EM Algorithm for Estimating the Parameters of the Multivariate Stable Distribution

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Abstract. Research of α-stable distributions is especially important nowadays, because they often occur in the analysis of financial data and information flows along computer networks. It has been found that financial data are often leptokurtic with a heavy-tailed distributions; many authors, e.g., Rachev, Mittnik (2000), Kabasinskas *et al*. (2012), Sakalauskas *et al*. (2013) have proved that the most often used normal distribution is not the most suitable way to analysis economic indicators and suggested to replace it with more general, for example, stable distributions. Since Rachev, Mittnik (2000), Kabasinskas *et al*. (2012), Sakalauskas *et al*. (2013) have estimated one-dimensional α-stable distributions a problem arises how to estimate multidimensional data. Maximum likelihood method for the estimation of multivariate α-stable distributions by using EM algorithm is presented in this work. Integrals included in the expressions of the estimates have been calculated using the Gaussian and Gauss-Laguerre quadrature formulas. The constructed model can be used in stock market data analysis.
Keywords: Gaussian and α-stable model, EM algorithm, Likelihood ratio test, Quadrature formulas.

1 Introduction

Stochastic processes can be modeled, estimated and predicted by probabilistic statistical methods, using the data that describes the course of the process. A number of empirical studies confirm that real commercial data are often characterized by skewness, kurtosis and heavy-tail (Janicki, Weron, 1993; Rachev, Mittnik, 2000; Samorodnitsky, Taqqu, 1994; etc.). Therefore, a well-known normal distribution does not always fit – for example, returns of stocks or risk factors are badly fitted by the normal distribution (Kabasinskas *et al*., 2009; Belovas *et al*., 2006). In this case, normal distributions are replaced with more general, for example, stable distributions, which allow to model both leptokurtic and asymmetric (Fielitz, Smith, 1972; Rachev, Mittnik, 2000; Kabasinskas *et al*., 2012; Sakalauskas *et al*., 2013). So stable distributions are the most often used in business and economics data analysis. Following to some experts, the α-stable distribution offers a reasonable improvement if not

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the best choice among the alternative distributions that have been proposed in the literature over the past four decades (e.g., Bertocchi *et al*., 2005; Hoechstoetter *et al*., 2005). However, a practical application of stable distributions is limited by the fact that their distribution and density functions are not expressed through elementary functions, except for a few special cases (Janicki, Weron, 1993; Rachev, Mittnik, 2000; Belovas *et al*., 2006). By the way, stable distributions have infinite variance (except for the normal case, when parameter of stability $\alpha = 2$). In this work the stable multivariate variables expression through normal multivariate vector with random variance, changing by a particular stable law, are used for the simulation.

Although the estimation of parameters of multivariate stable distributions has been discussed long time ago, the problem is not solved to the end yet (Press, 1972; Rachev, Xin, 1993; Nolan, 1998; Davydov, Paulauskas, 1999; Kring *et al*., 2009; Ogata, 2013). Maximum likelihood approach for the estimation of multivariate α-stable distribution by using EM algorithm is presented in this work.

In one-dimension case, random stable value is described by four parameters: stability $\alpha \in (0; 2]$, skewness $\beta \in [-1; 1]$, scale $\sigma > 0$ and position $\mu \in \Re^1$. Stable parameter α is the most important, which is essential for characterizing financial data, and parameter of scale σ can be also used to measure risk. Random variables, that are stable for a fixed number of random elements with respect to the composition, are called α-stable.

In one-dimension case, it is known that $s = \sqrt{s_1} \cdot s_2$, where:

s_1 – random stable variable with skewness parameter $\beta = 1$ and shape parameter $\alpha_1 < 1$;

s_2 – another random stable variable, independent of s_1, with skewness parameter $\beta = 0$ and shape parameter α_2;

s – random stable variable with skewness parameter $\beta = 0$ and shape parameter $\alpha = \alpha_1 \cdot \alpha_2$ (Rachev, Mittnik, 1993; Samorodnitsky, Taqqu, 1994; Ravishanker, Qiou, 1999).

While applying this method, it is usually chosen that s_2 be a random variable, which is normally distributed, i.e., $\alpha_1 = \dfrac{\alpha}{2}$ and $\alpha_2 = 2$. Random stable variable when $\alpha_1 < 1$ and $\beta = 1$ is called stable subordinator, and always obtains only positive values (Rachev, Mittnik, 1993; Ravishanker, Qiou, 1999).

This approach produces multidimensional random vector with dependent components, in which the heavy tailed data can be modeled (Nolan, 2007; Sakalauskas, Vaičiulytė, 2014). In this way, the multivariate stable symmetric vector can be expressed through normally distributed random vector, and α-stable variables (Ravishanker, Qiou, 1999; Rachev, Mittnik, 1993):

$$X = \mu + \sqrt{s_1} \cdot s_2, \tag{1}$$

where

s_1 – subordinator with parameter α ,

s_2 – random vector, distributed by *d*-variate normal law $N(0,\Omega)$,

μ – random vector of mean.

2 Estimators of maximum likelihood approach

Maximum likelihood (ML) approach allows us to obtain the values of parameter sets of model, which maximize the likelihood function for fixed independent uniformly distributed model data sample (Sakalauskas, 2010; Kabasinskas *et al.*, 2009; Ravishanker, Qiou, 1999). The higher the size is the higher is probability to obtain estimators, which will almost not differ from the actual parameter values.

Let's consider probability density of random vector created according to (1). Indeed, the density of the multivariate vector $N(\mu, s \cdot \Omega)$ is as follows:

$$f(x|\mu, s, \Omega) = \frac{s^{-\frac{d}{2}}}{(2\pi)^{\frac{d}{2}} \cdot |\Omega|^{\frac{1}{2}}} \cdot \exp\left[-\frac{(x-\mu)^T \cdot \Omega^{-1} \cdot (x-\mu)}{2 \cdot s}\right]. \tag{2}$$

Let us write down the probability density of α -stable subordinator (Rachev, Mittnik, 2000; Bogdan *et al.*, 2009):

$$f(s|\alpha) = \frac{\alpha \cdot s^{\frac{2}{\alpha-2}}}{2 \cdot |2-\alpha|} \int_{-1}^{1} (U_\alpha(y))^{\frac{\alpha}{2-\alpha}} \cdot \exp\left[-\left(\frac{U_\alpha(y)}{s}\right)^{\frac{\alpha}{2-\alpha}}\right] dy, \tag{3}$$

where, $s \geq 0$, and

$$U_\alpha(y) = \frac{\sin\left(\frac{\pi}{4} \cdot \alpha \cdot (y+1)\right) \cdot \cos\left(\frac{\pi}{4} \cdot (\alpha - (2-\alpha) \cdot y)\right)^{\frac{2-\alpha}{\alpha}}}{\cos\left(\pi \cdot \frac{y}{2}\right)^{\frac{2}{\alpha}} \cdot \cos\left(\frac{\pi \cdot \alpha}{4}\right)^{\frac{2}{\alpha}}}. \tag{4}$$

Thus, probability density of random vector under given parameters μ, Ω, α is expressed as bivariate integral

$$f\left(x\middle|\mu,\Omega,\alpha\right)=\frac{\left(\frac{\alpha}{2-\alpha}\right)}{2\cdot(2\pi)^{\frac{d}{2}}\cdot|\Omega|^{\frac{1}{2}}}\cdot\int_0^{\infty}\int_{-1}^{1}\exp\left[-\frac{1}{2}(x-\mu)^T\frac{\Omega^{-1}}{s}(x-\mu)-\frac{U_\alpha(y)}{s^{\frac{\alpha}{2-\alpha}}}\right]\times \times\frac{U_\alpha(y)}{s^{\frac{\alpha}{2-\alpha}+\frac{d}{2}}}dyds. \tag{5}$$

Let's consider the sample $X=\left(X^1,X^2,\ldots,X^K\right)$ consisting of independent d-variate stable vectors. The likelihood function by virtue of (5) is

$$\tilde{L}(X,\mu,\Omega,\alpha)=\prod_{i=1}^{K}f\left(X^i\middle|\mu,\Omega,\alpha\right)=\frac{\left(\frac{\alpha}{2-\alpha}\right)^K}{2^K\cdot(2\pi)^{\frac{K\cdot d}{2}}\cdot|\Omega|^{\frac{K}{2}}}\times \times\prod_{i=1}^{K}\int_0^{\infty}\int_{-1}^{1}\exp\left[-\frac{1}{2}\left(X^i-\mu\right)^T\frac{\Omega^{-1}}{s_i}\left(X^i-\mu\right)-s_i^{\frac{\alpha}{\alpha-2}}\cdot U_\alpha(y_i)\right]\cdot\frac{U_\alpha(y_i)}{s_i^{\frac{d}{2}+\frac{2}{2-\alpha}}}dy_i ds_i. \tag{6}$$

Denote

$$z_i=s_i^{\frac{\alpha}{\alpha-2}}\cdot U_\alpha(y_i). \tag{7}$$

The log-likelihood function now is as follows

$$L(X,\mu,\Omega,\alpha)=-\sum_{i=1}^{K}\ln\left(f\left(X^i\middle|\mu,\Omega,\alpha\right)\right)= =-\sum_{i=1}^{K}\ln\left(\int_0^{\infty}\exp\{-z_i\}\int_{-1}^{1}B\left(X^i,y_i,z_i,\mu,\Omega,\alpha\right)dy_i dz_i\right), \tag{8}$$

where

$$B\left(X^i,y_i,z_i,\mu,\Omega,\alpha\right)=\frac{1}{2\cdot(2\pi)^{\frac{d}{2}}\cdot|\Omega|^{\frac{1}{2}}\cdot U_\alpha(y_i)}\cdot z_i^{\frac{d\cdot(2-\alpha)}{2\cdot\alpha}}\times \times\exp\left\{-\frac{\left(X^i-\mu\right)^T\Omega^{-1}\left(X^i-\mu\right)}{2\cdot U_\alpha(y_i)}\cdot z_i^{\frac{2-\alpha}{\alpha}}\right\}. \tag{9}$$

Maximum likelihood estimators of multivariate α-stable distribution parameters μ, Ω for a given and fixed α are calculated by equating the likelihood function derivatives of optimized parameters to zero and solving the system of received equations:

$$\begin{cases} \dfrac{\partial L(X,\mu,\Omega,\alpha)}{\partial \mu} = -\sum_{i=1}^{K} \dfrac{\partial f\left(X^i \middle| \mu,\Omega,\alpha\right)}{\partial \mu} \cdot \dfrac{1}{f\left(X^i \middle| \mu,\Omega,\alpha\right)} = 0, \\ \dfrac{\partial L(X,\mu,\Omega,\alpha)}{\partial \Omega} = -\sum_{i=1}^{K} \dfrac{\partial f\left(X^i \middle| \mu,\Omega,\alpha\right)}{\partial \Omega} \cdot \dfrac{1}{f\left(X^i \middle| \mu,\Omega,\alpha\right)} = 0. \end{cases} \tag{10}$$

Let us denote the derivatives

$$\begin{aligned} &\frac{\partial B\left(X^i, y_i, z_i, \mu, \Omega, \alpha\right)}{\partial \mu} = -\frac{1}{2 \cdot (2\pi)^{\frac{d}{2}} \cdot |\Omega|^{\frac{1}{2}} \cdot U_\alpha(y_i)} \cdot z_i^{\frac{(d+2)(2-\alpha)}{2\cdot\alpha}} \times \\ &\times \Omega^{-1}\left(X^i - \mu\right) \cdot \exp\left\{ -\frac{\left(X^i-\mu\right)^T \Omega^{-1}\left(X^i-\mu\right)}{2 \cdot U_\alpha(y_i)} \cdot z_i^{\frac{2-\alpha}{\alpha}} \right\} = \\ &= \frac{\Omega^{-1}\left(X^i-\mu\right)}{U_\alpha(y_i)} \cdot z_i^{\frac{(2-\alpha)}{\alpha}} \cdot B\left(X^i, y_i, z_i, \mu, \Omega, \alpha\right), \end{aligned} \tag{11}$$

$$\begin{aligned} &\frac{\partial B\left(X^i, y_i, z_i, \mu, \Omega, \alpha\right)}{\partial \Omega} = \frac{1}{2 \cdot (2\pi)^{\frac{d}{2}} \cdot |\Omega|^{\frac{1}{2}} \cdot U_\alpha(y_i)} \cdot z_i^{\frac{(d+2)(2-\alpha)}{2\cdot\alpha}} \times \\ &\times \left(-\Omega^{-1} + \Omega^{-1} \cdot (x-\mu) \cdot (x-\mu)^T \cdot \Omega^{-1}\right) \cdot \exp\left\{ -\frac{\left(X^i-\mu\right)^T \Omega^{-1}\left(X^i-\mu\right)}{2 \cdot U_\alpha(y_i)} \cdot z_i^{\frac{2-\alpha}{\alpha}} \right\} = \\ &= \left(-\Omega^{-1} + \frac{\Omega^{-1} \cdot (x-\mu) \cdot (x-\mu)^T \cdot \Omega^{-1}}{U_\alpha(y_i)} \cdot z_i^{\frac{(2-\alpha)}{\alpha}} \right) \cdot B\left(X^i, y_i, z_i, \mu, \Omega, \alpha\right). \end{aligned} \tag{12}$$

Differentiating integrals by the parameters these derivatives are obtained:

$$h(X,\mu,\Omega,\alpha) = \frac{\sum_{i=1}^{K} \dfrac{X^i \cdot g_i}{f_i}}{\sum_{i=1}^{K} \dfrac{g_i}{f_i}}, \tag{13}$$

$$w(X,\mu,\Omega,\alpha) = \sum_{i=1}^{K} \frac{\left(X^i - \hat{\mu}\right)\left(X^i - \hat{\mu}\right)^T g_i}{f_i}. \tag{14}$$

The derivatives of log-likelihood function we can denote in this way:

$$\frac{\partial L}{\partial \mu} = \left(h(X, \mu, \Omega, \alpha) - \mu\right) \cdot \sum_{i=1}^{K} \frac{g_i}{f_i}, \tag{15}$$

$$\frac{\partial L}{\partial \Omega} = -K \cdot \Omega^{-1} + \Omega^{-1} \cdot w(X, \mu, \Omega, \alpha) \cdot \Omega^{-1}. \tag{16}$$

where

$$g(X, \mu, \Omega, \alpha) = \int_0^\infty \int_{-1}^{1} \frac{z^{\frac{2-\alpha}{\alpha}}}{U_\alpha(y)} \cdot B\left(X^i, y, z, \mu, \Omega, \alpha\right) \cdot \exp\{-z\} dydz, \tag{17}$$

$$f(X, \mu, \Omega, \alpha) = \int_0^\infty \int_{-1}^{1} B\left(X^i, y, z, \mu, \Omega, \alpha\right) \cdot \exp\{-z\} dydz. \tag{18}$$

Estimators of parameters satisfy equations of the fixed-point method:

$$\hat{\mu} = \frac{\sum_{i=1}^{K} \frac{X^i \cdot g_i}{f_i}}{\sum_{i=1}^{K} \frac{g_i}{f_i}}, \tag{19}$$

$$\hat{\Omega} = \frac{1}{K} \cdot \sum_{i=1}^{K} \frac{\left(X^i - \hat{\mu}\right)\left(X^i - \hat{\mu}\right)^T g_i}{f_i}, \tag{20}$$

The shape parameter α estimate is obtained by solving the minimization problem of one-dimensional likelihood function: $\hat{\alpha} = \arg \max_{0 \le \alpha \le 1} L\left(X, \hat{\mu}, \hat{\Omega}, \alpha\right)$. Golden section search method can be applied to the minimization.

3 Quadrature formulas

Integrals included in the expressions of the estimates can be calculated by integral calculation subprograms in mathematical systems MathCad, Maple and etc., or using the Gaussian and Gauss-Laguerre quadrature formulas (Ehrich, 2002; Stoer, Bulirsch, 2002; Kovvali, 2012; Casio Computer co., 2015).
Gauss-Laguerre quadrature formulas:

$$\int_0^\infty x^\alpha e^{-x} f(x) dx \cong \sum_{i=1}^{n} \omega_i f(x_i), \tag{21}$$

where $f(x_i)$ – integrated function, n – number of nodes, x_i – integration nodes, ω_i – fixed weights.
Gaussian quadrature formulas:

$$\int_{-1}^{1} f(\chi)d\chi \cong \sum_{i=1}^{m} \vartheta_i f(\chi_i), \tag{22}$$

where $f(\chi_i)$ – integrated function, m – number of nodes, χ_i – integration nodes, ϑ_i – fixed weights.

4 Computer modeling

Maximum likelihood parameters estimation by EM algorithm is an iterative process needing to choose initial values and perform a certain number of iterations while values in adjacent steps differ insignificantly. In order to test the behaviour of created algorithm, the experiments were made with financial statements – Total Current Assets, Total Assets, Total Current Liabilities, Total Liabilities – of 124 companies in the USA. According to the much shorter computing time (error of the likelihood function is only in the sixth sign), integrals were calculated using the Gaussian (22) and Gauss-Laguerre (21) quadrature formulas.
In this experiment, data consisted of 124 fourth-dimensional vectors with these sampling means and sampling covariances matrix:

$$\alpha = 1.5, \quad \mu = \begin{pmatrix} 1.044 \\ 2.046 \\ 0.37 \\ 0.873 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 1.175 & 2.024 & 0.486 & 0.911 \\ 2.024 & 5.225 & 1.038 & 2.572 \\ 0.486 & 1.038 & 0.418 & 0.667 \\ 0.911 & 2.572 & 0.667 & 1.953 \end{pmatrix}. \tag{23}$$

We have developed an algorithm, where alpha is minimized in each iteration. Fig. 1 shows that likelihood function is unimodal. Therefore, it can be applied to the minimization by the golden section search method.

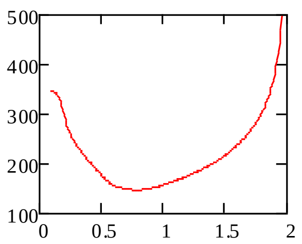

Fig. 1. Likelihood function dependence on alpha

Total 100 iterations by the described EM algorithm were performed. Fig. 2 shows the obtained parameters of α -stable law in dependence on the number of iterations. We see that the value of the likelihood function and the parameters estimates in a few iterations converge to the values, calculated with MathCad minimization subprogram.

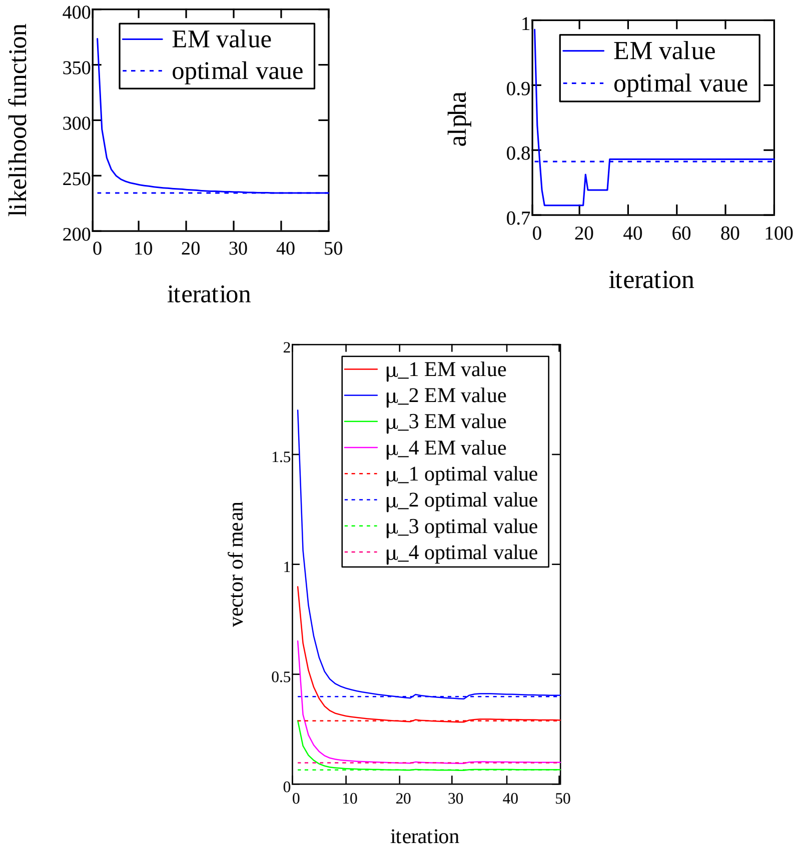

Fig. 2. Parameters dependence on the number of iterations

Further were generated $K = 100$ four-dimensional random α -stable law values with obtained parameters estimates and was performed a likelihood ratio test:

○ Parameters of the model were estimated by maximum likelihood method using practical data.

○ Then were generated a new sample by stable model whose parameters correspond to obtained estimates.

○ Further were calculated the empirical likelihood function and the likelihood function values of this sample, derived from practical data, empirical probability. If this probability is in the interval $\left(\frac{\alpha}{2},\ 1-\frac{\alpha}{2}\right)$, this is not a reason to reject the hypothesis about the data matching to the analyzed probability model, in a given case, to the α -stable law with reliability α .

○ Thus, the empirical probability of the test with financial balance data was 21.47 % (see, Fig. 3).

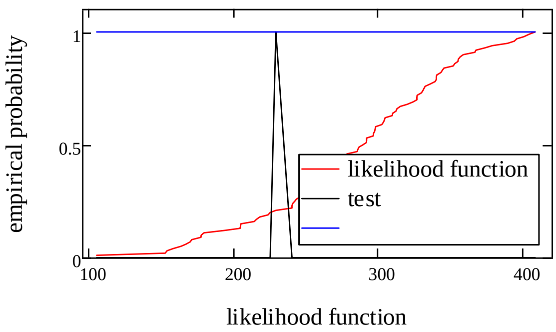

Fig. 3. Likelihood function test

Conclusions

1. The maximum likelihood method for the multivariate α -stable distribution was created in this work, which allows to estimate parameters of this distributions using EM algorithm.
2. α -stable distribution parameters estimators obtained by numerical simulation method are statistically adequate, because after a certain number of iterations, values of likelihood function and parameters convergent to the maximum likelihood values.
3. It was shown that this method realizes log-likelihood function golden section search, implementing it with EM algorithm.
4. This algorithm was applied for creation of model of balance data of USA companies. And it can be used creating financial models in stock market data analysis. Also algorithm can be used to test the systems of stochastic type and to solve other statistical tasks by using EM algorithm.

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