-\frac{n(T_i-p_i(1+10p_i))}{8}\right\}\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=n+1}^{\infty}|H_i^{*n}\{k\}|\\ &\quad=\sum_{k=n+1}^{T_in}|H_i^{*n}\{k\}|+\exp\left\{-\frac{n(T_i-p_i(1+10p_i))}{8}\right\}.\end{aligned} \tag{9.13}$$

The choice of T_i is determined by the values of n, p_i and expected precision of estimate. For example, let $p_1=0.25$, $n=100$, $T_1=5$. Then (9.12) is satisfied and

$$\left|\sum_{k=n+1}^{\infty}|H_1^{*n}\{k\}|-\sum_{k=n+1}^{5n}|H_1^{*n}\{k\}|\right| \leqslant \mathrm{e}^{-50}.$$

The following simulation results hold in total variation with the precision 10^{-6}.

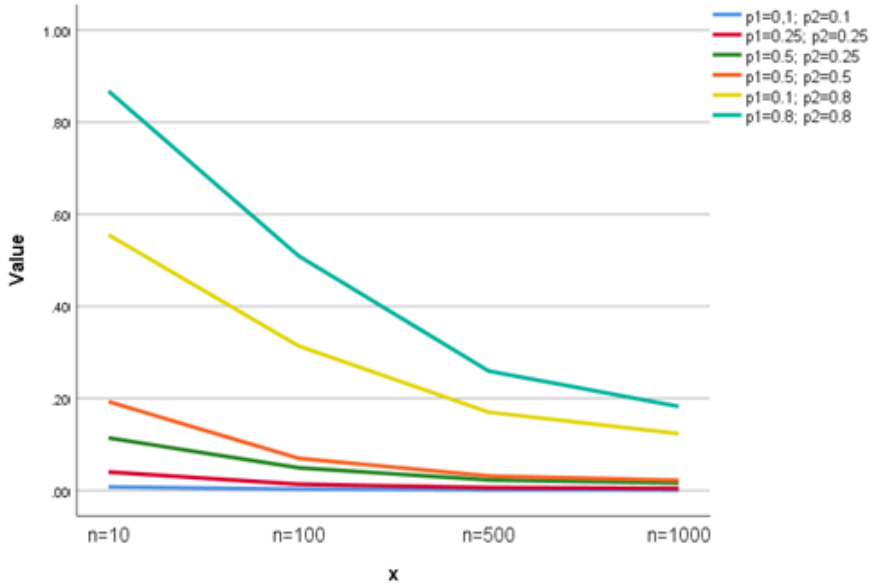

Figure 5: Graphic for the values of d.

p_1	p_2	$n = 10$	$n = 100$	$n = 500$	$n = 1000$
0.1	0.1	0.007778	0.002623	0.001161	0.000819
0.25	0.25	0.039974	0.013483	0.006018	0.004251
0.5	0.25	0.114048	0.049169	0.022802	0.016202
0.5	0.5	0.192787	0.069721	0.031357	0.022136
0.1	0.8	0.554983	0.313879	0.169952	0.123290
0.8	0.8	0.867300	0.509335	0.259375	0.182372

Table 1: The accuracy of approximation

The *r* code with some comments is given in section A.6. The accuracy of approximation can also be shown graphically.

Observe, that when both probabilities p_1 and p_2 are small, then the accuracy of approximation is very good even for the very small values of n. On the other hand, when at least on of the probabilities p_1, p_2 is close to 1, the accuracy of approximation improves quite slowly, when n grows.

Two dimensional Skellam approximation. Let $q_1 = 1 - 2p_1$, and $q_2 = 1 - 2p_2$,

$$\begin{aligned}\widehat{F}(t_1,t_2) &= (q_1+2p_1\cos t_1)(q_2+2p_2\cos t_2)\\ &= q_1q_2+p_1q_2\mathrm{e}^{\mathrm{i}t_1}+p_1q_2\mathrm{e}^{-\mathrm{i}t_1}+q_1p_2\mathrm{e}^{\mathrm{i}t_2}+q_1p_2\mathrm{e}^{-\mathrm{i}t_2}\\ &\quad +p_1p_2\mathrm{e}^{\mathrm{i}(t_1+t_2)}+p_1p_2\mathrm{e}^{\mathrm{i}(t_1-t_2)}+p_1p_2\mathrm{e}^{\mathrm{i}(-t_1+t_2)}\\ &\quad +p_1p_2\mathrm{e}^{-\mathrm{i}(t_1+t_2)}.\end{aligned}$$

Observe that

$$\operatorname{supp}F^{*n}=\{-n,-n+1,\ldots,n-1,n\}\times\{-n,-n+1,\ldots,n-1,n\}.$$

Skellam approximation for F^{*n} is defined by its characteristic function

$$\widehat{G}^n(t_1,t_2)=\exp\left\{-4np_1\sin^2(t_1/2)-4np_2\sin^2(t_2/2)\right\}.$$

Let

$$d_s:=\sum_{m=-n}^{n}\sum_{k=-n}^{n}|F^{*n}\{m,k\}-G^{*n}\{m,k\}|.$$

Then

$$\begin{aligned}\|F^{*n}-G^{*n}\|_{TV} &= d_s+G^{*n}\{\vec{x}\notin\operatorname{supp}F^{*n}\}\\ &= d_s+\left(1-\sum_{m=-n}^{n}\sum_{k=-n}^{n}G^{*n}\{m,k\}\right).\end{aligned}\tag{9.14}$$

As it turns out, for simulated examples the difference between d_s and $\|F^{*n}-G^{*n}\|_{TV}$ is $<3\cdot10^{-4}$ if $n=10$ and $<10^{-8}$ for larger n. Below we present simulated estimates for the total variation.

p_1	p_2	$F\{0,0\}$	$n=10$	$n=100$	$n=500$	$n=1000$
0.01	0.01	0.9604	0.00968	0.00504	0.00099	0.00049
0.10	0.10	0.64	0.05161	0.00497	0.00099	0.00049
0.25	0.25	0.25	0.05069	0.00498	0.00099	0.00049
0.25	0.45	0.40	0.11152	0.00499	0.00099	0.00049
0.10	0.45	0.08	0.11054	0.00498	0.00099	0.00049
0.45	45	0.01	0.13563	0.00499	0.00099	0.00049

Table 2: Skellam approximation in total variation

For calculation of $F^{*n}\{m,k\}$, $G^{*n}\{m.k\}$ we used numeric integration, which

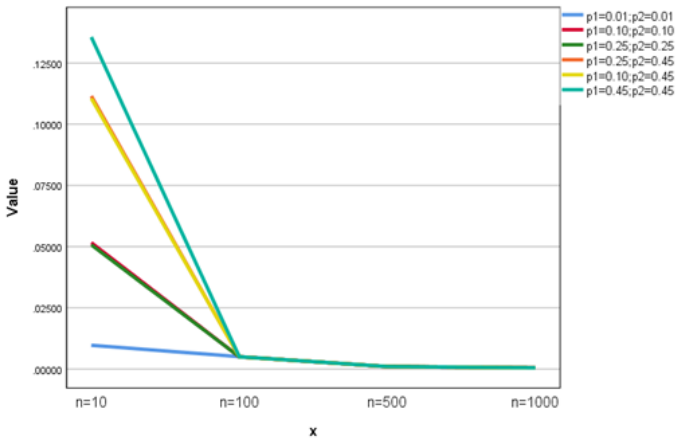

Figure 6: Graphic for Skellam approximation with $n = 10$.

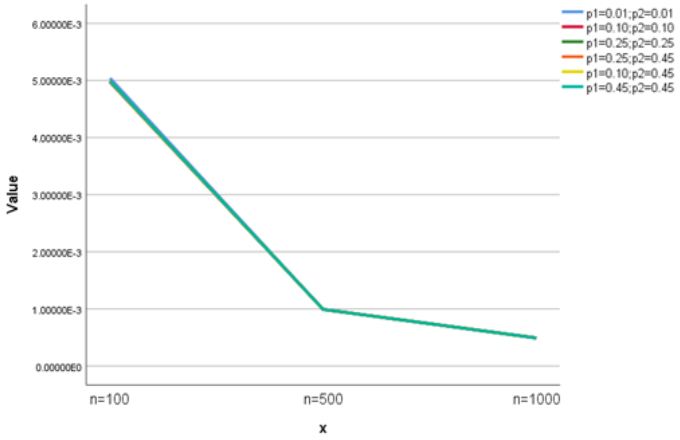

Figure 7: Graphic for Skellam approximation without $n = 10$.

has its own error (tolerance 10^{-5}). Therefore, we can conclude that, in this case, the accuracy of approximation depends on n, but practically does not depend on the choice of p_1, p_2. We also can see that the accuracy of approximation is $\approx 5/n$, which is almost twice better than the first estimate in Corollary 5.2.1 , which sis $8.66/n$.

We added two graphs for the Skellam approxiation. The first one includes $n = 10$, i.e. the case when the majority of differences still can be seen. These differences occur, because in some cases $np_1^2 < 1$. For the graph with $n \geqslant 100$ we can see that all estimates are virtually the same.

Conclusions

1. The accuracy of the Kolmogorov approximation for a sum of shifted independent identically distributed random vectors, obtained by constructing an asymptotic expansion for the accompanying CP law, is proved.

2. Structural simplifications to Hipp-type approximations for rare events are investigated. Estimates for the short asymptotic expansion and for the multivariate Skellam approximation in total variation are obtained.

3. General local, ℓ_r norm, and marginal non-uniform estimates for the Hipp-type approximations to the sums of independent lattice random vectors are obtained.

4. A total variation estimate of the distribution of the sum of independent, identically distributed symmetric lattice vectors, using the accompanying CP law and a Hipp-type SCP measure for non-rare events, is derived.

5. General local, ℓ_r norm, and marginal non-uniform estimates for the Hipp-type approximations to the sums of independent symmetric lattice random vectors are obtained.

6. General local, ℓ_r norm, total variation norm, and marginal non-uniform estimates for the Skellam approximation to the sums of independent symmetric random vectors are obtained.

7. A partial solution to the first Kolmogorov problem for a sequence of independent, identically distributed, lattice two-dimensional random vectors with finite fourth-order moments is proposed.

8. Recursive relations for two-dimensional Hipp-type approximations are obtained.

References

[1] Akpouea B.P., Angers J-F. (2017) Some contributions on the multivariate Poisson-Skellam probability distribution. *Commun. Stat. - Theory Methods*, **46**(1), 49–68. http://dx.doi.org/10.1080/03610926.2014.983806

[2] Aissaoui S.A., Genest C., Mesfioui M. (2017) A second look at inference for bivariate Skellam distributions. *Stat 2017*, **6**, 79–87. https://doi.org/10.1002/sta4.136

[3] Alzaid A.A., Omair M.A. (2010). On the Poisson difference distribution: inference and applications. *Bulletin of the Malaysian Mathematical Sciences Society. Second Series* **33**(1), 17–45. http://eudml.org/doc/244475

[4] Arak, T.V., Zaitsev, A.Yu. (1986) Uniform limit theorems for sums of independent random variables. Trudy Mat. Inst. Steklov (Monograph) 174 (Russian). Transl.: Proc. Steklov Inst. Math. 174, 3–214 (1988). https://www.mathnet.ru/rus/tm2143

[5] Barbour A.D., Hall P. (1984) On the rate of Poisson convergence. *Math. Proceedings Cambridge Philos. Soc.*, **95**, 473–480. https://doi.org/10.1017/S0305004100061806

[6] Barbour A.D., Holst L., Janson S. (1992) *Poisson Approximation.* Clarendon Press, Oxford, online edn, Oxford Academic, 31 Oct. (2023), https://doi.org/10.1093/oso/9780198522355.001.0001

[7] Barbour A.D., Xia A. (1999) Poisson perturbations. *ESAIM Probab. Statist.*, **3**, 131–150. http://eudml.org/doc/104253

[8] Barbour A. D., Chryssaphinou O., Vaggelatou E. (2001) Applications of compound Poisson approximation. In: Charalambides, C. A., Koutras, M. V. and Balakrishnan, N. (Eds.) *Probability and Statistical Models with Applications*, 41–62. Chapman & Hall. https://doi.org/10.1201/9781420036084

[9] Barbour A.D., Čekanavičius V. (2002). Total variation asymptotics for sums of independent integer random variables. *Ann. Probab.*, **30**(2), 509–545. https://doi.org/10.1214/aop/1023481001

[10] Barbour A.D. (2005) Multivariate Poisson-binomial approximation using Stein's method. In: Barbour A.D., Chen L.H.Y. (eds.) *Stein's Method and Applications. IMS Lecture Note Series*, vol. 5. https://doi.org/10.1142/9789812567673_0008

[11] Barbour A.D., Luczak M.J., Xia A. (2018) Multivariate approximation in total variation II: discrete normal approximation. *Ann. Probab.*, **46**, 1405–1440. https://doi.org/10.1214/17-aop1205

[12] Barbour A.D., Xia A. (2019) Multivariate approximation in total variation using local dependence. *Electronic J. Probab.*, **29**, paper no 27, 1–35. http: 10.1214/19-EJP284

[13] Bentkus V., Paulauskas V. (2004) Optimal error estimates in operator-norm approximations of semigroups. *Letters in Math. Physics.*, **68**, 131–138. https://doi.org/10.1023/B:MATH.0000045552.97035.87

[14] Bergström H. (1951) On asymptotic expansion of probability functions. *Skand. Aktuar.*, **1**, 1–34. https://doi.org/10.1080/03461238.1951.10432122

[15] Bhattacharya R.N., Ranga Rao R. (1976) *Normal approximation and asymptotic expansions*, Wiley. https://doi.org/10.1137/1.9780898719895

[16] Boutsikas M.V., Koutras M.V. (2002) On the number of overflown urns and excess balls in an allocation model with limited urn capacity. *J. Statist. Plan. Inf.*, **104**(2), 259–286. https://doi.org/10.1016/S0378-3758(01)00263-4

[17] Bulla J., Chesneau C., Kachour M. (2015) On the Bivariate Skellam Distribution. *Communications in Statistics - Theory and Methods*, **44**(21), 4552–4567. URL https://doi.org/10.1080/03610926.2013.837925.

[18] Čekanavičius V. (1990) On approximation of integer-valued measures by generalized Poisson measures. *In: Probabability Theory and Mathematical Statistics 2. Proceedings of the Fifth Vilnius Conference (1989)*, B.Grigelionis et al. (Eds.), VSP/Mokslas, ISBN: 90-6764-129-4, 228–237.

[19] Čekanavičius V. (1991) An approximation of mixtures of distributions. *Lith. Math. J.*, **31**(2), 243–257. https://doi.org/10.1007/BF00970821

[20] Čekanavičius V. (1993) Non-uniform theorems for discrete measures. *Lith. Math. J.*, **33**(2), 114–126. https://doi.org/10.1007/BF00975114

[21] Čekanavičius V. (1995) On asymptotic expansions in the first uniform Kolmogorov theorem. Statist. Probab. Lett. **25**(2), 145–151. https://doi.org/10.1016/0167-7152(94)00216-U

[22] Čekanavičius V. (1995) On multivariate Le Cam theorem and compound Poisson measures. Statist. Probab. Lett. **28**(1), 33–39 . https://doi.org/10.1016/0167-7152(95)00079-8

[23] Čekanavičius V. (1997) Asymptotic expansions in the exponent: a compound Poisson approach. *Adv. Appl. Probab.*, **29**(2), 374–387. https://doi.org/10.2307/1428008

[24] Čekanavičius V. (1998) Estimates in total variation for convolutions of compound distributions. *J. Lond. Math. Soc.*, **58**(3), 748–760. https://doi.org/10.1112/S0024610798006693

[25] Čekanavičius V. (1999) On compound Poisson approximations under moment restrictions. *Teor. Veroyatn. Primen.*, **44**(1), 74–86 (Russian). Transl.: *Theory Probab. Appl.* **44**(1), 18–28, 2000. https://doi.org/10.1137/S0040585X97977343

[26] Čekanavičius V. (2002) On multivariate compound distributions. *Teor. Veroyatn. Primen.*, **47**(3), 583–594 (Russian). Transl.: *Theory Probab. Appl.* **47**(3), 493–505, 2003. https://doi.org/10.4213/tvp3698

[27] Čekanavičius V., Roos B. (2006) Compound binomial approximations. *Ann. Inst. Stat. Math.*, **58**, 187–210. https://doi.org/10.1007/s10463-005-0018-4

[28] Čekanavičius V. (2016) *Approximation methods in Probability Theory*. Universitext: Springer. https://doi.org/10.1007/978-3-319-34072-2

[29] Čekanavičius V. , Vellaisamy P. (2021) Compound Poisson approximations in ℓ_p-norm for sums of weakly dependent vectors. *J. Theor. Probab.* **34**(4), 2241–2264. https://doi.org/10.1007/s10959-020-01042-9

[30] Čekanavičius, V., Novak, S.Y. (2024) *Compound Poisson approximation.* Chapman and Hall/CRC, New York. https://doi.org/10.1201/9781003478164

[31] Chryssaphinou O., Vaggelatou E. (2002) Compound Poisson approximation for a multiple runs in a Markov chain. *Ann. Inst. Stat. Math.*, **54**(2), 411–424. https://doi.org/10.1023/A:1022438422611

[32] Daly F. (2017) On magic factors in Stein's method for compound Poisson approximation. *Electron. Commun. Probab.*, **22**(67), 1–10. https://doi.org/10.1016/j.jspi.2011.01.004

[33] Eliseeva Yu.S. (2013) Multivariate estimates for the concentration function of weighted sums of independent identically distributed random variables. *Zap. Nauchn. Sem. POMI*, **412**, 121–137. Transl.: *J. Math. Sci.*, **204**(1), 78–89, 2015. https://doi.org/10.1007/s10958-014-2188-1

[34] Erhardsson T. (2005) Stein's method for Poisson and compound Poisson approximation. *In: An introduction to Stein's method. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., v. 4, 61–113.* Singapore: Singapore Univ. Press. https://doi.org/10.1142/9789812567680_0002

[35] Franken P. (1964) Approximation der verteilungen von summen unabhängiger nichtnegativer ganzzahliger zufallsgrössen durch Poissonsche verteilunged. *Math. Nachr.*, **27**(5-6), 303–340. https://doi.org/10.1002/mana.19640270507

[36] Gamkrelidze N.G. (1988) On the application of smoothness function in proving a local limit theorem. *Teor. Veroyatn. Primen.*, **33**(2), 373–376 (Russian). Transl.: *Theory Probab. Appl.*, **33**(2), 352–355, 1989. https://doi.org/10.1137/1133053

[37] Gan H.L., Kolaczyk E.D. (2018) Approximation of the difference of two Poisson-like counts by Skellam. *J. Appl. Probab.*, **55**(2), 416–430. https://doi.org/10.1017/jpr.2018.27

[38] Genest C., Mesfioui M. (2014) Bivariate extensions of Skellam's distribution. *Probability in the Engineering and Informational Sciences*, **28**, 401 – 417. https://doi.org/10.1017/S0269964814000072

[39] Genest C., Marceau E., Mesfioui M. (2003) Compound Poisson approximations for individual models with dependent risks. *Insurance Math. Econom.*, **32**(1), 73–91. https://doi.org/10.1016/S0167-6687(02)00205-6

[40] Geske M.X., Godbole A.P., Schaffner A.A., Skolnick A.M., Wallstrom G.L. (1995) Compound Poisson approximations for word patterns under Markovian hypotheses. *J. Appl. Probab.*, **32**(4), 877–892. https://doi.org/10.2307/3215201

[41] Götze F., Zaitsev A.Yu, Zaporozhets D. (2019) An improved multivariate version of Kolmogorov's second uniform limit theorem. *Zap. Nauchn. Sem. POMI*, **486**, 71–85 (Russian). Transl.: *J. Math. Sci.*, **258**, 782–792, 2021. https://doi.org/10.1007/s10958-021-05594-x

[42] Götze F., Zaitsev A.Yu. (2021) Convergence to infinite-dimensional compound Poisson distributions on convex polyhedra. *Zap. Nauchn. Sem. POMI* 501, 118–125 (Russian). Transl.: J. Math. Sci. 273(5), 732–737 (2023). https://doi.org/10.1007/s10958-023-06536-5

[43] Götze F., Zaitsev A.Yu. (2022) On alternative approximating distributions in the multivariate version of Kolmogorov's second uniform limit theorem. *Teor. Veroyatn. Primen.*, **67**(1), 3–22 (Russian) . Transl.: *Theory Probab. Appl.*, **67**(1), 1–16. https://doi.org/10.1137/S0040585X97T99071X

[44] Herrmann H. (1965) Variationsabstand zwischen der Verteilung einer Summe unabhängiger nichtnegativer ganzzahliger Zufallsgrössen und Poissonschen Verteilungen. *Math. Nachr.*, **29**(5), 265–289. https://doi.org/10.1002/mana.19650290502

[45] Hipp C. (1985) Approximation of aggregate claims distributions by compound Poisson distributions. *Insurance Math. Econom.*, **4**(4), 227–232. Correction: **6**(2), p. 165, 1987. https://doi.org/10.1016/0167-6687(85)90032-0; https://doi.org/10.1016/0167-6687(87)90026-6

[46] Hipp C. (1986) Improved approximations for the aggregate claims distribution in the individual model. *ASTIN Bull.*, **16**(2), 89–100. https://doi.org/10.2143/AST.16.2.2015001

[47] Hwang Y., Kim J.S., Kweon I.S. (2011) Difference-based image noise modeling using Skellam distribution. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **34**, 1329– 1341. https://doi.org/10.1109/TPAMI.2011.224.

[48] Jokubauskienė S., Čekanavičius V. (2025) Multivariate Hipp-type compound Poisson approximations for lattice distributions I. *Lith. Math. J.*, **65**(3), 337–356. https://doi.org/10.1007/s10986-025-09686-y

[49] Karlis D., Ntzoufras I. (2009) Bayesian modeling of football outcomes: Using the Skellam's distribution for the goal difference. *J. Manage. Math.* **20**, 133–146. https://doi.org/10.1093/imaman/dpn026

[50] Khintchine A.Y. (1938) *Limit laws for sums of random variables*. — Moscow: ONTI. (Russian) http://e-heritage.ru/Book/10075626

[51] Kolmogorov A.N. (1956) Two uniform limit theorems for sums of independent random variables. *Teor. Veroyatn. Primen.* **1**(4), 426–436 (Russian). Transl.: Theory Probab. Appl. 1(4), 384–394 (1956). https://doi.org/10.1137/1101030

[52] Kornya P. (1983) Distribution of aggregate claims in the individual risk theory model. *Society of Actuaries: Transactions*, **35**, 823–858. https://api.semanticscholar.org/CorpusID:122820362

[53] Kovach A. (1983) *Bergström Type Asymptotic Expansions for Integer-Valued Random Variables*.(Russian). PhD thesis, Vilnius University.

[54] Kruopis J. (1986) Precision of approximations of the generalized Binomial distribution by convolutions of Poisson measures. *Litovsk. Mat. Sb.*, **26**(1), 53–69 (Russian). Transl.: *Lith. Math. J.*, **26**(1), 37–49. https://doi.org/10.1007/BF00971345

[55] Kruopis J. (1986) Approximations for distributions of sums of lattice random variables I. *Litovsk. Mat. Sb.*, **26**(3), 455–467 (Russian). Transl.: : *Lith. Math. J.*, **26**(3), 234–244. https://doi.org/10.1007/BF01049463

[56] Kruopis J. and Čekanavičius V. (2014) Compound Poisson approximations for symmetric vectors. *J. Multivariate Anal.*, **123**, 30–42. https://doi.org/10.1016/j.jmva.2013.08.017

[57] Le Cam, L. (1965) On the distribution of sums of independent random variables. In: Neyman, J., Le Cam, L.M. (eds) Bernoulli 1713 Bayes 1763 Laplace 1813. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-49749-0_10

[58] Lishamol T., Veena G. (2022) A Retrospective Study on Skellam and Related Distributions. *Austrian Journal of Statistics*, **51**(1), 102–111. https://doi.org/10.17713/ajs.v51i1.1224

[59] Mikhailov V.G. (2001) Estimate of the accuracy of the compound Poisson approximation for the distribution of the number of matching patterns. *Teor. Veroyatn. Primen.*, **46**(4), 713–723 (Russian). Transl.: *Theory Probab. Appl.*, **46**(4), 667–675, 2002. https://doi.org/10.1137/S0040585X97979287

[60] Mattner L., Roos B. (2007) A shorter proof of Kanter's Bessel function concentration bound. *Probab. Theory Rel. Fields*, **139**, 191–205. https://doi.org/10.1007/s00440-006-0043-0

[61] Nagaev S.V., Khodzhabagyan S.S. (1996) On an estimate for the concentration function of sums of independent random variables. *Teor. Veroyatn. Primen.*, **41**(3), 655–665 (Russian). Transl.: *Theory Probab. Appl.*, **41**(3), 560–569, 1997. https://doi.org/10.1137/TPRBAU000041000003000532000001

[62] Novak S.Y. (2011) *Extreme value methods with applications to finance.* London: Chapman & Hall/CRC Press. ISBN 9781439835746

[63] Novak S.Y. (2017) On the length of the longest head run. *Statist. Probab. Lett.*, **130**, 111–114. http://dx.doi.org/10.1016/j.spl.2017.06.020

[64] Presman, E.L. (1973) On a multidimensional version of the Kolmogorov uniform limit theorem. Teor. Veroyatn. Primen. **18**(2), 396–402 (Russian). Transl.: Theory Probab. Appl. 18(2), 378–384 (1974). https://doi.org/10.1137/1118046

[65] Presman E.L. (1983) Approximation of Binomial distributions by infinitely divisible ones. *Teor. Veroyatn. Primen.*, **28**(2), 372–382 (Russian). Transl.: *Theory Probab. Appl.*, **28**(2), 393–403, 1984. https://doi.org/10.1137/1128033

[66] Presman E.L. (1985) Approximation in variation of the distribution of a sum of independent Bernoulli variables with a Poisson law. *Teor. Veroyatn. Primen.*, **30**(2), 391–396 (Russian). Transl.: *Theory Probab. Appl.*, **30**(2), 417–422, 1986. https://doi.org/10.1137/1130051

[67] Rave M., Kauermann G. (2024) The Skellam distribution revisited: Estimating the unobserved incoming and outgoing ICU COVID-19 patients on a regional level in Germany. *Statistical Modelling* **25**(3), 270–288. https://doi.org/10.1177/1471082X241235024

[68] Röllin A. (2007) Translated Poisson approximation using exchangeable pair coupling. *Ann. Appl. Probab.*, **17**(5-6), 1596–1614. https://doi.org/10.1214/105051607000000258

[69] Roos B. (1998) Metric multivariate Poisson approximation of the generalized multinomial distribution. *Teor. Veroyatn. Primen.*, **43**(2), 404–413. *Theory Probab. Appl.*, **43**(2), 306–316, 1999. https://doi.org/10.1137/S0040585X97976921

[70] Roos B. (1999) On the rate of multivariate Poisson convergence. *J. Multivar. Analysis*, **69**(1), 120–134. https://doi.org/10.1006/jmva.1998.1789

[71] Roos B. (2000) Binomial approximation to the Poisson binomial distribution: the Krawtchouk expansion. *Teor. Veroyatn. Primen.* **45**(2), 328–344. Reprinted in: Theory Probab. Appl. 45(2), 258–272 (2001). https://doi.org/10.1137/S0040585X9797821X

[72] Roos B. (2001) Sharp constants in the Poisson approximation. *Statist. Probab. Lett.*, **52**(2), 155–168. https://doi.org/10.1016/S0167-7152(00)00208-X

[73] Roos B. (2001) Multinomial and Krawtchouk approximations to the generalized multinomial distribution. *Teor. Veroyatn. Primen.*, **46**(1), 117–133. Reprinted in: *Theory Probab. Appl.* **46**(1), 103–117, 2002. https://doi.org/10.1137/S0040585X97978750

[74] Roos B. (2002) Kerstan's method in the multivariate Poisson approximation: an expansion in the exponent. *Teor. Veroyatn. Primen.* **43**(2), 397–402. Reprinted in: Theory Probab. Appl. 47(2), 358–363 (2003). https://doi.org/10.1137/TPRBAU000047000002000358000001

[75] Roos B. (2003) Poisson approximation via the convolution with Kornya-Presman signed measures. *Teor. Veroyatn. Primen.*, **48**(3), 628–632. Reprinted in: *Theor. Probab. Appl.*, **48**(3), 555–560, 2004. https://doi.org/10.1137/S0040585X97980646

[76] Roos B. (2005) On Hipp's compound Poisson approximations via concentration functions. *Bernoulli*, **11**(3), 533–557. https://doi.org/10.3150/bj/1120591188

[77] Roos B. (2007) On variational bounds in the compound Poisson approximation of the individual risk model. *Insurance: Mathematics and Economics*, **40**(3), 403–414. https://doi.org/10.1016/j.insmatheco.2006.06.003

[78] Roos B. (2017) Refined total variation bounds in the multivariate and compound Poisson approximation. *ALEA, Lat. Am. J. Probab. Math. Stat.* **14**, 337–360. https://doi.org/10.30757/ALEA.v14-19

[79] Skellam J.G. (1946) The frequency distribution of the difference between two Poisson variates belonging to different populations. *J. Royal Stat. Soc. : Ser. A*, 109–296.

[80] Šiaulys J., Čekanavičius V. (1988) Approximation of distributions of integer-valued additive functions by discrete charges I. *Litovsk. Mat. Sb.*, **28**(4), 795–810 (Russian). Transl.: *Lith. Math. J.*, **28**(4), 392–401. https://doi.org/10.1007/BF00972224

[81] Sundt B., Vernic R. (2009) *Recursions for Convolutions and Compound Distributions with Insurance Applications*. EAA Lecture Notes: Springer. ISBN 9783540929000

[82] Ushakov N.G. (1985) Upper estimates of maximum probability for sums of independent random vectors. *Teor. Veroyatn. Primen.*, **30**(1), 33–43 (Russian). Transl.: *Theory Prob. Appl.*, **30**(1), 38–49, 1986. https://doi.org/10.1137/1130004

[83] Upadhye N.S., Vellaisamy P., Čekanavičius V. (2017) On Stein operators for discrete approximations. *Bernoulli*, **23**(4A), 2828–2859. https://doi.org/10.3150/16-BEJ829

[84] Zaitsev A. Yu. (1980) The estimation of proximity of distribution of sequential sums of independent identically distributed random vectors. *Zap. Nauchn. Sem. LOMI* **97**, 83–87 (Russian). Transl.: J. Soviet Mathematics **24**(5), 536–539 (1984). https://doi.org/10.1007/BF01702330

[85] Zaitsev A.Yu.(1988) Estimates for the closeness of successive convolutions of multidimensional symmetric distributions. *Probab. Theory Rel. Fields* **79**, 175–200. https://doi.org/10.1007/BF00320918

[86] Zaitsev A.Yu. (1989) Multidimensional version of the second uniform limit theorem of Kolmogorov. *Teor. Veroyatn. Primen.*, **34**(1), 128–151 (Russian). Transl.: *Theory Prob. Appl.*, **34**(1), 108–128, 1990. https://doi.org/10.1137/1134003

[87] Zaitsev A.Yu. (1989) Approximation of convolutions of multi-dimensional symmetric distributions by accompanying laws. *Zap. Nauchn. Sem. LOMI AN SSSR* **177**, 55–72 (Russian). Transl.: J. Soviet Mathematics 61(1), 1859–1872 (1992). https://doi.org/10.1007/BF01362793

[88] Zaitsev A.Yu. (1991) An example of a distribution whose set of n-fold convolutions is uniformly separated from the set of infinitely divisible laws in distance in variation. *Teor. Veroyatn. Primen.*, **36**(2), 356–361 (Russian). Transl.: *Theory Probab. Appl.*, **36**(2), 419–425, 1992. https://doi.org/10.1137/1136052

[89] Zaitsev A. Yu. (2023) Estimates of stability with respect to the number of summands for distributions of successive sums of i.i.d. vectors. *Zap. Nauchn. Sem. POMI* **525**, 86–95 (Russian).

[90] Zaitsev A.Yu.(2024) On Proximity of Distributions of Successive Sums with Respect to the Prokhorov Distance. *Teor. Veroyatn. Primen.*, **69**(2), 272–284 (Russian). Transl.: *Theory Prob. Appl.*, **69**(2), 217–226, 2024. https://doi.org/10.1137/S0040585X97T991878

APPENDIX

A.1 One-dimensional estimates

Estimates for convolutions. The following lemma is Lemma 4.6 from [27].

Lemma A.1.1. *Let $P \in \mathscr{F}_1$ be a symmetric distribution concentrated on $\mathbb{Z}\setminus\{0\}$. Assume that P has a finite variance σ^2. Then for any $\lambda > 0$ and $j \in \mathbb{N}$*

$$\left\|(P-I)^{*j} * \mathrm{e}^{\lambda(P-I)}\right\|_{TV} \leqslant \frac{3.6\sqrt{1+\sigma}\, j^{j+1/4}}{\lambda^j \mathrm{e}^j},$$
$$\left\|(P-I)^{*j} * \mathrm{e}^{\lambda(P-I)}\right\|_{\infty} \leqslant 2\left(\frac{j+1/2}{\lambda \mathrm{e}}\right)^{j+1/2}.$$

The next lemma essentially is a partial case of (3.11) for $d = 1$, $k = 3,5$. However, since we need some intermediate estimates, we use a slightly different proof.

Lemma A.1.2. *Let $p \leqslant 1/3$, $n \in \mathbb{N}$. Then*

$$\left\|((1-p)I + pI_1)^{*n} - \exp\left\{n\sum_{j=1}^{3}\frac{(-1)^{j+1}p^j(I_1-I)^{*j}}{j}\right\}\right\|_{TV} \leqslant C\min\left(\frac{p^2}{n}, np^4\right), \quad \text{(A.1)}$$

$$\left\|((1-p)I + pI_1)^{*n} - \exp\left\{n\sum_{j=1}^{5}\frac{(-1)^{j+1}p^j(I_1-I)^{*j}}{j}\right\}\right\|_{TV} \leqslant C\min\left(\frac{p^3}{n^2}, np^6\right), \quad \text{(A.2)}$$

and, for $\tau_1, \tau_2, \ldots, \tau_5 \in [0,1]$,

$$\left\|\exp\left\{\frac{9np(I_1-I)}{10} + n\sum_{j=2}^{5}\frac{\tau_j(-1)^{j+1}p^j(I_1-I)^{*j}}{j}\right\}\right\|_{TV} \leqslant C. \quad \text{(A.3)}$$

Proof. We can write

$$n\sum_{j=2}^{5}\frac{(-1)^{j+1}\tau_j p^j(I_1-I)^{*j}}{j} = 0.298np(I_1-I)^{*2} * \Theta,$$

where

$$\begin{aligned}\|\Theta\|_{TV} &= \frac{1}{0.298}\left\|\sum_{j=2}^{5}\frac{(-1)^{j+1}\tau_j p^{j-1}(I_1-I)^{*(j-2)}}{j}\right\|_{TV}\\ &\leqslant \frac{1}{0.298}\sum_{j=2}^{5}\frac{p^{j-1}\|I_1-I\|_{TV}^{j-2}}{j}\\ &\leqslant \frac{1}{0.298}\sum_{j=2}^{5}\left(\frac{1}{3}\right)^{j-1}\frac{2^{j-2}}{j}<1.\end{aligned}$$

Estimate (A.24) now follows from Lemma A.2.5. Observe that, if $1\leqslant j\leqslant n$, then (A.24) allows to write

$$D^{*j}=\exp\left\{\frac{jp}{10}(I_1-I)\right\}*\tilde{\Theta}, \tag{A.4}$$

for some $\tilde{\Theta}$ and absolute constant C, such that $\|\tilde{\Theta}\|_{TV}\leqslant C$.

Next, observe that without loss of generality we can assume $n\geqslant 6$. Indeed, otherwise, from (A.4) and $\|F^{*n}\|_{TV}=1$ it follows that the RHS of (A.1) (resp. (A.2)) is majorized by $1+C\leqslant 6(1+C)n^{-1}$ (resp. $1+C\leqslant 36(1+C)n^{-2}$).

Applying (A.34) and (A.27) we obtain

$$\begin{aligned}((1-p)I+pI_1)-\exp\left\{n\sum_{j=1}^{3}\frac{(-1)^{j+1}p^j(I_1-I)^{*j}}{j}\right\} &= p^4(I_1-I)^{4*}*\tilde{\Theta}_1,\\ ((1-p)I+pI_1)-\exp\left\{n\sum_{j=1}^{5}\frac{(-1)^{j+1}p^j(I_1-I)^{*j}}{j}\right\} &= p^6(I_1-I)^{6*}*\tilde{\Theta}_2,\end{aligned}$$

where $\|\tilde{\Theta}_1\|_{TV},\|\tilde{\Theta}_2\|_{TV}\leqslant C$. Applying (A.32) for $k=0$ and taking into account (A.4) we see that

$$\begin{aligned}&\left\|((1-p)I+pI_1)^{*n}-\exp\left\{n\sum_{j=1}^{3}\frac{(-1)^{j+1}p^j(I_1-I)^{*j}}{j}\right\}\right\|_{TV}\\ &\quad\leqslant C\sum_{j=1}^{n}\Big\|p^4(I_1-I)^{*4}*((1-p)I+p_1I_1)^{*(n-j)}\\ &\qquad *\exp\{0.1(j-1)p(I_1-I)\}\Big\|_{TV}\\ &\quad\leqslant C\sum_{j\leqslant n/2}p^4\|(I_1-I)^{*4}*((1-p)I+p_1I_1)^{*(n-j)}\|_{TV}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j>n/2} p^4 \|(I_1-I)^{*4} * \exp\{0.1(j-1)p(I_1-I)\}\|_{TV} \\
\leqslant\ & Cnp^4 \|(I_1-I)^{*4} * ((1-p)I + p_1 I_1)^{*\lfloor n/2 \rfloor}\|_{TV} \\
&+Cnp^4 \|(I_1-I)^{*4} * \exp\{0.03np(I_1-I)\}\|_{TV}.
\end{aligned}
$$

To prove the first estimate in (A.1) it remains to apply (A.22) and (A.18). The second estimate follows directly from the fact that total variation of any distribution equals 1 and, therefore,

$$
\begin{aligned}
&\|(I_1-I)^{*4} * ((1-p)I + p_1 I_1)^{*\lfloor n/2 \rfloor}\|_{TV} \leqslant \|I_1 - I\|_{TV}^4 \leqslant 2^4; \\
&\|(I_1-I)^{*4} * \exp\{0.03np(I_1-I)\}\|_{TV} \leqslant \|I_1 - I\|_{TV}^4 \leqslant 2^4.
\end{aligned}
$$

The proof of (A.2) is very similar (one needs to replace $p^4(I_1-I)^{*4}$ by $p^6(I_1-I)^{*6}$) and, therefore, omitted. □

In [7], a new kind of estimate for the closeness of initial and shifted integer-valued distributions was proposed by Barbour and Xia. It was improved by Mattner and Roos in [60]. We formulate this inequality in terms of measures.

Lemma A.1.3. *Let $F \in \mathscr{F}_1$ be concentrated on integers, $n \in \mathbb{N}$. Then*

$$
\|F^{*n} * (I_1 - I)\|_{TV} \leqslant \frac{2\sqrt{2}}{\sqrt{\pi(1/4 + nv)}}. \tag{A.5}
$$

Here

$$
v = 1 - \frac{1}{2} \sum_{j=-\infty}^{\infty} |F\{j\} - F\{j-1\}|.
$$

Estimates for concentration functions. Let $h \geqslant 0$, $F \in \mathscr{F}_1$ and let $Q(F,h) = \sup_x F\{[x, x+h]\}$ denote Lévy's concentration function.

Lemma A.1.4. *Let $F \in \mathscr{F}_1$, $h > 0$, then*

$$
Q(\exp\{\lambda(F-I)\}, h) \leqslant 0.61/\sqrt{\lambda F\{|x| > h\}}, \tag{A.6}
$$

If in addition $\widehat{F}(t) \geqslant 0$ for all $t \in \mathbb{R}$, then

$$
h \int\limits_{|t|<1/h} \widehat{F}(t)\,\mathrm{d}t \leqslant 13 Q(F,h). \tag{A.7}
$$

For the proof of (A.6) see Proposition 3 and Remark 5 in [76]. The proof of (A.7) can be found in [4], Lemma 1.5.

Next lemma illustrates application of concentration function.

Lemma A.1.5. *Let $a_j \geqslant 0$, $j \in \mathbb{Z}$, $0 < \sum_{j\in\mathbb{Z}} a_j < \infty$. Then*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \exp\left\{ - \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j \sin^2(tj/2)\right\} \mathrm{d}t \leqslant 11.22\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j\Big)^{-1/2}. \quad (A.8)$$

Proof. We have

$$- \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j \sin^2(tj/2) = \sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{a_j}{2}(\cos(tj)-1) = \frac{a}{2}(Re\,\widehat{V}(t)-1),$$
$$a = \sum_{j\in\mathbb{Z}\setminus\{0\}} a_j,$$

where

$$\widehat{V}(t) = \frac{1}{2a}\sum_{j\in\mathbb{Z}\setminus\{0\}} a_j(\mathrm{e}^{\mathrm{i}tj} + \mathrm{e}^{-\mathrm{i}tj}).$$

Obviously, $V \in \mathscr{F}$ and $V\{\mathbb{Z}\setminus\{0\}\} = 1$ and we can apply (A.7) and (A.6):

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} \exp\{a(\widehat{V}(t)-1)/2\}\,\mathrm{d}t &\leqslant 13Q(\exp\{a(V-I)/2\}, 1/\pi) \\ &\leqslant 7.93(aV\{|x|>1/\pi\}/2)^{-1/2} = \frac{7.93\sqrt{2}}{\sqrt{a}},\end{aligned}$$

which proves the lemma. □

Estimate for the characteristic function. The next property was proved by Franken [35], see also Eq. (D.6) in [30].

Lemma A.1.6. *Let $0 \leqslant p \leqslant 1$, $F \in \mathscr{F}_1$. Then for any $t \in \mathbb{R}$*

$$|(1-p) + p\widehat{F}(t)| \leqslant \exp\{p(1-p)(Re\,\widehat{F}(t)-1)\}. \quad (A.9)$$

Formulas of inversion.
Usually inequalities relating norms and characteristic functions are called *formulas of inversion*. We present one such formula for total variation of integer valued r.v.s.

Lemma A.1.7. *Let $M \in \mathscr{M}_1^{\mathbb{Z}_+}$, $\sum_{k\in\mathbb{Z}_+} |kM\{k\}| < \infty$. Then for any $v \in \mathbb{R}$,*

$b>0$,

$$\|M\|_{TV}^2 \leqslant \frac{1+b\pi}{2\pi}\int\limits_{-\pi}^{\pi}\left(\left|\mathrm{e}^{-\mathrm{i}tv}\widehat{M}(t)\right|^2+b^{-2}\left|\left(\mathrm{e}^{-\mathrm{i}tv}\widehat{M}(t)\right)'\right|^2\right)\mathrm{d}t. \qquad (A.10)$$

The proof of Lemma A.1.7 is given in [66], see also [28], Lemma 5.1.

The following lemma allows to estimate lower bounds for Kolmogorov metric.

Lemma A.1.8. *Let M be a measure of finite variation concentrated on $\mathbb{Z}$, $a\in\mathbb{R}$, $b>1$. There exists an absolute constant $c>0$ such that*

$$\|M\|_{TV} \geqslant \|M\|_K \;\geqslant\; c\left|\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}\widehat{M}(t/b)\mathrm{e}^{-\mathrm{i}ta}\mathrm{d}t\right|, \qquad (A.11)$$

Estimate (A.4) remains valid if $\mathrm{e}^{-t^2/2}$ is replaced by $t\mathrm{e}^{-t^2/2}$.

The proof of Lemma A.1.8 can be found in [80] and [28], Lemma 10.4; see also Lemma D.15 in [30].

Estimates for the tails of CP distributions and SCP measures. The switch from SCP measure to CP distribution can be done in the following way. Let SCP measure P have the following characteristic function

$$\widehat{P}(t)=\exp\left\{\sum_{1\leqslant|\ell|\leqslant N}\lambda_\ell(\mathrm{e}^{\mathrm{i}\ell t}-1)\right\},\quad \lambda_\ell\in\mathbb{R},$$

and let W be CP distribution with the characteristic function

$$\widehat{W}(t)=\exp\left\{\sum_{1\leqslant|\ell|\leqslant N}|\lambda_\ell|(\mathrm{e}^{\mathrm{i}\ell t}-1)\right\}.$$

The following lemma is Lemma 6.4 from [9].

Lemma A.1.9. *For all $k\in\mathbb{Z}$*

$$|P\{k\}|\leqslant\exp\left\{\sum_{1\leqslant|\ell|\leqslant N}(|\lambda_\ell|-\lambda_\ell)\right\}W\{k\}.$$

The next lemma is a version of the so-called Bernstein's inequality, see [4], Theorem 4.1.

Lemma A.1.10. *Let $X_1, X_2, \ldots, X_n$ be independent r.v.s. for some $\tau > 0$ satisfying*

$$\mathbb{E}X_i = 0, \quad |\mathbb{E}X_1^m| \leqslant \frac{1}{2}\mathbb{E}X_i^2 \tau^{m-2} m!, \quad m = 2, 3, \ldots. \tag{A.12}$$

Let $S = X_1 + \ldots + X_n$, $\sigma^2 = Var\, S$. Then

$$\max\left(\mathbb{P}(S \geqslant x), \mathbb{P}(S \leqslant -x)\right) \leqslant \begin{cases} \mathrm{e}^{-x^2/4\sigma^2} & \text{for } 0 \leqslant x \leqslant \sigma^2/\tau, \\ \mathrm{e}^{-x/4\tau} & \text{for } x \geqslant \sigma^2/\tau. \end{cases}$$

Lemma A.1.11. *Let $\lambda > 0$, $F \in \mathscr{F}_1$, $F\{[-\tau, \tau]\} = 1$ and r.v. Y have the following characteristic function*

$$\mathbb{E}\mathrm{e}^{\mathrm{i}tY} = \exp\left\{-a\mathrm{i}t + \lambda\left(\widehat{F}(t) - 1\right)\right\}, \quad a = \lambda \int_{-\infty}^{\infty} xF\{\mathrm{d}x\}. \tag{A.13}$$

Then Y satisfies (A.12) with parameter τ.

Lamma's A.1.11 proof was given in the discussion after Lemma 4.6 in [4], p.p. 36–37.

A.2 Estimates following from one-dimensional results

Switching to one-dimensional estimates. The following simple well-known inequality allows to switch from compound measure to one-dimensional integer-valued measure:

$$\left\| \sum_{j=0}^{\infty} \alpha_j F^{*j} \right\|_{TV} \leqslant \sum_{j=0}^{\infty} |\alpha_j| = \left\| \sum_{j=0}^{\infty} \alpha_j I_j \right\|_{TV} \tag{A.14}$$

where now I_j is one-dimensional distribution concentrated at $j \in \mathbb{Z}_+$. Inequality (A.15) is well-known (for example, though not stated explicitly, it was used in [77, 78]). It can be proved in the following way:

$$\left\| \sum_{j=0}^{\infty} \alpha_j F^{*j} \right\|_{TV} \leqslant \sum_{j=0}^{\infty} |\alpha_j| \left\| F^{*j} \right\|_{TV} = \sum_{j=0}^{\infty} |\alpha_j| = \left\| \sum_{j=0}^{\infty} \alpha_j I_j \right\|_{TV}, \tag{A.15}$$

Here we used the fact that total variation of any distribution equals 1, that is $\|F^j\|_{TV} = 1$.

As an example of application of (A.15) let us consider $F \in \mathscr{F}_d$, $d \geqslant 1$,

$0 < p \leqslant 1/2$. Then

$$\begin{aligned}\|((1-p)I+pF)^{*n} - \exp\{np(F-I)\}\|_{TV} & \\ \leqslant \ \|((1-p)I+pI_1)^{*n} - \exp\{np(I_1-I)\}\|_{TV} & \\ \leqslant \ 2p\min(1,np). &\end{aligned}$$

where the last estimate follows from the well-known estimate for Poisson approximation to the binomial distribution, see [5] (and recall that total variation metric, which was used in [5] is half of the total variation norm).

Possibility to switch from d-dimensional to one-dimensional case holds for total variation metric (and norm). However it is not valid for local or ℓ_r, $r > 1$, norms. Moreover, if F is symmetric, then switching to one-dimensional case can significantly worsen the accuracy of estimate.

Estimates for convolutions.

Lemma A.2.1. *If $\lambda > 0$, then*

$$\sup_{F\in\mathscr{F}_d} \|(F-I)*\mathrm{e}^{\lambda(F-I)}\|_{TV} \ \leqslant \ \frac{\sqrt{2}}{\sqrt{\mathrm{e}\lambda}}, \tag{A.16}$$

$$\sup_{F\in\mathscr{F}_d} \|(F-I)^{*2}*\mathrm{e}^{\lambda(F-I)}\|_{TV} \ \leqslant \ \frac{3}{\mathrm{e}\lambda}, \tag{A.17}$$

$$\sup_{F\in\mathscr{F}_d} \|(F-I)^{*k}*\mathrm{e}^{\lambda(F-I)}\|_{TV} \ \leqslant \ \left(\frac{2k}{\mathrm{e}\lambda}\right)^{k/2}. \tag{A.18}$$

Lemma's results follow from (A.15) and one-dimensional results for $F = I_1$. Inequality (A.16) is Proposition A.2.7 in [6]. Estimate (A.17) was proved by Roos ([72], Lemma 3). Estimate (A.18) follows from (A.16) and (A.28).

If symmetric distribution is concentrated just at 3 points, then some analogue of Lemma A.2.1 holds.

Lemma A.2.2. *If $\lambda > 0$, $k \in \mathbb{N}$, $\vec{x} \in \mathbb{R}^d$, $\lambda > 0$. Then*

$$\|(I_{-\vec{x}}+I_{\vec{x}}-2I)*\exp\left(\lambda(I_{-\vec{x}}+I_{\vec{x}}-2I)\right)\|_{TV} \leqslant \frac{1}{\lambda}, \tag{A.19}$$

$$\left|\left\|(I_{-\vec{x}}+I_{\vec{x}}-2I)*\exp\left(\lambda(I_{-\vec{x}}+I_{\vec{x}}-2I)\right)\right\| - \frac{1}{\lambda}\sqrt{\frac{2}{\pi\mathrm{e}}}\right| \leqslant \frac{C}{\lambda\sqrt{\lambda}}, \tag{A.20}$$

$$\left|\left\|(I_{-\vec{x}}+I_{\vec{x}}-2I)^{*2}*\exp\left(\lambda(I_{-\vec{x}}+I_{\vec{x}}-2I)\right)\right\| - \frac{C^*}{\lambda^2}\right| \leqslant \frac{C}{\lambda^2\sqrt{\lambda}}. \tag{A.21}$$

Here

$$C^* = \mathrm{e}^{-3/2}\sqrt{\frac{3}{\pi}}\left(\mathrm{e}^{\sqrt{3/2}}\sqrt{3-\sqrt{6}}+\mathrm{e}^{-\sqrt{3/2}}\sqrt{3+\sqrt{6}}\right).$$

Estimate (A.19) follows from Lemma 4.6 in [27] and (A.15). For $d=1$ both estimates (A.20) and (A.21) were proved in [27], Lemma 4.7. In view of (A.15), the same proof holds for $d>1$.

Similar bounds are valid for the compound Binomial distribution $Bi(n,p) = (qI+pF)^{*n}$.

Lemma A.2.3. *Let $q=1-p$. If $0<p<1$, $j,n \in \mathbb{N}$, then*

$$\sup_{F\in\mathscr{F}_d} \|(F-I)^{*j} * (qI+pF)^{*n}\|_{TV} \leqslant \binom{n+j}{j}^{-1/2} (pq)^{-j/2}$$
$$\leqslant \sqrt{\mathrm{e}}\, j^{1/4}\left(\frac{n}{n+j}\right)^{n/2}\left(\frac{j}{(n+j)pq}\right)^{j/2}. \quad \text{(A.22)}$$

If $p \leqslant 1/4$, $\tau \in [0,1]$, then

$$\sup_{F\in\mathscr{F}_d} \left\|(qI+pF)^{*n} - \exp\left\{np(F-I) - 0.5np^2(F-I)^{*2}\right\}\right\|_{TV}$$
$$\leqslant Cp^{3/2}n^{-1/2}, \quad \text{(A.23)}$$

$$\sup_{F\in\mathscr{F}_d} \left\|\exp\left\{0.9np(F-I) - 0.5\tau np^2(F-I)^{*2}\right\}\right\|_{TV} \leqslant C. \quad \text{(A.24)}$$

Proof. Estimate (A.22) is a reformulation of Lemma 4 in [71]. Estimate (A.23) follows from Theorem 1 and Remark (c) in [74]. For estimate (A.24) observe that $3np^2/2 \cdot 0.9np \leqslant 0.4$ and, therefore, Lemma B.1 [30] can be applied. □

Next, we present generalization of Lemma B.2 from [30].

Lemma A.2.4. *Let $p_i \in (0,1)$, $(i=i,2,\ldots,n)$, $p_0 = \max_i p_i$, $\tau \in [0,1]$. Then*

$$\sup_{F\in\mathscr{F}_d}\left\|\exp\left\{\frac{(1+p_0)}{2}\sum_{i=1}^n p_i(F-I) - \frac{\tau}{2}\sum_{i=1}^n p_i^2(F-I)^{*2}\right\}\right\|_{TV} \leqslant \frac{3.5}{\sqrt{1-p_0}}.$$

Particularly, if $p_i \equiv p$, then

$$\sup_{F\in\mathscr{F}_d}\left\|\exp\left\{\frac{np(1+p)}{2}(F-I) - \frac{np^2\tau}{2}(F-I)^{*2}\right\}\right\|_{TV} \leqslant \frac{3.5}{\sqrt{1-p}}. \quad \text{(A.25)}$$

Proof. From (A.15) it follows that suffices to prove lemma for the one-dimensional case $F = I_1 \in \mathscr{F}_1$. Let

$$M := \frac{(1+p_0)}{2}\sum_{i=1}^{n} p_i(I_1 - I) - \frac{\tau}{2}\sum_{i=1}^{n} p_i^2 (I_1 - I)^{*2}.$$

We denote by i complex unit, that is, $\mathrm{i}^2 = -1$. Set $\mathrm{e}^{\mathrm{i}x} := \cos x + \mathrm{i}\sin x$ and let $\widehat{M}(t)$ be Fourier transform of M. Set $a = \sum_{i=1}^{n} p_i$, $\nu = a(1+p_0)/2$ and

$$\widehat{M}_c(t) := \widehat{M}(t) - \mathrm{i}t\nu = \frac{a(1+p_0)}{2}\left(\mathrm{e}^{\mathrm{i}t} - 1 - \mathrm{i}t\right) - \frac{\tau}{2}\sum_{i=1}^{n} p_i^2 (\mathrm{e}^{\mathrm{i}t} - 1)^2,$$

$$b = 2.634\sqrt{\frac{a}{1-p_0}}.$$

Then

$$\begin{aligned}\left|\mathrm{e}^{\widehat{M}_c(t)}\right| &\leqslant \exp\left\{-(1+p_0)a\sin^2(t/2) + \frac{p_0}{2}a|\mathrm{e}^{\mathrm{i}t} - 1|^2\right\} \\ &\leqslant \exp\left\{-(1-p_0)a\sin^2(t/2)\right\}.\end{aligned}$$

Observe that

$$|\widehat{M}_c'(t)| \leqslant \frac{a(1+p_0)}{2}|\mathrm{e}^{\mathrm{i}t} - 1| + ap_0|\mathrm{e}^{\mathrm{i}t} - 1| \leqslant 4a|\sin(t/2)|.$$

Therefore,

$$\left|\left(\mathrm{e}^{\widehat{M}_c(t)}\right)'\right|^2 \leqslant 16a^2\sin^2(t/2)\mathrm{e}^{-2(1-p_0)a\sin^2(t/2)}.$$

and by Lemma A.5.2

$$b\int_{-\pi}^{\pi}\left|\mathrm{e}^{\widehat{M}_c(t)}\right|^2 \mathrm{d}t + \frac{1}{b}\int_{-\pi}^{\pi}\left|\left(\mathrm{e}^{\widehat{M}_c(t)}\right)'\right|^2 \mathrm{d}t \leqslant \frac{21.54}{1-p_0}.$$

If $b > 2.5$, then lemma's statement follows from the formula of inversion (see (D.10) in [30])

$$\|\mathrm{e}^{M}\|_{TV}^2 \leqslant \frac{1+b\pi}{2\pi}\int\limits_{-\pi}^{\pi}\left(\left|\mathrm{e}^{\widehat{M}_c(t)}\right|^2 + b^{-2}\left|\left(\mathrm{e}^{\widehat{M}_c(t)}\right)'\right|^2\right)\mathrm{d}t.$$

and simple estimate

$$\frac{1+b\pi}{2\pi} \leqslant \frac{b}{2}\left(1 + \frac{2}{5\pi}\right) \leqslant 0.5637\, b.$$

If $b \leqslant 2.5$, then $a \leqslant 0.901(1-p_0)$, $a(1+3p_0) \leqslant 0.901(1-p_0)(1+3p_0) \leqslant 1.2014$ and

$$\|\mathrm{e}^M\|_{TV} \leqslant \exp\left\{\frac{a(1+p_0)}{2}\|I_1 - I\|_{TV} + \frac{ap_0}{2}\|I_1 - I\|_{TV}^2\right\} \leqslant \mathrm{e}^{a(1+3p_0)} \leqslant 3.33.$$

□

The next lemma allows to switch from SCP measure to CP distribution.

Lemma A.2.5. *Let $a, b > 0$, $0 < 3b/a \leqslant c < 1$ and let Θ denote a measure satisfying $\|\Theta\|_{TV} \leqslant 1$. Then there exists absolute constant C such that for any $P \in \mathscr{F}_d$*

$$\left\| \exp\{a(P-I) + b(P-I)^{*2} * \Theta\} \right\|_{TV} \leqslant C.$$

Lemma A.2.5 was proved for $P \in \mathscr{F}_1$ in [28], Lemma 2.5 (see also Lemma B.1 in [30]). However, it is not difficult to check that the same proof, which involves only general properties of total variation norm and (A.16) holds for $P \in \mathscr{F}_d$.

Lemma A.2.6. *Let $n, s \in \mathbb{N}$, $p \leqslant 1/4$. Then*

$$\sup_{V \in \mathscr{F}_d} \left\| \exp\left\{\frac{7np(V-I)}{8} + n\sum_{j=2}^{s} \frac{(-1)^{j+1} p^j (V-I)^{*j}}{j}\right\} \right\|_{TV} \leqslant C. \qquad \text{(A.26)}$$

Proof. It suffices to observe, that $\|V - I\|_{TV} \leqslant 2$ and, therefore,

$$\left\| \sum_{j=2}^{s} \frac{(-1)^{j+1} p^{j-1} (V-I)^{*(j-2)}}{j} \right\|_{TV} \leqslant \frac{p}{2} \sum_{j=2}^{\infty} (2p)^{j-2} \leqslant \frac{p}{2(1-2p)} \leqslant \frac{1}{4}.$$

Thus, we have

$$\exp\left\{\frac{7np(V-I)}{8} + \frac{np}{4}(V-I)^{*2} * \Theta\right\}$$

and Lemma A.2.5 can be applied. □

A.3 Norms and convolutions

Properties of norms. Let $M, V \in \mathscr{M}_d$, $F \in \mathscr{F}_d$. Then

$$\|M * V\|_{TV} \leqslant \|M\|_{TV} \|V\|_{TV}, \quad \|\exp\{M\}\|_{TV} \leqslant \exp\{\|M\|_{TV}\}, \quad \text{(A.27)}$$

$$\|M*V\|_K \leqslant \|M\|_{TV}\|V\|_K, \qquad \|F\|_{TV} = 1. \tag{A.28}$$

$$\|(F-I)^{*k} * \exp\{a(F-I)\}\|_{TV} \leqslant \left\|(F-I) * \exp\left\{\frac{a}{k}(F-I)\right\}\right\|_{TV}^{k} \tag{A.29}$$

Total variation norm of any CP distribution is equal to 1, since CP distribution also belongs to $\mathscr{F}_d$.

Taylor's expansion for exponential measure. The following expressions holds for finite measure M, see [28], Eq (1.33):

$$\mathrm{e}^M = I + M + \frac{M^{*2}}{2!} + \ldots + \frac{M^{*k}}{k!} + \frac{M^{*(k+1)}}{k!} * \int_0^1 \mathrm{e}^{\tau M}(1-\tau)^k \mathrm{d}\tau. \tag{A.30}$$

From (A.30) it follows that for any measures M, V

$$\mathrm{e}^M - \mathrm{e}^V = (M-V) * \int_0^1 \mathrm{e}^{\tau A + (1-\tau)V} \mathrm{d}\tau. \tag{A.31}$$

Bergström's identity. Let $M, V \in \mathscr{M}_d$; $n \in \mathbb{N}$, $k \in \{1, \ldots, n-1\}$. Then, see [14]:

$$\begin{aligned} M^{*n} &= V^{*n} + \sum_{m=1}^{k} \binom{n}{m} V^{*(n-m)} * (M-V)^{*m} \\ &\quad + (M-V)^{*(k+1)} * \sum_{m=k+1}^{n} \binom{m-1}{k} M^{*(n-m)} * V^{*(m-k-1)}. \end{aligned} \tag{A.32}$$

If $k = 0$, then (A.32) becomes

$$M^{*n} - V^{*n} = (M-V) * \sum_{m=1}^{n} M^{*(n-m)} * V^{*(m-1)}.$$

Bergstróm's identity is usually combined with

$$\sum_{m=k+1}^{n} \binom{m-1}{k} = \binom{n}{k+1}. \tag{A.33}$$

Inverse Hipp-type measure. The following identity relates initial measure with Hipp type approximation.

Lemma A.3.1. *Let $M \in \mathscr{M}_d$, $s \in \mathbb{N}$. Then*

$$(I+M) * \exp\left\{\sum_{j=1}^{s} \frac{(-1)^j M^{*j}}{j}\right\} - I$$
$$= (-1)^s M^{*(s+1)} * \int_0^1 \exp\left\{\sum_{j=1}^{s} \frac{(-1)^j \tau^j M^{*j}}{j}\right\} \tau^s \mathrm{d}\tau. \quad \text{(A.34)}$$

Lemma's proof is practically identical to the proof of (A.51) and, therefore, omitted.

A.4 Formulas of inversion

We recall that $\vec{x} = (x_1, x_2, \ldots, x_d)$.

Lemma A.4.1. *Let $M \in \mathscr{M}_d$ be concentrated on a subset of $\mathbb{Z}^d$. Then*

$$\|M\|_2^2 = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} |\widehat{M}(\vec{t})|^2 \, \mathrm{d}t_1 \ldots \mathrm{d}t_d, \quad \text{(A.35)}$$

$$\|M\|_\infty \leqslant \frac{1}{(2\pi)^d} \int\limits_{-\pi}^{\pi} \ldots \int\limits_{-\pi}^{\pi} |\widehat{M}(\vec{t})| \, \mathrm{d}t_1 \ldots \mathrm{d}t_d. \quad \text{(A.36)}$$

If, for some $j \in \{1,2,\ldots,d\}$, $\sum_{\vec{x}} |x_j| |M\{\vec{x}\}| < \infty$, then for any $a \in \mathbb{R}$

$$|x_j - a| \, |M\{\vec{x}\}| \leqslant \frac{1}{(2\pi)^d} \int\limits_{-\pi}^{\pi} \ldots \int\limits_{-\pi}^{\pi} \left| \frac{\partial}{\partial t_j} \left(\widehat{M}(\vec{t}) \mathrm{e}^{-\mathrm{i}at_j} \right) \right| \mathrm{d}t_1 \ldots \mathrm{d}t_d. \quad \text{(A.37)}$$

If, for some $j \in \{1,2,\ldots,d\}$, $\sum_{\vec{x}} x_j^2 |M\{\vec{x}\}| < \infty$, then for any $a \in \mathbb{R}$

$$(x_j - a)^2 \, |M\{\vec{x}\}| \leqslant \frac{1}{(2\pi)^d} \int\limits_{-\pi}^{\pi} \ldots \int\limits_{-\pi}^{\pi} \left| \frac{\partial^2}{\partial t_j^2} \left(\widehat{M}(\vec{t}) \mathrm{e}^{-\mathrm{i}at_j} \right) \right| \mathrm{d}t_1 \ldots \mathrm{d}t_d. \quad \text{(A.38)}$$

If, for some $j, m \in \{1,2,\ldots,d\}$, $\sum_{\vec{x}} |x_j x_m| |M\{\vec{x}\}| < \infty$, then for any $a \in \mathbb{R}$

$$|x_j - a| |x_m - b| \, |M\{\vec{x}\}|$$
$$\leqslant \frac{1}{(2\pi)^d} \int\limits_{-\pi}^{\pi} \ldots \int\limits_{-\pi}^{\pi} \left| \frac{\partial^2}{\partial t_j \partial t_m} \left(\widehat{M}(\vec{t}) \mathrm{e}^{-\mathrm{i}at_j - \mathrm{i}bt_m} \right) \right| \mathrm{d}t_1 \ldots \mathrm{d}t_d. \quad \text{(A.39)}$$

Proof. Essentially Lemma A.4.1 was already used in [15] (see, for example, formulas (21.27), (21.28) and (21.29)), see also Lemma 5.1 in [29]). The

proof is based on the following observation. If $M \in \mathscr{M}_d$ be concentrated on a subset of $\mathbb{Z}^d$, then its Fourier transform is 2π periodical: $\widehat{M}(t_1,t_2,\dots,t_d) = \widehat{M}(t_1+2\pi,t_2,\dots,t_d)$. Moreover $\widehat{M}(\pi,t_2,\dots,t_d) = \widehat{M}(-\pi,t_2,\dots,t_d)$ and for $\vec{x} \in \mathbb{Z}^d$

$$M\{\vec{x}\} = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(\vec{x},\vec{t})} \widehat{M}(\vec{t})\, \mathrm{d}t_1 \dots \mathrm{d}t_d. \tag{A.40}$$

Thus, for $x_j \neq a$,

$$\begin{aligned}
\int \widehat{M}(\vec{t})\mathrm{e}^{-\mathrm{i}tx_j}\mathrm{d}t_j &= \frac{-1}{\mathrm{i}(x_j-a)} \int \Big(\widehat{M}(\vec{t})\mathrm{e}^{-\mathrm{i}at_j}\Big)\mathrm{d}\mathrm{e}^{-\mathrm{i}(x_j-a)t_j} \\
&= \frac{\mathrm{i}}{(x_j-a)} \left(\Big(\widehat{M}(\vec{t})\mathrm{e}^{-\mathrm{i}at_j}\Big)\mathrm{e}^{-\mathrm{i}(x_j-a)t_j} - \int \mathrm{e}^{-\mathrm{i}(x_j-a)t}\mathrm{d}\Big(\widehat{M}(\vec{t})\mathrm{e}^{-\mathrm{i}at_j}\Big) \right) \\
&= \frac{\mathrm{i}}{(x_j-a)} \left(\widehat{M}(\vec{t})\mathrm{e}^{-\mathrm{i}tx_j} - \int \frac{\partial}{\partial t_j}\Big(\widehat{M}(\vec{t})\mathrm{e}^{-\mathrm{i}at_j}\Big)\mathrm{e}^{-\mathrm{i}(x_j-a)t_j}\mathrm{d}t_j \right).
\end{aligned}$$

Here

$$\frac{-1}{\mathrm{i}(x_j-a)} = \frac{-\mathrm{i}}{\mathrm{i}^2(x_j-a)} = \frac{\mathrm{i}}{(x_j-a)},$$

since $\mathrm{i}^2 = -1$. Note that $x_j \in \mathbb{Z}_+$, $\sin(x_j\pi) = 0$, and $\cos(x_j\pi) = \cos(-x_j\pi)$. Therefore

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}tx_j}\Big|_{-\pi}^{\pi} &= \cos(x_j\pi) + \mathrm{i}\sin(x_j\pi) - \cos(-x_j\pi) - \mathrm{i}\sin(x_j\pi) \\
&= \cos(x_j\pi) - \cos(-x_j\pi) = 0.
\end{aligned}$$

Similarly

$$\widehat{M}(t_1,\dots,t_{j-1},\pi,t_{j+1},\dots,t_k) = \widehat{M}(t_1,\dots,t_{j-1},-\pi,t_{j+1},\dots,t_k),$$

Therefore,

$$\widehat{M}(\vec{t})\mathrm{e}^{-\mathrm{i}at_j}\Big|_{-\pi}^{\pi} = 0$$

and

$$\begin{aligned}
&(x_j-a)M\{\vec{x}\} \\
&\quad = \frac{-i}{(2\pi)^d} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \frac{\partial}{\partial t_j}\Big(\widehat{M}(\vec{t})\mathrm{e}^{-\mathrm{i}at_j}\Big)\mathrm{e}^{-\mathrm{i}(x_j-a)t_j} \prod_{m\neq j}^{n} \mathrm{e}^{-\mathrm{i}x_m t_m}\, \mathrm{d}t_1 \dots \mathrm{d}t_d.
\end{aligned}$$

The last identity leads to (A.37). All other estimates are obtained similarly.

Estimate (A.35) is a version of Parseval's identity and can be proved

by applying (A.40) for $M * M^{(-)}\{\vec{0}\}$, where $M^{(-)}\{\vec{x}\} = M\{-\vec{x}\}$, so that $\widehat{M}^{(-)}(\vec{t}) = \widehat{M}(-\vec{t})$. □

Lemma A.4.2. *Let $W \in \mathscr{M}_2^{\mathbb{Z}}$, and $\sum_{(k,m)\in\mathbb{Z}^2}(k^2+m^2)|W\{(k,m)\}| < \infty$, then for any $\alpha,\beta \in \mathbb{R}$ and for any $\gamma_1,\gamma_2 > 0$,*

$$\begin{aligned}
\|W\|_{TV}^2 \;\leqslant\; & \frac{(1+\sqrt{\gamma_1}\pi)(1+\sqrt{\gamma_2}\pi)}{4\pi^2} \\
& \times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left(\left|\widehat{W}(\vec{t}\,)\right|^2 + \frac{1}{\gamma_1}\left|\frac{\partial}{\partial t_1}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2 \right. \\
& + \frac{1}{\gamma_2}\left|\frac{\partial}{\partial t_2}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2 \frac{1}{\gamma_1\gamma_2} \\
& \left. + \left|\frac{\partial^2}{\partial t_1\partial t_2}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2 \right)\mathrm{d}t_1\mathrm{d}t_2 \qquad \text{(A.41)}
\end{aligned}$$

Proof. Formula (A.41) can be proved exactly as one-dimensional its version from [66], see also Lemma 5.1 in [28]. For completeness, we give that proof there. Differentiating the Fourier transform

$$\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}t_1\alpha-\mathrm{i}t_2\beta} = \sum_{k,m} W\{(k,m)\}\mathrm{e}^{\mathrm{i}t_1(k-\alpha)+\mathrm{i}t_2(m-\beta)}$$

we establish that

$$\begin{aligned}
&\sum_{k,m\in\mathbb{Z}}(k-\alpha)W\{(k,m)\}\mathrm{e}^{\mathrm{i}t_1k+\mathrm{i}t_2m} = -\mathrm{i}\mathrm{e}^{\mathrm{i}t_1\alpha+\mathrm{i}t_2\beta}\frac{\partial}{\partial t_1}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right), \\
&\sum_{k,m\in\mathbb{Z}}(m-\beta)W\{(k,m)\}\mathrm{e}^{\mathrm{i}t_1k+\mathrm{i}t_2m} = -\mathrm{i}\mathrm{e}^{\mathrm{i}t_1\alpha+\mathrm{i}t_2\beta}\frac{\partial}{\partial t_2}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right), \\
&\sum_{k,m\in\mathbb{Z}}(k-\alpha)(m-\beta)W\{(k,m)\}\mathrm{e}^{\mathrm{i}t_1k+\mathrm{i}t_2m} \\
&\qquad\qquad = \quad -\mathrm{e}^{\mathrm{i}t_1\alpha+\mathrm{i}t_2\beta}\frac{\partial^2}{\partial t_1\partial t_2}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right).
\end{aligned}$$

Therefore Parseval's identity gives us

$$\sum_{k,m} W^2\{(k,m)\} \;=\; \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\widehat{W}(\vec{t}\,)\right|^2\mathrm{d}t_1\mathrm{d}t_1, \qquad \text{(A.42)}$$

$$\sum_{k,m}(k-\alpha)^2 W^2\{(k,m)\}$$
$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial t_1}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2 \mathrm{d}t_1\mathrm{d}t_1, \quad \text{(A.43)}$$

$$\sum_{k,m}(m-\beta)^2 W^2\{(k,m)\}$$
$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{\partial}{\partial t_2}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2 \mathrm{d}t_1\mathrm{d}t_1, \quad \text{(A.44)}$$

$$\sum_{k,m}(k-\alpha)(m-\beta)^2 W^2\{(k,m)\}$$
$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\frac{\partial^2}{\partial t_1\partial t_2}\left(\widehat{W}(\vec{t})\mathrm{e}^{-\mathrm{i}\alpha t_1-\mathrm{i}\beta t_2}\right)\right|^2 \mathrm{d}t_1\mathrm{d}t_1. \quad \text{(A.45)}$$

We also have

$$\sum_{k\in\mathbb{Z}}\left(1+\frac{(k-\alpha)^2}{\gamma_1}\right)^{-1} \leqslant 1+\pi\sqrt{\gamma_1}, \quad \sum_{m\in\mathbb{Z}}\left(1+\frac{(m-\beta)^2}{\gamma_2}\right)^{-1} \leqslant 1+\pi\sqrt{\gamma_2},$$

see (1.46) in [28]. Applying Hölder's inequality we obtain

$$\begin{aligned}\sum_{k,m}|W\{(k,m)\}| &= \sum_{k\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}|W\{(k,m)\}|\left(1+\frac{(k-\alpha)^2}{\gamma_1}\right)^{1/2}\right)\\ &\quad\times\left(1+\frac{(k-\alpha)^2}{\gamma_1}\right)^{-1/2}\\ &\leqslant \left(\sum_{k\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}|W\{(k,m)\}|\left(1+\frac{(k-\alpha)^2}{\gamma_1}\right)^{1/2}\right)^2(1+\pi\sqrt{\gamma_1})\right)^{1/2}\end{aligned}$$

and, therefore,

$$\begin{aligned}\|W\|_{TV}^2 &\leqslant (1+\pi\sqrt{\gamma_1})\sum_{k\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}|W\{(k,m)\}|\left(1+\frac{(k-\alpha)^2}{\gamma_1}\right)^{1/2}\right.\\ &\quad\left.\times\left(1+\frac{(m-\beta)^2}{\gamma_2}\right)^{1/2}\left(1+\frac{(m-\beta)^2}{\gamma_2}\right)^{-1/2}\right)^2\\ &\leqslant (1+\pi\sqrt{\gamma_1})(1+\pi\sqrt{\gamma_2})\\ &\quad\times\sum_{k,m\in\mathbb{Z}}W^2\{(k,m)\}\left(1+\frac{(k-\alpha)^2}{\gamma_1}\right)\left(1+\frac{(m-\beta)^2}{\gamma_2}\right). \quad \text{(A.46)}\end{aligned}$$

Substituting (A.42)–(A.45) into (A.46) we complete the proof of (A.41). □

For lower estimates let

$$\vec{a} = (a_1, a_2, \dots, a_d), \quad \vec{h} = (h_1, h_2, \dots, h_d), \quad \vec{t}_{\mathbf{h}} := \left(\frac{t_1}{h_1}, \frac{t_2}{h_2}, \dots, \frac{t_d}{h_d}\right)$$

and let

$$V(\vec{a}, \vec{h}) := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{m=1}^{d} \psi_m(t_m)\, \mathrm{e}^{-\mathrm{i}(\vec{a}, \vec{t}_{\mathbf{h}})}\, \widehat{M}(\vec{t}_{\mathbf{h}})\, \mathrm{d}t_1 \mathrm{d}t_2 \dots \mathrm{d}t_d.$$

Here

$$\psi_j(t_j) = \mathrm{e}^{-t_j^2/2} \qquad \text{or} \qquad \psi_j(t_j) = t_j \mathrm{e}^{-t_j^2/2}.$$

Next lemma is Lemma 5.3 from [29] and holds for both choices of $\psi_j(t_j)$.

Lemma A.4.3. *Let M be a finite measure concentrated on $\mathbb{Z}^d$. Then, for any $\vec{a} \in \mathbb{R}^d$ and $h_j \geqslant 1$, $(j = 1, 2, \dots, d)$, the following inequalities hold:*

$$\begin{aligned} \|M\|_\infty &\geqslant (4\sqrt{2\pi})^{-d} \prod_{j=1}^{d} h_j^{-1} |V(\vec{a}, \vec{h})|, \quad \|M\|_{TV} \geqslant (\sqrt{2\pi})^{-d} |V(\vec{a}, \vec{h})|, \\ \|M\|_r &\geqslant (\sqrt{2\pi})^{-d}\, 5^{-d(r-1)/r} \prod_{j=1}^{d} h_j^{-(r-1)/r} |V(\vec{a}, \vec{h})|, \quad (r > 1). \end{aligned}$$

A.5 Some mathematical facts

The following lemma contains easily verifiable relations (see also (1.32), (1.33) in [28]).

Lemma A.5.1. *Let $a \in \mathbb{R}$, $k \in \mathbb{N}$ and let b be real or complex number. Then*

$$\left| \mathrm{e}^{\mathrm{i}a} - 1 - (\mathrm{i}a) - \frac{(\mathrm{i}a)^2}{2!} - \dots - \frac{(\mathrm{i}a)^k}{k!} \right| \leqslant \frac{|a|^{k+1}}{(k+1)!}, \tag{A.47}$$

$$\mathrm{e}^b = 1 + b + \frac{b^2}{2!} + \dots + \frac{b^k}{k!} + \frac{b^{k+1}}{k!} \int_0^1 \mathrm{e}^{b\tau} (1-\tau)^k \, \mathrm{d}\tau. \tag{A.48}$$

The following lemma follows from Eq. (31) in [54] and [73] Lemma 6.

Lemma A.5.2. *Let $\lambda > 0$. Then*

$$\int_{-\pi}^{\pi} \sin^2(t/2) \mathrm{e}^{-2\lambda \sin^2(t/2)} \mathrm{d}t \leqslant \frac{\sqrt{\pi}}{\lambda^{3/2}}, \tag{A.49}$$

$$\int_{-\pi}^{\pi} \mathrm{e}^{-2\lambda \sin^2(t/2)} \mathrm{d}t \leqslant \frac{\pi}{\sqrt{3\lambda}} \left(1 + \sqrt{\frac{\pi}{2}}\right). \tag{A.50}$$

Lemma A.5.3. *Let x be any real or complex number, $k \in \mathbb{N}$. Then*

$$\left| (1+x) \exp\left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} x^j \right\} - 1 \right| \leqslant |x|^{k+1} \int_0^1 \exp\left\{ \sum_{j=1}^{k} \frac{\tau^j |x|^j}{j} \right\} \tau^k \mathrm{d}\tau.$$

Proof . We have

$$\begin{aligned}
&\left| (1+x) \exp\left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} x^j \right\} - 1 \right| \\
&= \left| \int_0^1 \left((1+x\tau) \exp\left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} x^j \tau^j \right\} \right)' \mathrm{d}\tau \right| \\
&= \left| \int_0^1 \exp\left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} x^j \tau^j \right\} \left(x + (1+x\tau) \sum_{j=1}^{k} (-1)^j x^j \tau^{j-1} \right) \mathrm{d}\tau \right| \\
&= \left| \int_0^1 \exp\left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} x^j \tau^j \right\} \right. \\
&\quad \left. \times \left(x - x + \sum_{j=2}^{k} (-1)^j x^j \tau^{j-1} - \sum_{j=1}^{k} (-1)^{j+1} x^{j+1} \tau^j \right) \mathrm{d}\tau \right| \\
&= \left| \int_0^1 \exp\left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} x^j \tau^j \right\} \left(\sum_{j=2}^{k} (-1)^j x^j \tau^{j-1} - \sum_{j=2}^{k+1} (-1)^j x^j \tau^{j-1} \right) \mathrm{d}\tau \right| \\
&= \left| \int_0^1 \exp\left\{ \sum_{j=1}^{k} \frac{(-1)^j}{j} x^j \tau^j \right\} (-1)^{k+2} x^{k+1} \tau^k \mathrm{d}\tau \right| \\
&\leqslant |x|^{k+1} \int_0^1 \exp\left\{ \sum_{j=1}^{k} \frac{\tau^j |x|^j}{j} \right\} \tau^k \mathrm{d}\tau.
\end{aligned} \tag{A.51}$$

□

For the completeness we formulate Hölder's inequality for sums by $\vec{x} \in \mathbb{R}^d$.

Lemma A.5.4. *Let $\vec{x} \in \mathbb{R}^d$. Then for any functions $a(\vec{x}), b(\vec{x}) \geqslant 0$ the following*

inequality holds

$$\sum_{\vec{x}} a(\vec{x})b(\vec{x}) \leqslant \left(\sum_{\vec{x}} a^2(\vec{x}) \sum_{\vec{x}} b^2(\vec{x})\right)^{1/2}. \tag{A.52}$$

provided that the RHS is finite.

Proof. The required inequality directly follows from one-dimensional Hölder's inequality. Indeed.

$$\begin{aligned}
\sum_{\vec{x}} a(\vec{x})b(\vec{x}) &= \sum_{x_1,x_2,\ldots,x_{d-1}} \sum_{x_d} a(\vec{x})b(\vec{x}) \\
&\leqslant \sum_{x_1,x_2,\ldots,x_{d-1}} \left(\sum_{x_d} a^2(\vec{x})\right)^{1/2} \left(\sum_{x_d} b^2(\vec{x})\right)^{1/2} \\
&\leqslant \sum_{x_1,x_2,\ldots,x_{d-2}} \left(\sum_{x_{d-1},x_d} a^2(\vec{x})\right)^{1/2} \left(\sum_{x_{d-1},x_d} b^2(\vec{x})\right)^{1/2} \leqslant \ldots \\
&\leqslant \left(\sum_{\vec{x}} a^2(\vec{x}) \sum_{\vec{x}} b^2(\vec{x})\right)^{1/2}.
\end{aligned}$$

□

Finally we present some formulas for the partial derivatives ($f_k = f_k(\vec{t})$, $g_k = g_k(\vec{t})$), provided partial derivatives exist:

$$\frac{\partial}{\partial t_j}\left(\prod_{i=1}^n f_i - \prod_{i=1}^n g_i\right) \tag{A.53}$$

$$= \sum_{i=1}^n \frac{\partial f_i}{\partial t_j}\left(\prod_{j\neq i}^n f_j - \prod_{j\neq i}^n g_j\right) + \sum_{i=1}^n \frac{\partial}{\partial t_j}(f_i - g_i)\prod_{j\neq i}^n g_j, \tag{A.54}$$

$$\begin{aligned}
&\frac{\partial^2}{\partial t_j^2}\left(\prod_{k=1}^b f_k - \prod_{k=1}^n g_k\right) \\
&\quad= \sum_{k=1}^n \frac{\partial^2 f_k}{\partial t_j^2}\left(\prod_{m\neq k}^n f_m - \prod_{m\neq k}^n g_m\right) + \sum_{k=1}^n \frac{\partial^2}{\partial t_j^2}(f_k - g_k)\prod_{m\neq k}^n g_m \\
&\qquad+ \sum_{k=1}^n \frac{\partial f_k}{\partial t_j}\sum_{m\neq k}^n \frac{\partial}{\partial t_j}(f_m - g_m)\prod_{l\neq k,m}^n g_l \\
&\qquad+ \sum_{k=1}^n \frac{\partial}{\partial t_j}(f_k - g_k)\sum_{m\neq k}^n \frac{\partial g_m}{\partial t_j}\prod_{l\neq k,m}^n g_l
\end{aligned}$$

$$+\sum_{k=1}^{n}\frac{\partial f_k}{\partial t_j}\sum_{m\neq k}^{n}\frac{\partial f_m}{\partial t_j}\Big(\prod_{l\neq j,k}^{n}f_l-\prod_{l\neq j,k}^{n}g_l\Big), \tag{A.55}$$

$$\begin{aligned}
&\frac{\partial^2}{\partial t_j\partial t_m}\Big(\prod_{k=1}^{n}f_k-\prod_{k=1}^{n}g_k\Big)\\
&\quad=\sum_{k=1}^{n}\frac{\partial^2 f_k}{\partial t_j\partial t_m}\Big(\prod_{l\neq k}^{n}f_l-\prod_{l\neq k}^{n}g_l\Big)+\sum_{k=1}^{n}\frac{\partial f_k}{\partial t_j}\frac{\partial}{\partial t_m}\Big(\prod_{l\neq k}^{n}f_l-\prod_{l\neq k}^{n}g_l\Big)\\
&\qquad+\sum_{k=1}^{n}\frac{\partial^2}{\partial t_j\partial t_m}(f_k-g_k)\prod_{l\neq k}^{n}g_l\\
&\qquad+\sum_{k=1}^{n}\frac{\partial}{\partial t_j}(f_k-g_k)\sum_{l\neq k}\frac{\partial g_l}{\partial t_m}\prod_{i\neq k,l}^{n}g_i.
\end{aligned} \tag{A.56}$$

A.6 Programm code for simulations

Hipp-type approximation of the second order. We will use recursive relation (9.3). Let

$$\begin{aligned}
\widehat{F}_1(t_1) &= \big(1-p_1+p_1\mathrm{e}^{\mathrm{i}t_1}\big)^n,\ \widehat{F}_2(t_2)=\big(1-p_2+p_2\mathrm{e}^{\mathrm{i}t_2}\big)^n,\\
\widehat{H}_1(t_1) &= \exp\Big\{n(p_1+p_1^2)(\mathrm{e}^{\mathrm{i}t_1}-1)-\frac{np_1^2}{2}(\mathrm{e}^{2\mathrm{i}t_1}-1)\Big\}\\
\widehat{H}_2(t_2) &= \exp\Big\{n(p_2+p_2^2)(\mathrm{e}^{\mathrm{i}t_2}-1)-\frac{np_2^2}{2}(\mathrm{e}^{2\mathrm{i}t_2}-1)\Big\},
\end{aligned}$$

so that

$$d=\sum_{m=0}^{n}\sum_{k=0}^{n}|F_1\{m\}F_2\{k\}-H_1\{m\}H_2\{k\}|=S_0+S_1+S_2+S.$$

Here

$$\begin{aligned}
S_0 &= |F_1\{0\}F_2\{0\}-H_1\{0\}H_2\{0\}|+|F_1\{0\}F_2\{1\}-H_1\{0\}H_2\{1\}|\\
&\quad+|F_1\{1\}F_2\{0\}-H_1\{1\}H_2\{0\}|+|F_1\{1\}F_2\{1\}-H_1\{1\}H_2\{1\}|,\\
S_1 &= \sum_{k=2}^{n}\big(|F_1\{0\}F_2\{k\}-H_1\{0\}H_2\{k\}|\\
&\quad+|F_1\{1\}F_2\{k\}-H_1\{1\}H_2\{k\}|\big),\\
S_2 &= \sum_{m=2}^{n}\big(|F_1\{m\}F_2\{0\}-H_1\{m\}H_2\{0\}|\\
&\quad+|F_1\{m\}F_2\{1\}-H_1\{m\}H_2\{1\}|\big),
\end{aligned}$$

$$S = \sum_{m=2}^{n}\sum_{k=2}^{n}|F_1\{m\}F_2\{k\} - H_1\{m\}H_2\{k\}|.$$

Further we denote

$$am0 = F_1\{0\}, \quad am1 = F_1\{1\}, \quad bk0 = F_2\{0\}, \quad bk1 = F_2\{1\},$$
$$z10 = H_1\{0\}, \quad z11 = H_1\{1\}, \quad z20 = H_2\{0\}, \quad z21 = H_2\{1\}.$$

We recall that

$$F_1\{k\} = \binom{n}{k}p_1^k(1-p_1)^{n-k}, \quad F_2\{m\} = \binom{n}{m}p_2^m(1-p_2)^{n-m}.$$

Below we present *r* program code for $n = 100, p_1 = 0.25, p_2 = 0.25$.

```
# Initial values
> n<-100; p1<-0.25; p2<-0.25
> l1<-n*p1*(1+p1); m1<-n*p2*(1+p2)
> l2<--n*p1*p1/2; m2<--n*p2*p2/2
> am0<-exp(-l1-l2); am1<-l1*am0
> bk0<-exp(-m1-m2); bk1<-m1*bk0
> z10<-(1-p1)^n; z11<-n*p1*(1-p1)^(n-1)
> z20<-(1-p2)^n; z21<-n*p2*(1-p2)^(n-1);
# S0
> S0<-abs(am0*bk0-z10*z20)+abs(am0*bk1-z10*z21)+abs(am1*bk0-
    z11*z20)+abs(am1*bk1-z11*z21)
# S1
> A2<-exp(-m1-m2); B2<-m1*A2; S1<-0
> for (k in 2:n){z2=choose(n,k)*p2^k*(1-p2)^(n-k); b0=(m1*B2
    +2*m2*A2)/k; A2<-B2; B2<-b0; S1=S1+abs(am0*b0-z10*z2)+
    abs(am1*b0-z11*z2)}
# S2
> A1<-exp(-l1-l2); B1<-l1*A1; S2<-0
> for (m in 2:n){z1=choose(n,m)*p1^m*(1-p1)^(n-m); a0=(l1*B1
    +2*l2*A1)/m; A1<-B1; B1<-a0; S2=S2+abs(bk0*a0-z20*z1)+
    abs(bk1*a0-z21*z1)}
# S
> A1<-exp(-l1-l2); B1<-l1*A1; S<-0
> for (m in 2:n){ A2<-exp(-m1-m2); B2<-m1*A2; z1=choose(n,m)
    *p1^m*(1-p1)^(n-m); a0=(l1*B1+2*l2*A1)/m; A1<-B1; B1<-a0
    ; for (k in 2:n){z2=choose(n,k)*p2^k*(1-p2)^(n-k); b0=(
    m1*B2+2*m2*A2)/k; A2<-B2; B2<-b0; S=S+abs(z1*z2-a0*b0)}}
#d
> d<-S0+S1+S2+S
> d
```

Changing values of n, p_1, p_2 we get the other results. The following simulation results hold

p_1	p_2	$n = 10$	$n = 100$	$n = 500$	$n = 1000$
0.1	0.1	0.0077784	0.0026235	0.0011614	0.0008194
0.25	0.25	0.0399749	0.0134833	0.0060180	0.0042510
0.5	0.25	0.1140486	0.0491694	0.0228023	0.0162027
0.5	0.5	0.1708265	0.0697217	0.0313571	0.0221368
0.1	0.8	0.3240923	0.3124936	0.1699524	0.1232903
0.8	0.8	0.4397247	0.5062850	0.2593756	0.1823728

Table 3: The values of d

For the total variation we use (9.11) and (9.13). Let

$$S3 = \sum_{k=0}^{n} |H_1\{k\}|, \quad S4 = \sum_{k=n}^{5n} |H_1\{k\}|.$$

Then

```
> library(Rmpfr)
>n<-100; p1=mpfr("0.5", 64); l1<-n*p1*(1+p1); m1<-n*p2*(1+p2
    ); am0<-exp(-l1-l2); am1<-l1*am0;
A1<-exp(-l1-l2); B1<-l1*A1; S3<-A1+B1;  for (m in 2:n){ a0=(
    l1*B1+2*l2*A1)/m; A1<-B1; B1<-a0; S3=S3+abs(a0)}; print(
    S3);
S4<-0; for (m in n:5*n){ a0=(l1*B1+2*l2*A1)/m; A1<-B1; B1<-
    a0; S4=S4+abs(a0)}; print(S4)
```

The following table gives estimates for $\sum_{k=0}^{n} |H_i\{k\}|$.

p_i	$n = 10$	$n = 100$	$n = 500$	$n = 10000$
0.1	<1.00003	<1.0000001	<1	<1.0000001
0.25	< 1.00188345	<1.0000003	<1.0000001	<1
0.5	<1.017936056	<1.0018115	<1.0000023	<1.0000001
0.8	<0.925947374	<1.1014326	<1.0373105	<1.0162221

Table 4: The values of $\sum_{k=0}^{n} |H_i\{k\}|$

We seek precision 10^{-6}. Therefore, for $n = 100; 500; 1000$ and

$p_1, p_2 < 0.8$, in (9.13) we take $T_i = 5$. Then

$$\begin{aligned}\|F^{*n} - H^{*n}\|_{TV} &= d + \sum_{k=0}^{n} |H_1^{*n}\{k\}| \sum_{m=n+1}^{5n} |H_2^{*n}\{m\}| \\ &+ \sum_{k=n+1}^{5n} |H_1^{*n}\{k\}| \sum_{m=0}^{n} |H_2^{*n}\{m\}| \\ &+ \sum_{k=n+1}^{5n} |H_1^{*n}\{k\}| \sum_{m=n+1}^{5n} |H_2^{*n}\{m\}| + r \end{aligned} \quad (A.57)$$

with $r < 6e^{-11}$. The following table gives estimates for $\sum_{k=n}^{5n} |H_i\{k\}|$.

p_i	$n = 10$	$n = 100$	$n = 500$	$n = 10000$
0.1		1.2686e-74	4.9854e-355	7.6471e-704
0.25		3.2093e-38	3.2851e-174	2.2263e-343
0.5		2.8692e-15	1.2885e-60	1.2645e-116
0.8				

Table 5: The values of $\sum_{k=n}^{5n} |H_i\{k\}|$

Taking into account tables A.4 and A.5 we see that $\|F^{*n} - H^{*n}\|_{TV} = d + \varepsilon$, where $\varepsilon \leqslant 10^{-7}$. Therefore, up to 10^{-14} the estimates for the total variation coincide with the estimates for d.

For $n = 10$ we take $T_1 = T_2 = 20$, so that by (9.13)

$$\begin{aligned}\|F^{*10} - H^{*10}\|_{TV} &= d + \sum_{k=0}^{10} |H_1^{*10}\{k\}| \sum_{m=11}^{200} |H_2^{*10}\{m\}| \\ &+ \sum_{k=11}^{200} |H_1^{*10}\{k\}| \sum_{m=0}^{10} |H_2^{*10}\{m\}| \\ &+ \sum_{k=11}^{200} |H_1^{*10}\{k\}| \sum_{m=11}^{200} |H_2^{*10}\{m\}| + r_1 \end{aligned} \quad (A.58)$$

with $r_1 < 1.126 \cdot 10^{-7}$.

For $\sum_{k=11}^{200} |H_1^{*10}\{k\}|$ we use

```
> library(Rmpfr)
>n<-10; p1=mpfr("0.5", 64); l1<-n*p1*(1+p1); l2<- -n*p1*p1/
    2; am0<-exp(-l1-l2); am1<-l1*am0;
A1<-exp(-l1-l2); B1<-l1*A1; S3<-A1+B1;  for (m in 2:n){ a0=(
    l1*B1+2*l2*A1)/m; A1<-B1; B1<-a0; S3=S3+abs(a0)}; print(
```

```
    S3);
S4<-0; for (m in n:200){ a0=(l1*B1+2*l2*A1)/m; A1<-B1; B1<-
    a0; S4=S4+abs(a0)}; print(S4)
```

with different values of p_i and get

$p_i = 0.1$	$p_i = 0.25$	$p_i = 0.5$	$p_i = 0.8$
1.94e-9	5.179e-6	0.0107302402069	0.2308841594187

Table 6: The values of $\sum_{k=11}^{200} |H_i\{k\}|$

Collecting above estimates, for $n = 10$, we get

p_1	p_2	$\|F^{*10} - H^{*10}\|_{TV}$
0.1	0.1	0.007778
0.25	0.25	0.039974
0.5	0.25	0.192787
0.1	0.8	0.554983
0.8	0.8	0.867300

Next we consider the case, when at least one $p_i = 0.8$. If $p_i = 0.8$, then we take $T_i = 10$, so that the remainder term is less than 10^{-10}. We also us the following estimates

$n = 100$	$n = 500$	$n = 1000$
0.00138400318416076407455	9.18e-10	3.13e-17

Table 7: The values of $\sum_{k=n+1}^{10n} |H_i\{k\}|$ for $p = 0.8$

Let $p_1 = p_2 = 0.8$, then

$$\begin{aligned} \|F^{*n} - H^{*n}\|_{TV} &= d + 2\sum_{k=0}^{n} |H_1^{*n}\{k\}| \sum_{m=n+1}^{10n} |H_2^{*n}\{m\}| \\ &\quad + \left(\sum_{k=n+1}^{10n} |H_1^{*n}\{k\}| \right)^2 + r_2, \end{aligned}$$

where $r_2 \leqslant 10^{-9}$. We can check, that up to estimates of order 10^{-9} , estimate for $\|F^{*n} - H^{*n}\|_{TV}$ and d coincide. For $n = 100$ we get

$$\|F^{*n} - H^{*n}\|_{TV} = d + 0.00305068 + \varepsilon = 0.509335 + \varepsilon, \quad \varepsilon < 10^{-7}.$$

For $p_1 = 0.1$, $p_2 = 0.8$ we similarly can write

$$\|F^{*n} - H^{*n}\|_{TV} = d + \sum_{k=0}^{n} |H_1\{k\}| \sum_{m=n+1}^{10n} |H_2\{m\}| + r_3, \quad r_3 < 10^{-7}.$$

If $n = 500$ or $n = 1000$ can check that the total variation and d differ less than by $3 \cdot 10^{-7}$. For $n = 100$ we get

$$\|F^{*100} - H^{*100}\|_{TV} = d + 0.001389 + \varepsilon = 0.313879 + \varepsilon, \quad \varepsilon < 10^{-7}.$$

In above we used recursions. Alternative approach is to use formula of inversion:

$$\begin{aligned} H_1\{m\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda_1(e^{-it_1}-1)+\lambda_2(e^{2it_1}-1)} e^{-it_1 m} dt_1 \\ &= Re\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda_1(e^{-it_1}-1)+\lambda_2(e^{2it_1}-1)} e^{-it_1 m} dt_1\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda_1(\cos t_1-1)+\lambda_2(\cos 2t_1-1)} \cos(\lambda_1 \sin t_1 + \lambda_2 \sin 2t_1 - t_1 m) dt_1, \\ H_2\{k\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mu_1(\cos t_2-1)+\mu_2(\cos 2t_2-1)} \cos(\mu_1 \sin t_2 + \mu_2 \sin 2t_2 - t_2 k) dt_2. \end{aligned}$$

Then

```
# initial values
> n<-10; p1<-0.25; p2<-0.25
> l1<-n*p1*(1+p1); m1<-n*p2*(1+p2)
> l2<--n*p1*p1/2; m2<--n*p2*p2/2
# d
>  d<-0
> for (m in 0:n){z1=choose(n,m)*p1^m*(1-p1)^(n-m); integrand
    <- function(t) exp(l1*(cos(t)-1)+l2*(cos(2*t)-1))*cos(l1
    *sin(t)+l2*sin(2*t)-t*m)/(2*pi); a= integrate(integrand,
     -pi,pi); for (k in 0:n){z2=choose(n,k)*p2^k*(1-p2)^(n-k
    ); integrand2<- function(t) exp(m1*(cos(t)-1)+m2*(cos(2*
    t)-1))*cos(m1*sin(t)+m2*sin(2*t)-t*k)/(2*pi); b=
    integrate(integrand2, -pi,pi);  d=d+abs(z1*z2-a$value*b$
    value)}}
> d
```

The difference of results can be seen in a table (blue for the estimate via Fourier transforms).

p_1	p_2	$n = 10$	$n = 100$	$n = 500$
0.1	0.1	0.007778	0.002623	0.001161
		0.007778	0.002639	0.001428
0.25	0.25	0.039974	0.013483	0.006018
		0.039974	0.013483	0.006235
0.5	0.25	0.114048	0.049169	0.022802
		0.114048	0.049169	0.022911
0.5	0.5	0.170826	0.069721	0.031357
		0.170826	0.069721	0.031357
0.1	0.8	0.324092	0.312493	0.169952
		0.324092	0.312502	0.170197
0.8	0.8	0.4397247	0.506285	0.259375
		0.4397247	0.506285	0.259595

We can see that, in principle, both approaches give very similar results. However, even for such not very large n values, recursions perform much faster. For $n = 1000$ the inverse of Fourier transform took a few hours to perform. On the other hand, if initial probabilities are so small that they are truncated to 0, the recursion approach becomes unapplicable, since then all subsequent probabilities become zeros. Therefore, we used an approach with high number of correct values:

```
> library(Rmpfr)
> n<-1000; p1=mpfr("0.5", 64); p2=mpfr("0.25",64)
```

Two dimensional Skellam approximation. We used formulas of inversion for $F^{*n}\{m,k\}, G^{*n}\{m,k\}$. Calculations for each values of n, p_1, p_2 took 1-48 hours.

```
> library(cubature)

> n<-10; p1<-0.25; p2<-0.25; q1<-1-2*p1; q2<-1-2*p2
> S<-0;  Z<-0; U<-0
> for (m in -n:n){
    integrandz1<- function(x) (q1+2*p1*cos(x[1]))^n*cos(m*x
    [1])/(2*pi);
    z1= hcubature(integrandz1, lowerLimit=-pi, upperLimit=pi
    );
    integranda1<- function(x) exp(2*n*p1*(cos(x[1])-1))*cos(
    x[1]*m)/(2*pi); a1= hcubature(integranda1,lowerLimit=-pi
    ,upperLimit=pi); for (k in -n:n){
```

```
9         integrandz2<- function(x) (q2+2*p2*cos(x[1]))^n*cos(
    k*x[1])/(2*pi);
10        z2= hcubature(integrandz2, lowerLimit=-pi,
    upperLimit=pi);
11        integranda2<- function(x) exp(2*n*p2*(cos(x[1])-1))*
    cos(x[1]*k)/(2*pi); a2= hcubature(integranda2,
    lowerLimit=-pi, upperLimit=pi); S=S+abs(z1$integral*z2$
    integral-a1$integral*a2$integral); Z=Z+ a1$integral*a2$
    integral; U<-U+z1$integral*z2$integral }}
12 > print(U)
13 > print(S)
14 > print(Z)
15 > var<-S+1-Z
16 >  print(var)
```

Summary In Lithuanian (Santrauka)

Įvadas

Tema ir aktualumas

Sudėtinės Puasono (CP) aproksinacijos turi platų teorinį ir praktinį pritaikomumą, išsamią apžvalgą žr. [8, 16, 30, 31, 32, 34, 39, 40, 59, 63, 68] ir ten pateiktuose literatūros šaltiniuose. CP aproksimacijos naudingos aktuarinėje matematikoje, biologijoje, žodžių skaičiavime, ekstremalių verčių teorijoje ir kt. CP aproksimacijos taip pat vaidina svarbų vaidmenį sprendžiant teorinės tikimybių teorijos problemas, tačiau absoliuti dauguma tyrimų atlikti vienmačiu atveju. Daugiamatės CP aproksimacijos yra nepalyginimai mažiau tirtos. Šioje disertacijoje vertinami daugiamačių CP ir ženklą keičiančių sudėtinių Puasono (SCP) aproksimacijų atsitiktinių vektorių sumoms tikslumai. Aproksimacijų pasirinkimą (lydinčiojo dėsnio, Hippo tipo aproksimacija, Skellam skirstinio) lėmė aktuarinės matematikos poreikiai ir geriausios be galo dalios aproksimacijos nepriklausomų atsitiktinių vektorių sumoms teorinė problema (pirmoji tolygioji Kolmogorovo teorema). Toliau aptarsime abi temas.

CP aproksimacijos ir aktuariniai modeliai. Sakykime, p_i yra tikimybė, kad i-toji rizika sukels nuostolį, o $V_i \in \mathscr{F}_d$ (čia ir toliau $\mathscr{F}_d$ yra visų d-mačių skirstinių klasė) yra sąlyginis nuostolių, susijusių su i-ąja rizika, skirstinys. Tada nuostolis turi skirstinį $(1-p_i)I + p_iV_i$; čia I yra išsigimęs nulio taške skirstinys; o *bendrasis n (nepriklausomų) nuostolių skirstinys* yra šių skirstinių sąsūka

$$\prod_{i=1}^{*n}((1-p_i)I + p_iV_i).$$

Natūralu laikyti, kad p_i yra mažos (nuostoliai yra reti). Jei laikysime, kad nuostolių kiekiai yra nepriklausomi ir vienodai pasiskirstę, tai $V_i \equiv V$.

Bendras išmokų pasiskirstymas vaidina labai svarbų vaidmenį aktuarinėje matematikoje, žr. diskusiją [81] 3 psl., tačiau apskaičiuoti bendrą išmokų pasiskirstymą ne visada lengva; net ir tada, kai V_i įgyja sveikuosius skaičius. Todėl naudojamos aproksimacijos, kurios yra lenviau pritaikomos. [81], 10 skyriuje parodyta, kad lydinčiojo CP dėsnio ir Hipp [46] bei Kornya [52] SCP aproksimacijas galima apskaičiuoti naudojant modifikuotą De Pril rekursiją.

[81], 14.1 skyriuje teigiama, kad daugiamačiai $V_i \in \mathscr{F}_d$ taip pat yra svarbūs aktuariniai modeliai, nes jie atspindi daugybę susijusių rizikų. Pavyzdžiui, V_i gali būti trimatis sveikatos, būsto ir automobilio draudimo rizikų pasiskirsty-

mas.

CP ir SCP aproksimacijų taikymas aktuariniame moksle paprastai susideda iš dviejų dalių:

a) aproksimacijos kokioje nors normoje tikslumo vertinimo;

b) algoritmų, skirtų skaitmeniškai apskaičiuoti dominančias charakteristikas.

Monografijoje [81] nagrinėjami skaičiavimo algoritmai. Šios disertacijos rezultatai susiję su aproksimacijos tikslumo įvertinimu. Įdomu tai, kad nors visiems [81] rekursiniams algoritmas reikalingos lokaliosiosios tikimybės, dauguma aproksimacijos rezultatų įrodyti pilnosios variacijos metrikoje. Yra akivaizdus lokaliųjų įverčių trūkumas.

CP aproksimacijos ir tolygiosios Kolmogorovo problemos. CP aproksimacijos vaidina lemiamą vaidmenį sprendžiant vadinamuosius Kolmogorovo uždavinius [51], susijusius su geriausiu įmanomu tikslumu, aproksimuojant visais be galo daliais skirstiniais. Vienmatis atvejis yra pilnai išspręstas, žr. Arako ir Zaicevo monografiją [4].

Daugiamačiu atveju yra sprendinys antrajai Kolmogorovo problemai, susijusiai su geriausia be galo dalia aproksimacija neivienodai paskirsčiusių nepriklausomų vektorių sumoms metrikose, silpnesnėse nei tolygioji, žr. [86]. Pirmoji daugiamatė tolygioji Kolmogorovo teorema turi tik dalinį sprendinį tolygiojoje metrikoje [64].

Simetriniai skirstiniai. Reikšminga disertacijos dalis skirta simetrinių gardelinių vektorių aproksimacijai. Viena vertus, gauti rezultatai gali būti traktuojami kaip Kolmogorovo problemos, susijusios su simetriškais vektoriais, tyrimas. Kita vertus, kaip parodyta [1, 2, 3, 17, 58, 67], simetrinius CP skirstinius galima susieti su tyrimais finansų, medicinos, sporto ir mokslo srityse. Nors simetriniai gardeliniai skirstiniai neturi akivaizdžios aktuarinės interpretacijos, [81] (žr. 12 skyrių) minima, kad atsitiktiniai vektoriai, įgyjantys neigiamas reikšmes, gali būti taikomi modeliuojant neigiamus nuostolių dydžius.

Tikslai ir uždaviniai

1. Pagerinti nepriklausomų vienodai pasiskirsčiusių atsitiktinių vektorių sumos Kolmogorovo aproksimacijos tikslumą, sudarant asimptotinį skleidinį lydinčiajam paslinktam CP dėsniui.

2. Ištirti retiems įvykiams skirtų Hippo tipo aproksimacijų struktūrinius supaprastinimus. Gauti trumpojo asimptotinio skleidinio ir daugiamatės Skelamo aproksimacijos įverčius pilnojoje variacijoje.

3. Gauti nepriklausomų gardelinių atsitiktinių vektorių sumų Hippo tipo aproksimacijų bendruosius lokalius, ℓ_r normos ir marginaliuosius netolygiuosius įverčius.

4. Rasti nepriklausomų vienodai pasiskirsčiusių simetrinių gardelinių vektorių sumos skirstinio pilnosios variacijos įvertį, pasitelkiant lydintįjį sudėtinį Puasono dėsnį ir Hippo tipo SCP matą neretiems įvykiams.

5. Gauti bendruosius lokaliuosius, ℓ_r normos ir marginaliuosius netolygiuosius Hippo tipo aproksimacijų įverčius nepriklausomų simetrinių gardelinių atsitiktinių vektorių sumoms.

6. Gauti bendruosius lokaliuosius, ℓ_r normos, pilnosios variacijos normos ir marginaliuosius netolygiuosius Skellamo aproksimacijos įverčius nepriklausomų simetrinių gardelinių atsitiktinių vektorių sumoms.

7. Pasiūlyti dalinį pirmojo Kolmogorovo uždavinio sprendinį nepriklausomų, vienodai pasiskirsčiusių, gardelinių dvimačių atsitiktinių vektorių, turinčių baigtinius 4 eilės momentus, sekai.

8. Gauti rekursinius sąryšius dvimatėms Hippo tipo aproksimacijoms.

Naujumas

Disertacijoje pateikiami (pastumtųjų) lydinčiųjų CP aproksimacijų ir Hippo tipo SCP aproksimacijų įverčiai atsitiktinių vektorių sumų skirstiniams, išplečiant vienmačius rezultatus daugiamačiu atveju. Patikslinti lokaliosios, Kolmogorovo, pilnosios variacijos ir ℓ_r ($r \geqslant 2$) normų įverčiai. Be to, pirmą kartą gauti netolygieji lokalieji bei marginalieji netolygieji lokalieji įverčiai. Įverčiai pilnosios variacijos normoje gauti taikant silpnesnes prielaidas. Šie rezultatai užpildo kritines spragas daugiamačių bendrų nuostolių skirstinių aktuariniuose modeliuose ir prisideda sprendžiant daugiamatį tolygųjį Kolmogorovo uždavinį. Visi rezultatai yra nauji, o kai kurie iš jų — nauji net ir vienmačiu atveju.

Metodai

Pagrindiniai disertacijoje naudojami įrodymų įrankiai yra sąsūkų ir metrikų savybės, charakteringųjų funkcijų metodas.

1 Pagrindinės sąvokos ir žymėjimai

Žymėjimai

Visų sveikųjų skaičių aibę žymėsime $\mathbb{Z}$, natūraliųjų skaičių aibę žymėsime $\mathbb{N} = \{1,2,\dots\}$, o $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$; $\mathbb{Z}^d$ yra aibių $\mathbb{Z}$ Dekarto sandauga d kartų: $\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}$. Vektoriams $\vec{x},\ \vec{y} \in \mathbb{R}^d$ žymėsime $\vec{x} \leqslant \vec{y}$, kai $x_i \leqslant y_i$ kiekvienam $i = 1,\dots,d$. Vektorių $\vec{x},\ \vec{t} \in \mathbb{R}^d$ skaliarinę sandaugą žymėsime $(\vec{x},\vec{t}) := x_1t_1 + x_2t_2 + \dots + x_dt_d$.

Tegu $L(\vec{X})$ žymi atsitiktinio vektoriaus (a.v.) $\vec{X}$ skirstinį. Tegu $\mathscr{F}_d$ žymi visų d-mačių skirstinių aibę. Simboliu $\mathscr{M}_d$ žymėsime visų baigtinių (ženklą keičiančių) d-mačių matų aibę. Akivaizdu, kad $\mathscr{F}_d \subset \mathscr{M}_d$, $d = 1,2,\dots$. Mato reikšmė aibėje A žymima $M\{A\}$; $M\{\vec{x}\} := M\{\{\vec{x}\}\}$.

Kai rašome, kad skirstinys F (matas M) yra sukoncentruotas aibėje $\mathbb{Z}^d$, turime omenyje, kad matas nėra sukoncentruotas $\mathbb{Z}^d$ hiperplokštumoje. Pavyzdžiui, dvimačiai skirstiniai, sukoncentruoti aibėse $\{(0,0);(1,1)\}$ arba $\{(0,0);(1,0);(2,0)\}$, šioje disertacijoje nėra nagrinėjami.

Tegu $I_{\vec{a}}$ žymi skirstinį, sukoncentruotą vektoriuje $\vec{a} \in \mathbb{R}^d$, o $I = I_{\vec{0}}$; čia $\vec{0} = (0,0,\dots,0) \in \mathbb{R}^d$.

Furje tranformacija. Jei $M \in \mathscr{M}_d$, tai jo Furje transformacija yra $\widehat{M}(t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(\vec{x},\vec{t})} M\{\mathrm{d}x\}$. Jei M sukoncentruotas aibėje $\mathbb{Z}^d$, tai jo Furje transformacija $\widehat{M}(\vec{t}) = \sum_{\vec{m} \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(\vec{m},\vec{t})} M\{\vec{m}\}$. Jei $F \in \mathscr{F}_d$, tuomet $\widehat{F}(\vec{t})$ yra F charakteringoji funkcija (ch.f.). Siekdami trumpesnio žymėjimo, toliau paprastai praleisime priklausomybę nuo $\vec{t}$ ir rašysime $\widehat{M}, \widehat{F}$ vietoj $\widehat{M}(\vec{t}), \widehat{F}(\vec{t})$.

Matų sąsūkos. Tegu $M,V \in \mathscr{M}_d$, tuomet jų sąsūka bet kuriai k-matei Borelio aibei apibrėžiama $(M*V)\{A\} = \int_{\mathbb{R}^d} M\{A-\vec{x}\}V\{\mathrm{d}\vec{x}\}$. Jei M ir V sukoncentruoti aibėje $\mathbb{Z}^d$, tai $(M*V)\{\vec{m}\} = \sum_{\vec{j} \in \mathbb{Z}^d} M\{\vec{m} - \vec{j}\}V\{\vec{j}\}$. Bet kokiems nepriklausomiems a.v. $\vec{X}$ ir $\vec{Y}$ jų sumos skirstinys yra skirstinių sąsūka: $\mathscr{L}(\vec{X}+\vec{Y}) = \mathscr{L}(\vec{X}) * \mathscr{L}(\vec{Y})$. Žinoma, kad $\widehat{M*V} = \widehat{M}\widehat{V}$.

Ženklą keičiančio mato eksponentė. Tegul $M \in \mathscr{M}_d$, tada $\exp\{M\} = \sum_{j \geqslant 0} M^{*j}/j!$; čia laipsniai M^{*j}, suprantami sąsūkos prasme, $M^{*0} = I$. Dviejų eksponentinių matų sąsūka taip pat yra eksponentinis matas: $\exp\{M\} * \exp\{V\} = \exp\{M+V\}$. Eksponentinio mato Furjė transformacija

yra Furjė transformacijos eksponentė: $\widehat{\exp\{M\}} = \exp\{\widehat{M}\}$.

Sudėtinis Puasono skirstinys. Tegu π_λ žymi Puasono atsitiktinį dydį su parametru λ, t. y. $\mathbb{P}(\pi_\lambda = m) = \mathrm{e}^{-\lambda}\lambda^m/m!$ $(m \in \mathbb{Z}_+)$. Sakykime, $\vec{X}, \vec{X}_1, \vec{X}_2, \ldots$ yra nepriklausomi vienodai pasiskirstę atsitiktiniai vektoriai (n.v.p. a.v.) su $\mathscr{L}(\vec{X}) = F$, nepriklausomi nuo π_λ. Tuomet sakome, kad $\vec{Y} = \sum_{j=0}^{\pi_\lambda} \vec{X}_j$ turi sudėtinį Puasono (CP) skirstinį su parametru λ. Turime, kad $\mathscr{L}(\vec{Y}) = \exp\{\lambda(F-I)\}$ ir $\vec{Y}$ ch.f. yra $\exp\{\lambda(\widehat{F}(\vec{t}) - 1)\}$.

Dviejų CP skirstinių sąsūka taip pat yra CP skirstinys. Pastebėtina, kad visi CP skirstiniai yra be galo dalūs. CP skirstinys $\exp\{F-I\}$ vadinamas skirstinio $F \in \mathscr{F}_d$ *lydinčiuoju dėsniu*.

Ženklą keičiantis Puasono matas. Jei $F \in \mathscr{F}_d$, tačiau $\lambda < 0$, tai $\exp\{\lambda(F-I)\}$ vadinamas ženklą keičiančiu sudėtiniu Puasono matu (SCP). SCP taip pat vadinsime matus $\exp\left\{\sum_{j=1}^{\infty} \lambda_j (F_j - I)\right\}$; čia $F_j \in \mathscr{F}_d$ ir bent vienas $\lambda_j < 0$.

Normos. Sakykime, $M \in \mathscr{M}_d$. Tuomet Kolmogorovo ir pilnosios variacijos normos atitinkamai apibrėžiamos

$$\|M\|_K = \sup_{\vec{x} \in \mathbb{R}_d} |M\{(-\infty, x_1] \times (-\infty, x_2] \times \ldots \times (-\infty, x_d]\}|,$$
$$\|M\|_{TV} = \int_{\mathbb{R}^d} 1 |M\{\mathrm{d}\vec{x}\}| = M^+\{\mathbb{R}^d\} + M^-\{\mathbb{R}^d\};$$

čia $M = M^+ - M^-$ yra mato M Hano dekompozicija. Matams $M \in \mathscr{M}_d$ sukoncentruotiems aibėje $\mathbb{Z}^d$ naudosime tolimesnius normų žymėjimus

$$\|M\|_{TV} = \sum_{\vec{x} \in \mathbb{Z}^d} |M\{\vec{x}\}|, \quad \|M\|_\infty = \sup_{\vec{x} \in \mathbb{Z}^d} |M\{\vec{x}\}|,$$
$$\|M\|_r = \left(\sum_{\vec{x} \in \mathbb{Z}^d} |M\{\vec{x}\}|^r \right)^{1/r}, \quad (r \geqslant 1).$$

CP ir SCP aproksimacijos

Pagrindinė mintis, vedanti prie CP ir SCP aproksimacijų, gali būti išreikšta tokiu būdu: tegu $F \in \mathscr{F}_d$, tuomet galime formaliai užrašyti (trumpumo dėlei praleidžiant priklausomybę nuo $\vec{t}$):

$$\widehat{F}^n = \mathrm{e}^{n \ln(1 + (\widehat{F} - 1))} = \exp\left\{ n(\widehat{F} - 1) - \frac{n(\widehat{F} - 1)^2}{2} + \frac{n(\widehat{F} - 1)^3}{3} + \cdots \right\}.$$

Paliekant pirmąjį eilutės narį laipsnio rodiklyje, gaunamas lydintysis CP skirstinys $\exp\{n(F-I)\}$. Laipsnio rodiklyje paliekant ilgesnę logaritminės eilutės dalį, gaunama Hippo tipo SCP aproksimacija:

$$\exp\left\{ n\sum_{k=1}^{s}\frac{(-1)^{k+1}(F-I)^{*k}}{k}\right\}.$$

Lydintysis dėsnis yra bene populiariausia CP aproksimacija, tyrinėta daugybėje mokslinių darbų. Atrodo, kad Presmanas [65] (binominio atsitiktinio dydžio atveju) ir Hippas [46] (bendruoju atveju) buvo pirmieji, panaudoję SCP aproksimacijas, pagrįstas aukščiau išdėstyta mintimi.

Hippo tipo aproksimacijos

Pateiksime žymėjimus, reikalingus Hipo tipo aprosimacijoms. Tegul $F_i \in \mathscr{F}_d$,

$$H_{i,k} = \exp\left\{\sum_{j=1}^{k}(-1)^{j+1}(F_i-I)^{*j}/j\right\}, \quad (i=1,\ldots,n)$$

toks, kad

$$\widehat{H}_{i,k} = \exp\left\{\sum_{j=1}^{k}(-1)^{j+1}(\widehat{F}_i-1)^{j}/j\right\}.$$

Žymėsime

$$\mathbf{F} = \mathscr{L}(\vec{S}_n) = \prod_{i=1}^{*n} F_i, \quad \mathbf{H}_{\mathrm{n,k}} = \prod_{i=1}^{*n} H_{i,k}.$$

2 Asimptotika Le Cam-Presman teoremoje

Vienas iš bendriausių tikimybių teorijos uždavinių yra nepriklausomų vienodai pasiskirsčiusių atsitiktinių dydžių sumos pasiskirstymo aproksimavimas be galo dalių skirstinių klase. Le Camo teorema daugiamačiu atveju teigia, kad bet kurios nepriklausomų vienodai pasiskirsčiusių atsitiktinių vektorių su postūmiu sumos skirstinys gali būti aproksimuotas lydinčiu dėsniu su tikslumu eilės $O(n^{-1/3})$. Parodėme, kad tinkamai parinkus asimptotinį skleidinį aproksimacijos greitį galima pagerinti. Šio skyriaus rezultatai susiję su pirmąja netolygia Kolmogorovo teorema daugiamačiu atveju.

Sakykime, kad F yra d-matis skirstinys. Pažymėkime

$$\Delta_{0,n}(\vec{\alpha},\mathsf{F})=0,\quad \Delta_{m,n}(\vec{\alpha},\mathsf{F}):=\sum_{j=1}^{m}\frac{1}{j!}\left(\frac{-n}{2}\right)^{j}(\mathsf{F}*I_{\vec{\alpha}}-I)^{*2j},$$

kai $m\in\mathbb{N},\vec{\alpha}\in\mathbb{R}^d$.

Teorema 1. *Tegul $m\in\{0,1,2,\dots\}$, $n\in\mathbb{N}$. Tada*

$$\sup_{\mathsf{F}\in\mathscr{F}_d}\inf_{\vec{\alpha}\in\mathbb{R}^d}\|\mathsf{F}^{*n}-I_{-n\vec{\alpha}}*\exp\{n(\mathsf{F}*I_{\vec{\alpha}}-I)\}*(I+\Delta_{m,n}(\vec{\alpha},\mathsf{F}))\|_K \leqslant\frac{C(d,m)}{n^{(m+1)/(2m+3)}}.$$

Matome, kad kiekvienam $m\in\{0,1,2,\dots\}$ reikalingas skirtingas poslinkis, t. y. $\vec{\alpha}=\vec{\alpha}(m)$.

3 Pilnosios variacijos aproksimacija

Pilnoji variacija yra bene populiariausias rodiklis, naudojamas skirstinių, kurių tikimybinė masė yra pakankama ties nuliu, apibūdinimui. Šiame skyriuje pateikiame keletą žinomų Hippo tipo aproksimacijų struktūrinių supaprastinimų.

Visi šio skyriaus rezultatai gauti darant prielaidą, kad atsitiktiniai vektoriai su gana didele tikimybe įgyja nulinę reikšmę. Gerai žinoma, kad aproksimacijai CP skirstiniu tokio apribojimo nereikia. Tačiau norint užtikrinti prasmingą aproksimaciją, kai $k\geqslant 3$, $\mathsf{F}_i\{\vec{0}\}$ negali būti labai artima 0.

Šiame skyriuje gauti nepriklausomų gardelinių atsitiktinių vektorių sumų Hippo tipo aproksimacijų bendrieji lokalieji, ℓ_r normos ir marginalieji netolygieji įverčiai.

4 Gardelinių skirstinių lokalieji įverčiai ir įverčiai ℓ_r metrikoje

Analizuokime d-mačius nepriklausomus a.v. $\vec{X}_i=(X_{i1},X_{i2},\dots,X_{id})$, $(i=1,\dots,n)$, įgyjančius reikšmes $\mathbb{Z}^d$ ir turinčius skirstinius $F_1,F_2\dots,F_n\in\mathscr{F}_d^Z$ (čia ir toliau $\mathscr{F}_d^Z$ žymi visų d-mačių skirstinių, įgyjančių reikšmes aibėje $\mathbb{Z}$, klasę). Mūsų tikslas — aproksimuoti $\vec{S}_n=\sum_{i=1}^n\vec{X}_i$ skirstinį Hippo tipo SCP matais.

Kad būtų trumpiau, rašysime $q_i=\mathbb{P}(\vec{X}_i=\vec{0})$, $p_i(\vec{x})=\mathbb{P}(\vec{X}_i=\vec{x})$, $i=1,2,\dots,n$. Toliau apibrėšime dydžius, naudojamus aproksimacijos tiks-

lumui įvertinti. Tegu $k \in \mathbb{N}$

$$c_{i,1} = c_{i,2} = q_i, \quad c_{i,k} = q_i - \frac{2(1-q_i)^2\left(1-(2(1-q_i))^{k-2}\right)}{2q_i-1}, \quad (k \geqslant 3),$$
$$\lambda_k(\vec{x}) = \max\left(1; \sum_{i=1}^{n} c_{i,k} p_i(\vec{x})\right), \Lambda_{i,k} = \sum_{\vec{x}} \frac{p_i(\vec{x})}{\sqrt{\lambda_k(\vec{x})}}, L_{i,k} = \sum_{\vec{x}} \frac{p_i(\vec{x})}{\lambda_k(\vec{x})}.$$

Koordinačių ašyse sukoncentruotoms tikimybėms ir susijusiems dydžiams naudosime tolimesnius žymėjimus. Tegu $k, m \in \mathbb{N}$

$$p^0_{i,1}(j) = p_i(j,0,\ldots,0), \quad p^0_{i,d}(j) = p_i(0,\ldots,0,j),$$
$$p^0_{i,m}(j) = p_i(0,\ldots,0,j,0,\ldots,0),$$
$$\lambda^0_{m,k} = \max\left(1; \sum_{i=1}^{n} c_{i,k} \sum_{j \in \mathbb{Z}\setminus\{0\}} p^0_{i,m}(j)\right), \quad \omega_{d,k} = \prod_{m=1}^{d} \lambda^0_{m,k},$$

$$b_1 = (\mathrm{e}^2+1)/4, \quad b_2 = 8.5, \quad \text{jei } k \geqslant 3, \text{ tai}$$
$$b_k = \frac{(-1)^k k!}{(8/3+2^{k-1})^{k+1}} \left(\mathrm{e}^{8/3+2^{k-1}} \sum_{m=0}^{k} \frac{(-1)^m \left(8/3+2^{k-1}\right)^m}{m!} - 1\right).$$

Tolimesnė teorema yra žinomų rezultatų apibendrinimas, tačiau galioja esant daug silpnesnėms prielaidoms.

Teorema 2. *Tegu $k \in \mathbb{N}$. Sakykime, $F_i \in \mathscr{F}_d^Z$, $0 < q_i < 1$ ir $c_{i,k} > 0$; $i = 1,\ldots,n$. Tada*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_\infty \leqslant 6^d b_k \left(\frac{2(k+1)}{\mathrm{e}}\right)^{(k+1)/2} \omega_{d,k}^{-1/2} \sum_{i=1}^{n} \Lambda_{i,k}^{k+1}. \qquad (1)$$

Akivaizdu, kad įvertis (1) yra prasmingas tik tada, kai visi $\lambda_k(\vec{x}) > 0$ ir $\lambda^0_{m,k} > 0$. Tuo atveju, kai $\lambda_k(\vec{x}) = 0$ arba $\lambda^0_{m,k} = 0$, laikoma, kad (1) dešinioji pusė lygi ∞, t. y. turime trivialų įvertį.

Toliau nagrinėsime pirmosios eilės asimptotinius skleidinius CP aproksimacijai $\mathbf{H}_{\mathrm{n},1}$ ir SCP aproksimacijai $\mathbf{H}_{\mathrm{n},2}$. Tegu

$$\mathscr{A}_1 = -\frac{1}{2}\sum_{i=1}^{n} (F_i - I)^{*2}, \qquad \mathscr{A}_2 = \frac{1}{3}\sum_{i=1}^{n} (F_i - I)^{*3}.$$

Teorema 3. *Sakykime,* $0 < q_i < 1$, $(i = 1, \ldots, n)$. *Tada*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},1} * (I + \mathscr{A}_1)\|_\infty \leqslant 6^d C \omega_{d,2}^{-1/2} \left(\sum_{i=1}^{n} \Lambda_{i,2}^3 + \left(\sum_{i=1}^{n} \Lambda_{i,2}^2 \right)^2 \right). \qquad (2)$$

Jei $1/2 < q_i < 1$, $(i = 1, 2, \ldots, n)$, *tai*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n},2} * (I + \mathscr{A}_2)\|_\infty \leqslant 6^d C \omega_{d,3}^{-1/2} \left(\sum_{i=1}^{n} \Lambda_{i,3}^4 + \left(\sum_{i=1}^{n} \Lambda_{i,3}^3 \right)^2 \right).$$

Įvertis (2) praplečia atitinkamą asimptotinį rezultatą iš [73], skirtą a.v., sukoncentruotiems $d+1$ taške, bendrajam gardelinių a.v. atvejui.

Toliau marginaliesiems momentams naudosime šį žymėjimą:

$$\mu_{i,j} = \sum_{\vec{x}} x_j p_i(\vec{x}), \quad \mu_{i,jm} = \sum_{\vec{x}} x_j x_m p_i(\vec{x}), \quad \mu_j = \sum_{i=1}^{n} \mu_{i,j},$$
$$\beta_{i,j} = \sum_{\vec{x}} |x_j| p_i(\vec{x}), \quad \beta_{i,jm} = \sum_{\vec{x}} |x_j||x_m| p_i(\vec{x}), \quad \beta_{i,j}^{(2)} = \sum_{\vec{x}} x_j^2 p_i(\vec{x}).$$

Mums nėra žinoma jokių netolygiaisiais lokaliaisiais rezultatais pagrįstų išvadų a.v. sumoms. Norėdami įtraukti atvejį $x_j = \mu_j$, rašome $|x_j - \mu_j|$ įverčio kairėje, o ne dešinėje pusėje.

Teorema 4. *Tegu* $k \in \mathbb{N}$, $F_i \in \mathscr{F}_d^Z$, $0 < q_i < 1$ *ir* $c_{i,k} > 0$, *kai* $i = 1, 2, \ldots, n$. *Sakykime,* $\beta_{i,j}^{(2)} < \infty$, *kai* $j, m \in \{1, 2, \ldots, d\}$. *Tada bet kokiam* $\vec{x} \in \mathbb{Z}^d$,

$$\begin{aligned} \left|x_j - \mu_j\right| \left|\mathbf{F}\{\vec{x}\} - \mathbf{H}_{\mathrm{n,k}}\{\vec{x}\}\right| \quad &\leqslant \quad C(d,k) \omega_{d,k}^{-1/2} \sum_{i=1}^{n} \beta_{i,j} \Lambda_{i,k}^k \\ &\quad + C(d,k) \omega_{d,k}^{-1/2} \sum_{i=1}^{n} \Lambda_{i,k}^{k+1} \sum_{l=1}^{n} \sqrt{\beta_{l,j}^{(2)} L_{l,k}}, \end{aligned}$$

ir

$$\begin{aligned} (x_j - \mu_j)^2 \left|\mathbf{F}\{\vec{x}\} - \mathbf{H}_{\mathrm{n,k}}\{\vec{x}\}\right| \leqslant C(d,k) \omega_{d,k}^{-1/2} \Bigg\{ &\sum_{i=1}^{n} \Lambda_{i,k}^{k+1} \sum_{m=1}^{n} \beta_{m,j}^{(2)} \\ &\times \left(1 + \sum_{l=1}^{n} L_{l,k} \right) + \sum_{i=1}^{n} \left(\beta_{i,j}^{(2)} \Lambda_{i,k}^k + \beta_{i,j}^2 \Lambda_{i,k}^{k-1} \right) \\ &+ \sum_{i=1}^{n} \beta_{i,j} \Lambda_{i,k}^k \sum_{m=1}^{n} \sqrt{\beta_{m,j}^{(2)} L_{m,k}} \Bigg\}. \end{aligned}$$

Jei $\beta_{i,j}^{(2)}, \beta_{i,m}^{(2)} < \infty, \beta_{i,jm} < \infty$, $(i = 1, \ldots, n)$, *tai*

$$
\begin{aligned}
&|x_j - \mu_j||x_m - \mu_m|\,|\mathbf{F}\{\vec{x}\} - \mathbf{H}_{\mathrm{n,k}}\{\vec{x}\}| \\
&\quad\leqslant\; C(d,k)\omega_{d,k}^{-1/2}\Bigg\{ \sum_{i=1}^{n} \Lambda_{i,k}^{k+1} \sum_{l=1}^{n} \left(\beta_{l,jm} + \beta_{l,j}\beta_{l,m}\right) \\
&\qquad + \sum_{l_1=1}^{n} \sqrt{\beta_{l_1,j}^{(2)} L_{l_1,k}} \left(\sum_{i=1}^{n} \beta_{i,m}\Lambda_{i,k}^{k} + \sum_{i=1}^{n} \Lambda_{i,k}^{k+1} \sum_{l_2=1}^{n} \sqrt{\beta_{l_2,m}^{(2)} L_{l_2,k}} \right) \\
&\qquad + \sum_{i=1}^{n} \left(\beta_{i,j}\beta_{i,m}\Lambda_{i,k}^{k-1} + \beta_{i,jm}\Lambda_{i,k}^{k} \right) + \sum_{i=1}^{n} \beta_{i,j}\Lambda_{i,k}^{k} \sum_{l=1}^{n} \sqrt{\beta_{l,m}^{(2)} L_{l,k}} \Bigg\}.
\end{aligned}
$$

Su lokaliaisiais įverčiais glaudžiai susiję įverčiai ℓ_r, $r \geqslant 2$ metrikoje. Atvejis $r = 1$ atitinka įverčius pilnosios variacijos metrikoje. Pažymėtina, kad pilnosios variacijos atveju, augant erdvės dimensijai d, tai gali turėti įtakos konstantoms, tačiau, be šio aspekto, dimensija neturi įtakos tikslumo eilei (žr. [74]). Mes parodome, kad kai $r \geqslant 2$, situacija yra kitokia, ir šie įverčiai yra labai panašūs į įverčius lokaliojoje metrikoje.

Teorema 5. *Tegu* $k \in \mathbb{N}$, $r \geqslant 2$ *ir tegu* $F_i \in \mathscr{F}_d^Z$, $0 < q_i < 1$ *ir* $c_{i,k} > 0$; $i = 1, \ldots, n$. *Tada*

$$
\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_r \leqslant 6^{d(r-1)/r} b_k \left(\frac{2(k+1)}{\mathrm{e}} \right)^{(k+1)/2} \omega_{d,k}^{-(r-1)/2r} \sum_{i=1}^{n} \Lambda_{i,k}^{k+1}.
$$

5 Simetrinių gardelinių skirstinių pilnosios variacijos įverčiai

Darbo [56] 3.2 teoremoje buvo daroma prielaida, kad visi gardeliniai skirstiniai yra sukoncentruoti koordinačių ašyse, turi baigtines dispersijas ir įgyja nulinę reikšmę su tikimybe, ne mažesne už $4/5$. Mes parodome, kad šios prielaidos gali būti gerokai sušvelnintos. Simetrinių d-mačių skirstinių klasę žymėsime $\mathscr{F}_d^S$ ir toliau nagrinėsime

$$
\vec{X}_{S_1} \sim F_{S_1} \in \mathscr{F}_d^S; \qquad \operatorname{supp} F_{S_1} \subset \mathbb{Z}^d, \qquad q_{S_1} := F_{S_1}\{\vec{0}\} \in (0,1). \tag{3}
$$

Prielaidos (3) yra bendresnės už tas, kurios naudotos [56], nes q_{S_1} gali būti pakankamai mažas, o nelygios nuliui tikimybės nėra apribotos tik koordinačių ašimis. Tegul $p_{\vec{x}} = \mathbb{P}(\vec{X}_{S_1} = \vec{x}) = F_{S_1}\{\vec{x}\}$ ir

$$
\delta(y) = \sum_{\vec{x}} \min\left(1; 4y p_{\vec{x}}\right).
$$

Minimoje literatūroje visas dėmesys skiriamas aproksimacijos eilei, o ne konstantų dydžiui. Žemiau esanti teorema suteikia bendrą supratimą, kokių konstantų galima tikėtis esant dideliems n.

Teorema 6. *Tegu galioja (3) prielaida. Tuomet bet kokiam* $n \in \mathbb{N}$

$$\|F_{S_1}^{*n} - \exp\{n(F_{S_1} - I)\}\|_{TV} \leqslant \frac{\delta^2(n/2)}{2n}\left(1 + \frac{C(\delta(nq_{S_1})+1)}{q_{S_1}^4\sqrt{n}}\right) + \frac{C(1-q_{S_1})^{9/2}}{q_{S_1}^5 n\sqrt{n}}.$$

Konkrečios (išreikštinės) konstantos gali būti gautos dviejų CP skirstinių, turinčių tą patį sudėtinį skirstinį F_{S_1}, skirtumui.

Teorema 7. *Tegul* $\operatorname{supp} F_{S_1} = \{\vec{0}, \pm\vec{x}_1, \pm\vec{x}_2, \ldots, \pm\vec{x}_N\}$*,* $N \geqslant d$ *ir galioja (3) prielaida. Tuomet bet kokioms* $a, b > 0$*, teisingas įvertis*

$$\|\exp\{a(F_{S_1} - I)\} - \exp\{b(F_{S_1} - I)\}\|_{TV} \leqslant \frac{|b-a|(N+2)}{\max(a,b)}.$$

6 Lokalieji įverčiai ir įverčiai ℓ_r metrikoje simetriškiems gardeliniams skirstiniams

Šiame skyriuje nepriklausomų gardelinių a.v., turinčių simetrinius pasiskirstymus, sumos skirstinys aproksimuojamas Hippo tipo SCP matais. Nustatomi viršutiniai ir apatiniai rėžiai lokaliai ir ℓ_r, $r \geqslant 1$, metrikoms. Taip pat gaunami marginalieji netolygieji lokalieji įverčiai.

Tarkime, kad $\vec{X}_{S_2,i} = (X_{i1}^{S_2}, X_{i2}^{S_2}, \ldots, X_{id}^{S_2})$, $(i = 1, \ldots, n)$ yra d-mačiai nepriklausomi a.v., įgyjantys reikšmes $\mathbb{Z}^d$ ir turintys simetrinius skirstinius $F_{S_2,1}, F_{S_2,2} \ldots, F_{S_2,n} \in \mathscr{F}_d^S$, t. y. $F_{S_2,i}\{\vec{x}\} = F_{S_2,i}\{-\vec{x}\}$, kai $\vec{x} \in \mathbb{Z}^d$, $i = 1, 2, \ldots, n$. Mūsų tikslas — aproksimuoti sumos $\vec{S}_n^\star = \sum_{i=1}^n \vec{X}_{S_2,i}$ skirstinį Hippo tipo SCP matais.

Tegul $i, j, m, k = 1, 2, \ldots, n$. Pažymėkime

$$p_{S_2,i}(\vec{x}) = \mathbb{P}(\vec{X}_{S_2,i} = \vec{x}), \quad q_{S_2,i} := \mathbb{P}(\vec{X}_{S_2,i} = \vec{0}) = p_{S_2,i}(\vec{0}),$$
$$\lambda_1^{S_2}(\vec{x}) = \max\left(1; \sum_{i=1}^n q_{S_2,i} p_{S_2,i}(\vec{x})\right), \quad \bar{L}_{i,1} = \sum_{\vec{x}} \frac{p_{S_2,i}(\vec{x})}{\lambda_1^{S_2}(\vec{x})},$$
$$\vec{p}_{i,1}^{\,0}(j) = p_{S_2,i}(j, 0, \ldots, 0), \quad \vec{p}_{i,m}^{\,0}(j) = p_{S_2,i}(0, \ldots, 0, j, 0, \ldots, 0),$$
$$\vec{p}_{i,d}^{\,0}(j) = p_{S_2,i}(0, \ldots, 0, j),$$

$$\bar{\lambda}^0_{m,1} = \max\left(1;\ \sum_{i=1}^{n} q_{S_2,i} \sum_{j\in\mathbb{Z}\setminus\{0\}} \bar{p}^0_{i,m}(j)\right);\quad \bar{\omega}_{d,1} = \prod_{m=1}^{d} \bar{\lambda}^0_{m,1}.$$

Teorema 8. *Tegul* $k \in \mathbb{N}$ *ir* $F_{S_2,i} \in \mathscr{F}_d^{\mathbb{Z}}$ *yra simetriniai,* $0 < q_{S_2,i} < 1$, $i = 1,\dots,n$, $r \geqslant 2$. *Tada*

$$\|\mathbf{F} - \mathbf{H}_{\mathrm{n,k}}\|_r \leqslant 3^{d(1-1/r)} 2^{d(1-3/2r)} \bar{\omega}_{d,1}^{(1-r)/2r} \mathrm{e}^{1-k} \left(2(k+1)\right)^{k+1} \sum_{i=1}^{n} \bar{L}_{i,1}^{k+1}.$$

Gautas įvertis galioja esant bendresnėms sąlygoms nei gerai žinomas įvertis, kadangi $k \geqslant 1$ ir nėra jokių papildomų specifinių struktūrinių prielaidų apie $F_{S_2,i}$ gardelės atramą. Pastebėkite, kad žinomo rezultato atveju dėmenų skirstiniai yra įvairūs tų pačių skirstinių, sukoncentruotų koordinačių ašyse, mišiniai.

7 Skellamo aproksimacijos simetrinių gardelinių vektorių sumoms

Tegul $\vec{X}, \vec{X}_1, \dots, \vec{X}_n$ yra nepriklausomi vienodai pasiskirstę a.v., įgyjantys reikšmes aibėje $\mathbb{R}^d$, $\mathscr{L}(\vec{X}) = F$. Taip pat laikysime, kad

1. Egzistuoja toks natūralusis $K \geqslant d$, kad

$$F = qI + \sum_{m=1}^{K} p_m F_m,\ q, p_1, \dots, p_m \in (0,1),\ \sum_{m=1}^{K} p_m = 1 - q. \tag{4}$$

2. Egzistuoja tokie nenuliniai vektoriai $\vec{y}_1, \vec{y}_2, \dots, \vec{y}_K \in \mathbb{R}^d$, kad

$$\operatorname{supp} F_m \subset \{j\vec{y}_m : j \in \mathbb{Z}\setminus\{0\}\},\ \ \operatorname{supp} F_m \cap \operatorname{supp} F_k = \emptyset,\ k \neq m. \tag{5}$$

3. Visi $\operatorname{supp} F_m \neq \emptyset$ ir F_m yra simetriniai:

$$F_m\{j\vec{y}_m\} = F_m\{-j\vec{y}_m\},\quad j \in \{1,2,\dots\};\quad m = 1,\dots,K. \tag{6}$$

4. Egzistuoja toks j_m^0, kad

$$F_m\{j_m^0\vec{y}_m\} > 0,\quad F_m\{(j_m^0+1)\vec{y}_m\} > 0. \tag{7}$$

Žymėjimų supaprastinimui laikysime, kad

$$q_{j,m} = F\{j\vec{y}_m\}, \quad U_m = I_{\vec{y}_m} - I, \quad \bar{U}_m = I_{-\vec{y}_m} - I.$$

Tuomet

$$\begin{aligned} F_m - I &= \sum_{k=1}^{\infty} q_{k,m}(I_{k\vec{y}_m} + I_{-k\vec{y}_m} - 2I) \\ &= \sum_{k=1}^{\infty} q_{k,m}\left((U_m + I)^{*k} + (\bar{U}_m + I)^{*k} - 2I\right). \end{aligned}$$

Skirstinio F lydintis dėsnis yra

$$\exp\{F - I\} = \exp\left\{\sum_{m=1}^{K} p_m \sum_{k=1}^{\infty} q_{k,m}\left((U_m+I)^{*k} - I + (\bar{U}_m+I)^{*k} - I\right)\right\}.$$

Mūsų tikslas — pakeisti lydintįjį dėsnį tam tikru paprastesnės struktūros CP skirstiniu. Atitinkamai pakeisime $(U_m+I)^{*k}-I$ ir $(\bar{U}_m+I)^{*k}-I$ į k^2U_m ir $k^2\bar{U}_m$. Tarkime, kad

$$\sigma_m^2 = \sum_{k=-\infty}^{\infty} k^2 F_m\{k\vec{y}_m\} = 2\sum_{k=1}^{\infty} k^2 q_{k,m}; \quad \mu_{j,m} = 2\sum_{k=1}^{\infty} k^j q_{k,m}.$$

Priminsime, kad σ_m^2 ir $\mu_{k,m}$ yra sveikaskaitės vienmatės skirstinio F_m „projekcijos" į $\vec{y}_m$ dispersija ir k-tasis momentas.

Panašiai kaip ir vienmatės Skellamo aproksimacijos atveju, Puasono skirstinių parametrus parenkame taip, kad abiejų skirstinių dispersijos sutaptų:

$$\begin{aligned} H &= \prod_{m=1}^{*K} H_m, \\ H_m &= \exp\left\{\frac{1}{2} p_m \sigma_m^2 (U_m + \bar{U}_m)\right\} = \exp\left\{-\frac{1}{2} p_m \sigma_m^2 (U_m * \bar{U}_m)\right\}. \end{aligned}$$

Naudosime specialiuosius dydžius:

$$v_m := 1 - \frac{1}{2}\sum_{j=1}^{\infty} |F_m\{j\} - F_m\{j-1\}| = 1 - \frac{1}{2}\sum_{j=1}^{\infty} |q_{j,m} - q_{j-1,m}|. \qquad (8)$$

Apskaičiuojant pilnosios variacijos atstumą tarp gardelinio skirstinio ir jo paslinktos versijos, darbe [7] buvo naudojamas v_m. Tikimybių teorijoje $1-v_m$ yra žinoma kaip glodinimo funkcija, žr. [36] ir ten nurodytus šaltinius.

Pažymėkime

$$\tilde{\Delta} := \frac{1}{q^{13/2}}\left(\sum_{m=1}^{K}\sqrt{\sigma_m}\right)^3 + \frac{1}{\sqrt{q}}\left(\sum_{m=1}^{K}\sigma_m^2\min\left(np_m,\frac{1}{v_m},\frac{1}{qq_{1,m}}\right)\right)^2 + \frac{1}{\sqrt{q}}\sum_{m=1}^{K}\frac{\mu_{4,m}-\sigma_m^2}{p_m}\left(\min\left(np_m,\frac{1}{v_m},\frac{1}{qq_{1,m}}\right)\right)^2.$$

Teoremos, esančios žemiau, rezultatas yra naujas net ir vienmačiu atveju, t. y. kai $d=1$.

Teorema 9. *Tarkime, kad F tenkina (4)–(7) sąlygas, $v_m>0$, $\mu_{4,m}<\infty$, $(m=1,\dots,N)$. Tuomet bet kokiam $n\in\mathbb{N}$*

$$\|F^{*n}-H^{*n}\|_{TV}\leqslant\frac{C\tilde{\Delta}}{n}.$$

8 Sudėtinio Puasono aproksimacija dvimačių gardelinių vektorių sekai

Tegul $\vec{X},\vec{X}_1,\vec{X}_2,\dots$ yra seka nepriklausomų, vienodai pasiskirsčiusių dvimačių a. v., sukoncentruotų $\mathbb{Z}^2$ poaibyje. Tarkime, kad $\vec{X}=(X_1,X_2)$ ir $\mathscr{L}(\vec{X})=F\in\mathscr{F}_2$, $F\{(j,l)\}=\mathbb{P}(X_1=j,X_2=l)$. Sakykime, kad

$$F\{(0,0)\}>0; \tag{9}$$

$$F\{(1,0)\}+F\{(-1,0)\}>0,\quad F\{(0,1)\}+F\{(0,-1)\}>0; \tag{10}$$

$$\mathbb{E}X_1^4+\mathbb{E}X_2^4<\infty. \tag{11}$$

Tegul $\vec{S}_n:=\vec{X}_1+\dots+\vec{X}_n$, taip kad $\mathscr{L}(\vec{S}_n)=F^{*n}$. Pastebėkime, kad $\vec{S}_n$ įgyja reikšmes iš $\mathbb{Z}^2$, todėl CP aproksimacija taip pat turi būti sukoncentruota $\mathbb{Z}^2$. Tegul

$$\delta_1:=n\mathbb{E}X_1-\lfloor n\mathbb{E}X_1\rfloor\in[0,1),\quad \delta_2:=n\mathbb{E}X_2-\lfloor n\mathbb{E}X_2\rfloor\in[0,1);$$

čia $\lfloor\cdot\rfloor$ žymi sveikąją dalį.

Naudojame CP aproksimaciją su charakteringąja funkcija:

$$\widehat{D}_n(\vec{t}) = \exp\left\{\mathrm{i}t_1\lfloor\mathbb{E}X_1\rfloor+\mathrm{i}t_2\lfloor\mathbb{E}X_2\rfloor+\delta_1(\mathrm{e}^{\mathrm{i}t_1}-1)+\delta_2(\mathrm{e}^{\mathrm{i}t_2}-1)+\frac{n}{6}(\widehat{F}(2\vec{t}\,)\widehat{F}^2(-\vec{t}\,)-1)\right\}. \tag{12}$$

Pastebėkime, kad (12) lygybe pateikta struktūra yra šiek tiek paprastesnė nei [25], esančioje 7 teoremoje, nors abiejose aproksimacijose yra ta pati išraiška $\hat{F}(2\vec{t})\hat{F}^2(-\vec{t})-1$, kurios naudojimą tiesiogiai įkvėpė Arako pirmojo Kolmogorovo uždavinio sprendimas. Eksponentinių matų terminais galime užrašyti:

$$\begin{aligned} D_n &= I_{(1,0)}^{*\lfloor n\mathbb{E}X_1\rfloor} * \exp\{\delta_1(I_{(1,0)}-I)\} * I_{(0,1)}^{*\lfloor n\mathbb{E}X_2\rfloor} * \exp\{\delta_2(I_{(0,1)}-I)\} \\ &\quad * \exp\left\{\frac{n}{6}(\mathscr{L}(2\vec{X}_1-\vec{X}_2-\vec{X}_3)-I)\right\}. \end{aligned}$$

Teorema 10. *Tegul galioja (9)–(11) prielaidos, $n\in\mathbb{N}$. Tuomet egzistuoja tokia konstanta $C_1(F)$, kad*

$$\|F^{*n}-D_n\|_{TV} \leqslant \frac{C_1(F)}{n}.$$

9 Dvimatės aproksimacijos ir neišspręstos problemos

Šiame skyriuje nagrinėjame atvejį, kai $d=2$, ir pateikiame įžvalgų dėl galimų sprendimų atviriesiems uždaviniams, suformuluotiems ankstesniuose skyriuose. Be to šiame skyriuje:

1. Gauti pilnosios variacijos įverčiai nevienodai pasiskirsčiusių gardelinių vektorių Hipo tipo CP aproksimacijai, esant žymiai silpnesnėms sąlygoms, nei buvo taikomos Roos'o darbuose [70, 77].

2. Gauti pilnosios variacijos įverčiai nevienodai pasiskirsčiusių simetrinių gardelinių vektorių Hipo tipo CP aproksimacijai, esant žymiai silpnesnėms sąlygoms, nei buvo taikomos Kruopio ir Čekanavičiaus darbe [56].

3. Išvestos kelios rekursinės formulės dvimatėms Hipo tipo aproksimacijoms.

4. Pateikti simuliacijų pavyzdžiai, rodantys labai gerą aproksimacijų tikslumą.

Pagrindinis įrodymų metodas yra charakteringųjų funkcijų metodas, kuris neužtikrina mažų konstantų verčių. Todėl klausimas apie absoliučiųjų konstantų didumą išlieka atviras, tačiau šiame skyriuje jis nenagrinėjamas.

Curriculum Vitae

Name: Simona
Surname: Jokubauskienė (née Staskevičiūtė)
Place of Birth: Marijampolė, Lithuania
E-mail: *simona.jokubauskiene@vdu.lt*

Education and Qualifications:

2013–to date	Vilnius univerity, Vilnius, Lithuania) *Ph.D. studies in Mathematics*
2011–2013	Vytautas Magnus University, Kaunas, Lithuania *Master's degree in Applied Mathematics*
2011–2013	Vytautas Magnus University, Kaunas, Lithuania *Teacher's qualification*
2007–2011	Vytautas Magnus University, Kaunas, Lithuania *Bachelor's degree in Mathematics and Its Applications*

Work experience:

2019–to date	*Lecturer* Department of Mathematics and Statistics, Vytautas Magnus University, Kaunas, Lithuania
2023–to date	*Junior Researcher* VMU STEAM Didactics Centre, Vytautas Magnus University, Kaunas, Lithuania
2016–2021	*Junior Researcher* Department of Mathematics and Statistics, Vytautas Magnus University, Kaunas, Lithuania
2014–2019	*Assistant* Department of Mathematics and Statistics, Vytautas Magnus University, Kaunas, Lithuania
2012–2013	*Senior Laboratory Assistant* Department of Mathematics and Statistics, Vytautas Magnus University, Kaunas, Lithuania
02/2012–03/2012	*Mathematics Teacher* VMU "Rasos" Gymnasium, Kaunas, Lithuania

Additional experience and projects:

2025–to date	*Project Researcher* Project "Fostering Research on Innovation through STEAM-Oriented Interdisciplinary Project Based Learning", Vytautas Magnus University, Kaunas, Lithuania
2020–to date	*Gifted Teacher* International Lithuanian and Israel program for gifted children "VMU Gifted", Vytautas Magnus University, Kaunas, Lithuania
2019–to date	*Project Lecturer* Project "Tęsk: ateik, tobulėk, prisidėk!" (Continue: Join, Grow, Contribute!) Vytautas Magnus University, Kaunas, Lithuania
2020–2021	*Project Researcher* Project "Broad-based university development in the context of transformation of universities' network", Vytautas Magnus University, Kaunas, Lithuania

Study subjects taught at Vytautas Magnus University:

1. MAT2004 "Discrete Mathematics",
   lecture and practical training | since 2016
2. MAT1013 "Probability Theory and Mathematical Statistics",
   part of practical training | since 2016
3. MAT4019 "Random Processes",
   lecture and practical training | since 2017
4. MAT1017 "Mathematical Analysis II",
   lecture | since 2019
5. MAT2018 "Probability Theory and Statistics"
   part of practical training | since 2021
6. MAT1021 "Mathematics 1" (taught in english),
   lecture and practical training | since 2022
7. MAT1022 "Mathematics 2" (taught in english),
   lecture and practical training | since 2023

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Vilniaus universiteto leidykla
Saulėtekio al. 9, III rūmai, LT-10222 Vilnius
El. p. info@leidykla.vu.lt, www.leidykla.vu.lt
bookshop.vu.lt, journals.vu.lt
Tiražas 15 egz.