VILNIUS UNIVERSITY

Gailė
PAUKŠTAITĖ

Generalized Green's functions for problems with nonlocal conditions

DOCTORAL DISSERTATION

Physical sciences,
Mathematics 01P

VILNIUS 2018

This dissertation was written between 2014 and 2018 at Vilnius university.

Academic supervisor:
prof. dr. Artūras Štikonas (Vilnius university, Physical Sciences, Mathematics – 01P)
Academic consultant:
prof. dr. Olga Štikonienė (Vilnius university, Physical Sciences, Mathematics – 01P)

VILNIAUS UNIVERSITETAS

Gailė
PAUKŠTAITĖ

Apibendrintosios Gryno funkcijos uždaviniams su nelokaliosiomis sąlygomis

DAKTARO DISERTACIJA

Fiziniai mokslai,
Matematika 01P

VILNIUS 2018

Disertacija rengta 2014–2018 metais Vilniaus universitete.

Mokslinis vadovas:
prof. dr. Artūras Štikonas (Vilniaus universitetas, fiziniai mokslai, matematika – 01P)
Mokslinė konsultantė:
prof. dr. Olga Štikonienė (Vilniaus universitetas, fiziniai mokslai, matematika – 01P)

Contents

Glossary of Notation

δ_{ij}, δ_i^j	–	Kronecker delta
$\varDelta$	–	special determinant, 38, 91, 182, 214
$\varDelta^{\mathrm{cl}}$	–	$\varDelta$ of classical problem
∇^k	–	k-th order finite difference, 158
∇_+, ∇_-	–	finite differences, 224
$\boldsymbol{\pi}_1$, $\boldsymbol{\pi}_2$	–	discretizing projectors, 150, 177, 227
$\overline{\omega}^h$	–	mesh, 119
ω^h	–	submesh, 119
ω^h_{n-m}	–	submesh, 171
$\overline{1,n}$	–	index set $\{1, 2, \ldots, n\}$
$\overline{k,m}$	–	index set $\{k, k+1, \ldots, m\}$
A^*	–	adjoint of A
A^{-1}	–	inverse of A
$A^\dagger$	–	Moore–Penrose inverse of A, 11, 12
$A^\top$	–	transposed of A
$\mathbb{C}$	–	complex field
$C[0,1]$	–	continuous functions on $[0,1]$
$C^m[0,1]$	–	functions with continuous m derivatives on $[0,1]$
$\mathbb{C}^{m\times n}$	–	$m \times n$ complex matrices
$\mathrm{col}_j M$	–	j-th column of M, 113
$\det A$	–	determinant of A
$\mathbf{e}^i$	–	unit vector, 112–113
$\boldsymbol{e}^i$	–	unit vector valued function, 93, 184
$\boldsymbol{E}^i$	–	unit matrix valued function, 214–215
$\langle f, u\rangle$	–	functional f value at the function u
$F(X_n)$	–	space of discrete functions on X_n, 112
$F^*(X_n)$	–	dual space of $F(X_n)$, 113
G, $\mathbf{G}$	–	Green's function, 50, 97, 202, 160, 188
$\boldsymbol{G}$	–	Green's matrix, 188, 217
G^a	–	Green's function for auxiliary problem, 42, 93, 116, 156, 184
G^c	–	Green's function for Cauchy problem, 39, 92, 116, 156, 183
G^{cl}	–	Green's function for classical problem
G^g	–	generalized Green's function, 63, 102, 126, 144, 201
G^h	–	modified discrete Green's function, 136
H, H(x)	–	Heaviside function, 48

$\mathcal{H}_1, \ldots, \mathcal{H}_4$	–	Hilbert spaces, 12, 142
$H^m[0,1]$	–	Sobolev space $W^{m,2}[0,1]$
$H^m(\overline{\omega}^h)$	–	discrete space, 136
$\boldsymbol{K}(x,y)$	–	Cauchy matrix, 183
$\boldsymbol{K}_{ij}$	–	discrete Cauchy matrix, 213
$L^2(\omega^h)$	–	discrete space, 136
$\langle L^{\cdot}, U_{\cdot j} \rangle$	–	summation, 113
$\langle \boldsymbol{L}, \boldsymbol{U}_{\cdot j}^{\cdot l} \rangle$	–	summation, 210
$\langle \boldsymbol{L}_{\cdot}, \boldsymbol{U}^{\cdot l} \rangle$	–	summation, 180
$M_{i\cdot} U_{\cdot j}$	–	summation, 113
$M^{\perp}$	–	orthogonal complement of M
$N(A)$	–	nullspace of A
$\boldsymbol{\mathcal{O}}(h^{\alpha})$	–	vectorial big O notation, 151
P_M	–	orthogonal projector onto M
$\mathbb{R}$	–	real field
$R(A)$	–	range of A
$\operatorname{rank} A$	–	rank of A
$\mathbb{R}^{m \times n}$	–	$m \times n$ real matrices
$\operatorname{row}_i M$	–	i-th row of M, 113
$\operatorname{span}(S)$	–	linear span of set S
u^c	–	unique solution to Cauchy problem, 51, 122, 161, 183
u^{cl}	–	solution to classical problem
u^g	–	general least squares solution, 56, 101, 124, 200, 220
u^o	–	minimum norm least squares solution, 56, 101, 124, 200
$v^1, \ldots, v^m$	–	biorthogonal fundamental system, 49, 97, 121, 160, 216
$v^{g,1}, \ldots, v^{g,m}$	–	generalized biorthogonal fundamental system, 63, 103
$W(y)$	–	Wronskian determinant, 40, 92
W_j	–	discrete Wronskian determinant, 116, 156
$\widetilde{W}(x,y)$, $\widetilde{W}_{ij}$	–	92, 156
X_n	–	index set $\{0, 1, \ldots, n\}$, 112
$y_0, y_1, \ldots, y_N$	–	discontinuity points, 53, 190
$\overline{z}$	–	complex conjugate of z
$z^1, \ldots, z^m$	–	fundamental system
$\boldsymbol{Z}(x)$	–	fundamental matrix and a point x, 181
$\mathbf{Z}_i$	–	discrete fundamental matrix at a point i, 212

Introduction

1 Formulation of the problem

In this thesis, we investigate linear differential and discrete problems with nonlocal conditions of a one variable. Since many mathematical problems, modelling processes and phenomena of the real life or taken for theoretical purposes only, do not have unique solutions, we consider problems in the *least squares sense*, where the existence of a unique *best approximate solution* is possible [6, Ben-Israel and Greville 2003]. This function is often called a *minimum norm least squares solution* and nowadays is one of the most popular objects of investigation.

Our aim is to describe the best approximate solution in a form related to the classical representation of the unique solution. Here the essential role is played by the concept of a *Green's function*. Indeed, if we know a Green's function, then a problem is considered as formally solved [22, Cabada 2014], [100, Roman 2011]. Thus, developing this analogy to a unique best approximate solution, we focus our study on a *generalized Green's function*, which describes a minimum norm least squares solution and extends the classical meaning of an ordinary Green's function.

2 Topicality of the problem

Topics about Green's functions are often popular. A Green's function manifests in many areas of science and, according to a context, is differently named.

In signal processing, it is known as the *impulse response* or *impulse response function* [12, Blackledge 2006]. Green's functions are used in acoustic and audio applications. Authors say [75, Marczuk and Majkut 2006] that the significant problem in room acoustic is evaluating an acoustic quality of projected and modernized rooms. They used a Green's function as a solution of the acoustic wave equation. Impulse response functions are also taken to investigate the ocean acoustics [17, Brooks and Gerstoft 2009] during a storm. There are modern applications of the impulse response analysis in radar, ultrasound imaging, digital signal processing [12, Blackledge 2006] and broadband internet connections [35, edited by Cooper and Madden 2004]. Seismologists use a Green's function as well and naturally call it by an *Earth's impulse response* [14, Bostock 2004]. Green's functions also appear in aerodynamics and aircraft configurations [45, Freedman and Tseng 1985]. Let us mention the application in economics, where impulse response functions are used to describe how the economy reacts over time to exogenous impulses, usually called *shocks*. Green's functions are also used for Black-Scholes model studying pricing options [38, Dorfleitner *et al.* 2008] and other applications [78, Y. Melnikov and M. Melnikov 2012].

Green's functions arise solving various problems in quantum mechanics as well [39, Economou 2006]. Here a particle such as an electron or a photon is described by the wave function. The dynamical behavior of the wave function is represented by a *propagator*, what is just the role played by a Green's function. A Green's function is also called a *two-point correlation function* since it is related to the probability of measuring a field at one point that it is sourced at a different point. In scientific literature we can also meet a *point-spread function* [120, Sheppard *et al.* 2014]. It is one more synonymous of a Green's function again.

In physics Green's functions are called by their rightful name in honour to the British mathematician George Green (1793–1841). In 1828 he wrote the article "An Essay on the Application of Mathematical Analysis to the Theories of Elasticity and Magnetism". Green was the first scientist who investigated a potential [47], that was later called a *Green's function*. Classically, a Green's function is understood as a kernel, which represents a solution to the differential problem of mathematical physics. However, the modern concept of a Green's function was introduced a little bit later by B. Riemann in [99, 1860].

Let us note that Green's functions are rather *distributions* than proper functions. According to [63, Kolmogorov and Fomin 1957], the concept of distributions originated in the work of Sobolev [122, 1936] for second order hyperbolic partial differential equations. However, the ideas were developed later in an extended form by Schwartz, who was awarded by the Fields medal for his work on distributions in 1950 [119].

The notion of a Green's function and other fundamental concepts in the theory of differential equations were formulated studying the classical problems of mathematical physics. Merely, the *classical boundary value problem* for a second order stationary differential equation

$$u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in [0,1], \tag{1}$$

$$\langle \kappa_1, u\rangle := \alpha_1 u(0) + \beta_1 u'(0) = 0, \tag{2}$$

$$\langle \kappa_2, u\rangle := \alpha_2 u(1) + \beta_2 u'(1) = 0, \tag{3}$$

where $a, b, f \in C[0,1]$ and $|\alpha_k| + |\beta_k| \neq 0$, $k = 1, 2$, is almost completely investigated and considered the classics in the theory of differential equations. *Classical boundary conditions*, describing this problem, relate functions u and u' at the same boundary points.

However, physics, mechanics and other natural sciences have been developed greatly during the last 50 years, and today they investigate such processes and phenomena that those mathematical models do not fit into the frames of the classical differential problem. For instance, we have the thermostat problems [61, Kalna and McKee 2004], heat conduction [62, Kamynin 1964] and bioreaction engineering [118, Schuegerl 1987] problems, and problems arising in electrochemistry [24, Choi and Chan 1992], microelectronics [21, Būda *et al.* 1985], biology [82, Nakhushev 1995], and other fields. In 2011 Special Issue for nonclassical conditions (27 articles) was published in the journal Boundary Value Problems [43].

We have just listed several examples of nonclassical problems but, in practice, there often arise problems where we cannot measure data directly at the boundary. Then we formulate additional conditions that link the solution u with its derivative u' to several different points or to the whole interval. Such conditions are called *nonlocal conditions.* If there appears a boundary point in nonlocal conditions, we name them *nonlocal boundary conditions.*

In 1969 Bitsadze and Samarskii published the paper [7] for an elliptic partial differential equation with nonlocal conditions, which influenced the appearance of many original articles [3, Ashyralyev 2008], [57, Infante 2003], [115, Sapagovas 2000], [123, Štikonas 2014], [126, Štikonas and Štikonienė 2009]. For the one dimensional case, we also consider such type nonlocal conditions

$$u(0) = \gamma_1 u(\xi_1) \qquad \text{or} \qquad u(1) = \gamma_2 u(\xi_2),$$

where $0 < \xi_1, \xi_2 < 1$, and naturally call them *Bitsadze–Samarskii conditions.* Il'in [56, 1976] and Moiseev [54, 1978] studied *multipoints boundary conditions*

$$u(0) = \sum_{i=1}^{m} \gamma_i u(\xi_i), \qquad u(1) = \sum_{i=1}^{n} \widetilde{\gamma}_i u(\xi_i),$$

where all $\gamma_i, \widetilde{\gamma}_i$ are real numbers and $\xi_i \in (0,1)$. Nonlinear boundary value problems with nonhomogenous multipoints boundary conditions were also investigated by L. Kong and Q. Kong in [64, 2010].

Moreover, Sapagovas with co-authors [33, 2004] investigated eigenvalues for differential equations with *nonlocal integral conditions*

$$u(0) = \gamma_1 \int_0^1 \alpha_1(x) u(x)\, dx, \qquad u(1) = \gamma_2 \int_0^1 \alpha_2(x) u(x)\, dx.$$

Spectral problems were also studied [135, Yurko and Yang 2014] for the second order differential problem with Stieltjes boundary conditions

$$\begin{aligned} -u'' + b(x)u &= \lambda u, \qquad x \in (0,T), \\ \langle L_k, u\rangle := \int_0^T & u(x)\, d\mu_k(x) = 0, \qquad k = 1,2, \end{aligned}$$

where $b \in L^1(0,T)$ is a complex valued function and μ_k are complex valued functions of bounded variation, continuous from the right for $x > 0$. Such boundary conditions can be rewritten in the following nonlocal form

$$\langle L_k, u\rangle := \gamma_k u(0) + \int_0^T u(x)\, d\,\widetilde{\mu}_k(x) = 0, \qquad k = 1,2$$

where γ_k denote finite limits $\gamma_k := \mu_k(0+) - \mu_k(0)$ but $\widetilde{\mu}_k$ are complex valued functions of bounded variations, continuous from the right for $x \geqslant 0$.

On the other hand, Chanane [23, 2009] considered Stieltjes boundary conditions involving derivatives

$$\langle L_k, u\rangle := \int_0^1 \big(u(x)\, d\,\mu_{k1}(x) + u'(x)\, d\,\mu_{k2}(x)\big) = 0, \qquad k = 1,2.$$

Here functions μ_{k1} and μ_{k2} are of bounded variations and the integration in understood in the Riemann–Stietljes sense again.

Let us note that all nonlocal conditions, given as examples above for a second order differential equation, are particular cases of nonlocal conditions $\langle L_k, u\rangle = 0$, $k = 1,2$, for some continuous linear functionals $L_k \in \big(C^1[0,1]\big)^*$. Indeed, according to Alt [2, 2016], every functional $L \in \big(C^1[0,1]\big)^*$ can be given by

$$\langle L, u\rangle := \gamma u(\xi) + \int_0^1 u'(x)\, d\mu(x) \tag{4}$$

for some $\gamma \in \mathbb{R}$, a point $\xi \in [0,1]$ and a regular bounded countably additive Borrel measure μ on $[0,1]$, i.e., $\mu \in \mathrm{rca}[0,1]$. Since the function μ is of bounded variation, it can have at most countably many discontinuities and need only be differentiable almost everywhere (a.e.). Hence, in practice, most nonlocal conditions (4) are considered of the form

$$\langle L, u\rangle := \sum_{i=1}^{\infty}\Big(a_i u(\xi_i) + b_i u'(\zeta_i)\Big) + \int_0^1 c(x)u(x) + d(x)u'(x)\,dx$$

for $\xi_i, \zeta_i \in [0,1]$, real numbers a_i, b_i and integrable functions $c, d \in L^1[0,1]$.

Nonlocal conditions for higher order differential problems are also studied. Bai [5, 2010] proved the existence of one or two positive solutions to the nonlocal fourth order boundary value problem

$$\begin{aligned} &u^{(4)} + \beta u'' = \lambda f(t, u, u''), \qquad t \in (0,1), \\ &u(0) = u(1) = \int_0^1 \alpha_1(t)u(t)\,dx, \qquad u''(0) = u''(1) = \int_0^1 \alpha_2(t)u''(t)\,dt, \end{aligned} \tag{5}$$

where $\alpha_1, \alpha_2 \in L^1[0,1]$, λ is a positive number and $f \in C\big([0,1]\times[0,\infty)\times(-\infty,0],[0;\infty)\big)$. The n-th order differential equation with one Stieltjes boundary condition

$$\begin{aligned} &u^{(n)}(t) + \lambda a(t) f(t, u(t)) = 0, \qquad t \in (0,1), \\ &u(0) = \ldots = u^{(n-2)}(0) = 0, \qquad u(1) = \int_0^1 u(s)\,d\,A(s) \end{aligned} \tag{6}$$

was widely studied in the work [49, Hao *et al.* 2015].

Let us mention works of Day [36, 37, 1982–1983], where nonlocal integral conditions for the heat equation with applications to thermoelasticity were investigated. Bitsadze and Samarskii formulated [7, 1969] the nonlocal problem for the elliptic differential equation, which is used in the plasma theory. Moreover, a mercury droplet in electric contact was investigated by Sapagovas [109, 110, 114, 1982–1984]. Here we recall the group of lithuanian mathematicians who productively deal with nonlocal problems: M. Sapagovas [111–113, 116], R. Čiegis [27–30], A. Štikonas and O. Štikonienė [31, 32]. In 2017, the triple - Sapagovas, Čiegis and Štikonas - were awarded by the lithuanian Research Council for the cycle of their works “Nonclassical problems and their solution methods (2002–2016)”.

Thereby, topics about Green's functions for nonlocal problems were also developed [26, Čiegis 1988], [117, Sapagovas and Čiegis 1987], [123, Štikonas 2014]. Let us mention works of Infante and Webb [58, 59, 2003], Lan [66, 2006], Ma [73, 74, 1998,2007], Sun [128, 2005], Troung with co-authors [129, 2008] and Zhao [136, 2007]. For instance, Sun and Zhang [127, 2008] examined the third order m-points boundary value problem

$$u'''(t) + f(t, u(t), u'(t), u''(t)) = 0, \qquad t \in (0, 1), \tag{7}$$
$$u(0) = u'(0) = 0, \qquad u''(1) = \sum_{i=1}^{m-2} \gamma_i u''(\xi_i),$$

and found the expression of a Green's function. Authors also obtained several properties of a Green's function and proved the existence of at least one solution. Let us recall the work of Xie with co-authors [134, Xie *et al.* 2009], where the representation of a Green's function was derived for the n-th order nonlocal problem

$$u^{(n)}(t) + h(t) f(t, u(t)) = 0, \qquad t \in [a, b], \tag{8}$$
$$u(a) = 0, \quad u'(a) = 0, \quad \ldots, \quad u^{(n-2)}(a) = 0, \quad u(b) = \gamma u(\xi).$$

Authors used a Green's function to prove the existence of the unique solution. They obtained properties of a Green's function as well.

We accentuate that weakly nonlinear differential problems (5)–(8) with a nonlinear function $f(t, u(t), u'(t), \ldots, u^{(n-1)})$ and the expressed highest order derivative $u^{(n)}$ in the differential equation are often met in nowadays literature. Here a part of a differential equation, omitting a nonlinear term $f(t, u(t), u'(t), \ldots, u^{(n)})$ and considered as a separate differential equation, is linear and provides many useful information. First, it is simpler to investigate. Second, it's Green's function $G(t, s)$ is often used to describe a solution of a weakly nonlinear differential equation, that is,

$$u(t) = \int_0^1 G(t, s) f\big(s, u(s), u'(s), \ldots, u^{(n-1)}(s)\big)\, ds.$$

Using the known expression and properties of a Green's function for the linear differential equation and properties of a nonlinear function f, there are often obtained various estimates, those help to prove the existence of solutions for weakly nonlinear problems. Here we recall another application of Green's functions of linear problems: they are also used in iterative methods for weakly nonlinear problems.

Thus, Green's functions for linear differential problems play the very important role in the analysis of nonlinear differential problems as well. We accentuate the work of Roman [100, 2011], where the author derived various direct representations of a Green's function for linear nonlocal problems with general conditions (4), studied its properties and illustrated these results by a variety of examples. Thus, applying methods of Roman's work [100, 2011], we can obtain the expression of a Green's function $G(x, y)$ and use it to represent the unique solution of a differential problem (1) with nonlocal conditions $\langle L_k, u\rangle = 0$, $k = 1, 2$, in the explicit form

$$u(x) = \int_0^1 G(x, y) f(y)\, dy, \qquad x \in [0, 1]. \tag{9}$$

However, there are many problems in practise, those have neither the unique solution nor a Green's function. Such *ill–possed problems* are so popular, since they describe processes and phenomena of the real life, and the analysis of these problems still remains the valuable investigation area. Their solutions are considered such functions, those satisfy some optimization conditions and can be written in the form (9) with a *generalized Green's function* instead of an ordinary Green's function.

The history of a *generalized Green's function* begins in 1904, where David Hilbert introduced a kernel of an integral invert operator for a consistent linear differential problem [53]. That integral representation extended the classical notion of a Green's function, which was used to describe the unique solution to problems of mathematical physics. Nowadays the concept of a generalized Green's function is applied to represent some optimal solution to mostly inconsistent differential problems [6, Ben-Israel and Greville 2003], [13, Boichuk and Samoilenko 2004].

For example, J. Locker [70, 1977] studied the characterization and obtained properties of a generalized Green's function, which describes a *minimum norm least squares solution* to the n-th order linear differential problem with two point boundary conditions. He also constructed the approximate sequence, converging to that minimizer [69, 1975]. Minimizers for various mathematical models were investigated in [50, Hasanov Hasanoğlu and Romanov 2017] as well.

There are many other authors, who investigated a generalized Green's function. Let us mention several famous works of Westfall [131, 1909], Bounitzky [15, 1909], Elliot [40, 41, 1928-1929], Reid [97, 98, 1931,1967], Bradley [16, 1966], Wyler [133, 1965] and Loud [71, 72, 1966,1970].

Generalized Green's functions were also studied by the lithuanian mathematician I. Matsionis. In 1973 he published the paper "A generalized Green's function" [77], where the n-th order differential equation

$$Lu := u^{(n)} + a_1(\lambda, x)u^{(n-1)} + \ldots + a_{n-1}(\lambda, x)u' + a_n(\lambda, x)u = f(x), \tag{10}$$

was considered. Here coefficients $a_1(\lambda, x), \ldots, a_n(\lambda, x)$ are bounded and continuous with respect to x on the interval $[a, b]$ but are analytic with respect to λ over the entire complex plane. Additionally, each function $a_k(\lambda, x)$ has continuous derivatives up to order $n-k$ ($k = \overline{1,n}$). Matsionis aspired to find the solution of the equation $\mathcal{L}u = f$ with $f \in L^2(a,b)$, which satisfies two point boundary conditions

$$\langle L_k, u\rangle = \sum_{l=1}^{n} \left(\alpha_{kl} u^{(l-1)}(a) + \beta_{kl} u^{(l-1)}(b)\right) = 0, \quad k = \overline{1,n}, \qquad (11)$$

with real constants α_{kl} and β_{kl}. He focused to study the problem (10)–(11) without the unique solution ($\varDelta(\lambda) = 0$). For the *consistent problem*, Matsionis took the general solution of the differential equation (10) in the form

$$u(x) = c_1 z^1(x,\lambda) + \ldots + c_n z^n(x,\lambda) + \int_a^b G(x,y,\lambda) f(y)\, dy,$$

where z^k, $k = \overline{1,m}$, is a fundamental system of the homogenous equation. In this case, he showed that a generalized Green's function can be found using the same method as in the case $\varDelta(\lambda) \neq 0$, where the problem has the unique solution.

However, Locker [70, 1977] studied a generalized Green's function for the n-th order differential problem $Lu = f$ with two point boundary conditions, which may be consistent (at least one solution) or inconsistent (no solutions). Precisely, he wrote the minimum norm least squares solution in the form $u^o = L^\dagger f$ using the generalized inverse operator $L^\dagger$ of the operator L, known as the *Moore–Penrose inverse*. Then applying the Riesz representation theorem for continuous linear functionals in the Hilbert space, he presented the minimum norm least squares solution in the form

$$u^o(x) = L^\dagger f(x) = \int_a^b G^g(x,y) f(y)\, dy$$

for all $x \in [a,b]$ and $f \in L^2(a,b)$. This method helped to introduce the kernel $G^g(x,y)$, which plays the role of a generalized Green's function. The author obtained properties of a generalized Green's function, those are analogical to well known properties of an ordinary Green's function if there exists the unique ordinary inverse L^{-1}. The minimum norm least squares solution for very relative differential problems was also considered by Loud [72, 1970].

Hestenes used the Green's function of the operator L^*L to find the Moore–Penrose inverse $L^\dagger$ [52, 1961]. Precisely, he studied the differential operator $L := d/dt$ with the domain

$$D(L) = \{u \in H^1[0,\pi] :\ u(0) = u(\pi) = 0\}.$$

It is the densely defined ($\overline{D(L)} = L^2[0,\pi]$) closed linear operator with the closed range

$$R(L) = \left\{ f \in L^2[0,\pi] : \int_0^\pi f(s)\,ds = 0 \right\} = \overline{R(L)}.$$

Hestenes obtained the minimum norm least squares solution

$$u^o(t) = L^\dagger f(t) = \int_0^t f(s)\,ds - \frac{t}{\pi}\int_0^\pi f(s)\,ds = \int_0^\pi G^g(t,s) f(s)\,ds, \qquad 0 \leqslant t \leqslant \pi$$

with the kernel

$$G^g(t,s) = \frac{\partial}{\partial s} G(t,s) = \frac{1}{\pi} \begin{cases} \pi - t, & s \leqslant t, \\ -t, & s > t, \end{cases}$$

which represents the generalized Green's function. Let us note that here $G(t,s)$ is the Green's function of the operator $L^*L := -d^2/dt^2$ with the domain $D(L^*L) = \{u \in H^2[0,\pi] : u(0) = u(\pi) = 0\}$.

Landesman considered analogous gradient problem [67, 1967]. He investigated the differential operator $\boldsymbol{L}u := (\partial u/\partial t_1, \partial u/\partial t_2)^\top$ with the domain

$$D(\boldsymbol{L}) = \left\{ u \in (H^1[0,\pi])^2 : \left\{ \begin{array}{l} u(0,t_2) = u(\pi,t_2) = 0 \text{ for a.e. } t_2 \in [0,\pi], \\ u(t_1,0) = u(t_1,\pi) = 0 \text{ for a.e. } t_1 \in [0,\pi] \end{array} \right\} \right\}.$$

The author took the Green's function

$$G(t_1,t_2,s_1,s_2) = \frac{4}{\pi^2} \sum_{m,n=1}^{\infty} \frac{1}{m^2+n^2} \sin(mt_1)\sin(nt_2)\sin(ms_1)\sin(ns_2),$$

where $0 \leqslant s_i, t_j \leqslant \pi$, of the operator $\boldsymbol{L}^*\boldsymbol{L}$, what is the negative of the Laplacian operator

$$\boldsymbol{L}^*\boldsymbol{L} = -\left(\frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_1^2} \right).$$

For $\boldsymbol{f} = (f_1, f_2)^\top \in L^2[0,\pi] \times L^2[0,\pi]$, he obtained the expression

$$u^o(t_1,t_2) = \boldsymbol{L}^\dagger \boldsymbol{f}(t_1,t_2) = \sum_{j=1}^{2} \int_0^\pi \int_0^\pi \frac{\partial}{\partial s_j} G(t_1,t_2,s_1,s_2) f_j(s_1,s_2)\,ds_1\,ds_2,$$

where

$$G^g(t_1,t_2,s_1,s_2) = \sum_{j=1}^{2} \frac{\partial}{\partial s_j} G(t_1,t_2,s_1,s_2)$$

represents the generalized Green's function for the gradient operator $\boldsymbol{L}$.

Moreover, Brown [18, 1974] investigated generalized inverses and generalized Green's matrices for differential systems with Stieltjes conditions

$$\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{f}, \qquad \int_0^1 d\boldsymbol{F}\,\boldsymbol{u} = \boldsymbol{0}. \tag{12}$$

Here $\boldsymbol{u}$ is n-dimensional absolutely continuous vector valued function, $\boldsymbol{A}$ is $n \times n$ continuous matrix on $[0, 1]$ and $\boldsymbol{F}$ is $m \times n$ matrix valued measure of bounded variation elementwise. According to Riesz representation theorem, Stieltjes conditions (12) describe general conditions for continuous $\boldsymbol{u}$ but, in practice, most nonlocal conditions are considered of the form

$$\sum_{i=1}^{\infty} \mathbf{A}_i \boldsymbol{u}_i(\xi_i) + \int_0^1 \boldsymbol{B}(x)\boldsymbol{u}(x)\,dx = 0.$$

Green's matrices and their properties for very relative problems were also studied by other authors [20, Bryan 1969], [60, Jones 1967] and [132, Whynurn 1942]. Moreover, Brown and Krall considered an adjoint problem and presented an eigenvalue expansion of a Green's matrix [19, 1974].

Nowadays the method to represent the minimum norm least squares solution using a generalized inverse is very popular. There is no doubt that generalized inverses are useful, applicable and significant tool in many areas of science, especially, for physicists who deal with optimization problems or data analysis. Even Ben-Israel and Greville in their book [6, 2003] said: "The observation that generalized inverses are like prose ("Good Heavens! For more than forty years I have been speaking prose without knowing it"- Molière, *Le Bourgeois Gentilhomme*) is nowhere truer than in the literature of linear operators".

It seems that the concept of a generalized inverse operator was first mentioned in 1903 by Fredholm in this paper [44], where a particular generalized inverse was obtained for an integral operator. Fredholm called this generalized inverse by the *pseudoinverse*, what is nowadays used to name generalized inverses, too. As we mentioned, Hilbert was the first scientist who investigated generalized inverse operators for differential problems. In 1904 he introduced [53] the concept of a generalized Green's function. Generalized inverses for integral and differential problems influenced the birth of generalized inverse matrices, whose existence was first obtained by Moore [80, 81, 1920,1935].

According to [6, Ben-Israel and Greville 2003], the concept of a generalized inverse matrix is understandable quite widely because each matrix, which has the following properties, can be considered as a generalized inverse of a matrix $\mathbf{A}$:

- it exists for a class of matrices which is larger then a class of nonsingular matrices;
- it has some properties of the usual inverse matrix;
- it is coincident with the usual inverse matrix if a matrix $\mathbf{A}$ is nonsingular.

A matrix may have a unique generalize inverse or even a lot of generalized inverses [6, Ben-Israel and Greville 2003]. However, Moore introduced the unique generalized inverse for every finite dimensional real or complex matrix $\mathbf{A} \in \mathbb{C}^{n\times m}$ (which may be singular!) and proved its existence. In 1951 Bjerhammar observed [9, 10], [11, 1958] that particular generalized inverses are related to "best fit" (*least squares*) solutions to systems of linear equations. While being a student, in 1955 Penrose showed [93] that the generalized inverse, earlier introduced by Moore, is the unique matrix $\mathbf{X} \in \mathbb{C}^{m\times n}$ satisfying all four *Penrose equations*

$$\mathbf{AXA} = \mathbf{A}, \qquad \mathbf{XAX} = \mathbf{X}, \qquad (\mathbf{AX})^* = \mathbf{AX}, \qquad (\mathbf{XA})^* = \mathbf{XA},$$

where $\mathbf{A}^*$ denotes the adjoint matrix of $\mathbf{A}$. This matrix is often denoted by $\mathbf{A}^\dagger$ and [94, Penrose 1956] is used to describe the *best approximate solution* to a linear system of equations $\mathbf{Au} = \mathbf{b}$ in the form $\mathbf{u}^o = \mathbf{A}^\dagger\mathbf{b}$. It is also called the minimum norm least squares solution since it minimizes the Euclidean norm of the residual

$$\|\mathbf{Au}^o - \mathbf{b}\| \leqslant \|\mathbf{Au} - \mathbf{b}\| \tag{13}$$

for all $\mathbf{u} \in \mathbb{C}^{n\times 1}$, and is smallest

$$\|\mathbf{u}^o\| < \|\mathbf{u}\|$$

among all vectors $\mathbf{u}$ giving the equality (13). Penrose rediscovered the Moore inverse, its applications seemed to be so useful and fruitful that this matrix was called the *Moore–Penrose inverse* in honour to both authors.

Ideas analogous to generalized inverse matrices were also developed in the theory of operators. Let us accentuate that generalized inverses of linear operators between Hilbert spaces have many similarities to the finite dimensional case.

For example, in 1956 Penrose [94] proved that the best approximate solution to the matrix equation $\mathbf{AUB} = \mathbf{C}$ is of the form $\mathbf{U}^o = \mathbf{A}^\dagger\mathbf{C}\mathbf{B}^\dagger$. The relative result for the operator equation $\boldsymbol{AUB} = \boldsymbol{C}$ in Hilbert spaces was also derived by Slavík [121, 1990]. His result is presented in the following theorem.

Theorem A (Slavík 1990, [121]). *Let* $\boldsymbol{A} : \mathcal{H}_3 \to \mathcal{H}_4$, $\boldsymbol{B} : \mathcal{H}_1 \to \mathcal{H}_2$ *be continuous linear operators with closed ranges but* $\boldsymbol{C} : \mathcal{H}_1 \to \mathcal{H}_4$ *is a Hilbert–Schmidt operator. Then the best approximate solution (with respect to the Hilbert–Schmidt norm) to the operator equation* $\boldsymbol{AUB} = \boldsymbol{C}$ *is given by* $\boldsymbol{U}^o = \boldsymbol{A}^\dagger \boldsymbol{C} \boldsymbol{B}^\dagger$.

The *Moore–Penrose inverse* $L^\dagger : \mathcal{H}_1 \to \mathcal{H}_2$ is the most famous generalized inverse of a linear operator $L : \mathcal{H}_2 \to \mathcal{H}_1$ between Hilbert spaces. As in the matrix case, it is the unique solution of the following operator equations

$$LL^\dagger L = L, \qquad L^\dagger L L^\dagger = L, \qquad (LL^\dagger)^* = LL^\dagger, \qquad (L^\dagger L)^* = L^\dagger L,$$

where L^* denotes the adjoint operator of the operator L. According to [6, Ben-Israel and Greville 2003], the Moore–Penrose inverse operator always exists for a continuous linear operator L with a closed range $R(L)$ and represents the best approximate solution $u^o = L^\dagger f$ for every $f \in \mathcal{H}_1$ to an equation $Lu = f$. This function minimizes the norm of the residual

$$\|Lu^o - f\|_{\mathcal{H}_1} = \inf_{u \in D(L)} \|Lu - f\|_{\mathcal{H}_1} \tag{14}$$

and is smallest

$$\|u^o\|_{\mathcal{H}_2} < \|u\|_{\mathcal{H}_2} \tag{15}$$

among all minimizers $u \in \mathcal{H}_2$ holding the equality (14). We can often met another names of the best approximate solution, that is, the *virtual solution* [130, Tseng 1956], the *least extremal solution* [6, Ben-Israel and Greville 2003], [121, Slavík 1990] or the *minimum norm least squares solution* [70, Locker 1977]. The Moore–Penrose inverse is used to solve various minimization problems of other type than (14)–(15) as well [96, Porter and Williams 1966].

The minimum norm least squares solutions are also used in modern financial modelling [1, Albrecher *et al.* 2009]. In recent years, Maroncelli and Rodríguez [76, 2013] investigated the minimum norm least squares solution to the boundary value problem with impulses

$$\begin{aligned} &\boldsymbol{u}'(t) = \boldsymbol{A}(t)\boldsymbol{u}(t) + \boldsymbol{f}(t) \quad a.e.\ [0,1] \\ &\boldsymbol{u}(t_i^+) - \boldsymbol{u}(t_i^-) = \mathbf{g}_i, \qquad i = 1, \dots, k, \end{aligned} \tag{16}$$

subject to classical boundary conditions

$$\mathbf{B}\boldsymbol{u}(0) + \mathbf{D}\boldsymbol{u}(1) = \mathbf{0}.$$

Here $0 < t_1 < t_2 < \ldots < t_k < 1$ are fixed points, $\boldsymbol{A}$ is $n \times n$ matrix valued function on $[0,1]$ whose elements are functions from $L^2([0,1])$ but

$\boldsymbol{f} : L^2[0,1] \to \mathbb{R}^n$. Moreover, $\mathbf{g}_i \in \mathbb{R}^n$ and $\mathbf{B}, \mathbf{D}$ are $n \times n$ matrices. Authors did analysis, which is so strongly related with ideas of generalized inverses and generalized Green's functions.

Impulsive differential problems (16) as well as differential equations with delay or systems of ordinary differential equations were also studied by Boichuk and Samoilenko [13, 2004]. Authors widely investigated real Fredholm boundary value problems with the operator equation $Lu = f$ and nonlocal conditions $\boldsymbol{\ell} u = \mathbf{g} \in \mathbb{R}^m$. Representing the problem into the vectorial form $\boldsymbol{L}u = \boldsymbol{f}$ with $\boldsymbol{L} = (L, \boldsymbol{\ell})^\top$ and $\boldsymbol{f} = (f, \mathbf{g})^\top$, they obtained a representation of the generalized inverse operator for the operator $\boldsymbol{L}$. Authors provided the expression of the generalized Green's matrix

$$\boldsymbol{G}^g(t,s) = \boldsymbol{G}^c(t,s) - \boldsymbol{Z}(t)\big(\boldsymbol{\ell}\boldsymbol{Z}\big)^{-}\boldsymbol{\ell}\boldsymbol{G}^c(\cdot,s),$$

where $\boldsymbol{Z}(t)$ is the fundamental matrix of the homogenous operator equation $Lu = 0$, $\boldsymbol{G}^c(t,s)$ is the Green's matrix of the corresponding Cauchy problem but $\big(\boldsymbol{\ell}\boldsymbol{Z}\big)^{-}$ is a particular generalized inverse of the matrix $\boldsymbol{\ell}\boldsymbol{Z}(\cdot)$ with real entries. This representation of the generalized Green's function is valid for a some class of solvable differential problems $\boldsymbol{L}u = \boldsymbol{f}$. Authors also derived solvability conditions for this problem, those can be given in the form

$$\mathbf{P}_d(\mathbf{g} - \boldsymbol{\ell} u^f) = \mathbf{0}.$$

Here u^f is a particular solution to the inhomogeneous equation $Lu = f$ but $d := m - \operatorname{rank}\big(\boldsymbol{\ell}\boldsymbol{Z}(\cdot)\big)$ and $\mathbf{P}_d$ is $d \times m$ matrix composed of all d linearly independent rows of the projector $\mathbf{P}_{N((\boldsymbol{\ell}\boldsymbol{Z})^*)}$.

Boichuk and Samoilenko also investigated a linear system of difference equations with nonlocal conditions

$$\boldsymbol{u}(i+1) = \boldsymbol{A}(i)\boldsymbol{u}(i) + \boldsymbol{f}(i+1), \quad i = \overline{n_0, N}, \qquad \boldsymbol{\ell}\boldsymbol{u} = \mathbf{g} \in \mathbb{R}^m$$

and derived the expression of the generalized discrete Green's matrix

$$\boldsymbol{G}^g(i,j) = \boldsymbol{G}^c(i,j) - \boldsymbol{Z}(i,n_0)(\boldsymbol{\ell}\boldsymbol{Z}(\cdot,n_0))^{\dagger}\boldsymbol{\ell}\boldsymbol{G}^c(\cdot,j).$$

Here

$$\boldsymbol{G}^c(i,j) = \begin{cases} \boldsymbol{Z}(i,j), & n_0 \leqslant j \leqslant i \leqslant N, \\ 0, & j > i \end{cases}$$

is the discrete Green's matrix (by authors called the *Cauchy matrix*) for the Cauchy problem for difference equations. The generalized discrete Green' matrix describes all solutions of consistent difference system with nonlocal conditions, which satisfies the following solvability conditions

$$\mathbf{P}_d\Bigg(\mathbf{g} - \sum_{j=n_0+1}^{N} \boldsymbol{\ell}\boldsymbol{G}^c(\cdot, j)\boldsymbol{f}(j)\Bigg) = \mathbf{0},$$

where $\mathbf{P}_d$ is the matrix composed of all d linearly independent rows of the projector onto $N((\boldsymbol{\ell}\boldsymbol{Z}(\cdot, n_0))^*)$ as above.

Discrete problems and their Green's functions were also studied by Roman [100, 2011] and Štikonas [102,103, 2011], Liu with co-authors [68, 2010], Ghanbary [46, 2007], Chung and Yau [25, 2000] and other authors. Discrete problems is the very useful and important investigation area because, in general, an explicit solution or some optimization solution of the differential nonlocal problem cannot be found analytically. Since computer programming science is nowadays widely developed, various numerical methods have been investigated and applied to differential problems [4, Bachvalov *et al.* 2011], [104, Samarskii 2001]. Authors often discuss on differential and discrete aspects of problems [101, Roman 2011], [51, Hernandez-Martinez *et al.* 2011], [54, Il'in and Moiseev 1987]. Direct and iterative solution methods for differential and discrete problems were considered in monographs [106, Samarskii and Gulin 1989], [105, Samarskii and Nikolaev 1978] as well.

Convergence analysis of the unique discrete solution to the solution of a differential problem is often discussed. Most results and convergence conditions are formulated for problems having the usual inverse. The importance and application worth of the minimum norm least squares solution is significant. However, during the preparation of this dissertation and looking for some literature on convergence, it seems that the analogous convergence theory of problems having only the Moore–Penrose is not widely developed (is it developed at all?). There are many open questions about the convergence of the discrete minimizer to the minimizer of the differential problem, those require systematical studies.

Thus, this doctoral dissertation discuss on parallel results for differential and discrete problems with nonlocal conditions. Here some sufficient convergence conditions will be provided and illustrated by the convergence of discrete minimizers. Obtained results can be used for more detailed studies of the converge analysis. Due to quite large volume of this doctoral dissertation, questions about converge are only touched.

3 Aims and problems

The target of this dissertation is to obtain the representation of the minimum norm least squares solution to ordinary differential problems with nonlocal conditions. To realize this idea, we had to study the following problems.

1) To assure the existence of the minimum norm least squares solution to the differential nonlocal problem, given in the vectorial form.
2) To obtain solvability conditions, those answer if the minimizer is an exact solution to the problem or only an approximate solution.
3) To express the minimum norm least squares solution using the unique exact solution to other relative differential problem.
4) To derive the expression of the generalized Green's function, which represents the minimizer.

Since differential problems cannot always be solved analytically, we examined discrete problems with nonlocal conditions as well. Thus, the following problems were also implemented.

5) To obtain solvability conditions, those answer if the discrete minimum norm least squares solution is an exact solution to the discrete problem or only an approximate solution.
6) To express the discrete minimizer using the unique exact solution to other relative discrete problem.
7) To find the representation of the generalized discrete Green's function, which describes the discrete minimizer.
8) To study the convergence of the discrete minimizer to the minimizer of the differential problem and obtain some sufficient convergence conditions.

4 Methods

In this thesis, we used popular methods from linear algebra, functional analysis (as Riesz representation theorem for continuous linear functionals in Hilbert spaces, Sobolev embedding theorem), differential equations and optimization problems. Ordinary and generalized inverse methods were also applied to describe solutions and study their properties. Let us accentuate the Green's function method, which was used to obtain the representation of the generalized Green's function. For discrete systems in Chapter 6, the method of variation of parameters is taken to derive the expression of the discrete Green's matrix, too.

5 Actuality and novelty

Since the Moore–Penrose inverse has many applications and is widely investigated in scientific literature, some results, obtained in this work, are relative with the features derived by other authors as well. However, most of the results presented in this thesis are original and nowhere published. Provided information is actual in physics, mechanics, economics, biology and other areas of science, where the representation of the best approximate solution to the mathematical problem of the real life can be obtained. Derived solvability conditions can answer if the considering minimizer is an exact solution or not. Moreover, in this thesis all results are studied for differential problems as well as discrete problems in parallel, where all properties and representations are compared for problems with the unique solution and without it. Let us accentuate that this information can be a profitable background to investigate the convergence of the discrete minimizer to the minimizer of a differential problem, what seems not being systematically studied yet. Explicit representations of generalized Green's functions can be used to solve linear or nonlinear problems, or investigate the existence of solutions to nonlinear problems as well.

6 Dissemination of results

Results of the research were presented in 14 conferences, where half of them are international:

1) *MMA2013* (Mathematical Modelling and Analysis in 2013), Tartu, Estonia, May 27-30, 2013, "Generalized Green's functions for discrete boundary value problems";
2) *MMA2014*, Druskininkai, Lithuania, May 26–29, 2014, "Ordinary and generalized Green's functions for discrete nonlocal problems";
3) *MMA2015*, Sigulda, Latvia, May 26-29, 2015, "Generalized Greens functions for m-th order discrete nonlocal problems";
4) *MMA2016*, Tartu, Estonia, June 1-4, 2016, "Generalized Green's functions to the differential nonlocal problems";
5) *NUMTA2016* (Numerical Computations: Theory and Algorithms in 2016), Pizzo Calabro, Italy, June 19–25, 2016, "The minimum norm least squares solution to the discrete nonlocal problems";
6) *MMA2017*, Druskininkai, Lithuania, May 30–June 2, 2017, "Green's matrices for first order differential systems with nonlocal conditions";
7) *MMA2018*, Sigulda, Latvia, May 29–June 1, 2018, "The minimum norm least squares solution to differential nonlocal problems".

Other results were presented in the national conference of the Lithuanian Mathematical Society (LMD):

8) *LMD*, Klaipėda, Lithuania, June 11–12, 2012, "Generalized Green's functions for second order discrete problem with nonlocal conditions";
9) *LMD*, Vilnius, Lithuania, June 19–20, 2013, "Investigation of matrix nullity for the second order discrete problem with nonlocal conditions";
10) *LMD*, Vilnius, Lithuania, June 26–27, 2014, "General classification of the nullity for the second order discrete problems with nonlocal conditions";
11) *LMD*, Kaunas, Lithuania, June 16–17, 2015, "Nullity for the second order discrete problem with nonlocal multipoint boundary conditions";
12) *LMD*, Vilnius, Lithuania, June 20–21, 2016, "Nullspace of the m-th order discrete problem with nonlocal conditions";
13) *LMD*, Vilnius, Lithuania, June 21–22, 2017, "On the convergence of the minimizer for second order discrete nonlocal problems";
14) *LMD*, Kaunas, Lithuania, June 18-19, 2018, "Green's Matrices for Differential Systems with Nonlocal Conditions".

7 Publications

Research results are published in 11 papers, where 5 of them are in the *Web of Science* list of the *Clarivate Analytics* data base. In journals with Citation Index, there are published 4 works:

1) G. Paukštaitė, A. Štikonas, *Generalized Green's functions for the second-order discrete problems with nonlocal conditions*, Lith. Math. J., **54**(2): 203-219, 2014, https://doi.org/10.1007/s10986-014-9238-8
2) G. Paukštaitė, A. Štikonas, "Ordinary and generalized Green's functions for the second order discrete nonlocal problems", Bound. Value Probl., **2015**:**207**, 1-19, 2015, https://doi.org/10.1186/s13661-015-0474-6
3) G. Paukštaitė, A. Štikonas, "Green's Matrices for First Order Differential Systems with Nonlocal Conditions", Math. Model. Anal., **22**(2): 213-227, 2017, https://doi.org/10.3846/13926292.2017.1291456
4) G. Paukštaitė, A. Štikonas, *Generalized Green's functions for mth-order discrete nonlocal problems*, Lith. Math. J., **57**(1):109-127, 2017, https://doi.org/10.1007/s10986-017-9346-3

The one paper appeared in the proceedings of the international conference "Numerical Computations: Theory and Algorithms" (NUMTA2016):

5) A. Štikonas, G. Paukštaitė, *The minimum norm least squares solution to the discrete nonlocal problems*, AIP Conf. Proc. **1776**, 090039 (2016), https://doi.org/10.1063/1.4965403

Below we listed other papers, those appeared in referenced publications "Proceedings of the Lithuanian Mathematical Society":

6) G. Paukštaitė, A. Štikonas, *Generalized Green's functions for second-order discrete boundary-value problems with nonlocal boundary conditions*, Liet. matem. rink. Proc. LMS, Ser. A, **53**: 96-101, 2012.
7) G. Paukštaitė, A. Štikonas, *Investigation of matrix nullity for the second order discrete nonlocal boundary value problem*, Liet. matem. rink. Proc. LMS, Ser. A, **54**: 49-54, 2013.
8) G. Paukštaitė, A. Štikonas, *Classification of the nullity for the second order discrete nonlocal problems*, Liet. matem. rink. Proc. LMS, Ser. A, **55**: 40-45, 2014.
9) G. Paukštaitė, A. Štikonas, *Nullity of the second order discrete problem with nonlocal multipoint boundary conditions*, Liet. matem. rink. Proc. LMS, Ser A, **56**: 72-76, 2015.
10) G. Paukštaitė, A. Štikonas, *Nullspace of the m-th order discrete problem with nonlocal conditions*, Liet. matem. rink. Proc. LMS, Ser A, **57**: 59-64, 2016.
11) G. Paukštaitė, A. Štikonas, *The minimizer for the second order differential problem with one nonlocal condition*, Liet. matem. rink. Proc. LMS, Ser A, **58**: 28-33, 2017.

8 Structure of the dissertation and main results

This dissertation is composed of the introduction, six chapters, general conclusions and the bibliography. Each chapter begins with the introduction and ends with the conclusions as well. Below we formulate main results of each section.

8.1 Second order differential problems with nonlocal conditions

In Chapter 1, a second order differential problem with nonlocal conditions of a one variable is considered, that is,

$$\begin{aligned} &\mathcal{L}u := u'' + a(x)u' + b(x)u = f(x), \quad x \in [0,1], \\ &\langle L_k, u \rangle = g_k,\ k = 1, 2, \end{aligned} \tag{17}$$

where $u \in H^2[0,1]$, $a, b \in C[0,1]$, $f \in L^2[0,1]$ but $L_k \in \left(C^1[0,1]\right)^*$ and $g_k \in \mathbb{R}$. We introduce the operator $\boldsymbol{L} := (\mathcal{L}, L_1, L_2)^\top$ and study the problem (17) in the vectorial form $\boldsymbol{L}u = \boldsymbol{f}$ with the right hand side $\boldsymbol{f} = (f, g_1, g_2)^\top$. If the problem has the unique solution ($\varDelta \neq 0$), it is of the form

$$u = \boldsymbol{L}^{-1}\boldsymbol{f} = \int_0^1 G(x,y) f(y)\, dy + g_1 v^1 + g_2 v^2,$$

where $G(x,y)$ is the *Green's function* but functions v^1, v^2 – the *biorthogonal fundamental system* for the problem (17). If $\varDelta = 0$ and $\boldsymbol{f} \in R(\boldsymbol{L})$, the problem (17) has a lot of solutions. Thus, we studied the range representation for a nonlocal problem (17) without the unique solution ($\varDelta = 0$). For different values of the nullity $d := \dim N(\boldsymbol{L})$, the range is also differently represented.

Lemma 1. *(Lemma 1.3 in Chapter 1)*

1) If $d = 2$, then for all $f \in L^2[0,1]$ we have

$$R(\boldsymbol{L}) = \left\{ \left(f; \int_0^1 \langle L_1, G^c(\cdot, y)\rangle f(y)\, dy; \int_0^1 \langle L_2, G^c(\cdot, y)\rangle f(y)\, dy \right)^\top \right\}.$$

2) If $d = 1$ and $k_1 = 1$, then for all $f \in L^2[0,1]$ and $g_2 \in \mathbb{R}$ we have

$$R(\boldsymbol{L}) = \left\{ \left(f; g_2\langle L_1, v^2\rangle + \int_0^1 \langle L_1, G^a(\cdot, y)\rangle f(y)\, dy; g_2 \right)^\top \right\}.$$

3) If $d = 1$ and $k_1 = 2$, then for all $f \in L^2[0,1]$ and $g_1 \in \mathbb{R}$

$$R(\boldsymbol{L}) = \left\{ \left(f; g_1; g_1\langle L_2, v^1\rangle + \int_0^1 \langle L_2, G^a(\cdot, y)\rangle f(y)\, dy \right)^\top \right\}.$$

Here $G^c(x,y)$ is the Green's function for the second order Cauchy problem; $G^a(x,y)$ is the Green's function and $\{v^1, v^2\}$ is the biorthogonal fundamental system to the problem $\mathcal{L}u = f$ with the original condition $\langle L_{3-k_1}, u\rangle = 0$ and condition $\langle \ell, u\rangle = 0$, replacing $\langle L_{k_1}, u\rangle = 0$. Here $\langle \ell, u\rangle = 0$ is selected such that this auxiliary problem has the unique solution ($\varDelta \neq 0$).

In this lemma, for each case with $d = 1$, the number k_1 denotes the "*dependent*" equation $\langle L_{k_1}, u\rangle = g_{k_1}$ in the system $\boldsymbol{L}u = \boldsymbol{f}$.

From Lemma 1 we obtained two corollaries, where the first provides the structure of the nullspace to the adjoint problem.

Corollary 2. *(Corollary 1.4 in Chapter 1) The following three statements are valid:*

1) $N(\boldsymbol{L}^*) = \operatorname{span}\left\{\left(-\langle L_1, G^c(\cdot,x)\rangle; 1; 0\right)^\top,\ \left(-\langle L_2, G^c(\cdot,x)\rangle; 0; 1\right)^\top\right\}$ *if* $d = 2$;

2) $N(\boldsymbol{L}^*) = \operatorname{span}\left\{\left(-\langle L_1, G^a(\cdot,x)\rangle; 1; -\langle L_1, v^2\rangle\right)^\top\right\}$ *if* $d = 1$ *and* $k_1 = 1$;

3) $N(\boldsymbol{L}^*) = \operatorname{span}\left\{\left(-\langle L_2, G^a(\cdot,x)\rangle; -\langle L_2, v^1\rangle; 1\right)^\top\right\}$ *if* $d = 1$ *and* $k_1 = 2$.

Another corollary presents solvability conditions for a problem without the unique solution.

Corollary 3. *(Solvability conditions; Corollary 1.5 in Chapter 1) The problem* (17) *with* $\Delta = 0$ *is solvable if and only if the conditions are valid:*

1) $\int_0^1 \langle L_1, G^c(\cdot,y)\rangle f(y)\,dy = g_1,\ \int_0^1 \langle L_2, G^c(\cdot,y)\rangle f(y)\,dy = g_2$ *for* $d = 2$;

2) $g_2\langle L_1, v^2\rangle + \int_0^1 \langle L_1, G^a(\cdot,y)\rangle f(y)\,dy = g_1$ *for* $d = 1$ *and* $k_1 = 1$;

3) $g_1\langle L_2, v^1\rangle + \int_0^1 \langle L_2, G^a(\cdot,y)\rangle f(y)\,dy = g_2$ *for* $d = 1$ *and* $k_1 = 2$.

Further in this chapter, Roman's results [100, 2011] ($u \in C^2[0,1]$ and $f \in C[0,1]$) are adopted to provide representations of the unique solution $u \in H^2[0,1]$ and a Green's function for a problem with $f \in L^2[0,1]$ and the unique solution ($\Delta \neq 0$). Then we assure the existence of the minimum norm least squares solution

$$u^o = \boldsymbol{L}^\dagger \boldsymbol{f} = \int_0^1 G^g(x,y) f(y)\,dy + g_1 v^{g,1} + g_2 v^{g,2},$$

where $G^g(x,y)$ is the *generalized Green's function* but functions $v^{g,1}, v^{g,2}$ – the *biorthogonal fundamental system* for the problem (17). This minimizer always exists. We derive its representations and the expression of the generalized Green's function, study their properties.

For instance, in [100, 2011], Roman obtained the representation of the unique solution

$$u = u^c + (g_1 - \langle L_1, u^c\rangle)v^1 + (g_2 - \langle L_2, u^c\rangle)v^2$$

of the second order problem (17) using the unique solution u^c of the Cauchy problem. Here v^1, v^2 is the biorthogonal fundamental system of the original problem (17). For general case, where a problem may have the unique solution or not, we derived the analogous expression of the minimum norm least squares solution.

Corollary 4. *(Corollary 1.20 in Chapter 1) The minimum norm least squares solution to the second order problem with nonlocal conditions is of the form*

$$u^o = u^c - P_{N(\boldsymbol{L})}u^c + (g_1 - \langle L_1, u^c\rangle)v^{g,1} + (g_2 - \langle L_2, u^c\rangle)v^{g,2}.$$

Here $v^{g,1}$, $v^{g,2}$ is the generalized biorthogonal fundamental system of (17).

The similar relation between the minimizer and the unique solution is also valid for other two relative problems

$$\begin{aligned} &\mathcal{L}u = f, && \mathcal{L}v = f, \\ &\langle \widetilde{L}_k, u\rangle = \widetilde{g}_k, \quad k = 1, 2, && \langle L_k, v\rangle = g_k, \quad k = 1, 2, \end{aligned} \tag{18}$$

where $f \in L^2[0,1]$, $u, v \in H^2[0,1]$ but functionals $\widetilde{L}_k$ and $L_k,\ k = 1, 2$, from $(C^1[0,1])^*$ may be different.

Theorem 5. *(Theorem 1.19 in Chapter 1) If the first problem* (18) *has the unique solution u, then the minimum norm least squares solution for the second problem* (18) *is given by*

$$u^o = u - P_{N(\boldsymbol{L})}u + (g_1 - \langle L_1, u\rangle)v^{g,1} + (g_2 - \langle L_2, u\rangle)v^{g,2}.$$

Below we formulate the representation of the generalized Green's function.

Lemma 6. *(Lemma 1.22 in Chapter 1) The generalized Green's function for the second order problem with nonlocal conditions* (17) *is of the form*

$$G^g(x,y) = G^c(x,y) - P_{N(\boldsymbol{L})}G^c(x,y) - \sum_{k=1}^{2}\langle L_k, G^c(\cdot,y)\rangle v^{g,k}(x).$$

Analogous relation is also obtained for two relative problems (18).

Theorem 7. *(Theorem 1.24 in Chapter 1) If the first problem* (18) *has the Green's function $G(x,y)$, then the generalized Green's function $G^g(x,y)$ of the second problem* (18) *is given by*

$$G^g(x,y) = G(x,y) - P_{N(\boldsymbol{L})}G(x,y) - \langle L_1, G(\cdot,y)\rangle v^{g,1}(x) - \langle L_2, G(\cdot,y)\rangle v^{g,2}(x),$$

for all $x \in [0,1]$ and a.e. $y \in [0,1]$.

On the one hand, Lemma 6 gives the representation of the generalized Green's function, which is always applicable since the Cauchy problem always has the Green's function $G^c(x,y)$. On the other hand, it is very useful to apply Theorem 7. Indeed, we can express the generalized Green's function via the Green's function of other "difficult" nonlocal problem making less calculations. For instance, let us take the differential problem with nonlocal boundary conditions

$$\begin{aligned} \mathcal{L}u &:= u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in [0,1], \\ \langle L_k, u\rangle &:= \langle \kappa_k, u\rangle - \gamma_k \langle \varkappa_k, u\rangle = g_k, \quad k = 1,2. \end{aligned} \tag{19}$$

Here functionals $\kappa_k,\ k=1,2$, describe classical parts of conditions (19) and $\varkappa_k,\ k=1,2$, represent fully nonlocal parts of conditions (19). If parameters γ_1, γ_2 vanish, the problem becomes classical.

Corollary 8. *(Corollary 1.28 in Chapter 1) If the classical problem* (19) *(*$\gamma_1, \gamma_2 = 0$*) has the Green's function* $G^{\mathrm{cl}}(x,y)$*, then the generalized Green's function of the nonlocal boundary value problem* (19) *is given by*

$$G^g(x,y) = G^{\mathrm{cl}}(x,y) - P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x,y) + \sum_{k=1}^{2} \gamma_k \langle \varkappa_k, G^{\mathrm{cl}}(\cdot, y)\rangle v^{g,k}(x),$$

for all $x \in [0,1]$ *and a.e.* $y \in [0,1]$.

Thus, only fully nonlocal parts of conditions (19) are used in calculations of $G^g(x,y)$ above if we know the Green's function of the classical problem.

The last section of this chapter is devoted to study examples of minimizers and generalized Green's functions.

8.2 m-th order differential problems with nonlocal conditions

In Chapter 2, all results of Chapter 1 are generalized to m-th order differential problems with nonlocal conditions

$$\begin{aligned} &\mathcal{L}u := u^{(m)} + a_{m-1}(x)u^{(m-1)} + \ldots + a_1(x)u' + a_0(x)u = f(x), \quad x \in [0,1], \\ &\langle L_k, u\rangle = g_k, \quad k = \overline{1,m}, \end{aligned} \tag{20}$$

where $u \in H^m[0,1]$, $a_1, \ldots, a_{m-1} \in C[0,1]$, $f \in L^2[0,1]$, $L_k \in \left(C^{m-1}[0,1]\right)^*$ and $g_k \in \mathbb{R}$. Indeed, we obtained the analogous range representation (Lemma 2.3) as well as the composition of the nullspace for the adjoint problem (Corollary 2.4) and solvability conditions (Corollary 2.5). We formulated properties of the unique solution. As in Chapter 1, such information directed the way how to write expressions of the minimizer and the generalized Green's function. For instance, below we provide descriptions of the minimizer and the generalized Green's function, those are always applicable.

Corollary 9. *(Corollary 2.16 in Chapter 2) The minimum norm least squares solution to the m-th order problem is of the form*

$$u^o = u^c - P_{N(\boldsymbol{L})}u^c + (g_1 - \langle L_1, u^c\rangle)v^{g,1} + \ldots + (g_m - \langle L_m, u^c\rangle)v^{g,m}.$$

The following expression of the generalized Green's function is always valid.

Lemma 10. *(Lemma 2.18 in Chapter 2) The generalized Green's function of the m-th order nonlocal problem is given by*

$$G^g(x,y) = G^c(x,y) - P_{N(\boldsymbol{L})}G^c(x,y) - \sum_{k=1}^{m} \langle L_k, G^c(\cdot, y)\rangle v^{g,k}(x).$$

Other features for the minimizer (Theorem 2.15, Corollary 2.22) and the generalized Green's function (Theorem 2.20, Corollary 2.23) are also analogous.

8.3 Second order discrete problems with nonlocal conditions

In Chapter 3, second order discrete problems with nonlocal conditions

$$\begin{aligned}&(\mathcal{L}u)_i := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad a_i^0, a_i^2 \neq 0, \quad i \in X_{n-2},\\ &\langle L_k, u\rangle = g_k, \quad k = 1, 2\end{aligned} \tag{21}$$

are studied in parallel to Chapter 1. Here we take discrete functions a^0, a^1, a^2, $f \in F(X_{n-2})$, $L_k \in F^*(X_n)$ and $g_k \in \mathbb{C}$, where $F(X_n)$ denotes the set of complex valued functions defined on the set $X_n := \{0, 1, 2, \ldots, n\}$. We rewrite the problem (21) in the matrix form $\mathbf{Au} = \mathbf{b}$ and obtain analogues results to Chapter 1. First, we similarly describe the range composition for different values of the nullity $d := \dim N(\mathbf{A})$ as given below.

Lemma 11. *(Lemma 3.1 in Chapter 3)*

1) If $d = 2$, then for all $f \in F(X_{n-2})$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-2};\ \sum_{j=0}^{n-2} \langle L_1^{\cdot}, G_{\cdot j}^c\rangle f_j;\ \sum_{j=0}^{n-2} \langle L_2^{\cdot}, G_{\cdot j}^c\rangle f_j \right)^\top \right\}.$$

2) If $d = 1$ and $k_1 = 1$, then for all $f \in F(X_{n-2})$ and $g_2 \in \mathbb{C}$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-2};\ g_2\langle L_1, v^2\rangle + \sum_{j=0}^{n-2} \langle L_1^{\cdot}, G_{\cdot j}^a\rangle f_j;\ g_2 \right)^\top \right\}.$$

3) If $d = 1$ and $k_1 = 2$, then for all $f \in F(X_{n-2})$ and $g_1 \in \mathbb{C}$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-2};\ g_1;\ g_1\langle L_2, v^1\rangle + \sum_{j=0}^{n-2} \langle L_2^{\cdot}, G^a_{\cdot j}\rangle f_j \right)^\top \right\}.$$

Here $G^c \in F(X_n \times X_{n-2})$ is the discrete Green's function for the discrete Cauchy problem. Other discrete Green's function $G^a \in F(X_n \times X_{n-2})$ and the biorthogonal fundamental system v^1, v^2 are taken for the problem $\mathcal{L}u = f$ with the original condition $\langle L_{3-k_1}, u\rangle = 0$ and condition $\langle \ell, u\rangle = 0$, replacing $\langle L_{k_1}, u\rangle = 0$. Here $\langle \ell, u\rangle = 0$ is selected such that for this auxiliary problem $\Delta \neq 0$.

Similarly to Chapter 1, k_1 denotes the number of the "*dependent*" equation $\langle L_{k_1}, u\rangle = g_{k_1}$ in the system $\mathbf{Au} = \mathbf{b}$. Let us take $\mathbf{e}^0 = (1; 0; \ldots; 0)^\top$, $\mathbf{e}^1 = (0; 1; \ldots; 0)^\top, \ldots,\ \mathbf{e}^n = (0; 0; \ldots; 1)^\top$ and formulate the following result.

Corollary 12. *(Corollary 3.2 in Chapter 3) Such statements are valid:*

1) if $d = 2$, then $N(\mathbf{A}^*)$ is spanned by two vectors

$$\mathbf{w}^1 = -\sum_{j=0}^{n-2} \overline{\langle L_1^{\cdot}, G^c_{\cdot j}\rangle} \mathbf{e}^j + \mathbf{e}^{n-1}, \qquad \mathbf{w}^2 = -\sum_{j=0}^{n-2} \overline{\langle L_2^{\cdot}, G^c_{\cdot j}\rangle} \mathbf{e}^j + \mathbf{e}^n;$$

2) if $d = 1$ and $k_1 = 1$, then $N(\mathbf{A}^*)$ is spanned by the vector

$$\mathbf{w} = -\sum_{j=0}^{n-2} \overline{\langle L_1^{\cdot}, G^a_{\cdot j}\rangle} \mathbf{e}^j + \mathbf{e}^{n-1} - \overline{\langle L_1, v^2\rangle} \mathbf{e}^n;$$

3) if $d = 1$ and $k_1 = 2$, then $N(\mathbf{A}^*)$ is spanned by the vector

$$\mathbf{w} = -\sum_{j=0}^{n-2} \overline{\langle L_2^{\cdot}, G^a_{\cdot j}\rangle} \mathbf{e}^j - \overline{\langle L_2, v^1\rangle} \mathbf{e}^{n-1} + \mathbf{e}^n.$$

We also derive solvability conditions for the discrete problem (21).

Corollary 13. *(Solvability conditions; Corollary 3.32 in Chapter 3) The problem* (3.1)–(3.2) *with $\Delta = 0$ is solvable if and only if the conditions are valid:*

1) $\sum_{j=0}^{n-2} \langle L_1^{\cdot}, G^c_{\cdot j}\rangle f_j = g_1$, $\sum_{j=0}^{n-2} \langle L_2^{\cdot}, G^c_{\cdot j}\rangle f_j = g_2$ for $d = 2$;

2) $g_2\langle L_1, v^2\rangle + \sum_{j=0}^{n-2} \langle L_1^{\cdot}, G^a_{\cdot j}\rangle f_j = g_1$ for $d = 1$ and $k_1 = 1$;

3) $g_1\langle L_2, v^1\rangle + \sum_{j=0}^{n-2}\langle L_2^{\cdot}, G^a_{\cdot j}\rangle f_j = g_2$ *for* $d = 1$ *and* $k_1 = 2$.

Expressions of the unique discrete solution are analogously formulated and applied to obtain representations of the discrete minimum norm least squares solution $\mathbf{u}^o = \mathbf{A}^\dagger\mathbf{b}$. For example, the discrete minimizer is always described the unique solution u^c to the discrete Cauchy problem.

Corollary 14. *(Corollary 3.14 in Chapter 3) The minimum norm least squares solution to the problem* (21) *is of the form*

$$u^o = u^c - P_{N(A)}u^c + (g_1 - \langle L_1, u^c\rangle)v^{g,1} + (g_2 - \langle L_2, u^c\rangle)v^{g,2}.$$

Here $v^{g,1}$, $v^{g,2}$ *is the generalized biorthogonal fundamental system of* (21).

The similar relation between the minimizer and the unique solution is also valid for two relative discrete problems (18). We formulate this relation below.

Theorem 15. *(Theorem 3.12 in Chapter 3) If the first discrete problem* (18) *has the unique exact solution* $u \in F(X_n)$, *then the minimum norm least squares solution* $u^o \in F(X_n)$ *of the other discrete problem* (18) *is given by*

$$u^o = u - P_{N(A)}u + v^{g,1}(g_1 - \langle L_1, u\rangle) + v^{g,2}(g_2 - \langle L_2, u\rangle).$$

We also obtain the expression of the generalized discrete Green's function.

Corollary 16. *(Corollary 3.17 in Chapter 3) The generalized discrete Green's function* $G^g \in F(X_n \times X_{n-2})$ *to the problem* (21) *is always given by*

$$G^g_{ij} = G^c_{ij} - (P_{N(A)})_{i\cdot}G^c_{\cdot j} - v_i^{g,1}\langle L_1^{\cdot}, G^c_{\cdot j}\rangle - v_i^{g,2}\langle L_2^{\cdot}, G^c_{\cdot j}\rangle, \quad i \in X_n,\ j \in X_{n-2}.$$

The analogous relation is also valid for two discrete relative problems (18), where the first problem has the discrete Green's function $G \in F(X_n \times X_{n-2})$.

Theorem 17. *(Theorem 3.16 in Chapter 3) The generalized discrete Green's function* $G^g \in F(X_n \times X_{n-2})$ *of the second discrete problem* (18) *is of the form*

$$G^g_{ij} = G_{ij} - (P_{N(A)})_{i\cdot}G_{\cdot j} - v_i^{g,1}\langle L_1^{\cdot}, G_{\cdot j}\rangle - v_i^{g,2}\langle L_2^{\cdot}, G_{\cdot j}\rangle, \quad i \in X_n,\ j \in X_{n-2}.$$

This result is applied to obtain the representation of the generalized discrete Green's function for the problem with nonlocal boundary conditions using the discrete Green's function of the classical problem (Corollary 3.19 in Chapter 3).

The target of this chapter is to obtain a convergence of the discrete minimum norm least squares solution to the minimizer of a differential problem. Thus, here we discuss on the minimization problem in two different discrete spaces and obtain literally similar results as above for a discrete minimizer (Corollaries 3.24–3.26) and the generalized discrete Green's function (Theorem 3.27 with Corollaries 3.28–3.29). We provided examples and obtained sufficient convergence conditions.

Theorem 18. *(Sufficient convergence conditions; Theorem 3.33 in Chapter 3) Let the following approximations*

$$\begin{aligned}
&\mathbf{A}(\boldsymbol{\pi}_1 u) = \boldsymbol{\pi}_2 \boldsymbol{L} u + \boldsymbol{\mathcal{O}}(h^\alpha),\\
&\mathbf{P}_{H^2(\overline{\omega}^h),N(A)}(\boldsymbol{\pi}_1 u) = \boldsymbol{\pi}_1(\boldsymbol{P}_{N(\boldsymbol{L})} u) + \boldsymbol{\mathcal{O}}(h^\alpha),\\
&\mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2,R(A)}\mathbf{b} = \boldsymbol{\pi}_2(\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{f}) + \boldsymbol{\mathcal{O}}(h^\alpha)
\end{aligned}$$

be valid for some $\alpha > 0$. *If* $\sup_{n\in\mathbb{N}} \|\mathbf{A}^\dagger\|_{H^2(\overline{\omega}^h),\, L^2(\omega^h)\times\mathbb{R}^2} < +\infty$, *then the minimizer* $\mathbf{u}^o \in H^2(\overline{\omega}^h)$ *of the discrete problem* (21) *converges to the minimizer* $u^o \in H^2[0,1]$ *of the differential problem* (17), *i.e.,*

$$\|\mathbf{u}^o - \boldsymbol{\pi}_1 u^o\|_{C(\overline{\omega}^h)} = \max_{x_i\in\overline{\omega}^h} |u_i^o - u^o(x_i)| \to 0 \qquad \text{if} \quad n\to\infty.$$

Here we took the projection operator $\boldsymbol{\pi}_1 : H^2[0,1] \to H^2(\overline{\omega}^h)$ discretizing a function $u \in H^2[0,1]$ on the mesh $\overline{\omega}^h$ pointwise

$$\boldsymbol{\pi}_1 u = \big(u(x_0), u(x_1), \ldots, u(x_n)\big)^\top.$$

Another projector $\boldsymbol{\pi}_2 : L^2[0,1]\times\mathbb{R}^2 \to L^2(\omega^h)\times\mathbb{R}^2$ is not of some special form.

Let us note that Chapter 3 is written for discrete complex problems, those were not considered for differential problems in previous chapters. First, we are motivated by the fact that a differential problem (17) can be approximated by a complex discrete problem. Second, it seems that the theory of generalized inverses for complex matrices is more wider developed and more easier applied than for complex operators. Third, most authors study complex matrices instead of the real case only.

8.4 m-th order discrete problems with nonlocal conditions

In Chapter 4, we generalize results of Chapter 3 studying m-th order discrete problems with nonlocal conditions

$$
\begin{aligned}
&(\mathcal{L}u)_i := a_i^m u_{i+m} + \ldots + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \qquad a_i^0, a_i^m \neq 0, \quad i \in X_{n-m},\\
&\langle L_k, u\rangle = g_k, \quad k = \overline{1,m},
\end{aligned}
$$

where $a^0, \ldots, a^m, f \in F(X_{n-m})$ but $L_k \in F^*(X_n)$, $g_k \in \mathbb{C}$ and $n \geqslant m$. We provide the range representation (Lemma 4.1), the composition of the nullspace for the adjoint problem (Corollary 4.2) and solvability conditions (Corollary 4.3). Afterwards, we derive expressions of the discrete minimum norm least squares solution (Theorem 4.11, Corollaries 4.12, 4.16) and the generalized discrete Green's function (Theorem 4.14, Corollaries 4.15, 4.17). We also discuss on the minimization problem in two different discrete spaces and obtain literally similar results as above for the discrete minimizer (Corollaries 4.21–4.23) and the generalized discrete Green's function (Theorem 4.24 with Corollaries 4.25–4.26). Lastly in Chapter 4, we formulate sufficient convergence conditions (Theorem 4.32) of the discrete minimizer to the minimizer of the differential problem (20).

8.5 First order differential systems with nonlocal conditions

In Chapter 5, we study a first order differential system with nonlocal conditions

$$
\begin{aligned}
&\frac{d\,u^k}{dx} = \sum_{l=1}^{m} a^{kl}(x)u^l + f^k(x), \quad x \in [0,1],\\
&\sum_{l=1}^{m} \langle L_{kl}, u^l\rangle = g_k, \quad k = \overline{1,m},
\end{aligned}
\tag{22}
$$

given in the short form $\boldsymbol{\mathcal{L}u} = \boldsymbol{f}$, $\langle \boldsymbol{L}_k, \boldsymbol{u}\rangle = g_k$, $k = \overline{1,m}$. Here we take real numbers g_k and functions $u^k \in H^1[0,1], f^k \in L^2[0,1]$, $a^{kl} \in C[0,1]$, $L_{kl} \in C^*[0,1]$. We write the system in the vectorial form $\boldsymbol{Lu} = \boldsymbol{b}$ and investigate analogously as in previous chapters. Indeed, we obtain the range representation (Lemma 5.3) as well as the composition of the nullspace for the adjoint system (Corollary 5.4) and formulate solvability conditions (Corollary 5.5). Then expressions of unique solutions (Lemma 5.7) and the Green's matrix (Lemma 5.9) are derived.

In this Chapter, we also study the relation between the m-th order scalar differential problem (20) and equivalent first order system (22). First, we obtain the representation of the Green's function via Green's matrix.

Corollary 19. *(Corollary 5.12 in Chapter 5) The Green's function for the problem* (20) *can be represented by the function from the Green's matrix of the system* (22)

$$G(x,y) = G^{1m}(x,y).$$

Second, the Green's matrix can also be characterized by the Green's function.

Lemma 20. *(Lemma 5.14 in Chapter 5) The Green's matrix can be described by the Green's function of the equivalent scalar problem as below*

$$G^{kl}(x,y) = -\frac{\partial^{m+k-l-1}}{\partial x^{k-1}\partial y^{m-l}}G(x,y) - \sum_{i=0}^{m-l-1}(-1)^i\frac{\partial^i}{\partial y^i}\big(a_{l+i}(y)\frac{\partial^{k-1}}{\partial x^{k-1}}G(x,y)\big)$$

for $k,l=\overline{1,m}$.

Afterwards, we focus on the system with nonlocal conditions (22), which may not have the unique solution. We solve such problem in the least squares sense and obtain the representation of the minimum norm least squares solution.

Corollary 21. *(Corollary 5.19 in Chapter 5) The minimum norm least squares solution to the system* (22) *is of the form*

$$\boldsymbol{u}^o = \boldsymbol{u}^c - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{u}^c + (g_1 - \langle \boldsymbol{L}_1, \boldsymbol{u}^c\rangle)\boldsymbol{v}^{g,1} + \ldots + (g_m - \langle \boldsymbol{L}_m, \boldsymbol{u}^c\rangle)\boldsymbol{v}^{g,m}.$$

This expression is always valid since the Cauchy system always has the unique solution $\boldsymbol{u}^c$ as well as the Green's matrix $\boldsymbol{G}^c(x,y)$. Thus, the generalized Green's matrix is also similarly represented.

Lemma 22. *(Lemma 5.21 in Chapter 5) The generalized Green's matrix for the system* (22) *is given by*

$$\boldsymbol{G}^g(x,y) = \boldsymbol{G}^c(x,y) - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^c(x,y) - \sum_{k=1}^{m}\boldsymbol{v}^{g,k}(x)\langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot,y)\rangle.$$

Similar relations are also valid if we take other relative problem to the system (22) instead of the Cauchy problem.

Theorem 23. *(Theorem 5.18 in Chapter 5) If the problem* $\boldsymbol{\mathcal{L}u} = \boldsymbol{f}$*,* $\langle \widetilde{\boldsymbol{L}}_k, \boldsymbol{u}\rangle = \widetilde{g}_k$*,* $k = \overline{1,m}$*, has the unique solution* $\boldsymbol{u}$*, then the minimizer for the system* (22) *is given by*

$$\boldsymbol{u}^o = \boldsymbol{u} - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{u} + \sum_{k=1}^{m}(g_k - \langle \boldsymbol{L}_k, \boldsymbol{u}\rangle)\boldsymbol{v}^{g,k}.$$

The generalized Green's matrix is also similarly described.

Theorem 24. *(Theorem 5.23 in Chapter 5) If the problem* $\boldsymbol{\mathcal{L}u} = \boldsymbol{f}$*,* $\langle \widetilde{\boldsymbol{L}}_k, \boldsymbol{u}\rangle = \widetilde{g}_k$*,* $k = \overline{1,m}$*, has the ordinary Green's matrix* $\boldsymbol{G}(x,y)$*, then the generalized Green's matrix for the problem* (22) *is given by*

$$\boldsymbol{G}^g(x,y) = \boldsymbol{G}(x,y) - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}(x,y) - \sum_{k=1}^{m}\boldsymbol{v}^{g,k}(x)\langle \boldsymbol{L}_k, \boldsymbol{G}(\cdot,y)\rangle,$$

for all $x \in [0,1]$ *and a.e.* $y \in [0,1]$*.*

This theorem is applied to problems with nonlocal boundary conditions $\langle \boldsymbol{L}_k, \boldsymbol{u}\rangle := \langle \boldsymbol{\kappa}_k, \boldsymbol{u}\rangle - \gamma_k\langle \boldsymbol{\varkappa}_k, \boldsymbol{u}\rangle = g_k$, $k = \overline{1,m}$. Here functionals κ_k describe classical parts but $\boldsymbol{\varkappa}_k$ represent fully nonlocal parts of conditions. For vanishing parameters γ_k, conditions becomes classical. If the classical problem (all $\gamma_k = 0$ in (22)) has the Green's matrix $\boldsymbol{G}^{\text{cl}}$, then we can obtain the following relation.

Lemma 25. *(Lemma 5.24 in Chapter 5) The generalized Green's matrix for the system* (22) *with nonlocal boundary conditions is given by*

$$\boldsymbol{G}^g(x,y) = \boldsymbol{G}^{\text{cl}}(x,y) - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^{\text{cl}}(x,y) + \sum_{k=1}^{m}\gamma_k\boldsymbol{v}^{g,k}(x)\langle \boldsymbol{\varkappa}_k, \boldsymbol{G}^{\text{cl}}(\cdot,y)\rangle$$

for all $x \in [0,1]$ *and a.e.* $y \in [0,1]$*.*

8.6 First order discrete systems with nonlocal conditions

In Chapter 6, we study first order discrete systems with nonlocal conditions

$$\begin{aligned}(\mathcal{L}U)_i^k &:= u_{i+1}^k - \sum_{l=1}^{m} a_i^{kl}u_i^l = f_i^k, \quad i \in X_{n-1},\\ \sum_{l=1}^{m}\langle L_{kl}, u^l\rangle &= g_k, \quad k = \overline{1,m},\end{aligned} \tag{23}$$

simply given by $\boldsymbol{\mathcal{L}U} = \boldsymbol{F}$, $\langle \boldsymbol{L}_k, \boldsymbol{U}\rangle = g_k$, $k = \overline{1,m}$. Here functions $\boldsymbol{U} = (u^1,\dots,u^m)^\top \in F^m(X_n)$, $\boldsymbol{F} = (f^1,\dots,f^m)^\top \in F^m(X_{n-1})$, $a^{kl} \in F(X_{n-1})$,

$g_k \in \mathbb{C}$ and $L_{kl} \in F^*(X_n)$. Taking the short description $\boldsymbol{AU} = \boldsymbol{B}$ of the problem, we formulate the range representation (Lemma 6.1) and the nullspace composition of the adjoint system (Corollary 6.2), formulate solvability conditions (Corollary 6.3). Obtained results resemble corresponding results from Chapter 5, where we studied first order differential systems with nonlocal conditions.

Moreover, we get familiar representations of the unique solution (Lemmas 6.4, 6.5) and the discrete Green's matrix (Lemma 6.6) as well. Afterwards, the system without the unique solution is investigated. Here we study the minimum norm least squares solution $\boldsymbol{U}^o = \boldsymbol{A}^\dagger \boldsymbol{B}$ and obtain its expressions.

Corollary 26. *(Corollary 6.12 in Chapter 6) The minimizer of the system* (23) *is always given by*

$$\boldsymbol{U}^o = \boldsymbol{U}^c - \boldsymbol{P}_{N(\boldsymbol{A})}\boldsymbol{U}^c + \sum_{k=1}^{m}(g_k - \langle \boldsymbol{L}_k, \boldsymbol{U}^c\rangle)\boldsymbol{V}^{g,k}.$$

Here $\boldsymbol{U}^c$ is the unique solution to the Cauchy system, which always exists. Generalization of this statement is given below.

Theorem 27. *(Theorem 6.11 in Chapter 6) If the problem $\boldsymbol{\mathcal{L}U} = \boldsymbol{F}$, $\langle \widetilde{\boldsymbol{L}}_k, \boldsymbol{U}\rangle = \widetilde{g}_k$, $k = \overline{1,m}$, has the unique solution $\boldsymbol{U}$, then the minimum norm least squares solution for the problem* (23) *is of the form*

$$\boldsymbol{U}^o = \boldsymbol{U} - \boldsymbol{P}_{N(\boldsymbol{A})}\boldsymbol{U} + \sum_{k=1}^{m}(g_k - \langle \boldsymbol{L}_k, \boldsymbol{U}\rangle)\boldsymbol{V}^{g,k}.$$

We also investigate the generalized discrete Green's matrix, which is always described by the discrete Green's matrix $\boldsymbol{G}^c$ of the discrete Cauchy problem.

Lemma 28. *(Lemma 6.15 in Chapter 6) The generalized discrete Green's matrix for the system* (23) *is of the form*

$$G_{ij}^{g,kl} = G_{ij}^{c,kl} - \left(P_{N(\boldsymbol{A})}G^c\right)_{ij}^{kl} - \sum_{\ell=1}^{m} v_i^{g,\ell,k}\langle \boldsymbol{L}_\ell, \boldsymbol{G}_{\cdot j}^{c,\cdot l}\rangle,$$

where $i \in X_n$, $j \in X_{n-1}$ and $k, l = \overline{1,m}$.

This formula is always applicable to find the generalized discrete Green's matrix. However, the following equality may be more practical.

Theorem 29. *(Theorem 6.16 in Chapter 6) If the problem* $\boldsymbol{\mathcal{L}U} = \boldsymbol{F}$, $\langle \widetilde{\boldsymbol{L}}_k, \boldsymbol{U}\rangle = \widetilde{g}_k$, $k = \overline{1,m}$, *has the discrete Green's matrix* $\boldsymbol{G}$, *then the generalized discrete Green's matrix for the system* (23) *is given by*

$$G_{ij}^{g,kl} = G_{ij}^{kl} - \left(P_{N(\boldsymbol{A})}G\right)_{ij}^{kl} - \sum_{\ell=1}^{m} v_i^{g,\ell,k}\langle \boldsymbol{L}_\ell, \boldsymbol{G}_{\cdot j}^{\cdot l}\rangle$$

for all $i \in X_n$, $j \in X_{n-1}$, $k, l = \overline{1,m}$.

Let us illustrate the worth of the application of this lemma. We take nonlocal boundary conditions for the system (23), those are $\langle \boldsymbol{L}_k, \boldsymbol{u}\rangle := \langle \boldsymbol{\kappa}_k, \boldsymbol{u}\rangle - \gamma_k\langle \boldsymbol{\varkappa}_k, \boldsymbol{u}\rangle = g_k$, $k = \overline{1,m}$.

Corollary 30. *(Corollary 6.17 in Chapter 6) Let the classical system* (23) *(all* $\gamma_k = 0$*) has the discrete Green's matrix* $\boldsymbol{G}^{\mathrm{cl}}$. *Then the generalized discrete Green's matrix for the system with nonlocal boundary conditions (*$\gamma_k \in \mathbb{R}$*) is of the form*

$$G_{ij}^{g,kl} = G_{ij}^{\mathrm{cl},kl} - \left(P_{N(\boldsymbol{A})}G^{\mathrm{cl}}\right)_{ij}^{kl} + \sum_{\ell=1}^{m} v_i^{g,\ell,k}\gamma_\ell\langle \boldsymbol{\varkappa}_\ell, \boldsymbol{G}_{\cdot j}^{\mathrm{cl},\cdot l}\rangle.$$

The final goal of this chapter is to obtain the convergence of the discrete minimizer to the minimizer of the differential system (22). Sufficient convergence conditions are formulated below.

Theorem 30. *(Sufficient convergence conditions; Theorem 6.19 in Chapter 6) Let the following approximations*

$$\boldsymbol{A}(\boldsymbol{\pi}_1\boldsymbol{u}) = \boldsymbol{\pi}_2\boldsymbol{L}\boldsymbol{u} + \boldsymbol{\mathcal{O}}(h^\alpha), \qquad \boldsymbol{P}_{N(\boldsymbol{A})}(\boldsymbol{\pi}_1\boldsymbol{u}) = \boldsymbol{\pi}_1\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{u} + \boldsymbol{\mathcal{O}}(h^\alpha),$$
$$\boldsymbol{P}_{R(\boldsymbol{A})}\boldsymbol{B} = \boldsymbol{\pi}_2\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{f} + \boldsymbol{\mathcal{O}}(h^\alpha)$$

be valid for some $\alpha > 0$. *If* $\sup_{n\in\mathbb{N}} \|\mathbf{A}^\dagger\|_{1,2} < +\infty$, *then the discrete minimizer* $\boldsymbol{U}^o$ *converges to the minimizer* $\boldsymbol{u}^o \in \left(H^1[0,1]\right)^m$ *of the differential system* (22) *as given below*

$$\|\boldsymbol{U}^o - \boldsymbol{\pi}_1\boldsymbol{u}^o\|_{(C(\overline{\omega}^h))^m} := \max_{x_i\in\overline{\omega}^h,\ k=\overline{1,m}} |u_i^{o,k} - u^{o,k}(x_i)| \to 0 \qquad \text{if} \quad n \to \infty.$$

Finally, we formulate general conclusions of this thesis.

Let us accentuate that results in all chapters are analogous and their proofs are similar. Thus, all proofs are given only in Chapter 1 but in other chapters many proofs will be omitted. In this thesis, Roman's work [100, 2011] will be often cited because there are collected many useful facts and results about Green's functions and nonlocal conditions.

9 Acknowledgements

I am so sincerely thankful to my supervisor prof. dr. Artūras Štikonas for his support and collaboration during the studies. My supervisor believed in me and gave the freedom to do many works in the way as I understand and make decisions personally as my parents did. I appreciate it very much because then I can think originally and work creatively, what influenced the realization of my potential in mathematics.

My greatest thanks is also to the wise person, my beloved friend Jonas Markūnas, with whom I had long lasting evening discussions on mathematics and its meaning for our life.

I am also thankful to my parents and fellows for supporting me financially and emotionally.

I am so happy for the possibility to study such interesting subject as mathematics. So, I also say thanks to my lectures and the personal of the faculty of Mathematics and informatics at Vilnius university.

Chapter 1

Second order differential problems with nonlocal conditions

1 Introduction

In this chapter, we consider second order differential problems with two nonlocal conditions

$$\mathcal{L}u := u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in [0,1], \tag{1.1}$$

$$\langle L_k, u \rangle = g_k, \quad k = 1,2, \tag{1.2}$$

defined on the real Sobolev space $H^2[0,1]$. Here $a, b \in C[0,1]$, $f \in L^2[0,1]$ are real functions, $\mathcal{L} : H^2[0,1] \to L^2[0,1]$, $g_k \in \mathbb{R}$ and $L_k \in \big(C^1[0,1]\big)^*$, $k = 1,2$. According to [2, Alt 2016], each continuous linear functional $L \in \big(C^1[0,1]\big)^*$ is represented by

$$\langle L, u \rangle := \gamma u(\xi) + \int_0^1 u'(x)\, d\mu(x) \tag{1.3}$$

for some $\gamma \in \mathbb{R}$, a point $\xi \in [0,1]$ and a regular bounded countably additive Borrel measure μ on $[0,1]$, i.e., $\mu \in \mathrm{rca}[0,1]$. Since the function μ is of bounded variation, it can have at most countably many discontinuity points and need only be differentiable almost everywhere (a.e.). Hence, in practice, most nonlocal conditions (1.3) are considered of the form

$$\langle L, u \rangle := \sum_{i=1}^{\infty} \Big(a_i u(\xi_i) + b_i u'(\zeta_i) \Big) + \int_0^1 c(x)u(x) + d(x)u'(x)\, dx \tag{1.4}$$

for $\xi_i, \zeta_i \in [0,1]$, real numbers a_i, b_i and integrable functions $c, d \in L^1[0,1]$. In examples, we also investigate problems (1.1)–(1.2) with nonlocal conditions (1.2) of this form (1.4).

The structure of this chapter is as follows. First, we represent a differential problem with nonlocal conditions into the equivalent vectorial form and derive properties of the vectorial operator. Then the problem with the unique solution is considered. Various representations of the unique solution and a Green's function are given. Afterwards, we study the nonlocal problem without the unique solution. Here we introduce a *minimum norm least squares solution*, which is a unique solution to the problem (1.1)–(1.2) in the least squares sense. Properties and representations of this minimizer are derived. A *generalized Green's function*, which describes a minimum norm least squares solution, is also considered. The last section is devoted to present examples of a minimizer as well as its generalized Green's function. One example of the minimizer is published in the paper [92, Paukštaitė and Štikonas 2017]. Finally, we formulate the basic conclusions.

2 The vectorial problem

The problem (1.1)–(1.2) can be rewritten into the equivalent vectorial form

$$\boldsymbol{L}u = \boldsymbol{f} \tag{2.1}$$

with $\boldsymbol{L} := (\mathcal{L}, L_1, L_2)^\top$ and $\boldsymbol{f} = (f, g_1, g_2)^\top \in L^2[0,1] \times \mathbb{R}^2$. For the Hilbert space $L^2[0,1] \times \mathbb{R}^2$, we use the inner product

$$(\boldsymbol{f}, \tilde{\boldsymbol{f}}) = \int_0^1 f(x)\tilde{f}(x)\,dx + g_1 \cdot \tilde{g}_1 + g_2 \cdot \tilde{g}_2$$

and take the norm

$$\|\boldsymbol{f}\| = \|\boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^2} = (\boldsymbol{f}, \boldsymbol{f})^{1/2} = \sqrt{\|f\|^2_{L^2[0,1]} + |g_1|^2 + |g_2|^2},$$

here $\boldsymbol{f}, \tilde{\boldsymbol{f}} \in L^2[0,1] \times \mathbb{R}^2$. The Sobolev embedding theorem [42, Evans 2010] says that $H^2[0,1] \subset C^1[0,1]$ and the inequality

$$\|u\|_{C^1[0,1]} \leqslant C\|u\|_{H^2[0,1]}, \qquad \forall u \in H^2[0,1], \tag{2.2}$$

is valid for a particular constant C independent on a chosen u. Thus, $(C^1[0,1])^* \subset (H^2[0,1])^*$. Since functionals $L_k \in \big(C^1[0,1]\big)^*$, $k = 1,2$, belong to the dual space $(H^2[0,1])^*$ and $\mathcal{L}$ is defined on $H^2[0,1]$, then the vectorial operator $\boldsymbol{L}$ maps one Hilbert space $H^2[0,1]$ to another Hilbert space $L^2[0,1] \times \mathbb{R}^2$.

Lemma 1.1. *The operator* $\boldsymbol{L}: H^2[0,1] \to L^2[0,1] \times \mathbb{R}^2$ *is the continuous linear operator with the domain* $D(\boldsymbol{L}) = H^2[0,1]$.

Proof. Since the operator $\mathcal{L}$ and functionals $L_k \in (H^2[0,1])^*,\ k = 1,2$, are defined on entire $H^2[0,1]$, then the operator $\boldsymbol{L}$ is defined on entire $H^2[0,1]$ as well. Moreover, the operator $\boldsymbol{L}$ is linear, because the operator $\mathcal{L}$ and functionals $L_k,\ k = 1,2$, are linear.

Now we show that $\boldsymbol{L}$ is continuous. From the triangle inequality, we have $\|\mathcal{L}u\|_{L^2[0,1]} \leqslant \|u''\|_{L^2[0,1]} + \|a\|_{C[0,1]}\|u'\|_{L^2[0,1]} + \|b\|_{C[0,1]}\|u\|_{L^2[0,1]} \leqslant C_o\|u\|_{H^2[0,1]}$ for all $u \in H^2[0,1]$. Since functionals $L_k \in (H^2[0,1])^*,\ k = 1,2$, then $|\langle L_k, u\rangle| \leqslant \|L_k\| \cdot \|u\|_{H^2[0,1]}$ for all $u \in H^2[0,1]$ and finite nonnegative numbers $\|L_k\|,\ k = 1,2$, those denote norms of functionals $L_k : H^2[0,1] \to \mathbb{R}$. Thus, from these estimates we obtain the following inequality

$$\|\boldsymbol{L}u\| = \left(\|\mathcal{L}u\|^2_{L^2[0,1]} + |\langle L_1, u\rangle|^2 + |\langle L_2, u\rangle|^2\right)^{\frac{1}{2}} \leqslant C_1\|u\|_{H^2[0,1]}$$

for $C_1 = C_o + \|L_1\| + \|L_2\|$ and all $u \in H^2[0,1]$, what means that the operator $\boldsymbol{L}$ is continuous. □

2.1 Nullspace of the operator $\boldsymbol{L}$

In this subsection, we discuss on the nullspace $N(\boldsymbol{L})$ of the vectorial operator **L**.

We find the nullspace $N(\boldsymbol{L})$ solving the homogenous system $\boldsymbol{L}u = \mathbf{0}$, i.e., $\mathcal{L}u = 0$ and $\langle L_k, u\rangle = 0,\ k = 1,2$. The differential equation $\mathcal{L}u = 0$, or explicitly $u'' + a(x)u' + b(x)u = 0$, has the general solution $u = c_1z^1 + c_2z^2$ with $c_k \in \mathbb{R}$ and $z^k \in C^2[0,1],\ k = 1,2$, since the equation $u'' + a(x)u' + b(x)u = 0$ with continuous coefficients has two classical fundamental solutions $z^1, z^2 \in C^2[0,1]$. Substituting the general solution into conditions $\langle L_k, u\rangle = 0,\ k = 1,2$, we obtain the system

$$\begin{aligned} c_1\langle L_1, z^1\rangle + c_2\langle L_1, z^2\rangle = 0, \\ c_1\langle L_2, z^1\rangle + c_2\langle L_2, z^2\rangle = 0, \end{aligned}$$

from where constants c_1, c_2 are chosen. Thus, $N(\boldsymbol{L}) \subset N(\mathcal{L}) = \operatorname{span}\{z^1, z^2\} \subset C^2[0,1]$ and the nullity $d := \dim N(\boldsymbol{L}) \in \{0,1,2\}$.

Denoting the determinant

$$\varDelta := \begin{vmatrix} \langle L_1, z^1\rangle & \langle L_1, z^2\rangle \\ \langle L_2, z^1\rangle & \langle L_2, z^2\rangle \end{vmatrix},$$

we separate the following cases:

- $d = 0 \Leftrightarrow \mathit{\Delta} \neq 0$. Then $N(\boldsymbol{L})$ is trivial. It is obvious.

- $d = 2 \Leftrightarrow$ if $\mathit{\Delta} = 0$ with all $\langle L_k, z^l \rangle = 0$ for $k, l = 1, 2$. Then the general solution to $\boldsymbol{L}u = \mathbf{0}$ depends on two arbitrary constants c_1, c_2 and $N(\boldsymbol{L}) = \operatorname{span}\{z^1, z^2\}$. Thus, the solution to $\boldsymbol{L}u = \mathbf{0}$ is equivalent to the solution to the differential equation $\mathcal{L}u = 0$ only.

- $d = 1 \Leftrightarrow$ if $\mathit{\Delta} = 0$ and exists at least one value $\langle L_k, z^l \rangle \neq 0$. Let us say $\langle L_{k_2}, z^l \rangle \neq 0$ emphasizing the number k_2 of the functional. Then we can solve one constant c_l but other c_{3-l} remains arbitrary. Here the one condition $\langle L_{k_1}, u \rangle = 0$ ($k_1 = 3 - k_2$) is dependent since it gives no additional information how to find the arbitrary constant c_{3-l}. Thus, the solution to the problem $\boldsymbol{L}u = \mathbf{0}$ is equivalent to the solution of the simplified problem $\mathcal{L}u = 0$, $\langle L_{k_2}, u \rangle = 0$.

Since an operator $\boldsymbol{L}$ is continuous and linear [65, Kreyszig 1978], then it is closed and has the closed nullspace $N(\boldsymbol{L})$. Thus, we have $\overline{N(\boldsymbol{L})} = N(\boldsymbol{L})$. Then the Sobolev space $H^2[0,1]$ can be represented by the direct sum of orthogonal subspaces as follows

$$H^2[0,1] = N(\boldsymbol{L}) \oplus N(\boldsymbol{L})^{\perp}. \tag{2.3}$$

2.2 Range of the operator $\boldsymbol{L}$

We also derived the following results about the range of the vectorial operator **L**.

Theorem 1.2. *The range $R(\boldsymbol{L})$ of the operator $\boldsymbol{L}$ is closed.*

Proof. The range of the operator $\boldsymbol{L}$ is the set

$$R(\boldsymbol{L}) = \{\boldsymbol{f} = (f, g_1, g_2)^{\top} \in L^2[0,1] \times \mathbb{R}^2 : \boldsymbol{L}u = \boldsymbol{f} \text{ for some } u \in H^2[0,1]\}.$$

Let us arbitrarily take the sequence $\{\boldsymbol{f}_n\} \subset R(\boldsymbol{L}) \subset L^2[0,1] \times \mathbb{R}^2$ converging in the space $L^2[0,1] \times \mathbb{R}^2$, i.e., $\boldsymbol{f}_n \to \boldsymbol{f} \in L^2[0,1] \times \mathbb{R}^2$. If $\boldsymbol{f} \in R(\boldsymbol{L})$ or, equivalently, there exists such $u \in D(\boldsymbol{L})$ that $\boldsymbol{L}u = \boldsymbol{f}$, then the set $R(\boldsymbol{L})$ is closed. Since $\{\boldsymbol{f}_n\} \subset R(\boldsymbol{L})$, then there exist such sequence $\{u_n\} \subset D(\boldsymbol{L})$ that

$$\boldsymbol{L}u_n = \boldsymbol{f}_n, \quad n \in \mathbb{N}. \tag{2.4}$$

From the continuity of $\boldsymbol{L}$, we have

$$\boldsymbol{f} = \lim_{n \to +\infty} \boldsymbol{f}_n = \lim_{n \to +\infty} \boldsymbol{L}u_n = \boldsymbol{L}\big(\lim_{n \to +\infty} u_n\big). \tag{2.5}$$

If the limit $u := \lim\limits_{n\to+\infty} u_n$ belongs to $H^2[0,1]$, which is coincident with the domain of $\boldsymbol{L}$, then $\boldsymbol{f} = \boldsymbol{L}u$. We divide the proof into several steps.

1) Let us take the extended form of the problem (2.4):

$$\mathcal{L}u_n = f_n, \quad \langle L_1, u_n\rangle = g_{1n}, \quad \langle L_2, u_n\rangle = g_{2n}. \tag{2.6}$$

It is solvable, since $\boldsymbol{f}_n \in R(\boldsymbol{L})$. First, the general solution to the equation $\mathcal{L}u_n = f_n$ is of the form

$$u_n = c_1 z^1 + c_2 z^2 + u^{f_n}, \qquad c_1, c_2 \in \mathbb{R}. \tag{2.7}$$

Here z^1 and z^2 are the fundamental system of solutions to the homogenous equation $\mathcal{L}u = 0$ but u^{f_n} is one particular solution to the nonhomogenous equation $\mathcal{L}u = f_n$. As discussed in Subsection 2.1, functions $z^1, z^2 \in C^2[0,1]$ are classical solutions to the equation $\mathcal{L}u = 0$. Moreover, only the last member u^{f_n} of the representation (2.7) depends on the number n. Constants c_1, c_2 do not depend on n, since the general solution to the homogenous equation $\mathcal{L}u = 0$ is of the form $c_1 z^1 + c_2 z^2$ without the number n.

2) For every $n \in \mathbb{N}$, we take the function

$$u^{f_n}(x) = \int\limits_0^1 G^c(x,y) f_n(y)\, dy, \qquad f_n \in L^2[0,1], \tag{2.8}$$

where $G^c(x,y)$ is the Green's function to the Cauchy problem

$$\mathcal{L}u = f, \qquad u(0) = 0, \qquad u'(0) = 0. \tag{2.9}$$

The representation of the Green's function $G^c(x,y)$ is given by

$$G^c(x,y) = \frac{1}{W(y)} \begin{cases} z^1(y)z^2(x) - z^1(x)z^2(y), & 0 \leqslant y \leqslant x, \\ 0, & x \leqslant y \leqslant 1, \end{cases}$$

where $W(y) := W[z^1, z^2](y)$ is the Wronskian of the fundamental system z^1, z^2 at the point $y \in [0,1]$. So, this Green's function $G^c(x,y)$ always exists for the problem (2.9) and has the following properties:

a) $G^c(x,y)$ is continuous on $0 \leqslant x, y \leqslant 1$;

b) $G^c(x,y)$ is C^2 in x except the diagonal $x = y$;

c) $(\partial/\partial x)G^c(y+0,y) - (\partial/\partial x)G^c(y-0,y) = 1$;

d) $\mathcal{L}G^c(\cdot, y) = 0$ except the diagonal $x = y$;

e) $G^c(0,y) = 0$ and $(\partial/\partial x)G^c(0,y) = 0$.

Applying these properties, we derive representations of weak derivatives

$$(u^{f_n})'(x) = \int_0^x \frac{\partial}{\partial x} G^c(x,y) f_n(y)\, dy + \int_x^1 \frac{\partial}{\partial x} G^c(x,y) f_n(y)\, dy, \tag{2.10}$$

$$(u^{f_n})''(x) = \int_0^x \frac{\partial^2}{\partial x^2} G^c(x,y) f_n(y)\, dy + \int_x^1 \frac{\partial^2}{\partial x^2} G^c(x,y) f_n(y)\, dy + f_n(x), \tag{2.11}$$

where $(\partial/\partial x)G^c$ and $(\partial^2/\partial x^2)G^c$ are classical partial derivatives of the Green's function. Let us note that $G^c(x,y)$ is H^1 in x with the continuous partial derivative $(\partial/\partial x)G^c(x,y)$, which has the jump on the diagonal $x = y$. Thus, we can also write

$$(u^{f_n})'(x) = \int_0^1 \frac{\partial}{\partial x} G^c(x,y) f_n(y)\, dy, \tag{2.12}$$

where $(\partial/\partial x)G^c(x,y)$ is now and further understood as the weak derivative if it is not said otherwise.

Now we can directly verify that the function u^{f_n} is the solution to the problem $\mathcal{L}u = f_n$, i.e., it belongs to $H^2[0,1]$ and satisfies the equation $\mathcal{L}u = f_n$.

3) Substituting the general solution $u_n = c_1 z^1 + c_2 z^2 + u^{f_n}$ of the equation $\mathcal{L}u_n = f_n$ into nonlocal conditions (2.6) and rewriting, we get the system

$$\begin{aligned} c_1\langle L_1, z^1\rangle + c_2\langle L_1, z^2\rangle &= g_{1n} - \langle L_1, u^{f_n}\rangle,\\ c_1\langle L_2, z^1\rangle + c_2\langle L_2, z^2\rangle &= g_{2n} - \langle L_2, u^{f_n}\rangle. \end{aligned}$$

Since the problem (2.6) is solvable, then this system is solvable as well and its solutions c_{1n}, c_{2n} depend on the number $n \in \mathbb{N}$. If the determinant of the system $\Delta \neq 0$, then we can find constants

$$c_{1n} = \frac{\begin{vmatrix} g_{1n} - \langle L_1, u^{f_n}\rangle & \langle L_1, z^2\rangle \\ g_{2n} - \langle L_2, u^{f_n}\rangle & \langle L_2, z^2\rangle \end{vmatrix}}{\Delta}, \qquad c_{2n} = \frac{\begin{vmatrix} \langle L_1, z^1\rangle & g_{1n} - \langle L_1, u^{f_n}\rangle \\ \langle L_2, z^2\rangle & g_{2n} - \langle L_2, u^{f_n}\rangle \end{vmatrix}}{\Delta}$$

uniquely. If $\Delta = 0$, we cannot solve constants uniquely, and there are possible two cases. First, all $\langle L_i, z^j\rangle = 0$, $i,j = 1,2$. Then the general solution to the system depends on two arbitrary constants, i.e., $c_1, c_2 \in \mathbb{R}$. We take particular solutions $c_{1n} = c_{2n} = 0$, $n \in \mathbb{N}$. Second, at least one value $\langle L_i, z^j\rangle \neq 0$. For instance, let us say that $\langle L_1, z^1\rangle \neq 0$. Then we can solve one constant

$$c_{1n} = \frac{g_{1n} - \langle L_1, u^{f_n}\rangle - c_{2n}\langle L_1, z^2\rangle}{\langle L_1, z^1\rangle}, \qquad c_{2n} \in \mathbb{R}.$$

We choose arbitrary constant $c_{2n} = 0,\ n \in \mathbb{N}$, and obtain particular solutions c_{1n}, c_{2n} again. So, for all cases we constructed the particular solution $u_n = c_{1n}z^1 + c_{2n}z^2 + u^{f_n},\ n \in \mathbb{N}$, for the problem (2.6).

4) Denoting $u^f = \int_0^1 G^c(x,y)f(y)\,dy$ for the limit function $f \in L^2[0,1]$, we take the difference

$$u^{f_n} - u^f = \int_0^1 G^c(x,y)\big(f_n(y) - f(y)\big)\,dy$$

and obtain the following estimate

$$\begin{aligned}\|u^{f_n} - u^f\|_{L^2[0,1]} &= \left(\int_0^1 \left|\int_0^1 G^c(x,y)\big(f_n(y) - f(y)\big)\,dy\right|^2 dx\right)^{1/2}\\ &\leqslant \left(\int_0^1 \left(\int_0^1 |G^c(x,y)|\cdot|f_n(y) - f(y)|\,dy\right)^2 dx\right)^{1/2}\\ &\leqslant C\left(\int_0^1 \left(\int_0^1 |f_n(y) - f(y)|\,dy\right)^2 dx\right)^{1/2}\end{aligned}$$

$$= C\int_0^1 |f_n(y) - f(y)|\,dy \leqslant C\|f_n - f\|_{L^2[0,1]} \leqslant C\|\boldsymbol{f}_n - \boldsymbol{f}\| \to 0.$$

Here we evaluated the Green's function by the finite constant C since the continuous on the entire unit square Green's function is bounded. For the last integral, we used the Cauchy–Schwartz inequality.

Similarly, differences $(u^{f_n})^{(i)} - (u^f)^{(i)},\ i = 1,2$, are equal to (2.10) and (2.11), respectively, where f_n is replaced by $f_n - f$. Then applying the triangle inequality and the continuity properties of G^c, given in the part 2) of this proof, we obtain other estimates

$$\|(u^{f_n})^{(i)} - (u^f)^{(i)}\|_{L^2[0,1]} \leqslant C\|\boldsymbol{f}_n - \boldsymbol{f}\| \to 0, \qquad i = 1,2.$$

Last three estimates show that $u^{f_n} \to u^f$ in the space $H^2[0,1]$, since $\boldsymbol{f}_n \to \boldsymbol{f}$ in the space $L^2[0,1] \times \mathbb{R}^2$ if $n \to \infty$.

5) Moreover, sequences c_{1n} and c_{2n} also converge, i.e., exist their limits $c_{kn} \to c_k \in \mathbb{R},\ k = 1,2$. It is obvious if $c_{kn} = 0$, since then $c_{kn} \to c_k = 0$. Otherwise, c_{1n} and c_{2n} depend on numbers $g_{kn} - \langle L_k, u^{f_n}\rangle$, those sequences converge to $g_k - \langle L_k, u^f\rangle \in \mathbb{R}$, respectively, since $g_{kn} \to g_k \in \mathbb{R}$ and the

continuity of functionals implies $\lim_{n\to+\infty}\langle L_k, u^{f_n}\rangle = \langle L_k, \lim_{n\to+\infty} u^{f_n}\rangle = \langle L_k, u^f\rangle$, $k = 1, 2$. Thus, the limits c_1 and c_2 always are finite numbers.

6) Finally, from the parts 4) and 5), we get

$$u_n = c_{1n}z^1 + c_{2n}z^2 + u^{f_n} \to c_1 z^1 + c_2 z^2 + u^f =: u$$

if $n \to +\infty$ in the space $H^2[0,1]$, since $z^1, z^2 \in C^2[0,1]$, $c_{kn} \to c_k \in \mathbb{R}$ and $u^{f_n} \to u^f$ in the space $H^2[0,1]$. It means that $\lim_{n\to+\infty} u_n = u \in H^2[0,1]$. From (2.5) follows $\boldsymbol{f} = \boldsymbol{L}u$, which means $\boldsymbol{f} \in R(\boldsymbol{L})$. Since the sequence $\{\boldsymbol{f}_n\} \subset R(\boldsymbol{L})$ is chosen arbitrarily, then $R(\boldsymbol{L})$ is closed. □ □

Below we provide the direct representation of the range $R(\boldsymbol{L})$.

Lemma 1.3.

1) If $d = 2$, then for all $f \in L^2[0,1]$ we have

$$R(\boldsymbol{L}) = \left\{\left(f; \int_0^1 \langle L_1, G^c(\cdot,y)\rangle f(y)\,dy; \int_0^1 \langle L_2, G^c(\cdot,y)\rangle f(y)\,dy\right)^\top\right\}.$$

2) If $d = 1$ and $k_1 = 1$, then for all $f \in L^2[0,1]$ and $g_2 \in \mathbb{R}$ we have

$$R(\boldsymbol{L}) = \left\{\left(f; g_2\langle L_1, v^2\rangle + \int_0^1 \langle L_1, G^a(\cdot,y)\rangle f(y)\,dy; g_2\right)^\top,\right\}.$$

3) If $d = 1$ and $k_1 = 2$, then for all $f \in L^2[0,1]$ and $g_1 \in \mathbb{R}$ we have

$$R(\boldsymbol{L}) = \left\{\left(f; g_1; g_1\langle L_2, v^1\rangle + \int_0^1 \langle L_2, G^a(\cdot,y)\rangle f(y)\,dy\right)^\top\right\}.$$

Here $G^a(x,y)$ is the Green's function and $\{v^1, v^2\}$ is the biorthogonal fundamental system to the problem $\mathcal{L}u = f$ with the original condition $\langle L_{3-k_1}, u\rangle = 0$ and condition $\langle \ell, u\rangle = 0$, replacing $\langle L_{k_1}, u\rangle = 0$. Here $\langle \ell, u\rangle = 0$ is selected such that for this auxiliary problem $\Delta \neq 0$.

Proof. 1) We take the general solution

$$u = c_1 z^1 + c_2 z^2 + \int_0^1 G^c(x,y) f(y)\,dy$$

to the consistent differential equation (1.1). Putting it into nonlocal conditions (1.2), we get the system

$$\begin{aligned} c_1\langle L_1, z^1\rangle + c_2\langle L_1, z^2\rangle &= g_1 - \textstyle\int_0^1 \langle L_1, G^c(\cdot,y)\rangle f(y)\,dy,\\ c_1\langle L_2, z^1\rangle + c_2\langle L_2, z^2\rangle &= g_2 - \textstyle\int_0^1 \langle L_2, G^c(\cdot,y)\rangle f(y)\,dy. \end{aligned}$$

Here we applied the property (2.12) and used the Fubini's theorem for measure spaces to change the order of integration in representations of functionals $L_k,\ k=1,2$, as below

$$\langle L,\int_0^1 G^c(x,y)f(y)\,dy\rangle = \gamma\int_0^1 G^c(\xi,y)f(y)\,dy + \int_0^1\int_0^1 \frac{\partial}{\partial x}G^c(x,y)f(y)\,dy\,d\mu(x)$$
$$= \int_0^1\Big(\gamma G^c(\xi,y)+\int_0^1\frac{\partial}{\partial x}G^c(x,y)\,d\mu(x)\Big)f(y)\,dy = \int_0^1\langle L, G^c(\cdot,y)\rangle f(y)\,dy.$$

Since $d=2$, then all $\langle L_k, z^j\rangle = 0$ and we get $g_1=\int_0^1\langle L_1,G^c(\cdot,y)\rangle f(y)\,dy$ and $g_2=\int_0^1\langle L_2,G^c(\cdot,y)\rangle f(y)\,dy$. Thus, the range $R(\boldsymbol{L})$ is composed of functions $\boldsymbol{f}=(f;g_1;g_2)^\top=\big(f;\int_0^1\langle L_1,G^c(\cdot,y)\rangle f(y)\,dy;\int_0^1\langle L_2,G^c(\cdot,y)\rangle f(y)\,dy\big)^\top$ with an arbitrary $f\in L^2[0,1]$.

2) Since $d=1$, the one condition $\langle L_1,u\rangle = g_1$ (here $k_1=1$) can be omitted as dependent in the consistent problem (1.1)–(1.2). Thus, the problem (1.1)–(1.2) has two independent equations $\mathcal{L}u=f$ and $\langle L_2,u\rangle = g_2$. We can choose such condition $\langle \ell,u\rangle = 0$ that the problem $\mathcal{L}u=f,\ \langle \ell,u\rangle = 0,\ \langle L_2,u\rangle=g_2$ has $\Delta\neq 0$. Then this special problem has the Green's function $G^a(x,y)$ and the fundamental system v^1,v^2, satisfying $\langle L_2,v^k\rangle=\delta_2^k$ and $\langle \ell,v^k\rangle=\delta_1^k$ for $k=1,2$. In example, $\langle \ell,u\rangle=0$ can always be one of independent conditions $u(0)=0$ or $u'(0)=0$. For details see [100, Roman 2011] or Section 2.

As in the part 1) of the proof, we take the general solution

$$u = c_1v^1+c_2v^2+\int_0^1 G^a(x,y)f(y)\,dy$$

to the consistent differential equation (1.1). Putting it into nonlocal conditions (1.2), we analogously get the system

$$\begin{aligned} c_1\langle L_1,v^1\rangle + c_2\langle L_1,v^2\rangle &= g_1-\textstyle\int_0^1\langle L_1,G^a(\cdot,y)\rangle f(y)\,dy,\\ c_1\langle L_2,v^1\rangle + c_2\langle L_2,v^2\rangle &= g_2-\textstyle\int_0^1\langle L_2,G^a(\cdot,y)\rangle f(y)\,dy.\end{aligned}$$

Since $\langle L_2,v^2\rangle = 1,\ \langle L_2,v^1\rangle=0$ and $\langle L_2,G^a(\cdot,y)\rangle=0$ a.e., then $c_2=g_2$. On the other hand, $\Delta=0$ for the problem (1.1)–(1.2) can be rewritten in the form $\langle L_1,v^1\rangle\langle L_2,v^2\rangle = \langle L_1,v^2\rangle\langle L_2,v^1\rangle$ for the particular fundamental system v^1,v^2. From here follows that $\langle L_1,v^1\rangle=0$. Then the condition $c_1\langle L_1,v^1\rangle + c_2\langle L_1,v^2\rangle = g_1-\int_0^1\langle L_1,G^a(\cdot,y)\rangle f(y)\,dy$ can be rewritten as $g_1=g_2\langle L_1,v^2\rangle+\int_0^1\langle L_1,G^a(\cdot,y)\rangle f(y)\,dy$. Finally, the range $R(\boldsymbol{L})$ representation is given by the function $\boldsymbol{f}=(f;g_1;g_2)^\top=(f;g_2\langle L_1,v^2\rangle+\int_0^1\langle L_1,G^a(\cdot,y)\rangle f(y)\,dy;g_2)^\top$ with arbitrary $g_2\in\mathbb{R}$ and $f\in L^2[0,1]$.

3) The proof is obtained similarly. □

According to [6, Ben-Israel and Greville 2003], properties of $\boldsymbol{L}$ implies the closeness of $N(\boldsymbol{L}^*)$, where $\boldsymbol{L}^* : L^2[0,1] \times \mathbb{R}^2 \to H^2[0,1]$ is the adjoint operator of $\boldsymbol{L}$. Then the nullspace and range theorem gives $N(\boldsymbol{L}^*) = R(\boldsymbol{L})^\perp$, which representation can also be derived in the following forms.

Corollary 1.4. *The following three statements are valid:*

1) $N(\boldsymbol{L}^*) = \operatorname{span}\left\{\big(-\langle L_1, G^c(\cdot,x)\rangle; 1; 0\big)^\top,\ \big(-\langle L_2, G^c(\cdot,x)\rangle; 0; 1\big)^\top\right\}$ *if* $d = 2$;

2) $N(\boldsymbol{L}^*) = \operatorname{span}\left\{\big(-\langle L_1, G^a(\cdot,x)\rangle; 1; -\langle L_1, v^2\rangle\big)^\top\right\}$ *if* $d = 1$ *and* $k_1 = 1$;

3) $N(\boldsymbol{L}^*) = \operatorname{span}\left\{\big(-\langle L_2, G^a(\cdot,x)\rangle; -\langle L_2, v^1\rangle; 1\big)^\top\right\}$ *if* $d = 1$ *and* $k_1 = 2$.

Proof. 1) We have the orthogonality condition $(\boldsymbol{f}, \tilde{\boldsymbol{f}}) = 0$ for all $\boldsymbol{f} \in R(\boldsymbol{L})$ and $\tilde{\boldsymbol{f}} = (\tilde{f}, \tilde{g}_1, \tilde{g}_2)^\top \in R(\boldsymbol{L})^\perp = N(\boldsymbol{L}^*)$, i.e.,

$$\int_0^1 f(x)\tilde{f}(x)\,dx + \tilde{g}_1 \int_0^1 \langle L_1, G^c(\cdot,x)\rangle f(x)\,dx + \tilde{g}_2 \int_0^1 \langle L_2, G^c(\cdot,x)\rangle f(x)\,dx = 0$$

with arbitrary $f \in L^2[0,1]$. From the rewritten form

$$\int_0^1 \big(\tilde{f}(x) + \tilde{g}_1\langle L_1, G^c(\cdot,x)\rangle + \tilde{g}_2\langle L_2, G^c(\cdot,x)\rangle\big) f(x)\,dx = 0,$$

we get the condition $\tilde{f}(x) + \tilde{g}_1\langle L_1, G^c(\cdot,x)\rangle + \tilde{g}_2\langle L_2, G^c(\cdot,x)\rangle = 0$ or $\tilde{f}(x) = -\tilde{g}_1\langle L_1, G^c(\cdot,x)\rangle - \tilde{g}_2\langle L_2, G^c(\cdot,x)\rangle$ valid with every $\tilde{g}_1, \tilde{g}_2 \in \mathbb{R}$. Thus, the subspace $R(\boldsymbol{L})^\perp$ is composed of following functions $\tilde{\boldsymbol{f}} = (-\tilde{g}_1\langle L_1, G^c(\cdot,x)\rangle - \tilde{g}_2\langle L_2, G^c(\cdot,x)\rangle; \tilde{g}_1; \tilde{g}_2)^\top$ for all $\tilde{g}_1, \tilde{g}_2 \in \mathbb{R}$, those are generated by two linearly independent vector valued functions $\boldsymbol{w}^1 = \big(-\langle L_1, G^c(\cdot,x)\rangle; 1; 0\big)^\top$ and $\boldsymbol{w}^2 = \big(-\langle L_2, G^c(\cdot,x)\rangle; 0; 1\big)^\top$.

2) We write the orthogonality condition $(\boldsymbol{f}, \tilde{\boldsymbol{f}}) = 0$ in the explicit form

$$\int_0^1 \big(\tilde{f}(x) + \tilde{g}_1\langle L_1, G^a(\cdot,x)\rangle\big) f(x)\,dx + g_2 \cdot \big(\tilde{g}_1\langle L_1, v^2\rangle + \tilde{g}_2\big) = 0$$

for every $g_2 \in \mathbb{R}$ and $f \in L^2[0,1]$. Since g_2 and f obtain values independently, we take $g_2 = 0$, afterwards $f = 0$ and as in the part 1) of the get two conditions $\tilde{f}(x) + \tilde{g}_1\langle L_1, G^a(\cdot,x)\rangle = 0$ and $\tilde{g}_1\langle L_1, v^2\rangle + \tilde{g}_2 = 0$. Rewriting we have $\tilde{f}(x) = -\tilde{g}_1\langle L_1, G^a(\cdot,x)\rangle$ and $\tilde{g}_2 = -\tilde{g}_1\langle L_1, v^2\rangle$. Thus, the nullspace $N(\boldsymbol{L}^*)$ is represented by $\tilde{\boldsymbol{f}} = (-\tilde{g}_1\langle L_1, G^a(\cdot,x)\rangle; \tilde{g}_1; -\tilde{g}_1\langle L_1, v^2\rangle)^\top$ with an arbitrary $\tilde{g}_1 \in \mathbb{R}$, generated by the one vector function $\boldsymbol{w} = \big(-\langle L_1, G^a(\cdot,x)\rangle; 1; -\langle L_1, v^2\rangle\big)^\top$.

3) The proof of the last statement is derived analogously. □

First, from Corollary 1.4, we have that $d = \dim N(\boldsymbol{L})$ and $d^* := \dim N(\boldsymbol{L}^*)$ are equal. Second, applying Corollary 1.4 to the Fredholm alternative theorem, we get the solvability conditions to a problem (1.1)–(1.2) without the unique solution ($\mathit{\Delta} = 0$).

Corollary 1.5. *(Solvability conditions) The problem* (1.1)–(1.2) *with* $\mathit{\Delta} = 0$ *is solvable if and only if the conditions are valid:*

1) $\int_0^1 \langle L_1, G^c(\cdot, y)\rangle f(y)\,dy = g_1,\ \int_0^1 \langle L_2, G^c(\cdot, y)\rangle f(y)\,dy = g_2$ *for* $d = 2$*;*

2) $g_2\langle L_1, v^2\rangle + \int_0^1 \langle L_1, G^a(\cdot, y)\rangle f(y)\,dy = g_1$ *for* $d = 1$ *and* $k_1 = 1$*;*

3) $g_1\langle L_2, v^1\rangle + \int_0^1 \langle L_2, G^a(\cdot, y)\rangle f(y)\,dy = g_2$ *for* $d = 1$ *and* $k_1 = 2$*.*

Example 1.6. *Let us consider the differential problem with one initial and other nonlocal Bitsadze–Samarskii condition*

$$-u'' = f(x), \quad x \in [0, 1], \tag{2.13}$$

$$u(0) = g_1, \qquad u(1) = \gamma u(\xi) + g_2, \tag{2.14}$$

where $\gamma \in \mathbb{R},\ \xi \in (0, 1)$*. This problem does not have the unique solution if* $\mathit{\Delta} := 1 - \gamma\xi = 0$*, i.e.,* $\gamma\xi = 1$*. Indeed, we take the general solution to the differential equation* (2.13)

$$u = c_1 + c_2 x + \int_0^1 G^{\mathrm{cl}}(x, y) f(y)\,dy, \tag{2.15}$$

where $c_1, c_2 \in \mathbb{R}$ *are arbitrary constants and*

$$G^{\mathrm{cl}}(x, y) = \begin{cases} y(1 - x), & y \leqslant x, \\ x(1 - y), & y \geqslant x \end{cases}$$

is the ordinary Green's function to the classical problem (2.13)–(2.14) *with* $\gamma = 0$ *[100, Roman 2011]. Substituting* (2.15) *into conditions* (2.13)*, we get* $c_1 = g_1$ *and*

$$(1 - \gamma\xi)c_2 = (\gamma - 1)g_1 + g_2 + \gamma \int_0^1 G^{\mathrm{cl}}(\xi, y) f(y)\,dy. \tag{2.16}$$

If $\gamma\xi = 1$*, we cannot solve* c_2 *uniquely and obtain the unique solution to the problem* (2.13)–(2.14)*. For the consistent problem* (2.13)–(2.14) *with* $\gamma\xi = 1$*, we get the condition* $g_2 = (1 - \gamma)g_1 - \gamma \int_0^1 G^{\mathrm{cl}}(\xi, y) f(y)\,dy$ *and derive the representation of the range*

$$R(\boldsymbol{L}) = \left\{ \left(f; g_1; (1 - \gamma)g_1 - \gamma \int_0^1 G^{\mathrm{cl}}(\xi, y) f(y)\,dy \right)^{\top} \right\}, \tag{2.17}$$

where $f \in L^2[0,1]$, $g_1 \in \mathbb{R}$.

On the other hand, we obtain the same representation of the range $R(\boldsymbol{L})$ *using Lemma 1.3. Indeed, we take the fundamental system* $z^1 = 1$, $z^2 = x$ *of the homogenous equation* (2.13) *and get* $\langle L_1, z^1 \rangle = z^1(0) = 1 \neq 0$. *Thus,* $d = 1$ *and* $k_1 = 2$, $k_2 = 1$. *Then we formulate the auxiliary problem* $-u'' = f$, $u(0) = 0$, $u(1) = 0$ *for which* $\langle \ell, u \rangle := u(1) = 0$ *and* $\Delta = 1 \neq 0$. *We note that this problem is obtained from the problem* (2.13)–(2.14) *taking* $\gamma = 0$ *as previous. It has the Green's function* $G^{\mathrm{cl}}(x,y)$ *and the biorthogonal fundamental system* $v^1 = 1 - x$, $v^2 = x$. *Substituting* G^{cl} *and* v^1 *into the part* 3) *of Lemma 1.3, we get the same range representation* (2.17).

Furthermore, Corollary 1.4 provides the function

$$\boldsymbol{w}(x) = \Big(\gamma G^{\mathrm{cl}}(\xi, x); \gamma - 1; 1\Big)^\top = \left(\begin{cases} x(1/\xi - 1), & x \leqslant \xi, \\ 1 - x, & x \geqslant \xi \end{cases}; \frac{1}{\xi} - 1; 1\right)^\top,$$

which generates the nullspace $N(\boldsymbol{L}^*)$. *Finally, we get the solvability condition*

$$g_2 = (1-\gamma)g_1 - \gamma \int_0^1 G^{\mathrm{cl}}(\xi, y) f(y)\, dy,$$

or explicitly

$$g_2 = (1-\gamma)g_1 - \gamma \int_0^\xi y f(y)\, dy - \int_\xi^1 f(y)\, dy + \int_0^1 y f(y)\, dy, \qquad (2.18)$$

for the problem (2.13)–(2.14) *with* $\Delta = 0$, *that gives* $\gamma = 1/\xi$ *in formulas above. However, we leave the notion of* γ *since such representations are more easily comparable with analogous expressions for higher order problems and analogous discrete problems.*

Example 1.7. *Here we investigate the differential problem with two Bitsadze–Samarskii conditions*

$$-u'' = f(x), \quad x \in [0,1], \qquad (2.19)$$

$$u(0) = \gamma_1 u(\xi_1) + g_1, \qquad u(1) = \gamma_2 u(\xi_2) + g_2, \qquad (2.20)$$

where γ_1, γ_2 *are real numbers and* $\xi_1, \xi_2 \in (0,1)$.

This problem does not have the unique solution if $\Delta = 0$. *Thus, we take the fundamental system* $z^1 = 1$, $z^2 = x$ *and calculate*

$$\Delta = \begin{vmatrix} 1 - \gamma_1 & 1 - \gamma_2 \\ -\gamma_1 \xi_1 & 1 - \gamma_2 \xi_2 \end{vmatrix} = 1 - \gamma_1(1 - \xi_1) - \gamma_2 \xi_2 - \gamma_1 \gamma_2 (\xi_1 - \xi_2),$$

what gives the relation among parameters $\gamma_1(1-\xi_1) + \gamma_2\xi_2 + \gamma_1\gamma_2(\xi_1 - \xi_2) = 1$ *if* $\Delta = 0$ *in this example.*

Now we are going to represent the range $R(\boldsymbol{L})$, find the function generating the nullspace $N(\boldsymbol{L}^*)$ and provide the solvability condition for the problem with $\varDelta = 0$. Let us note that the case with $\gamma_1 = 0$ is fully investigated in Example 1.6. So, now we consider the problem (2.19)–(2.20) *with $\gamma_1 \neq 0$ only.*

First, we observe that here $\gamma_2\xi_2 \neq 1$. Indeed, if we have otherwise, i.e., $\gamma_2\xi_2 = 1$, then $\varDelta = \gamma_1\xi_1(\xi_2 - 1)/\xi_2 \neq 0$ since $\xi_1, \xi_2 \in (0,1)$ and $\gamma_1 \neq 0$. Second, the condition $\langle L_2, z^2\rangle := z^2(1) - \gamma_2 z^2(\xi_2) = 1 - \gamma_2\xi_2$ does not vanish. It means that $d = 1$ for the problem (2.19)–(2.20) *and we can take $k_1 = 1$ with $k_2 = 2$.*

Now we formulate the auxiliary problem

$$
\begin{gathered}
-u'' = f(x), \quad x \in [0,1], \\
u(0) = 0, \qquad u(1) = \gamma_2 u(\xi_2)
\end{gathered}
$$

with $\widetilde{\varDelta} := 1 - \gamma_2\xi_2 \neq 0$. Let us take its biorthogonal fundamental system

$$
v^1 = \frac{1 - \gamma_2\xi_2 + (\gamma_2 - 1)x}{1 - \gamma_2\xi_2}, \qquad v^2 = \frac{x}{1 - \gamma_2\xi_2}
$$

and the Green's function

$$
G^a(x,y) = \begin{cases} y(1-x), & y \leqslant x, \\ x(1-y), & y \geqslant x, \end{cases} + \frac{\gamma_2 x}{1 - \gamma_2\xi_2} \begin{cases} y(1-\xi_2), & y \leqslant \xi_2, \\ \xi_2(1-y), & y \geqslant \xi_2, \end{cases} \tag{2.21}
$$

derived by Roman [100, 2011].

So, using Lemma 1.3, we obtain the range representation

$$
R(\boldsymbol{L}) = \left\{ \left(f; g_2 \frac{\gamma_1\xi_1}{\gamma_2\xi_2 - 1} - \gamma_1 \int_0^1 G^a(\xi_1, y) f(y)\, dy; g_2 \right)^{\top} \right\}
$$

with every $f \in L^2[0,1]$ and $g_2 \in \mathbb{R}$. Moreover, the vector valued function

$$
\boldsymbol{w} = \left(\gamma_1 G^a(\xi_1, x); 1; \frac{\gamma_1\xi_1}{1 - \gamma_2\xi_2} \right)^{\top}
$$

generates the nullspace $N(\boldsymbol{L}^*)$ as Corollary 1.4 says. Then we know the solvability condition

$$
g_1 = \frac{\gamma_1\xi_1}{\gamma_2\xi_2 - 1} \cdot g_2 - \gamma_1 \int_0^1 G^a(\xi_1, y) f(y)\, dy
$$

for the problem (2.19)–(2.20) *without the unique solution ($\varDelta = 0$).*

Example 1.8. *Let us now take such problem with two nonlocal conditions*

$$-u'' = f(x), \quad x \in [0,1], \tag{2.22}$$

$$u(0) = -2\int_0^1 (2-3x)u(x)\,dx + g_1, \qquad u'(1) = u'(\xi) + g_2, \tag{2.23}$$

depending on parameter $\xi \in [0,1)$.

For this problem, we have $\varDelta \equiv 0$ *with every* $\xi \in [0,1)$. *Indeed, taking the fundamental system* $z^1 = 1$, $z^2 = x$, *we get all* $\langle L_k, z^l\rangle = 0$. *It gives* $d = 2$. *Now we take the Green's function*

$$G^c(x,y) = \begin{cases} y - x, & y \leqslant x, \\ 0, & y \geqslant x, \end{cases} \tag{2.24}$$

for the Cauchy problem $-u'' = f$, $u(0) = 0$, $u'(0) = 0$ *and from Lemma 1.3 obtain the range representation*

$$R(\boldsymbol{L}) = \left\{ \left(f; \int_0^1 y(1-y)^2 f(y)\,dy; -\int_\xi^1 f(y)\,dy \right)^\top, \quad \forall\, f \in L^2[0,1] \right\}.$$

Moreover, Corollary 1.4 gives another composition

$$N(\boldsymbol{L}^*) = \left\{ \big(-x(1-x)^2; 1; 0\big)^\top, \quad \big(\mathrm{H}(x-\xi); 0; 1\big)^\top \right\}, \tag{2.25}$$

where

$$\mathrm{H}(x) := \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0 \end{cases}$$

is the Heaviside function. Lastly, we know solvability conditions

$$g_1 = \int_0^1 y(1-y)^2 f(y)\,dy, \qquad g_2 = \int_\xi^1 f(y)\,dy$$

for the problem (2.22)–(2.23) *with every* $\xi \in [0,1)$ *value and all functions* $f \in L^2[0,1]$.

3 Problem with the unique solution (case $\varDelta \neq 0$)

Substituting the general solution

$$u = c_1 z^1 + c_2 z^2 + \int_0^1 G^c(x,y) f(y)\,dy$$

of the equation (1.1) into nonlocal conditions (1.2), we get the system

$$\begin{array}{l} c_1\langle L_1, z^1\rangle + c_2\langle L_1, z^2\rangle = g_1 - \int_0^1 \langle L_1, G^c(\cdot,y)\rangle f(y)\,dy, \\ c_1\langle L_2, z^1\rangle + c_2\langle L_2, z^2\rangle = g_2 - \int_0^1 \langle L_2, G^c(\cdot,y)\rangle f(y)\,dy. \end{array} \tag{3.1}$$

If $\Delta \neq 0$, we can solve constants c_1, c_2 uniquely and obtain the representation of the unique solution to the problem (1.1)–(1.2), simply denoted by $\boldsymbol{L}u = \boldsymbol{f}$. The unique solution also has the form $u = \boldsymbol{L}^{-1}\boldsymbol{f}$ for every $\boldsymbol{f} \in L^2[0,1] \times \mathbb{R}^2$, where $\boldsymbol{L}^{-1} : L^2[0,1] \times \mathbb{R}^2 \to H^2[0,1]$ is the inverse operator of $\boldsymbol{L} : H^2[0,1] \to L^2[0,1] \times \mathbb{R}^2$. Now we are going to investigate the structure of $\boldsymbol{L}^{-1}$ following the earlier work done by Roman [100, 2011], where the problem (1.1)–(1.2) with $f \in C[0,1]$ and the classical unique solution $u \in C^2[0,1]$ was considered. Here we will adopt her results taking $f \in L^2[0,1]$ and looking for $u \in H^2[0,1]$. This information helps us to make generalizations for problems without the unique solution ($\Delta = 0$) and directs the form how obtained results in the following section should be represented.

3.1 Representation of the inverse operator

To fulfill our plan, let us first select the particular fundamental system v^1, v^2 satisfying the biorthogonality conditions $\langle L_k, v^j \rangle = \delta_k^j$ for $k, j = 1, 2$. These functions v^1, v^2 are also known as the *biorthogonal fundamental system* [100, Roman 2011]. They are unique solutions to the problems

$$\begin{array}{cc} \mathcal{L}v^1 = 0, & \mathcal{L}v^2 = 0, \\ \langle L_1, v^1 \rangle = 1,\ \langle L_2, v^1 \rangle = 0, & \langle L_1, v^2 \rangle = 0,\ \langle L_2, v^2 \rangle = 1, \end{array} \tag{3.2}$$

respectively, and can be expressed as follows

$$v^1 = \frac{\begin{vmatrix} z^1(x) & \langle L_2, z^1 \rangle \\ z^2(x) & \langle L_2, z^2 \rangle \end{vmatrix}}{\Delta}, \qquad v^2 = \frac{\begin{vmatrix} \langle L_1, z^1 \rangle & z^1(x) \\ \langle L_1, z^2 \rangle & z^2(x) \end{vmatrix}}{\Delta}$$

using any other fundamental system $\{z^1, z^2\}$. The biorthogonal fundamental system v^1, v^2 directly gives us the constants

$$c_k = g_k - \int_0^1 \langle L_k, G^c(\cdot, y) \rangle f(y)\, dy, \quad k = 1, 2$$

from the system (3.1) and represents the unique solution

$$u = c_1 v^1 + c_2 v^2 + \int_0^1 G^c(x, y) f(y)\, dy.$$

Substituting here the found constants, we use the Fubini's theorem in measure spaces and rewrite the unique solution into the form

$$u = \int_0^1 G(x, y) f(y)\, dy + g_1 v^1(x) + g_2 v^2(x) \tag{3.3}$$

for all $f \in L^2[0,1]$ and $g_1, g_2 \in \mathbb{R}$. Since the kernel

$$G(x,y) := G^c(x,y) - \langle L_1, G^c(\cdot,y)\rangle v^1(x) - \langle L_2, G^c(\cdot,y)\rangle v^2(x) \tag{3.4}$$

is called *the Green's function* for the problem (1.1)–(1.2) [100, Roman 2011], we introduce *the Green's operator* by

$$Gf = \int_0^1 G(x,y) f(y)\, dy$$

and get the operator representation of the unique solution

$$u = Gf + g_1 v^1 + g_2 v^2 \tag{3.5}$$

for all $f \in L^2[0,1]$ and $g_1, g_2 \in \mathbb{R}$.

On the other hand, we recall $u = \boldsymbol{L}^{-1}\boldsymbol{f}$ with every $\boldsymbol{f} \in L^2[0,1] \times \mathbb{R}^2$ and get the following structure of the inverse operator

$$\boldsymbol{L}^{-1} = \left(G,\ v^1,\ v^2\right) : L^2[0,1] \times \mathbb{R}^2 \to H^2[0,1]. \tag{3.6}$$

Here $G : L^2[0,1] \to H^2[0,1]$ and $v^1, v^2 \in H^2[0,1]$ (precisely, $v^1, v^2 \in C^2[0,1]$ according to Subsection 2.1) are also characterized by the inverse operator as given below

$$Gf = \boldsymbol{L}^{-1}(f,0,0)^\top, \qquad v^1 = \boldsymbol{L}^{-1}(0,1,0)^\top, \qquad v^2 = \boldsymbol{L}^{-1}(0,0,1)^\top. \tag{3.7}$$

3.2 Properties of the unique solution

Authors [100, Roman 2011], [125, Štikonas and Roman 2009] obtained various representations of the unique solution to the nonlocal problem (1.1)–(1.2), where $f \in C[0,1]$. For example, the unique solution, if it exists, is always described by the unique solution

$$u^c = \int_0^1 G^c(x,y) f(y)\, dy$$

to the Cauchy problem (2.9) as follows

$$u = u^c + (g_1 - \langle L_1, u^c\rangle) v^1 + (g_2 - \langle L_2, u^c\rangle) v^2. \tag{3.8}$$

This representation is also valid if $f \in L^2[0,1]$ and $u \in H^2[0,1]$. Indeed, we can take the extended form of the Green's function (3.4) in the formula (3.5), group functions as desired and change the order of integration for functionals.

Since the Cauchy problem (2.9) always has the unique solution u^c, the representation (3.8) is always applicable. On the other hand, sometimes we know the unique solution to another differential problem, that is not the Cauchy problem, and desire for some possible representation as (3.8). Applying the proof, given by Roman [100, 2011] with $f \in C[0,1]$, we answer that the unique solutions of two relative problems

$$\begin{aligned} &\mathcal{L}u = f, &&\mathcal{L}v = f, \\ &\langle \widetilde{L}_k, u\rangle = \widetilde{g}_k, \quad k=1,2, &&\langle L_k, v\rangle = g_k, \quad k=1,2, \end{aligned} \tag{3.9}$$

where $f \in L^2[0,1]$, $u \in H^2[0,1]$ but functionals $\widetilde{L}_k$ and L_k, $k=1,2$, may be different, are analogously related. We formulate this statement below.

Corollary 1.9. *For unique solutions to the problems* (3.9)*, the following equality is always satisfied*

$$v = u + (g_1 - \langle L_1, u\rangle)v^1 + (g_2 - \langle L_2, u\rangle)v^2.$$

Let us note that here we used the biorthogonal fundamental system v^1, v^2 for the second problem (3.9) only. Furthermore, conditions $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$ for both problems, respectively, are fulfilled. Applying this corollary, we get the relation between biorthogonal fundamental systems for these problems (3.9) as well.

Corollary 1.10. *Let $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$ for problems* (3.9)*. Then their biorthogonal fundamental systems $\widetilde{v}^1, \widetilde{v}^2$ and v^1, v^2 are related by*

$$\begin{pmatrix} \langle L_1, \widetilde{v}^1\rangle & \langle L_2, \widetilde{v}^1\rangle \\ \langle L_1, \widetilde{v}^2\rangle & \langle L_2, \widetilde{v}^2\rangle \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \widetilde{v}^1 \\ \widetilde{v}^2 \end{pmatrix}.$$

Proof. First, let us take $f = 0$, $\widetilde{g}_1 = g_1 = 1$ and $\widetilde{g}_2 = g_2 = 0$ for problems (3.9). According to Corollary 1.9, their solutions are $\widetilde{v}^1$ and v^1, respectively, are linked with the equality

$$v^1 = \widetilde{v}^1 + (1 - \langle L_1, \widetilde{v}^1\rangle)v^1 - \langle L_2, \widetilde{v}^1\rangle v^2,$$

that can be rewritten in the form $\langle L_1, \widetilde{v}^1\rangle v^1 + \langle L_2, \widetilde{v}^1\rangle v^2 = \widetilde{v}^1$. Afterwards taking $f = 0$, $\widetilde{g}_1 = g_1 = 0$ and $\widetilde{g}_2 = g_2 = 1$ for problems (3.9), we obtain other equality $\langle L_1, \widetilde{v}^2\rangle v^1 + \langle L_2, \widetilde{v}^2\rangle v^2 = \widetilde{v}^2$. Together they confirm the statement of this corollary. □

3.3 Properties of the Green's function

Let us analyze the representation (3.4) of the Green's function. First, the measure μ_k can have at most countably many discontinuity points $y_0, y_1, y_2, \ldots$. Thus, functions

$$\langle L_k, G^c(\cdot, y)\rangle = \gamma G^c(\xi, y) + \int_0^1 \frac{\partial}{\partial x} G^c(x,y)\, d\mu_k(x), \qquad k=1,2,$$

may have countably many discontinuities as well, what means that the Green's function $G(x,y)$ for every fixed $x \in [0,1]$ may have countably many discontinuities at $y = y_0, y_1, y_2, \ldots$. In other words, the square $[0,1] \times [0,1]$ may be divided into $N \in \overline{\mathbb{N}}$ rectangular domains (each rectangle $x \in [0,1]$, $y_{l-1} < y < y_l$, $l = \overline{1,N}$, $y_0 = 0$, $y_N = 1$), where $G(x,y)$ in continuous in y. In each rectangular domain, we apply the properties of the Green's function $G^c(x,y)$ (given in the proof of Theorem 1.2) and properties (3.2) of $v^1, v^2 \in C^2[0,1]$ (see Subsection 2.1). From here, we obtain the following properties of the Green's function.

Corollary 1.11. *For $y \neq y_0, y_1, y_2, \ldots$ with any $x \in [0,1]$, we have:*

1) $G(x,y)$ is continuous in (x,y);

2) $G(x,y)$ is C^2 in x except the diagonal $x = y$;

3) $(\partial/\partial x)G(y+0,y) - (\partial/\partial x)G(y-0,y) = 1$;

4) $\mathcal{L}G(\cdot,y) = 0$ except the diagonal $x = y$;

5) $\langle L_k, G(\cdot,y)\rangle = 0$, $k = 1,2$.

Moreover, a Green's function $G(x,y)$ for the problem (1.1)–(1.2) can also be represented by a Green's function $\widetilde{G}(x,y)$ for another relative differential problem [100, Roman 2011]. We present the following relation.

Proposition 1.12. *For the problems (3.9) with $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$, their Green's functions $\widetilde{G}(x,y)$ and $G(x,y)$, respectively, are linked with the equality*

$$G(x,y) = \widetilde{G}(x,y) - \langle L_1, \widetilde{G}(\cdot,y)\rangle v^1(x) - \langle L_2, \widetilde{G}(\cdot,y)\rangle v^2(x),$$

for all $x \in [0,1]$ and a.e. $y \in [0,1]$.

Proof. We prove it analogously as Roman did [100, 2011] for problems with $f \in C[0,1]$ and classical solutions from $C^2[0,1]$. □

3.4 Applications to nonlocal boundary conditions

Let us consider the problem with nonlocal boundary conditions

$$\mathcal{L}u := u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in [0,1], \tag{3.10}$$

$$\langle L_k, u\rangle := \langle \kappa_k, u\rangle - \gamma_k \langle \varkappa_k, u\rangle = g_k, \quad k = 1,2. \tag{3.11}$$

Here functionals $\kappa_k,\ k=1,2$, describe classical parts of conditions (3.11), whereas γ_1,γ_2 are parameters holding fully nonlocal conditions, represented by functionals $\varkappa_k,\ k=1,2$. In example,

$$\langle L_k,u\rangle := u(0)-\gamma_1 u'(\xi) = g_1, \qquad \langle L_k,u\rangle := u'(1)-\gamma_2\int_0^1 \alpha(x)u(x)\,dx = g_2,$$

where $\xi\in(0,1)$ and $\alpha\in L^1[0,1]$.

This problem (3.10)–(3.11) is special, because, for vanishing parameters $\gamma_1,\gamma_2=0$, it becomes classical. The unique solution u^{cl} to the classical problem is widely investigated, its Green's function G^{cl} is practically known. So, we want to know, how the unique solution u of the problem with nonlocal boundary conditions (3.10)–(3.11) is related to the unique solution u^{cl} of the classical problem. According to Roman [100, 2011], we apply Corollary 1.9, do simplifications and present the desired relation below

$$u = u^{\mathrm{cl}} + \gamma_1\langle\varkappa_1,u^{\mathrm{cl}}\rangle v^1 + \gamma_2\langle\varkappa_2,u^{\mathrm{cl}}\rangle v^2,$$

where v^1,v^2 is the biorthogonal fundamental system to the problem with nonlocal boundary conditions (3.10)–(3.11). We note that only functionals $\varkappa_1,\varkappa_2$ for fully nonlocal conditions here are used. Indeed, this solution is coincident with the solution u^{cl} to the classical problem if parameters γ_1,γ_2 vanish.

Furthermore, Green's functions for the problem (3.10)–(3.11) and the classical problem ($\gamma_1,\gamma_2=0$) are similarly linked with the equality

$$G(x,y) = G^{\mathrm{cl}}(x,y) + \gamma_1\langle\varkappa_1,G^{\mathrm{cl}}(\cdot,y)\rangle v^1(x) + \gamma_2\langle\varkappa_2,G^{\mathrm{cl}}(\cdot,y)\rangle v^2(x) \quad (3.12)$$

for all $x\in[0,1]$ and a.e. $y\in[0,1]$. From here we can also obtain the following property.

Corollary 1.13. *If fully nonlocal conditions for the problem* (3.10)–(3.11) *are of the form*

$$\langle\varkappa,u\rangle := \sum_{j=1}^{\infty}\gamma_j u(\xi_j) + \int_0^1 \alpha(x)u(x)\,dx \quad (3.13)$$

with $\gamma_j\in\mathbb{R}$, $\xi_j\in(0,1)$ *and* $\alpha\in L^1[0,1]$, *then the Green's function* $G(x,y)$ *is continuous on the entire square* $0\leqslant x,y\leqslant 1$ *as well as its partial derivatives* $(\partial/\partial x)G(x,y)$ *and* $(\partial^2/\partial x^2)G(x,y)$ *except the diagonal* $x=y$.

Proof. First, the Green's function $G^{\mathrm{cl}}(x,y)$ is continuous on the entire unit square. Its partial derivatives $(\partial/\partial x)G^{\mathrm{cl}}(x,y)$ and $(\partial^2/\partial x^2)G^{\mathrm{cl}}(x,y)$ are also continuous except the diagonal $x = y$ [100, Roman 2011]. Substituting it into conditions (3.13), we get that $\langle \varkappa, G^{\mathrm{cl}}(\cdot, y)\rangle$ is a continuous function for every $y \in [0, 1]$. According to Subsection 2.1, the biorthogonal fundamental system $v^1, v^2 \in C^2[0, 1]$. Then the right hand side of the representation (3.12) is also continuous on the square $0 \leqslant x, y \leqslant 1$. From here we also see that its partial derivatives $(\partial/\partial x)G(x, y)$ and $(\partial^2/\partial x^2)G(x, y)$ are continuous on the entire domain $0 \leqslant x, y \leqslant 1$ except the diagonal $x = y$. □

Remark 1.14. We can also derive that the Green's function $G(x, y)$ is continuous on the entire unit square $0 \leqslant x, y \leqslant 1$ and its partial derivatives in x are continuous except the diagonal $x = y$ if κ_k, $k = 1, 2$, represent initial conditions (instead of classical conditions) and fully nonlocal conditions for the problem (3.10)–(3.11) are of the form (3.13) with $\xi_j \in (0, 1]$. Here we take $G^c(x, y)$ instead $G^{\mathrm{cl}}(x, y)$ in the expression (3.12). Since $G^c(x, y)$ is also continuous on the entire unit square and is C^2 in x except $x = y$, we similarly obtain the proof.

According to Corollary 1.13, the expression of the Green's function (3.12) gives continuous partial derivatives

$$\frac{\partial^i}{\partial x^i}G(x, y) = \frac{\partial^i}{\partial x^i}G^{\mathrm{cl}}(x, y) + \sum_{k=1}^{2} \gamma_1 \langle \varkappa_k, G^{\mathrm{cl}}(\cdot, y)\rangle (v^k)^{(i)}(x)$$

for $i = 1, 2$, on the entire unit square except the diagonal $x = y$. Moreover, the weak derivatives of the unique solution (3.3) can be described using these classical derivatives of the Green's function, that is,

$$u' = \int_0^x \frac{\partial}{\partial x}G(x, y)f(y)\, dy + \int_x^1 \frac{\partial}{\partial x}G(x, y)f(y)\, dy + \sum_{k=1}^{2} g_k (v^k)'(x)$$

and

$$u'' = \int_0^x \frac{\partial^2}{\partial x^2}G(x, y)f(y)\, dy + \int_x^1 \frac{\partial^2}{\partial x^2}G(x, y)f(y)\, dy + f(x) + \sum_{k=1}^{2} g_k (v^k)''(x).$$

Here the first derivative u' is classical since $u \in H^2[0, 1] \subset C^1[0, 1]$. Let us note that u'' is also continuous and $u \in C^2[0, 1]$ if $f \in C[0, 1]$.

4 The unique minimizer (case $\Delta = 0$)

We have just considered the unique solution to the nonlocal problem (1.1)–(1.2) with $\Delta \neq 0$, presented its properties and representations for the Green's function $G(x, y)$. If $\varDelta = 0$, then the problem $\boldsymbol{L}u = \boldsymbol{f}$ is not uniquely solvable and we cannot obtain such results.

However, here a unique solution also exists but in a little bit different sense. Precisely, we are going to look for the unique function, which minimizes the norm of the residual $\boldsymbol{L}u - \boldsymbol{f}$ to the problem (1.1)–(1.2) and is "*smallest*" among all minimizers of the residual. It is obvious that this minimizer is coincident with the unique solution $u = \boldsymbol{L}^{-1}\boldsymbol{f}$ if it exists. Such an extended interpretation of the problem, introducing the minimization, always gives the meaning for a *unique solution–minimizer* to the problem (1.1)–(1.2), which may have the unique exact solution ($\varDelta \neq 0$) or not ($\varDelta = 0$).

In this section, we derive properties and representations for such minimizer and its Green's function. To our amazement, we will obtain literally similar and relative results as in Section 3.

4.1 The minimum norm least squares solution

If $\varDelta = 0$ for the system (3.1), we cannot solve constants c_1, c_2 uniquely and obtain the unique solution to the differential problem (1.1)–(1.2), simply denoted by $\boldsymbol{L}u = \boldsymbol{f}$. Here the representation $u = \boldsymbol{L}^{-1}\boldsymbol{f}$ is also invalid.

Now the problem (1.1)–(1.2) has a lot of solutions (consistent problem) or has no solutions (inconsistent problem). For both cases, let us look for the least minimizer u^o among all functions $u^g \in H^2[0, 1]$, minimizing the residual

$$\|\boldsymbol{L}u^g - \boldsymbol{f}\| = \inf_{u \in H^2[0,1]} \|\boldsymbol{L}u - \boldsymbol{f}\|, \tag{4.1}$$

i.e.,

$$\|u^o\| < \|u^g\| \qquad \forall\ u^g \neq u^o. \tag{4.2}$$

Minimization steps (4.1)–(4.2) for the consistent problem means that we select the unique solution, which has the minimum $H^2[0, 1]$ norm among all solutions. If the problem is inconsistent, we think about an approximate solution. If there are several approximate solutions, we choose the one of the minimum norm again.

Since $\boldsymbol{L}$ is the continuous linear operator with the closed range $R(\boldsymbol{L})$ [6, Ben-Israel and Greville 2003], such a minimizer u^o for the differential problem (1.1)–(1.2) always exists and is unique. It is often called *the minimum norm least squares solution*. Other authors also use names as *the approximate solution*, *the virtual solution*, *the least extremal solution* and etc.

Let us note that every *least squares solution*, which is the minimizer of (4.1), can be represented by the minimum norm least squares solution

$$u^g = u^o + P_{N(\boldsymbol{L})}c \tag{4.3}$$

for some $c \in H^2[0,1]$, whereas the minimum norm least squares solution is always equal to

$$u^o = P_{N(\boldsymbol{L})^\perp} u^g. \tag{4.4}$$

Here $P_{N(\boldsymbol{L})}$ and $P_{N(\boldsymbol{L})^\perp}$ denote orthogonal projectors onto the nullspace $N(\boldsymbol{L})$ and its orthogonal complement $N(\boldsymbol{L})^\perp$, respectively. We accentuate that $P_{N(\boldsymbol{L})}$ is the zero operator for the problem (2.1) with the unique solution ($d = 0$) because here $N(\boldsymbol{L})$ is trivial. Otherwise ($d > 0$), the projection onto the nullspace can be calculated as below

$$P_{N(\boldsymbol{L})}c = \sum_{l=1}^{d} z^l(x)(z^l, c)_{H^2[0,1]} \tag{4.5}$$

for every function $c \in H^2[0,1]$, where z^l, $l = \overline{1,d}$, is the orthonormal basis of the nullspace $N(\boldsymbol{L})$ with respect to the inner product in $H^2[0,1]$.

If the problem (4.1)–(4.2), or simply $\boldsymbol{L}u = \boldsymbol{f}$, has the unique solution $u = \boldsymbol{L}^{-1}\boldsymbol{f}$, then it is coincident with the minimizer u^o. For $\Delta = 0$, the problem $\boldsymbol{L}u = \boldsymbol{f}$ does not have the inverse $\boldsymbol{L}^{-1} : L^2[0,1] \times \mathbb{R}^2 \to H^2[0,1]$ but we have a decision. Since $\boldsymbol{L} : H^2[0,1] \to L^2[0,1] \times \mathbb{R}^2$ is the continuous linear operator with the closed range $R(\boldsymbol{L})$, there exists the unique operator $\boldsymbol{L}^\dagger : L^2[0,1] \times \mathbb{R}^2 \to H^2[0,1]$ satisfying the four operator equations

$$\boldsymbol{L}\boldsymbol{L}^\dagger\boldsymbol{L} = \boldsymbol{L}, \quad \boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{L}^\dagger = \boldsymbol{L}^\dagger, \quad (\boldsymbol{L}\boldsymbol{L}^\dagger)^* = \boldsymbol{L}\boldsymbol{L}^\dagger, \quad (\boldsymbol{L}^\dagger\boldsymbol{L})^* = \boldsymbol{L}^\dagger\boldsymbol{L}.$$

The operator $\boldsymbol{L}^\dagger$ is often called *the Moore–Penrose inverse* of the operator $\boldsymbol{L}$ [6, Ben-Israel and Greville 2003]. Below we list several basic properties of the Moore–Penrose inverse:

1) $\boldsymbol{L}^\dagger = \boldsymbol{L}^{-1}$, if $\boldsymbol{L}^{-1}$ exists ($\Delta \neq 0$);
2) $(\boldsymbol{L}^\dagger)^\dagger = \boldsymbol{L}$;
3) $(\boldsymbol{L}^\dagger)^* = (\boldsymbol{L}^*)^\dagger$;
4) $N(\boldsymbol{L}^\dagger) = N(\boldsymbol{L}^*)$;
5) $R(\boldsymbol{L}^\dagger) = R(\boldsymbol{L}^*)$;
6) $R(\boldsymbol{L}) = N(\boldsymbol{L}^*)^\perp$;
7) $\boldsymbol{P}_{R(\boldsymbol{L})} = \boldsymbol{L}\boldsymbol{L}^\dagger$ and $\boldsymbol{P}_{R(\boldsymbol{L}^*)} = \boldsymbol{L}^\dagger\boldsymbol{L}$.

Moreover, we accentuate the following properties.

Lemma 1.15. *The Moore–Penrose inverse* $\boldsymbol{L}^\dagger$ *for the problem* (4.1)–(4.2) *is the continuous linear operator with the domain* $D(\boldsymbol{L}^\dagger) = L^2[0,1]\times\mathbb{R}^2$ *and the range* $R(\boldsymbol{L}^\dagger) = N(\boldsymbol{L})^\perp$.

Proof. It follows from the definition of the Moore–Penrose inverse and its well known properties [6, Ben-Israel and Greville 2003]. From there, $D(\boldsymbol{L}^\dagger) := R(\boldsymbol{L}) \oplus R(\boldsymbol{L})^\perp = \overline{R(\boldsymbol{L})} \oplus R(\boldsymbol{L})^\perp = L^2[0,1] \times \mathbb{R}^2$ since $R(\boldsymbol{L})$ is closed (see Theorem 1.2). Moreover, $R(\boldsymbol{L}^\dagger) := D(\boldsymbol{L}) \cap N(\boldsymbol{L})^\perp = N(\boldsymbol{L})^\perp$ because $D(\boldsymbol{L})$ is coincident with the entire space $H^2[0,1]$ (see the Lemma 1.1). □

The most important property for our investigation says that the Moore–Penrose inverse describes the minimizer to the problem (4.1)–(4.2) for all $\boldsymbol{f} \in L^2[0,1] \times \mathbb{R}^2$ in the form

$$u^o = \boldsymbol{L}^\dagger \boldsymbol{f} \tag{4.6}$$

similarly as the unique solution is represented by $u = \boldsymbol{L}^{-1}\boldsymbol{f}$, if it exists. Furthermore, the minimizer (4.6) to the problem $\boldsymbol{L}u = \boldsymbol{f}$, which may be consistent ($\boldsymbol{f} \in R(\boldsymbol{L})$) or inconsistent ($\boldsymbol{f} \notin R(\boldsymbol{L})$), is always the minimizer to a consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{f}$ [6, Ben-Israel and Greville 2003].

To illustrate the nature of minimizers, we discuss on two examples below.

Example 1.16. *The following problem*

$$\begin{aligned} -u'' = 0, &\quad x \in [0,1], \\ u(0) = \gamma, &\qquad u'(1) - \gamma u'(1/2) = 0 \end{aligned}$$

depends on one parameter $\gamma \in \mathbb{R}$. *We can directly obtain its unique solution* $u = \gamma$ *if* $\gamma \neq 1$. *Here* $\gamma \neq 1$ *means the condition* $\Delta \neq 0$. *In Figure 1.1, unique solutions are presented by solid blue horizontal lines.*

The horizontal dashed line denotes the area $u = 1$ *for* $\gamma = 1$, *where the problem does not have the unique solution. Here we obtain a lot of solutions* $u = 1 + cx$, $c \in \mathbb{R}$, *those can be represented by rays going through the point* $(0,1)$. *So, the problem with* $\gamma = 1$ ($\Delta = 0$) *is consistent and its general least squares solution is also of the form* $u^g = 1 + cx$, $c \in \mathbb{R}$. *Taking particular values of the constant c, we get particular solutions (rays).*

Let us now find the unique solution of the minimum $H^2[0,1]$ *norm. We minimize the square of its* $H^2[0,1]$ *norm*

$$\|u^g\|^2_{H^2[0,1]} = \int_0^1 (1+cx)^2\,dx + \int_0^1 c^2\,dx = \frac{4}{3}c^2 + c + 1.$$

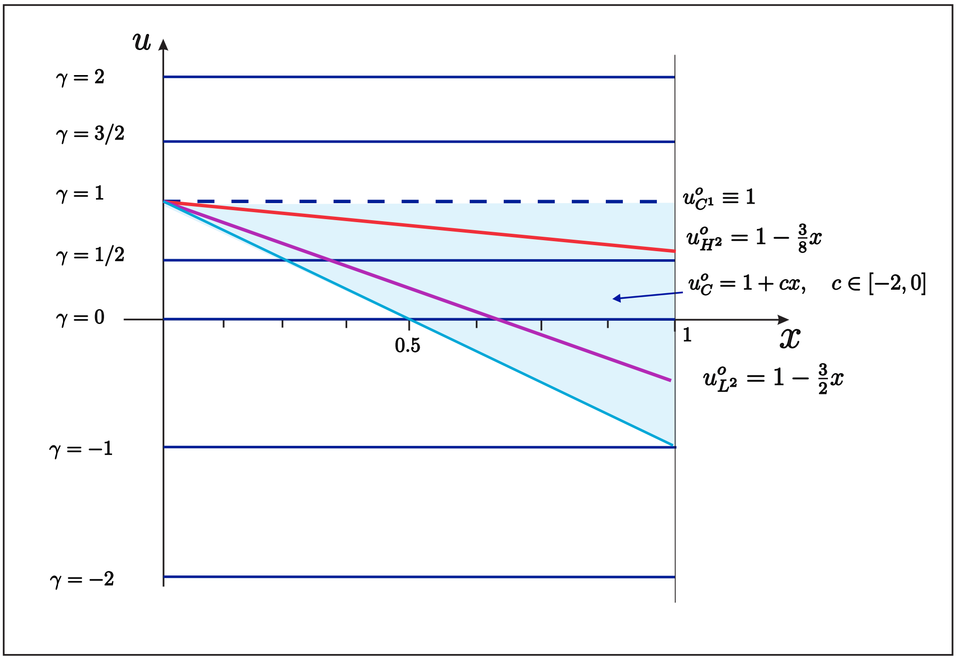

Fig. 1.1. Unique solutions ($\gamma \neq 1$) and minimizers ($\gamma = 1$).

Vanishing the first derivative, we get the equality $8c/3 + 1 = 0$ *and solve* $c = -3/8$. *Since the second derivative is positive with this* c *value, we indeed obtain the minimum. So, the calculated function*

$$u^o_{H^2} = 1 - \frac{3}{8}x$$

is the unique minimizer (red ray in Figure 1.1), introduced in this subsection. Furthermore, we can get another least squares solution of the minimum $L^2[0,1]$ *norm (violet ray)*

$$u^o_{L^2} = 1 - \frac{3}{2}x$$

as well as $C^1[0,1]$ *norm, that is* $u^o_{C^1}$*=1 (see Figure 1.1).*

So, obtained minimizers differ from each other in the part cx *only. Indeed,* cx *with an arbitrary constant* c *represents* $P_{N(\mathbf{L})}c$, $c \in H^2[0,1]$ *in the formula* (4.3). *As we see,* $u^o_{C^1} = u^o_{H^2} + c^o x$ *with* $c^o = 3/8$.

However, minimizing the $C[0,1]$ *norm*

$$\|u^g\|_{C[0,1]} = \begin{cases} 1, & -2 \leqslant c \leqslant 0, \\ |1 + c|, & \text{otherwise}, \end{cases}$$

we get the interval $c \in [-2, 0]$ *and functions* $\widetilde{u}^o = 1 + cx$ *with the minimum* $C[0,1]$ *norm* $\|\widetilde{u}^o\|_C = 1$. *Here we do not have the uniqueness. All rays, those* *lie on the light blue area in Figure 1.1, are minimizers in* $C[0,1]$ *norm. We also see that obtained minimizers* $u^o_{H^2}$, $u^o_{L^2}$ *and* $u^o_{C^1}$ *also satisfy this requirement, i.e., their* $C[0,1]$ *norms are equal to* 1.

This example leads to the following remark. Since the Sobolev embedding theorem gives $H^2[0,1] \subset C^1[0,1]$, then each least squares solution $u^g \in H^2[0,1]$ is also from $C^1[0,1]$. Thus, we can look for the minimum $C[0,1]$ (or $C^1[0,1]$) norm solution among all least squares solutions (4.3). The minimum $C[0,1]$ norm solution u^o_C differs from the minimizer $u^o = u^o_{H^2}$ with a particular function $c^o \in H^2[0,1]$ in

$$u^o_C = u^o + P_{N(\boldsymbol{L})}c^o.$$

If $\Delta \neq 0$, then $N(\boldsymbol{L})$ is trivial and $u^o_C = u^o_{H^2}$. Otherwise, $d > 0$ and we obtain two equivalent representations

$$P_{N(\boldsymbol{L})}c = \sum_{l=1}^{d} z^l(x)(z^l, c)_{H^2[0,1]} = \sum_{l=1}^{d} c_l z^l(x)$$

for $c \in H^2[0,1]$ and $c_l \in \mathbb{R}$. So, we have the following relation

$$u^o_C = u^o + \sum_{l=1}^{d} c^o_l z^l$$

for particular constants $c^o_l = (z^l, c^o)_{H^2[0,1]} \in \mathbb{R}$, $l = \overline{1,d}$. Since the function $c^o \in H^2[0,1]$ is practically unknown, we use another way as in Example 1.16. Precisely, minimizing the $C[0,1]$ norm of the general least squares solution

$$u^g = u^o + \sum_{l=1}^{d} c_l z^l,$$

we find the particular values of constants $c_l = c^o_l$ and know the minimizer u^o_C. However, here we cannot guarantee the uniqueness of the function with the minimum $C[0,1]$ (or $C^1[0,1]$) norm differently to the minimizer of the minimum $H^2[0,1]$ norm! We have the unique minimizer in $H^2[0,1]$ norm because the norm in the Hilbert space is the strictly convex functional but norms $C[0,1]$ and $C^1[0,1]$ are not strictly convex.

Example 1.17. *Let us now take another problem*

$$\begin{gathered} -u'' = 0, \quad x \in [0,1], \\ u(0) = 0, \qquad u'(1) - \gamma u'(1/2) = 1, \end{gathered}$$

which for $\gamma = 1$ is inconsistent. Otherwise, we obtain the unique solution $u = x/(1-\gamma)$ taking $\gamma \neq 1$. For the inconsistent problem with $\gamma = 1$, we can find the minimizer u^o, that is also the minimizer to the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}(0,0,1)^\top$. Here we denote $\boldsymbol{e}^2 = (0,0,1)^\top$ and calculate

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2 = \boldsymbol{e}^2 - P_{N(\boldsymbol{L}^*)}\boldsymbol{e}^2 = \boldsymbol{e}^2 - \frac{(\boldsymbol{v},\boldsymbol{e}^2)}{\|\boldsymbol{v}\|^2} = \frac{1}{3}\big(2\mathrm{H}(x-0,5);0;1\big)^\top.$$

Here we used the function $\boldsymbol{w} = (-\mathrm{H}(x-0,5);0;1)^\top$, which according to Corollary 1.4 ($d=1$, $k_1 = 2$ and the auxiliary problem: $-u'' = f$, $u(0) = 0$, $u'(1) = 0$), generates the nullspace $N(\boldsymbol{L}^*)$. Now, we solve the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2$, that is given in the explicit form

$$-u'' = \frac{2}{3}\mathrm{H}(x-0,5), \qquad u(0) = 0, \qquad u'(1) - \gamma u'(0,5) = \frac{1}{3}.$$

It has the general solution

$$u^g = cx + \frac{1}{12}\begin{cases} 4x, & x \leqslant 1/2, \\ -1 + 8x - 4x^2, & x \geqslant 1/2, \end{cases}$$

depending on one arbitrary constant $c \in \mathbb{R}$ ($d = 1$). Minimizing its $H^2[0,1]$ norm, we find the minimizer

$$u^o_{H^2} = -\frac{201}{64}x + \frac{1}{12}\begin{cases} 4x, & x \leqslant 1/2, \\ -1 + 8x - 4x^2, & x \geqslant 1/2. \end{cases}$$

For details, see Section 5. Moreover, minimizing the $C[0,1]$ norm of the general solution

$$\|u^g\|_{C[0,1]} = \begin{cases} \frac{3}{4}c^2 + c + \frac{1}{4}, & \frac{\sqrt{10}-4}{3} < c < 0, \\ \left|c + \frac{1}{4}\right|, & \textit{otherwise}, \end{cases}$$

we get the constant $c = (\sqrt{10} - 4)/3$ and the unique minimizer

$$u^o_C = \frac{\sqrt{10}-4}{3}x + \frac{1}{12}\begin{cases} 4x, & x \leqslant 1/2, \\ -1 + 8x - 4x^2, & x \geqslant 1/2. \end{cases}$$

Similarly, we find another unique minimizer

$$u^o_{C^1} = \frac{1}{12}\begin{cases} 4x, & x \leqslant 1/2, \\ -1 + 8x - 4x^2, & x \geqslant 1/2. \end{cases}$$

4.2 Generalized Green's function

Developing the parallel to Subsection 3.1, let us now investigate the structure of the Moore–Penrose inverse operator $\boldsymbol{L}^\dagger$. First, since $\boldsymbol{f} = (f, g_1, g_2)^\top = (f, 0, 0)^\top + (0, g_1, 0)^\top + (0, 0, g_2)^\top$ and $\boldsymbol{L}^\dagger$ is linear, we get the following composition of the minimum norm least squares solution

$$u^o = \boldsymbol{L}^\dagger \boldsymbol{f} = \boldsymbol{L}^\dagger (f, 0, 0)^\top + \boldsymbol{L}^\dagger (0, g_1, 0)^\top + \boldsymbol{L}^\dagger (0, 0, g_2)^\top = G^g f + g_1 v^{g,1} + g_2 v^{g,2},$$

where we denoted

$$G^g f := \boldsymbol{L}^\dagger (f, 0, 0)^\top, \quad v^{g,1} := \boldsymbol{L}^\dagger (0, 1, 0)^\top, \quad v^{g,2} := \boldsymbol{L}^\dagger (0, 0, 1)^\top. \tag{4.7}$$

From $\boldsymbol{L}^\dagger \boldsymbol{f} \in H^2[0,1]$, we have $G^g : L^2[0,1] \to H^2[0,1]$ is a continuous linear operator (since $\boldsymbol{L}^\dagger$ is continuous and linear) and $v^{g,1}, v^{g,2} \in H^2[0,1]$.

Now and further we focus our investigation on the following representations of the Moore–Penrose inverse

$$\boldsymbol{L}^\dagger = (G^g,\ v^{g,1},\ v^{g,2})$$

and the minimum norm least squares solution

$$u^o = G^g f + g_1 v^{g,1} + g_2 v^{g,2}, \quad \forall\ f \in L^2[0,1], \quad g_1, g_2 \in \mathbb{R}. \tag{4.8}$$

Precisely, for every fixed $x \in [0,1]$, we have $G^g f(x) \in \mathbb{R}$, because the function $G^g f \in H^2[0,1] \subset C^1[0,1]$ and, furthermore, $\|G^g f\|_{C^1[0,1]} \leqslant C \, \|G^g f\|_{H^2[0,1]}$ for some finite constant C according to the Sobolev embedding theorem [42, Evans 2010]. As Locker did [70, 1977], for every fixed $x \in [0,1]$, we also define the linear functional $F : L^2[0,1] \to \mathbb{R}$ by

$$\langle F, f \rangle = G^g f(x), \qquad \forall\ f \in L^2[0,1].$$

It is continuous because it is bounded

$$\begin{aligned} |\langle F, f \rangle| &= |G^g f(x)| \leqslant \sup_{x \in [0,1]} |G^g f(x)| \leqslant \|G^g f\|_{C^1[0,1]} = \|\boldsymbol{L}^\dagger (f, 0, 0)^\top\|_{C^1[0,1]} \\ &\leqslant C \cdot \|\boldsymbol{L}^\dagger (f, 0, 0)^\top\|_{H^2[0,1]} \leqslant C \cdot \|\boldsymbol{L}^\dagger\| \cdot \|f\|_{L^2[0,1]} \end{aligned}$$

for the finite constant $C \cdot \|\boldsymbol{L}^\dagger\|$ and every $f \in L^2[0,1]$. Then according to the Riesz representation theorem for continuous linear functionals in the Hilbert space [65, Kreyszig 1978], there exists the unique function $G^g(x, \cdot) \in L^2[0,1]$ that F can be represented by the inner product in the space $L^2[0,1]$ as follows

$$\langle F, f \rangle = G^g f(x) = \int_0^1 G^g(x, y) f(y) dy, \quad \forall f \in L^2[0,1],$$

and this equality is valid for every $x \in [0,1]$. Thus, the minimum norm least squares solution (4.8) has the representation

$$u^o(x) = \int_0^1 G^g(x,y)f(y)dy + g_1 v^{g,1}(x) + g_2 v^{g,2}(x) \tag{4.9}$$

for all $f \in L^2[0,1]$, $g_1, g_2 \in \mathbb{R}$ and $x \in [0,1]$. For the particular case $\varDelta \neq 0$ investigated in Section 3, the problem (1.1)–(1.2) has the unique solution of the form

$$u(x) = \int_0^1 G(x,y)f(y)dy + g_1 v^1(x) + g_2 v^2(x), \tag{4.10}$$

where $G(x,y)$ is the Green's function and v^1, v^2 are the biorthogonal fundamental system of the problem (1.1)–(1.2). According to the similarity, we call the kernel $G^g(x,y)$ – *the generalized Green's function* and the functions $v^{g,1}, v^{g,2}$ – *the generalized biorthogonal fundamental system* for the nonlocal problem (1.1)–(1.2).

So, for $\varDelta \neq 0$, we have that $\boldsymbol{L}^\dagger = \boldsymbol{L}^{-1}$, the minimum norm least squares solution u^o is coincident with the unique solution u, the generalized Green's function $G^g(x,y)$ is coincident with the ordinary Green's function $G(x,y)$, the biorthogonal fundamental system v^1, v^2 is coincident with the generalized biorthogonal fundamental system $v^{g,1}, v^{g,2}$.

Thus, now we are naturally interested, if they also have similar properties, and discuss on this question below.

4.3 Properties of minimizers

Let us begin our discussion investigating the minimum norm least squares solution, where we derive its properties. Firstly, we obtain the following characterization for the generalized biorthogonal fundamental system that is the analogue of (3.2).

Theorem 1.18. *Every function $v^{g,1}, v^{g,2} \in H^2[0,1]$ is the minimum norm least squares solution to the corresponding problem*

$$\begin{array}{cc} \mathcal{L}v^{g,1} = 0, & \mathcal{L}v^{g,2} = 0, \\ \langle L_1, v^{g,1}\rangle = 1,\ \langle L_2, v^{g,1}\rangle = 0, & \langle L_1, v^{g,2}\rangle = 0,\ \langle L_2, v^{g,2}\rangle = 1. \end{array} \tag{4.11}$$

Proof. The minimum norm least squares solution to the problem (1.1)–(1.2) is of the form (4.8). Taking $f = 0$ and $g_1 = 1$, $g_2 = 0$, from (4.8) we obtain the minimum norm least squares solution $u^o = v^{g,1}$ to the first problem

(4.11). Similarly, choosing $f = 0$ and $g_1 = 0$, $g_2 = 1$, the formula (4.8) simplifies to $u^o = v^{g,2}$, which now is the minimum norm least squares solution to the second problem (4.11). $\square$

Let us now consider two relative problems (3.9), where the first problem has the unique solution, i.e., the condition $\widetilde{\Delta} \neq 0$ is valid. Here and further $G^g(x, y)$ is the generalized Green's function and $v^{g,1}, v^{g,2}$ are the generalized biorthogonal fundamental system to the second problem (3.9), which may have the unique solution ($\Delta \neq 0$) or not ($\Delta = 0$).

Theorem 1.19. *If the first problem* (3.9) *has the unique solution u, then the minimum norm least squares solution for the second problem* (3.9) *is given by*

$$u^o = u - P_{N(\boldsymbol{L})}u + (g_1 - \langle L_1, u\rangle)v^{g,1} + (g_2 - \langle L_2, u\rangle)v^{g,2}.$$

Proof. Let us take the difference $w = u^o - u$ between the minimum norm least squares solution u^o to the second problem (3.9) and the unique exact solution u to the first problem of (3.9). Now we will show that w is the least squares solution to the problem

$$\mathcal{L}w = 0, \quad \langle L_k, w\rangle = g_k - \langle L_k, u\rangle, \quad k = 1, 2, \tag{4.12}$$

which can be written in the unexpanded form $\boldsymbol{L}w = \widetilde{\boldsymbol{f}}$ with the right hand side $\widetilde{\boldsymbol{f}} = (0, g_1 - \langle L_1, u\rangle, g_2 - \langle L_2, u\rangle)^\top$. Since u^o is the minimum norm least squares solution to the problem (2.1) with the right hand side $\boldsymbol{f} = (f, g_1, g_2)^\top$, then

$$\|\boldsymbol{L}u^o - \boldsymbol{f}\| = \inf_{v\in H^2[0,1]} \|\boldsymbol{L}v - \boldsymbol{f}\|. \tag{4.13}$$

We rewrite the norm as follows $\|\boldsymbol{L}v - \boldsymbol{f}\| = \|\boldsymbol{L}v + (\boldsymbol{L}u - \boldsymbol{L}u) - \boldsymbol{f}\| = \|\boldsymbol{L}(v - u) - (-\boldsymbol{L}u + \boldsymbol{f})\| = \|\boldsymbol{L}(v - u) - \widetilde{\boldsymbol{f}}\|$ for all $v \in H^2[0, 1]$. From here we have

$$\|\boldsymbol{L}(u^o - u) - \widetilde{\boldsymbol{f}}\| = \inf_{v\in H^2[0,1]} \|\boldsymbol{L}(v - u) - \widetilde{\boldsymbol{f}}\|$$

or, denoting $z = v - u \in H^2[0, 1]$, obtain

$$\|\boldsymbol{L}w - \widetilde{\boldsymbol{f}}\| = \inf_{z\in H^2[0,1]} \|\boldsymbol{L}z - \widetilde{\boldsymbol{f}}\|.$$

Thus w is the least squares solution to the problem (4.12) and, according to the formulas (4.3)–(4.4), it can be represented by the minimum norm least squares solution $w^o = \boldsymbol{L}^\dagger\widetilde{\boldsymbol{f}}$. Precisely, rewriting the equality $w =$

$P_{N(\boldsymbol{L})^\perp} w + P_{N(\boldsymbol{L})} w$, which is valid according to Subsection 2.1, we obtain $w = \boldsymbol{L}^\dagger \widetilde{\boldsymbol{f}} + P_{N(\boldsymbol{L})} w$. Since $w = u^o - u$, then

$$u^o = u + \boldsymbol{L}^\dagger \widetilde{\boldsymbol{f}} + P_{N(\boldsymbol{L})} w. \tag{4.14}$$

Now we take the composition $u = P_{N(\boldsymbol{L})^\perp} u + P_{N(\boldsymbol{L})} u$ and rewrite the representation (4.14) into the form $u^o = P_{N(\boldsymbol{L})^\perp} u + \boldsymbol{L}^\dagger \widetilde{\boldsymbol{f}} + P_{N(\boldsymbol{L})} u^o$, because $w + u = u^o$. From (4.4), we have $u^o \in N(\boldsymbol{L})^\perp$. Then $P_{N(\boldsymbol{L})} u^o = 0$ and the previous representation simplifies to $u^o = P_{N(\boldsymbol{L})^\perp} u + \boldsymbol{L}^\dagger \widetilde{\boldsymbol{f}}$. Rewriting into the extended form, we obtain $u^o = u - P_{N(\boldsymbol{L})} u + (g_1 - \langle L_1, u\rangle) v^{g,1} + (g_2 - \langle L_2, u\rangle) v^{g,2}$. □

As the unique solution (3.8) is always represented by the unique solution u^c to the Cauchy problem (2.9), similarly we can describe the minimum norm least squares solution.

Corollary 1.20. *The minimum norm least squares solution to the problem* (2.1)–(2.2) *can always be represented by the unique exact solution u^c to the Cauchy problem* (2.9) *as follows*

$$u^o = u^c - P_{N(\boldsymbol{L})} u^c + (g_1 - \langle L_1, u^c\rangle) v^{g,1} + (g_2 - \langle L_2, u^c\rangle) v^{g,2}.$$

Proof. It follows from Theorem 1.19 since the Cauchy problem (2.9) always has the unique solution u^c. □

Do generalized biorthogonal fundamenal systems for problems (3.9) also have a property similar to Corollary 1.10? We can provide the following answer.

Corollary 1.21. *Let $\widetilde{\Delta} \neq 0$ for the first problem* (3.9). *Then the biorthogonal fundamental system $\widetilde{v}^1, \widetilde{v}^2$ of the first problem and the generalized biorthogonal fundamental system $v^{g,1}, v^{g,2}$ of the second problem* (3.9) *are related by*

$$\begin{pmatrix} \langle L_1, \widetilde{v}^1\rangle & \langle L_2, \widetilde{v}^1\rangle \\ \langle L_1, \widetilde{v}^2\rangle & \langle L_2, \widetilde{v}^2\rangle \end{pmatrix} \begin{pmatrix} v^{g,1} \\ v^{g,2} \end{pmatrix} = \begin{pmatrix} P_{N(\boldsymbol{L})^\perp} \widetilde{v}^1 \\ P_{N(\boldsymbol{L})^\perp} \widetilde{v}^2 \end{pmatrix}.$$

Proof. The proof is coincident with the proof of Corollary 1.10, where we apply Theorem 1.19. □

Let us note that, for $\Delta \neq 0$, we get the trivial nullspace $N(\boldsymbol{L}) = \{0\}$. Here the orthogonal projector $P_{N(\boldsymbol{L})}$ vanishes in all formulas above. Furthermore, the generalized biorthogonal fundamental system $v^{g,1}, v^{g,2}$ is coincident with the biorthogonal fundamental system v^1, v^2. So, we get that all results from this subsection are coincident with the corresponding results, formulated in Section 3.1 for the problem (1.1)–(1.2) with $\Delta \neq 0$.

4.4 Properties of a generalized Green's function

Let us begin with a representation of a generalized Green's function, which is analogous to the definition of an ordinary Green's function (3.4).

Lemma 1.22. *The generalized Green's function for the problem* (1.1)–(1.2) *is of the form*

$$G^g(x,y) = G^c(x,y) - P_{N(\boldsymbol{L})}G^c(x,y) - \sum_{k=1}^{2}\langle L_k, G^c(\cdot,y)\rangle v^{g,k}(x).$$

Proof. Let us now consider two problems (3.9) with $\widetilde{g}_k = g_k = 0,\ k = 1,2$. The first problem (3.9) has the unique solution $u^c = G^c f(x)$ but the second problem (3.9) has the minimizer $u^o = G^g f(x)$. Corollary 1.20 provides the equality

$$u^o = G^c f(x) - P_{N(\boldsymbol{L})}G^c f(x) - \langle L_1, G^c f\rangle v^{g,1}(x) - \langle L_2, G^c f\rangle v^{g,2}(x). \quad (4.15)$$

We will rewrite it. First, we have $G^c f(x) = \int_0^1 G^c(x,y)f(y)\,dy$.

Second, applying the Fubini's theorem in measure spaces for conditions (1.3), we get $\langle L_k, G^c f\rangle = \int_0^1 \langle L_k, G^c(\cdot,y)\rangle f(y)\,dy$, $k = 1,2$.

Third, if the second problem (3.9) has the unique solution ($\Delta \neq 0$), then $P_{N(\boldsymbol{L})}$ is the zero operator and $P_{N(\boldsymbol{L})}G^c f(x) = 0$. Otherwise, the formula (4.5) gives the projection $P_{N(\boldsymbol{L})}G^c f(x) = P_{N(\boldsymbol{L})}u^c = \sum_{l=1}^{d} z^l(x)(z^l,u^c)_{H^2[0,1]}$. We take weak derivatives of the function $u^c = \int_0^1 G^c(x,y)f(y)\,dy$, i.e.,

$$(u^c)' = \int_0^x (\partial/\partial x)G^c(x,y)f(y)\,dy + \int_x^1 (\partial/\partial x)G^c(x,y)f(y)\,dy,$$
$$(u^c)'' = \int_0^x (\partial^2/\partial x^2)G^c(x,y)f(y)\,dy + \int_x^1 (\partial^2/\partial x^2)G^c(x,y)f(y)\,dy + f(x).$$

Substituting these expressions into inner products $(z^l,u^c)_{H^2[0,1]} = (z^l,u^c)_{L^2[0,1]}$ $+((z^l)',(u^c)')_{L^2[0,1]} + ((z^l)'',(u^c)'')_{L^2[0,1]}$, we change here the order of integration and have the representation of the projection

$$P_{N(\boldsymbol{L})}G^c f(x) = \int_0^1 P_{N(\boldsymbol{L})}G^c(x,y)f(y)\,dy$$

with the kernel

$$P_{N(\boldsymbol{L})}G^c(x,y) = \sum_{l=1}^{d} z^l(x)\Big(\big(z^l,G^c(\cdot,y)\big)_{H^2[0,y]} + \big(z^l,G^c(\cdot,y)\big)_{H^2[y,1]} + (z^l)''(y)\Big).$$

Substituting obtained integral representations into (4.15), we get

$$u^o = \int_0^1 \Big(G^c(x,y) - P_{N(\boldsymbol{L})}G^c(x,y) - \sum_{k=1}^{2}\langle L_k, G^c(\cdot,y)\rangle v^{g,k}(x)\Big) f(y)\,dy,$$

which is also equal to $u^o = \int_0^1 G^g(x,y)f(y)\,dy$. From here we obtain the statement of this lemma. $\square$

Below we present properties of a generalized Green's function, those are analogues of properties, listed in Corollary 1.11 for a Green's function.

Corollary 1.23. *For $y \neq y_0, y_1, y_2, \ldots$ with any $x \in [0,1]$, we have:*

1) $G^g(x,y)$ is continuous in (x,y);

2) $G^g(x,y)$ is H^2 in x except the diagonal $x = y$;

3) $(\partial/\partial x)G^g(y+0,y) - (\partial/\partial x)G^g(y-0,y) = 1$;

4) $\mathcal{L}G^g(\cdot,y) = -\langle L_1, G^c(\cdot,y)\rangle \mathcal{L}v^{g,1} - \langle L_2, G^c(\cdot,y)\rangle \mathcal{L}v^{g,2}$ except $x = y$;

5) $\langle L_k, G^g(\cdot,y)\rangle = \langle L_k, G^c(\cdot,y)\rangle - \langle L_1, G^c(\cdot,y)\rangle \cdot \langle L_k, v^{g,1}\rangle - \langle L_2, G^c(\cdot,y)\rangle \cdot \langle L_k, v^{g,2}\rangle$ for $k = 1,2$.

Proof. It follows from the representation of $G^g(x,y)$, given in Lemma 1.22. First, functions $v^{g,1}$, $v^{g,2} \in H^2[0,1] \subset C^1[0,1]$. Second, the Green's function $G^c(x,y)$ is continuous and belongs to C^2 in x except the diagonal $x = y$. Third, functions z^l, $l = \overline{1,d}$, representing the kernel $P_{N(\boldsymbol{L})}G^c(x,y)$, are classical solutions $z^l \in C^2[0,1]$ (see Subsection 2.1). Thus, this representation of $G^g(x,y)$ implies that the generalized Green's function $G^g(x,y)$ is $H^2[0,1] \subset C^1[0,1]$ in x except the diagonal $x = y$ and discontinuity points $y = y_0, y_1, \ldots$. Then

$$\frac{\partial G^g(x,y)}{\partial x} = \frac{\partial G^c(x,y)}{\partial x} - \frac{\partial P_{N(\boldsymbol{L})}G^c(x,y)}{\partial x} - \sum_{k=1}^{2} \langle L_k, G^c(\cdot,y)\rangle (v^{g,k})'(x)$$

for $x \in [0,1]$ and $y \neq y_0, y_1, \ldots$. Taking $x = y+0$ and $x = y-0$, we obtain the difference

$$\frac{\partial G^g(y+0,y)}{\partial x} - \frac{\partial G^g(y-0,y)}{\partial x} = \frac{\partial G^g(x,y)}{\partial x}\bigg|_{x=y-0}^{x=y+0} = \frac{\partial G^c(x,y)}{\partial x}\bigg|_{x=y-0}^{x=y+0}$$
$$- \frac{\partial P_{N(\boldsymbol{L})}G^c(x,y)}{\partial x}\bigg|_{x=y-0}^{x=y+0} - \sum_{k=1}^{2} \langle L_k, G^c(\cdot,y)\rangle (v^{g,k})'(x)\big|_{x=y-0}^{x=y+0}.$$

Since we have $P_{N(\boldsymbol{L})}G^c(\cdot,y), v^{g,1}, v^{g,2} \in C^1[0,1]$, then

$$\frac{\partial P_{N(\boldsymbol{L})}G^c(x,y)}{\partial x}\bigg|_{x=y-0}^{x=y+0} = 0, \quad (v^{g,1})'(x)\big|_{x=y-0}^{x=y+0} = 0, \quad (v^{g,2})'(x)\big|_{x=y-0}^{x=y+0} = 0.$$

The Green's function G^c has the jump $(\partial/\partial x)G^c(x,y)|_{x=y-0}^{x=y+0} = 1$. Substituting the differences, we obtain the jump of the generalized Green's function, which is equal to 1 as well.

Let us now consider the expression $\mathcal{L}G^g(\cdot,y)$. It is equal to

$$\mathcal{L}G^g(\cdot,y) = \mathcal{L}G^c(\cdot,y) - \mathcal{L}P_{N(\boldsymbol{L})}G^c(\cdot,y) - \langle L_1, G^c(\cdot,y)\rangle \mathcal{L}v^{g,1} \\ - \langle L_2, G^c(\cdot,y)\rangle \mathcal{L}v^{g,2} - \langle L_1, G^c(\cdot,y)\rangle \mathcal{L}v^{g,1} - \langle L_2, G^c(\cdot,y)\rangle \mathcal{L}v^{g,2}$$

if $x \neq y$ since $\mathcal{L}G^c(\cdot,y) = 0$ and $\mathcal{L}P_{N(\boldsymbol{L})}G^c(\cdot,y) = 0$ except the diagonal $x = y$. The proof of the last statement is analogous. □

We can describe a generalized Green's function using other ordinary Green's function as given below.

Theorem 1.24. *If $\widetilde{\Delta} \neq 0$ for the first problem* (3.9)*, then its Green's function $G(x,y)$ and the generalized Green's function $G^g(x,y)$ of the second problem* (3.9) *are related by the equality*

$$G^g(x,y) = G(x,y) - P_{N(\boldsymbol{L})}G(x,y) - \langle L_1, G(\cdot,y)\rangle v^{g,1}(x) - \langle L_2, G(\cdot,y)\rangle v^{g,2}(x),$$

for all $x \in [0,1]$ and a.e. $y \in [0,1]$.

Proof. We obtain the proof similarly as Lemma 1.22 is proved. □

In this lemma, we used the kernel $P_{N(\boldsymbol{L})}G(x,y)$ of an operator $P_{N(\boldsymbol{L})}G : L^2[0,1] \to H^2[0,1]$, which vanishes if $\Delta \neq 0$. For $\Delta = 0$, we rewrite inner products in the expression

$$P_{N(\boldsymbol{L})}Gf(x) = \sum_{l=1}^{d} z^l(x)(z^l, Gf)_{H^2[0,1]} = \int_0^1 P_{N(\boldsymbol{L})}G(x,y)f(y)\,dy$$

and get the kernel representation

$$P_{N(\boldsymbol{L})}G(x,y) = \sum_{l=1}^{d} z^l(x)\Bigg(\sum_{j=1,\ j\neq M}^{N} \big(z^l, G(\cdot,y)\big)_{H^2[y_{j-1},y_j]} \\ + \big(z^l, G(\cdot,y)\big)_{H^2[y_{M-1},y]} + \big(z^l, G(\cdot,y)\big)_{H^2[y,y_M]} + (z^l)''(y)\Bigg).$$

Here $y_0, y_1, \dots$ are discontinuity points of a Green's function $G(x,y)$ but $[y_{M-1}; y_M]$ denotes an interval, where a fixed y belongs.

Remark 1.25. Proposition 1.12 generates the sequence of Green's functions, where one Green's function is represented by other Green's function. It gives the possibility to construct Green's functions for more and more complicated problems with nonlocal conditions. However, from Theorem 1.24, we obtain that the generalized Green's function can be expressed using that sequence of ordinary Green's functions.

Remark 1.26. Let us discuss a little bit on the representation of $P_{N(\boldsymbol{L})}G(x, y)$ for the problem (3.10)–(3.11) with the operator $\mathcal{L}u := -u''$ and nonlocal conditions of the form (3.13). First, there the nullspace $N(\boldsymbol{L}) \subset \{1; x\}$ is composed of such orthonormal functions z^l, $l = \overline{1, d}$, in the space $H^2[0, 1]$ that $(z^l)'' = 0$. Second, Corollary 1.13 says that the Green's function is continuous on the entire domain $0 \leqslant x, y \leqslant 1$, its partial derivative $(\partial/\partial x)G(x, y)$ is also continuous except the diagonal. It means that the Green's function has the weak derivative $(\partial/\partial x)G(x, y)$ on the entire unit square. So, $P_{N(\boldsymbol{L})}G(x, y)$ simplifies to the following expression

$$P_{N(\boldsymbol{L})}G(x, y) = \sum_{l=1}^{d} z^l(x)(z^l, G(\cdot, y))_{H^1[0,1]}. \tag{4.16}$$

We will use it considering examples in the last section of this chapter.

So, we again get results those are analogous to the properties of the Green's function, given in Subsection 3.3. Furthermore, if $\Delta \neq 0$, then $N(\boldsymbol{L}) = \{0\}$ and the function $P_{N(\boldsymbol{L})}G^g(x, y)$ vanish in all expressions of this subsection. Thus, here obtained results are coincident with the properties from Subsection 3.3, where the particular case $\Delta \neq 0$ was considered.

4.5 Applications to nonlocal boundary conditions

Analogous properties are also valid for the problem with nonlocal boundary conditions (3.10)–(3.11), investigated in Subsection 3.4. Firstly, we provide the representation of the unique minimizer u^o via the unique solution u^{cl} to the classical problem ($\gamma_1, \gamma_2 = 0$ in (3.11)) as follows.

Corollary 1.27. *If the classical problem* (3.10)–(3.11) *(*$\gamma_1, \gamma_2 = 0$*) has the unique solution* u^{cl}*, then the minimum norm least squares solution to the nonlocal boundary value problem* (3.10)–(3.11) *is of the form*

$$u^o = u^{\text{cl}} - P_{N(\boldsymbol{L})}u^{\text{cl}} + \gamma_1\langle\varkappa_1, u^{\text{cl}}\rangle v^{g,1} + \gamma_2\langle\varkappa_2, u^{\text{cl}}\rangle v^{g,2}.$$

Proof. We obtain this corollary applying Theorem 1.19 with $\langle L_k, u^{\text{cl}}\rangle = g_k - \gamma_k\langle\varkappa_k, u^{\text{cl}}\rangle$, since u^{cl} satisfies conditions $\langle\kappa_k, u^{\text{cl}}\rangle = g_k$, $k = 1, 2$. □

Second, the generalized Green's function for the problem with nonlocal boundary conditions (3.10)–(3.11) can also be similarly represented.

Corollary 1.28. *If the classical problem* (3.10)–(3.11) *($\gamma_1, \gamma_2 = 0$) has the Green's function $G^{\mathrm{cl}}(x, y)$, then the generalized Green's function of the nonlocal problem* (3.10)–(3.11) *is given by*

$$G^g(x, y) = G^{\mathrm{cl}}(x, y) - P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x, y) + \sum_{k=1}^{2} \gamma_k \langle \varkappa_k, G^{\mathrm{cl}}(\cdot, y) \rangle v^{g,k}(x),$$

for all $x \in [0, 1]$ and a.e. $y \in [0, 1]$.

Proof. It follows from Theorem 1.24, since we have $\langle \kappa_k, G^{\mathrm{cl}}(\cdot, y) \rangle = 0$ and $\langle L_k, G^{\mathrm{cl}}(\cdot, y) \rangle = -\gamma_k \langle \varkappa_k, G^{\mathrm{cl}}(\cdot, y) \rangle$ for $k = 1, 2$. □

Let us accentuate the boundary value problem (3.10)–(3.11) with nonlocal parts $\varkappa_k$ of the form (3.13) since here we can obtain nice smoothness properties.

Proposition 1.29. *If $f \in C[0, 1]$, then the boundary value problem* (3.10)–(3.11) *with* (3.13) *has the minimizer $u^o \in C^2[0, 1]$.*

Proof. If $\varDelta \neq 0$, then the boundary value problem has the unique solution, which is from $C^2[0, 1]$ for $f \in C[0, 1]$. Such problem was investigated in Section 3. Let us now consider the case $\varDelta = 0$.

The minimizer u^o to the problem $\boldsymbol{L} u = \boldsymbol{f}$ is also the minimizer to the consistent problem $\boldsymbol{L} u = \widetilde{\boldsymbol{f}}$ with the right hand side $\widetilde{\boldsymbol{f}} = (\widetilde{f}, \widetilde{g}_1, \widetilde{g}_2)^\top = \boldsymbol{P}_{R(\boldsymbol{L})} \boldsymbol{f}$. This consistent problem has a lot of solutions from $H^2[0, 1]$. If $\widetilde{f} \in C[0, 1]$, then these solutions are from $C^2[0, 1]$, where u^o is a particular solution. So, since $R(\boldsymbol{L})$ and $N(\boldsymbol{L}^*)$ are closed, we calculate the projection

$$\boldsymbol{P}_{R(\boldsymbol{L})} \boldsymbol{f} = \boldsymbol{f} - \boldsymbol{P}_{R(\boldsymbol{L})^\perp} \boldsymbol{f} = \boldsymbol{f} - \boldsymbol{P}_{N(\boldsymbol{L}^*)} \boldsymbol{f} = \boldsymbol{f} - \sum_{j=1}^{d} \frac{\boldsymbol{w}^j}{\|\boldsymbol{w}^j\|^2} (\boldsymbol{w}^j, \boldsymbol{f}) \quad (4.17)$$

with every $\boldsymbol{f} \in L^2[0, 1] \times \mathbb{R}^2$. Here $\boldsymbol{w}^j$, $j = \overline{1, d}$ is the orthogonal basis of $N(\boldsymbol{L}^*)$. Using Corollary 1.4, we can always obtain these functions. From this corollary we also get that each vector valued function $\boldsymbol{w}^j \in C[0, 1] \times \mathbb{R}^2$. Indeed, let us consider the case $d = 2$. Since the Green's function $G^c(x, y)$ for the Cauchy problem (2.9) is continuous on the entire unit square $0 \leqslant x, y \leqslant 1$ and nonlocal boundary conditions (3.11) have a special form with (3.13), we get $\langle L_k, G^c(\cdot, x) \rangle \in C[0, 1]$, $k = 1, 2$. Then we recall $f \in C[0, 1]$ and from (4.17) obtain $\widetilde{\boldsymbol{f}} = \boldsymbol{P}_{R(\boldsymbol{L})} \boldsymbol{f} \in C[0, 1] \times \mathbb{R}^2$. So, from here $\widetilde{f} \in C[0, 1]$ and, as we noted above, the minimizer u^o is from $C^2[0, 1]$.

The case $d = 1$ is investigated analogously. Precisely, now in Corollary 1.4 we use the Green's function $G^a(x, y)$ of the problem $\mathcal{L} u = f$, $\langle L_{k_2}, u \rangle = 0$,

$\langle \ell, u \rangle = 0$. Here $\langle \ell, u \rangle = 0$ can be selected as $\langle \kappa_{k_1}, u \rangle = 0$ or $u(0) = 0$, or $u'(0) = 0$, or $u(1) = 0$, or $u'(1) = 0$, which gives $\varDelta \neq 0$ for this auxiliary problem. Then Corollary 1.13 and Remark 1.14 says that the Green's function $G^a(x, y)$ for such auxiliary problem is continuous on the entire unit square as well. Now analogously as previous, we get $u^o \in C^2[0, 1]$. □

From here we also obtain the following quality of the generalized biorthogonal fundamental system.

Corollary 1.30. *For the problem* (3.10)–(3.11)*, where fully nonlocal parts of conditions are of the form* (3.13)*, we have* $v^{g,1}, v^{g,2} \in C^2[0, 1]$.

Proof. It follows directly from Proposition 1.29 since $f = 0$ is continuous. □

Similarly applying Corollary 1.28 and Corollary 1.30 with continuity properties of $G^{\mathrm{cl}}(x, y)$ on the unit square, we can derive the property for the generalized Green's function.

Corollary 1.31. *For the problem* (3.10)–(3.11) *with* (3.13)*, the generalized Green's function* $G^g(x, y)$ *is continuous on the entire unit square* $0 \leqslant x, y \leqslant 1$ *as well as its partial derivatives* $(\partial/\partial x)G^g(x, y)$ *and* $(\partial^2/\partial x^2)G^g(x, y)$ *except the diagonal* $x = y$.

From here we also get that the generalized Green's function for the problem (3.10)–(3.11), where fully nonlocal parts ar of the form (3.13), has classical partial derivatives as below

$$\frac{\partial^i G^g(x, y)}{\partial x^i} = \frac{\partial^i G^{\mathrm{cl}}(x, y)}{\partial x^i} - \frac{\partial^i P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x, y)}{\partial x^i} + \sum_{k=1}^{2} \gamma_k \langle \varkappa_k, G^{\mathrm{cl}}(\cdot, y) \rangle (v^{g,k})^{(i)}(x)$$

for $i = 1, 2$, except the diagonal $x = y$. Moreover, we can obtain weak derivatives of the minimizer (4.9) using these classical partial derivatives of the generalized Green's function, i.e.,

$$(u^o)' = \int_0^x \frac{\partial}{\partial x} G^g(x, y) f(y)\, dy + \int_x^1 \frac{\partial}{\partial x} G^g(x, y) f(y)\, dy + \sum_{k=1}^{2} g_1 (v^{g,k})'(x)$$

and

$$\begin{aligned}(u^o)'' = &\int_0^x \frac{\partial^2}{\partial x^2} G^g(x, y) f(y)\, dy \\ &+ \int_x^1 \frac{\partial^2}{\partial x^2} G^g(x, y) f(y)\, dy + f(x) + \sum_{k=1}^{2} g_k (v^{g,k})''(x).\end{aligned}$$

Here $(u^o)'$ is continuous because $u^o \in H^2[0,1] \subset C^1[0,1]$. According to Proposition 1.29, we have the minimizer u^o from $C^2[0,1]$ if $f \in C[0,1]$.

Remark 1.32. Let us note that Proposition 1.29 and Corollaries 1.30–1.31 are also valid if $\langle \kappa_1, u \rangle := u(0)$ and $\langle \kappa_2, u \rangle := u'(0)$ represent initial conditions instead of classical conditions.

5 Examples of minimizers

This section is devoted to illustrate obtained theoretical results and get more familiar with the minimizer and its generalized Green's function. Here we investigate minimum norm least squares solutions to second order differential problems with several popular nonlocal conditions of Bitsadze–Samarskii and integral type. Representations of generalized Green's functions and biorthogonal fundamental systems will also be discussed.

5.1 Problem with one Bitsadze–Samarskii condition

Let us now find the minimum norm least squares solution to the problem

$$-u'' = f(x), \quad x \in [0,1], \tag{5.1}$$

$$u(0) = g_1, \qquad u(1) = \gamma u(\xi) + g_2, \tag{5.2}$$

where γ is real and $\xi \in (0,1)$. We focus on the problem without the unique solution, i.e., $\Delta = 0$ or $\gamma\xi = 1$, because the case $\Delta \neq 0$ is investigated by Roman [100, 2011]. Thus, we have $\gamma = 1/\xi$ in all formulas below if it is not said otherwise.

The minimizer is always given by

$$u^o(x) = \int_0^1 G^g(x,y) f(y)\, dy + g_1 v^{g1}(x) + g_2 v^{g2}(x) \tag{5.3}$$

with every $f \in L^2[0,1]$ and $g_1, g_2 \in \mathbb{R}$. From Corollary 1.28, we obtain the representation of the generalized Green's function

$$G^g(x,y) = G^{\mathrm{cl}}(x,y) - P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x,y) + \gamma v^{g,2}(x) G^{\mathrm{cl}}(\xi, y). \tag{5.4}$$

Here $G^{\mathrm{cl}}(x,y)$ is the Green's function to the classical problem (5.1)–(5.2) ($\gamma = 0$), its expression is given in Example 1.6. Let us note that, for $\Delta \neq 0$ ($\gamma \neq 1/\xi$), the Green's function has the similar representation

$$G(x,y) = G^{\mathrm{cl}}(x,y) + \gamma v^2(x) G^{\mathrm{cl}}(\xi, y),$$

where $v^2 = x/(1-\gamma\xi)$ [100, Roman 2011].

Another component of (5.4) is the kernel $P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x,y)$. We discussed on its representation in Remark 1.26. Since here $d = \dim N(\boldsymbol{L}) = 1$ (recall Example 1.6) and $x \in N(\boldsymbol{L})$ generates the nullspace but $x'' = 0$, the formula (4.16) gives

$$P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x,y) = \frac{x}{\|t\|^2_{H^2[0,1]}}(t, G^{\mathrm{cl}}(t,y))_{H^1[0,1]} = \frac{1}{8}\cdot xy(1-y^2), \qquad (5.5)$$

where t denotes the variable of integration.

Now we are going to find v^{g2}, which is the minimizer to the problem $\boldsymbol{L}u = (0,0,1)^\top$. This problem is inconsistent but the function v^{g2} is also the minimizer to the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}(0,0,1)^\top$. Let us denote $\boldsymbol{e}^2 = (0,0,1)^\top$. According to (4.17), we can calculate the projection as follows

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2 = \boldsymbol{e}^2 - \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2}(\boldsymbol{w}, \boldsymbol{e}^2),$$

where the function

$$\boldsymbol{w}(x) = \Big(\gamma G^{\mathrm{cl}}(\xi,x); \gamma-1; 1\Big)^\top = \left(\begin{cases} x(\gamma-1), & x \leqslant \xi, \\ 1-x, & x \geqslant \xi \end{cases}; \gamma-1; 1\right)^\top$$

generates the nullspace $N(\boldsymbol{L}^*)$. For details, you can recall Example 1.6. So, we obtain

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2 = \frac{1}{\|\boldsymbol{w}\|^2}\Big(-\gamma G^{\mathrm{cl}}(\xi,x); 1-\gamma; \|\boldsymbol{w}\|^2 - 1\Big)^\top$$

with the denominator $\|\boldsymbol{w}\|^2 = \gamma^2\int_0^1 (G^{\mathrm{cl}}(\xi,y))^2\,dy + (1-\gamma)^2 + 1 = (\xi^2 + 3)(\gamma-1)^2/3 + 1 = (\xi^2+3)(1-\xi)^2/(3\xi^2) + 1$ since $\gamma = 1/\xi$.

Let us solve the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2$, given in the extended form

$$-u'' = -\gamma G^{\mathrm{cl}}(\xi,x)/\|\boldsymbol{w}\|^2, \quad x \in [0,1], \qquad (5.6)$$

$$u(0) = (1-\gamma)/\|\boldsymbol{w}\|^2, \qquad (5.7)$$

$$u(1) - \gamma u(\xi) = 1 - 1/\|\boldsymbol{w}\|^2. \qquad (5.8)$$

We obtain the general solution to the differential equation (5.6), that is

$$u = c_1 + c_2 x - \frac{\gamma}{\|\boldsymbol{w}\|^2}\int_0^1 G^{\mathrm{cl}}(x,y)G^{\mathrm{cl}}(\xi,y)\,dy.$$

Substituting it into the condition (5.7), we get $c_1 = (1-\gamma)/\|\boldsymbol{w}\|^2$. Since the problem (5.6)–(5.8) is consistent but $d = 1$, the last condition (5.8) is satisfied trivially. Thus, the general least squares solution

$$u^g = \frac{1-\gamma}{\|\boldsymbol{w}\|^2} + cx - \frac{\gamma}{\|\boldsymbol{w}\|^2}\int_0^1 G^{\mathrm{cl}}(x,y)G^{\mathrm{cl}}(\xi,y)\,dy$$

depends on one arbitrary constant $c \in \mathbb{R}$.

Now we are going to use the formula $v^{g2}(x) = P_{N(\boldsymbol{L})^\perp} u^g$, that is also given by $v^{g2}(x) = u^g - P_{N(\boldsymbol{L})} u^g$. First, $x \in N(\boldsymbol{L})$ gives $P_{N(\boldsymbol{L})} cx = c \cdot P_{N(\boldsymbol{L})} x = cx$. Second, from the formula (4.5), we calculate the projection $P_{N(\boldsymbol{L})} 1 = x(x,1)_{H^2[0,1]}/\|x\|^2_{H^2[0,1]} = 3x/8$ and have

$$P_{N(\boldsymbol{L})} \frac{1-\gamma}{\|\boldsymbol{w}\|^2} = \frac{1-\gamma}{\|\boldsymbol{w}\|^2} P_{N(\boldsymbol{L})} 1 = \frac{1-\gamma}{\|\boldsymbol{w}\|^2} \cdot \frac{3}{8} x.$$

Third, taking the representation of the kernel (5.5), we obtain

$$P_{N(\boldsymbol{L})} \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\mathrm{cl}}(x,y) G^{\mathrm{cl}}(\xi,y)\, dy = \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x,y) G^{\mathrm{cl}}(\xi,y)\, dy$$
$$= \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 \frac{1}{8} x \cdot y(1-y^2) G^{\mathrm{cl}}(\xi,y)\, dy.$$

Substituting these expressions above, we get the minimizer

$$v^{g2}(x) = \frac{1-\gamma}{\|\boldsymbol{w}\|^2} + c^o x - \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\mathrm{cl}}(x,y) G^{\mathrm{cl}}(\xi,y)\, dy$$

with the particular constant

$$c^o = \frac{3}{8} \cdot \frac{\gamma - 1}{\|\boldsymbol{w}\|^2} + \frac{1}{8} \cdot \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 y(1-y^2) G^{\mathrm{cl}}(\xi,y)\, dy.$$

Simplifying, we obtain the following representation

$$v^{g2}(x) = \frac{1-\gamma}{\|\boldsymbol{w}\|^2} + c^o x - \frac{1}{6\|\boldsymbol{w}\|^2} \begin{cases} \xi(1-\xi)(2-\xi)x + (1-\gamma)x^3, & x \leqslant \xi, \\ -\xi^2 + (2+\xi^2)x - 3x^2 + x^3, & x \geqslant \xi, \end{cases}$$

with the constant

$$c^o = \frac{3(\gamma-1)}{8 \cdot \|\boldsymbol{w}\|^2} - \frac{1}{160\|\boldsymbol{w}\|^2} (7 + 180\xi - 190\xi^2 + 3\xi^4 - 48\xi^5),$$

or, recalling $\gamma = 1/\xi$, explicitly

$$v^{g,2} = \frac{1}{160} \cdot \frac{\xi}{(\xi^3+3)(\xi-1)^2 + 3\xi^2} \Bigg(3(48\xi^6 - 3\xi^5 + 190\xi^3 - 180\xi^2 - 67\xi + 60)x$$
$$+ 480(1-\xi) - 80 \begin{cases} \xi(1-\xi)(2-\xi)x + (\xi-1)x^3, & x \leqslant \xi, \\ -\xi^3 + \xi(2+\xi^2)x - 3\xi x^2 + \xi x^3, & x \geqslant \xi \end{cases} \Bigg).$$

So, we found the minimizer to the problem $\boldsymbol{L}u = \boldsymbol{e}^2$. Substituting the obtained v^{g2} expression as well as the Green's function $G^{\mathrm{cl}}(x,y)$ and $P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x,y)$

into (5.4), we know the full representation of the generalized Green's function $G^g(x,y)$, that is given below

$$G^g(x,y) = \begin{cases} y(1-x) & y \leqslant x, \\ x(1-y), & y \geqslant x, \end{cases} - \frac{1}{8}xy(1-y^2)$$
$$+ \frac{1}{160}\frac{1}{(\xi^3+3)(\xi-1)^2+3\xi^2}\Bigg(3(48\xi^6 - 3\xi^5 + 190\xi^3 - 180\xi^2 - 67\xi + 60)x$$
$$+ 480(1-\xi) + 80\begin{cases}(1-\xi)(2-\xi)x + (\xi-1)x^3, & x \leqslant \xi, \\ -\xi^3 + \xi(2+\xi^2)x - 3\xi x^2 + \xi x^3, & x \geqslant \xi,\end{cases}\Bigg) \times$$
$$\times \begin{cases} y(1-\xi) & y \leqslant \xi, \\ \xi(1-y), & y \geqslant \xi. \end{cases}$$

However, to get the full representation of the minimizer (5.3), we still need to find the function v^{g1}. Since the problem (5.1)–(5.2) with $\gamma = 0$ always has the unique solution ($\widetilde{\Delta} = 1 \neq 0$), we take its biorthogonal fundamental system $\widetilde{v}^1 = 1 - x$, $\widetilde{v}^2 = x$ and apply Corollary 1.21. From there we get the equality $v^{g1} + (1-\gamma)v^{g2} = P_{N(\boldsymbol{L})^\perp}\widetilde{v}^1$. Since $P_{N(\boldsymbol{L})^\perp}\widetilde{v}^1 = \widetilde{v}^1 - P_{N(\boldsymbol{L})}\widetilde{v}^1 = 1 - 3x/8$, we have the following expression

$$v^{g1} = (\gamma - 1)v^{g2} + 1 - \frac{3}{8}x.$$

Substituting the obtain representation of v^{g2}, we find v^{g1} and know all representations of $G^g(x,y)$, v^{g1} and v^{g2}. So, now we can always calculate the minimum norm least squares solution u^o with every right hand side by the formula (5.3).

Let us finally remark that here, investigating the problem (5.1)–(5.2) with $\Delta = 0$, we obtained a nice generalization for the case $\Delta \neq 0$. First, from the obtained representations we see functions $v^{g1}, v^{g2} \in C^2[0,1]$, what also confirms the claim of Corollary 1.30. Second, the generalized Green's function $G^g(x,y)$, given by (5.4), is continuous on the entire domain $0 \leqslant x, y \leqslant 1$. Its partial derivatives $(\partial^i/\partial x^i)G^g(x,y)$ for $i = 1,2$ are continuous on the unit square except the diagonal $x = y$ as well. We note that those features are also stated in Corollary 1.31. So, the generalized Green's function $G^g(x,y)$ and the generalized biorthogonal fundamental system v^{g1}, v^{g2} have strongly related representations and the same considered smoothness properties as the Green's function $G(x,y)$ with the biorthogonal fundamental system v^1, v^2 have.

5.2 Problem with two Bitsadze–Samarskii conditions

Let us now take the differential problem with two Bitsadze–Samarskii conditions

$$-u'' = f(x), \quad x \in [0,1], \tag{5.9}$$

$$u(0) = \gamma_1 u(\xi_1) + g_1, \qquad u(1) = \gamma_2 u(\xi_2) + g_2, \tag{5.10}$$

where γ_1, γ_2 are real numbers and $\xi_1, \xi_2 \in (0,1)$. We are going to find its minimizer (5.3) if the problem does not have the unique solution.

As given in Example 1.7, now we have the condition $\varDelta = 0$ that can be described by the relation among parameters $\gamma_1(1-\xi_1)+\gamma_2\xi_2+\gamma_1\gamma_2(\xi_1-\xi_2) = 1$. The case $\gamma_1 = 0$ is investigated in Subsection 5.1. Thus, now we focus our study on the problem (5.9)–(5.10) with $\gamma_1 \neq 0$. Here we recall Example 1.7, where we formulated the auxiliary problem

$$\begin{gathered} -u'' = f(x), \quad x \in [0,1], \\ u(0) = 0, \qquad u(1) - \gamma_2 u(\xi_2) = 0 \end{gathered} \tag{5.11}$$

and assured that $\widetilde{\varDelta} := 1 - \gamma_2\xi_2 \neq 0$ for this auxiliary problem. Then we take its Green's function $G^a(x,y)$ (see (2.21)) in Lemma 1.24 and get the representation of the generalized Green's function

$$G^g(x,y) = G^a(x,y) - P_{N(\boldsymbol{L})}G^a(x,y) + \gamma_1 G^a(\xi_1,y)v^{g,1}(x) \tag{5.12}$$

for the problem (5.9)–(5.10).

First, Remark 1.26 gives the kernel $P_{N(\boldsymbol{L})}G^a(x,y)$. Since here $d = \dim N(\boldsymbol{L}) = 1$ and $z^1 := \gamma_1\xi_1 + (1-\gamma_1)x \in N(\boldsymbol{L})$ generates the nullspace but $(z^1)'' = 0$, we get

$$\begin{aligned} P_{N(\boldsymbol{L})}G^a(x,y) &= \frac{z^1}{\|z^1\|^2_{H^2[0,1]}}(z^1, G^a(\cdot,y))_{H^1[0,1]} \\ &= \frac{\gamma_1\xi_1 + (1-\gamma_1)x}{2\gamma_1^2\xi_1^2 + 2\gamma_1\xi_1(1-\gamma_1) + 3(1-\gamma_1)^2} \cdot \frac{1}{3}F(y), \end{aligned}$$

where

$$\begin{aligned} F(y) =& y(y-1)\big(3\gamma_1\xi_1(y-3) + (\gamma_1 - 1)(1+y)\big) \\ &+ \frac{3\gamma_2\big(2\gamma_1\xi_1 + 3(1-\gamma_1)\big)}{1-\gamma_2\xi_2}\begin{cases} y(1-\xi_2), & y \leqslant \xi_2, \\ \xi_2(1-y), & y \geqslant \xi_2. \end{cases} \end{aligned}$$

Second, we are going to find the function v^{g1}. It is the minimizer to the problem $\boldsymbol{L}u = (0,1,0)^\top$ as well to the consistent problem $\boldsymbol{L}u =$

$\boldsymbol{P}_{R(\boldsymbol{L})}(0,1,0)^\top$. Denoting $\boldsymbol{e}^1 = (0,1,0)^\top$, we use the formula (4.17) to calculate the projection as follows

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1 = \boldsymbol{e}^1 - \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2}(\boldsymbol{w}, \boldsymbol{e}^1).$$

Here we took the function

$$\boldsymbol{w}(x) = \left(\gamma_1 G^a(\xi_1, x); 1; \frac{\gamma_1\xi_1}{1-\gamma_2\xi_2}\right)^\top$$

from Example 1.7, which generates the nullspace $N(\boldsymbol{L}^*)$. Then we get

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1 = \frac{1}{\|\boldsymbol{w}\|^2}\left(-\gamma_1 G^a(\xi_1, x); \|\boldsymbol{w}\|^2 - 1; \frac{\gamma_1\xi_1}{\gamma_2\xi_2 - 1}\right)^\top$$

with the denominator $\|\boldsymbol{w}\|^2 = \gamma_1^2 \int_0^1 (G^a(\xi_1, y))^2\,dy + (\gamma_1\xi_1)^2/(1-\gamma_2\xi_2)^2 + 1$. Now we can solve the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1$, that is given by

$$-u'' = -\gamma_1 G^a(\xi_1, x)/\|\boldsymbol{w}\|^2, \quad x \in [0,1], \tag{5.13}$$

$$u(0) - \gamma_1 u(\xi_1) = 1 - 1/\|\boldsymbol{w}\|^2, \tag{5.14}$$

$$u(1) - \gamma_2 u(\xi_2) = \gamma_1\xi_1/(\gamma_2\xi_2 - 1)\|\boldsymbol{w}\|^2. \tag{5.15}$$

Let us choose such fundamental system $z^1 = \gamma_1\xi_1 + (1-\gamma_1)x$, $z^2 = x/(1-\gamma_2\xi_2)$ and write the general solution to the differential equation (5.13) in the form

$$u = c_1 z^1 + c_2 z^2 - \frac{\gamma_1}{\|\boldsymbol{w}\|^2}\int_0^1 G^a(x,y)G^a(\xi,y)\,dy.$$

Substituting it into condition (5.15), we solve $c_2 = \gamma_1\xi_1/(\gamma_2\xi_2 - 1)\|\boldsymbol{w}\|^2$ directly since $z^1 \in N(\boldsymbol{L})$ and the Green's function for the problem (5.11) satisfies $G(1,y) - \gamma_2 G(\xi_2, y) = 0$. We do not use the condition (5.14) because for the consistent problem it is satisfied trivially. So, the general least squares solution

$$u^g = c\big(\gamma_1\xi_1 + (1-\gamma_1)x\big) - \frac{\gamma_1\xi_1 x}{(1-\gamma_2\xi_2)^2\|\boldsymbol{w}\|^2} - \frac{\gamma_1}{\|\boldsymbol{w}\|^2}\int_0^1 G^a(x,y)G^a(\xi_1,y)\,dy$$

depends on one arbitrary constant $c \in \mathbb{R}$. Then, we use the formula $v^{g1}(x) = P_{N(\boldsymbol{L})^\perp} u^g$ and obtain the minimizer

$$v^{g1}(x) = c^o\big(\gamma_1\xi_1 + (1-\gamma_1)x\big) - \frac{\gamma_1\xi_1 x}{(1-\gamma_2\xi_2)^2\|\boldsymbol{w}\|^2} - \frac{\gamma_1}{\|\boldsymbol{w}\|^2}\int_0^1 G^a(x,y)G^a(\xi_1,y)\,dy,$$

where

$$c^o = \frac{3\gamma_1^2\xi_1^2 + 8\xi_1(1-\gamma_1) + \gamma_1(1-\gamma_2\xi_2)^2\int_0^1 F(y)G(\xi_1,y)\,dy}{3(1-\gamma_2\xi_2)^2\big(2\gamma_1^2\xi_1^2 + 2\xi_1(1-\gamma_1) + 3(1-\gamma_1)^2\big)\|\boldsymbol{w}\|^2}.$$

Third, we need to find another function v^{g2}. Taking the biorthogonal fundamental system $\widetilde{v}^1 = \big(1-\gamma_2\xi_2+(\gamma_2-1)x\big)/(1-\gamma_2\xi_2)$, $\widetilde{v}^2 = x/(1-\gamma_2\xi_2)$ for the problem (5.11), we apply Corollary 1.21 and get the relation

$$v^{g2} = \frac{\gamma_1\xi_1}{1-\gamma_2\xi_2}v^{g1} + P_{N(\boldsymbol{L})^\perp}\widetilde{v}^2.$$

Now we calculate

$$\begin{aligned} P_{N(\boldsymbol{L})^\perp}\widetilde{v}^2 &= \widetilde{v}^2 - P_{N(\boldsymbol{L})}\widetilde{v}^2 \\ &= \frac{x}{1-\gamma_2\xi_2} - \frac{\gamma_1\xi_1+3(1-\gamma_2)}{1-\gamma_2\xi_2}\cdot\frac{\gamma_1\xi_1+(1-\gamma_1)x}{2(\gamma_1\xi_1)^2+2\gamma_1\xi_1(1-\gamma_1)+3(\gamma_1-1)^2}, \end{aligned}$$

and we know the full representation of v^{g2}.

Finally, substituting obtained v^{g1}, $G^a(x,y)$ and $P_{N(\boldsymbol{L})}G^a(x,y)$ expressions into (5.12), we find the generalized Green's function $G^g(x,y)$ for the problem (5.9)–(5.10). Together the minimum norm least squares solution (5.3) can also be formally considered as known.

5.3 Problem with one integral condition

Here we consider the differential problem with one integral condition

$$-u'' = f(x), \quad x\in[0,1], \tag{5.16}$$

$$u(0) = g_1, \qquad u(1) = \gamma\int_0^\xi u(x)\,dx + g_2, \tag{5.17}$$

where γ is a any real number but $\xi\in(0,1]$. Let us note that spectral properties for such problem was studied by Pečiulytė [95, 2007].

We investigate the problem without the unique solution again and, simplifying $\varDelta = 0$, obtain the relation $\gamma\xi^2 = 2$. So, we have $\gamma = 2/\xi^2$ in all formulas. Similarly as in previous examples, we get $d = 1$, $k_1 = 2$, $k_2 = 1$ and formulate the auxiliary problem, i.e., $-u'' = f$, $u(0) = 0$ and $u(1) = 0$ with classical conditions only. It has the biorthogonal fundamental system $v^1 = 1-x$, $v^2 = x$ and the Green's function $G^{\mathrm{cl}}(x,y)$, presented in Example 1.6. So, now we use Corollary 1.28, where the representation of the generalized Green's function is given

$$G^g(x,y) = G^{\mathrm{cl}}(x,y) - P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x,y) + \gamma\int_0^\xi G^{\mathrm{cl}}(t,y)\,dt\, v^{g,2}(x) \tag{5.18}$$

for the problem (5.16)–(5.17). Since $x\in N(\boldsymbol{L})$ generates the nullspace, the kernel $P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x,y)$ is of the form (5.5).

Further, we need to find the function v^{g2}. It is the minimizer to the problem $\boldsymbol{L}u = \boldsymbol{e}^2$ or the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2$. First, Corollary 1.4 gives the function

$$\boldsymbol{w}(x) = \left(\gamma \int_0^{\xi} G^{\text{cl}}(t,x)\,dt; \gamma\xi - 1; 1\right)^{\top},$$

which generates the nullspace $N(\boldsymbol{L}^*)$. Then we calculate the projection

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2 = \boldsymbol{e}^2 - \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2}(\boldsymbol{w}, \boldsymbol{e}^2) = \frac{1}{\|\boldsymbol{w}\|^2}\left(-\gamma \int_0^{\xi} G^{\text{cl}}(t,x)\,dt; 1 - \gamma\xi; \|\boldsymbol{w}\|^2 - 1\right)^{\top}$$

with the denominator $\|\boldsymbol{w}\|^2 = \gamma^2 \int_0^1 \big(\int_0^{\xi} G^{\text{cl}}(x,y)\,dx\big)^2 dy + (\gamma\xi - 1)^2 + 1$.

Now we solve consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2$, that is

$$-u'' = -\gamma \int_0^{\xi} G^{\text{cl}}(t,x)\,dt \,/ \|\boldsymbol{w}\|^2, \quad x \in [0,1], \tag{5.19}$$

$$u(0) = (1 - \gamma\xi)/\|\boldsymbol{w}\|^2, \tag{5.20}$$

$$u(1) - \gamma \int_0^{\xi} u(x)\,dx = 1 - 1/\|\boldsymbol{w}\|^2.$$

Substituting the general solution of the differential equation (5.19)

$$u = c_1 + c_2 x - \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\text{cl}}(x,y) \int_0^{\xi} G^{\text{cl}}(t,y)\,dt\,dy$$

into the condition (5.20), we get $c_1 = (1 - \gamma\xi)/\|\boldsymbol{w}\|^2$. So, the general least squares solution is of the form

$$u^g = \frac{1 - \gamma\xi}{\|\boldsymbol{w}\|^2} + cx - \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\text{cl}}(x,y) \int_0^{\xi} G^{\text{cl}}(t,y)\,dt\,dy$$

with one arbitrary constant $c \in \mathbb{R}$. Now applying the formula $v^{g2}(x) = P_{N(\boldsymbol{L})^{\perp}} u^g$, we find the minimizer

$$v^{g2}(x) = \frac{1 - \gamma\xi}{\|\boldsymbol{w}\|^2} + c^o x - \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\text{cl}}(x,y) \int_0^{\xi} G^{\text{cl}}(t,y)\,dt\,dy$$

with the constant

$$c^o = \frac{1}{8 \cdot \|\boldsymbol{w}\|^2}\left(3\gamma\xi - 3 + \gamma \int_0^1 y(1 - y^2) \int_0^{\xi} G^{\text{cl}}(t,y)\,dt\,dy\right).$$

Now substituting obtained v^{g2} and $P_{N(\boldsymbol{L})}G^{\text{cl}}(x,y)$ expressions into (5.18), we know the full representation of the generalized Green's function.

Lastly, we obtain the function v^{g1} using Corollary 1.21. Precisely, we use the biorthogonal fundamental system $\widetilde{v}^1 = 1 - x$ and $\widetilde{v}^2 = x$ of the auxiliary problem in that corollary and simplifying get

$$v^{g1} = (\gamma\xi - 1)v^{g2} + 1 - \frac{3}{8}x.$$

Found expressions of the generalized Green's function $G^g(x, y)$ and the generalized biorthogonal fundamental system v^{g1}, v^{g2} describes the minimum norm least squares solution

$$u^o = \int_0^1 G^g(x, y) f(y)\, dy + g_1 v^{g1}(x) + g_2 v^{g2}(x)$$

for the problem (5.16)–(5.17) with every right hand side $f \in L^2[0, 1]$, $g_1, g_2 \in \mathbb{R}$ if this problem does not have the unique solution, i.e., $\gamma\xi^2 = 2$ ($\varDelta = 0$). This minimizer is the exact solution to the problem (5.16)–(5.17) if this problem is solvable. From Corollary 1.5, we know the solvability condition that is given below

$$g_2 = (1 - \gamma\xi)g_1 - \gamma \int_0^1 \int_0^\xi G^{\mathrm{cl}}(x, y)\, dx\, f(y)\, dy.$$

5.4 Problem with other integral condition

Let us now investigate another the differential problem with an integral condition

$$-u'' = f(x), \quad x \in [0, 1], \tag{5.21}$$

$$u(0) = g_1, \qquad u(1) = \gamma \int_0^1 \alpha(x) u(x)\, dx + g_2, \tag{5.22}$$

where γ is a real number and $\alpha \in L^1[0, 1]$. Here $\varDelta = 0$ means the equality $1 = \gamma \int_0^1 x\alpha(x)\, dx$. So, we have $\gamma = 1/\int_0^1 x\alpha(x)\, dx$ in all formulas and solve this problem almost identically as the previous example.

Indeed, we again obtain $d = 1$, $k_1 = 2$, $k_2 = 1$, formulate the same auxiliary problem with classical conditions only, which has the Green's function $G^{\mathrm{cl}}(x, y)$. Moreover, Corollary 1.4 gives the function

$$\boldsymbol{w}(x) = \left(\gamma \int_0^1 \alpha(t) G^{\mathrm{cl}}(t, x)\, dt; \gamma \int_0^1 \alpha(t)\, dt - 1; 1\right)^\top,$$

generating the nullspace $N(\boldsymbol{L}^*)$.

The expression of the generalized Green's function for the problem (5.21)–(5.22) is also analogous

$$G^g(x, y) = G^{\mathrm{cl}}(x, y) - P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x, y) + \gamma \int_0^1 \alpha(t) G^{\mathrm{cl}}(t, y)\, dt\, v^{g,2}(x), \tag{5.23}$$

where the kernel $P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x,y)$ is of the form (5.5). The generalized biorthogonal fundamental system is presented below

$$v^{g1} = \Big(\gamma \int_0^1 \alpha(t)\,dt - 1\Big)v^{g2} + 1 - \frac{3}{8}x,$$

$$v^{g2}(x) = \frac{1-\gamma\int_0^1 \alpha(t)\,dt}{\|\boldsymbol{w}\|^2} + c^o x - \frac{\gamma}{\|\boldsymbol{w}\|^2}\int_0^1 G^{\mathrm{cl}}(x,y)\int_0^1 \alpha(t)G^{\mathrm{cl}}(t,y)\,dt\,dy,$$

where

$$c^o = \frac{1}{8\cdot\|\boldsymbol{w}\|^2}\Big(3\int_0^1 \alpha(t)\,dt - 3 + \gamma\int_0^1 y(1-y^2)\int_0^\xi G^{\mathrm{cl}}(t,y)\,dt\,dy\Big).$$

Using calculated expressions of the generalized Green's function $G^g(x,y)$ and the generalized biorthogonal fundamental system v^{g1}, v^{g2}, we can find the minimum norm least squares solution

$$u^o = \int_0^1 G^g(x,y)f(y)\,dy + g_1 v^{g1}(x) + g_2 v^{g2}(x)$$

for the problem (5.16)–(5.17) with every right hand side $f \in L^2[0,1]$, $g_1, g_2 \in \mathbb{R}$. This minimizer is the exact solution to the problem (5.21)–(5.22) if this problem is consistent. Corollary 1.5 gives us the solvability condition in the following form

$$g_2 = \Big(1-\gamma\int_0^1 \alpha(x)\,dx\Big)g_1 - \gamma\int_0^1\int_0^1 \alpha(x)G^{\mathrm{cl}}(x,y)\,dx\,f(y)\,dy$$

for the problem (5.21)–(5.22) with $\Delta = 0$, what means $\int_0^1 x\alpha(x)\,dx = 1$.

5.5 Problem with two integral conditions

Let us now take the differential problem with two integral conditions

$$-u'' = f(x), \quad x \in [0,1], \tag{5.24}$$

$$u(0) = \gamma_1\int_0^1 \alpha_1(x)u(x)\,dx + g_1, \quad u(1) = \gamma_2\int_0^1 \alpha_2(x)u(x)\,dx + g_2, \tag{5.25}$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$ and, for simplicity, $\alpha_1, \alpha_2 \in L^1[0,1]$ are positive functions on $[0,1]$.

We note that the problem with $\gamma_1 = 0$ was investigated in Example 5.4. So, now we are interested to develop the case with $\gamma_1 \neq 0$. Here the inequality $1 \neq \gamma_2\int_0^1 x\alpha_2(x)\,dx$ is also valid. Let us assure it.

First, we rewrite the condition $\Delta = 0$, where the problem does not have the unique solution, in the following form

$$\gamma_1\int_0^1 (1-x)\alpha_1(x)\,dx - \gamma_2\int_0^1 x\alpha_2(x)\,dx - \gamma_1\gamma_2\int_0^1\int_0^1 \alpha_1(x)\alpha_2(y)(x-y)\,dx\,dy = 1.$$

Let us take the fundamental system $z^1 = 1$, $z^2 = x$ and calculate

$$\langle L_2, z^1\rangle = 1 - \gamma_2 \int_0^1 \alpha_2(x)\,dx, \qquad \langle L_2, z^2\rangle = 1 - \gamma_2 \int_0^1 x\alpha_2(x)\,dx.$$

Now we note that two equalities $\langle L_2, z^1\rangle = 0$ and $\langle L_2, z^2\rangle = 0$, or equivalently $\gamma_2 \int_0^1 \alpha_2(x)\,dx = 1$ and $\gamma_2 \int_0^1 x\alpha_2(x)\,dx = 1$, are invalid at once. It is obvious for $\gamma_2 = 0$. Otherwise, we have $\gamma_2 \int_0^1 \alpha_2(x)\,dx = \gamma_2 \int_0^1 x\alpha_2(x)\,dx$ or $\int_0^1 (1-x)\alpha_2(x)\,dx = 0$. However, $\int_0^1 (1-x)\alpha_2(x)\,dx > 0$ for the positive function α_2. Thus, $\langle L_2, z^1\rangle = 0$ and $\langle L_2, z^2\rangle = 0$ do not valid together. In other words,

$$1 = \gamma_2 \int_0^1 \alpha_2(x)\,dx, \qquad 1 = \gamma_2 \int_0^1 x\alpha_2(x)\,dx$$

cannot be fulfilled together. If one condition is valid, then another is not. It gives $d = 1$ and $k_2 = 2$ with $k_1 = 1$.

Now we recall the inequality $1 \neq \gamma_2 \int_0^1 x\alpha_2(x)\,dx$. If we have the equality, then

$$\begin{aligned}\varDelta = &\begin{vmatrix} 1 - \gamma_1 \int_0^1 \alpha_1(x)\,dx & 1 - \gamma_2 \int_0^1 \alpha_2(x)\,dx \\ -\gamma_1 \int_0^1 \alpha_1(x)\,dx & 1 - \gamma_2 \int_0^1 x\alpha_2(x)\,dx \end{vmatrix} \\ =&\gamma_1 \int_0^1 x\alpha_1(x)\,dx \cdot \left(1 - \gamma_2 \int_0^1 \alpha_2(y)\,dy\right) \neq 0,\end{aligned}$$

since $\gamma_1 \neq 0$, $\int_0^1 x\alpha_1(x)\,dx > 0$ for positive α_1 and $\gamma_2 \int_0^1 \alpha_2(y)\,dy \neq 1$ for $1 = \gamma_2 \int_0^1 x\alpha_2(x)\,dx$. It gives the contradiction because we study the case $\varDelta = 0$. So, for $\varDelta = 0$ with $\gamma_1 \neq 0$, the inequality $1 \neq \gamma_2 \int_0^1 x\alpha_2(x)\,dx$ must also be satisfied.

Now formulate the auxiliary problem

$$\begin{aligned} &-u'' = f(x), \quad x \in [0,1], \\ u(0) =&0, \qquad u(1) = \gamma_2 \int_0^1 \alpha_2(x)u(x)\,dx, \end{aligned}$$

where $\widetilde{\varDelta} = 1 - \gamma_2 \int_0^1 t\alpha_2(t)\,dt \neq 0$. It has the biorthogonal fundamental system

$$v^1 = \frac{1 - \gamma_2 \int_0^1 t\alpha_2(t)\,dt - \big(1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt\big)x}{1 - \gamma_2 \int_0^1 t\alpha_2(t)\,dt}, \quad v^2 = \frac{x}{1 - \gamma_2 \int_0^1 t\alpha_2(t)\,dt}$$

and the Green's function

$$\begin{aligned} G^a(x,y) = &\begin{cases} y(1-x), & y \leqslant x, \\ x(1-y), & y \geqslant x, \end{cases} \\ &+ \frac{\gamma_2 x}{1 - \gamma_2 \int_0^1 t\alpha_2(t)\,dt}\left((1-y)\int_0^y t\alpha_2(t)\,dt + y\int_y^1 (1-t)\alpha_2(t)\,dt\right). \end{aligned}$$

Using this Green's function in Lemma 1.24, we get the representation of the generalized Green's function

$$G^g(x,y) = G^a(x,y) - P_{N(\boldsymbol{L})}G^a(x,y) + \gamma_1 \int_0^1 \alpha_1(t)G^a(t,y)\,dt\,v^{g,1}(x) \quad (5.26)$$

for the problem (5.24)–(5.25).

First, Remark 1.26 gives the kernel $P_{N(\boldsymbol{L})}G^a(x,y)$. Since here $d = \dim N(\boldsymbol{L}) = 1$ and

$$z^1 := \gamma_1 \int_0^1 t\alpha_1(t)\,dt + \left(1 - \gamma_1 \int_0^1 \alpha_1(t)\,dt\right)x \in N(\boldsymbol{L})$$

generates the nullspace but $(z^1)'' = 0$, we can always use the formula

$$P_{N(\boldsymbol{L})}G^a(x,y) = \frac{z^1}{\|z^1\|^2_{H^2[0,1]}}(z^1, G^a(\cdot,y))_{H^1[0,1]}.$$

Second, we are going to find the function v^{g1}, which is the minimizer to the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1$. Now we use the formula (4.17) to calculate the projection as follows

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1 = \boldsymbol{e}^1 - \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2}(\boldsymbol{w}, \boldsymbol{e}^1).$$

Here we take the function

$$\boldsymbol{w}(x) = \left(\gamma_1 \int_0^1 \alpha_1(t)G^a(t,x)\,dt; 1; \frac{\gamma_1 \int_0^1 t\alpha_1(t)\,dt}{1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt}\right)^\top,$$

which, according to Corollary 1.4, generates the nullspace $N(\boldsymbol{L}^*)$. Then

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1 = \frac{1}{\|\boldsymbol{w}\|^2}\left(-\gamma_1 \int_0^1 \alpha_1(t)G^a(t,x)\,dt; \|\boldsymbol{w}\|^2 - 1; -\frac{\gamma_1 \int_0^1 t\alpha_1(t)\,dt}{1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt}\right)^\top$$

where the denominator is equal to $\|\boldsymbol{w}\|^2 = \gamma_1^2 \int_0^1 \left(\int_0^1 \alpha_1(x)G^a(x,y)\,dx\right)^2 dy + \left(\gamma_1 \int_0^1 t\alpha_1(t)\,dt\right)^2 / \left(1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt\right)^2 + 1$.

Now we can solve the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1$, that is,

$$-u'' = -\gamma_1 \int_0^1 \alpha_1(t)G^a(t,x)\,dt/\|\boldsymbol{w}\|^2, \quad x \in [0,1], \quad (5.27)$$

$$u(0) - \gamma_1 \int_0^1 \alpha_1(x)u(x)\,dx = 1 - 1/\|\boldsymbol{w}\|^2, \quad (5.28)$$

$$u(1) - \gamma_2 \int_0^1 \alpha_2(x)u(x)\,dx = -\gamma_1 \int_0^1 t\alpha_1(t)\,dt/(1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt)\|\boldsymbol{w}\|^2. \quad (5.29)$$

Let us choose the particular fundamental system

$$z^1 = \gamma_1 \int_0^1 t\alpha_1(t)\,dt + \Big(1 - \gamma_1 \int_0^1 \alpha_1(t)\,dt\Big)x, \quad z^2 = \frac{x}{1 - \gamma_2 \int_0^1 t\alpha_2(t)\,dt}$$

and write the general solution of the differential equation (5.24) in the form

$$u = c_1 z^1 + c_2 z^2 - \frac{\gamma_1}{\|\boldsymbol{w}\|^2} \int_0^1 G^a(x,y) \int_0^1 G^a(t,y)\,dt\,dy.$$

We substitute it into (5.29) and solve the constant $c_2 = -\gamma_1 \int_0^1 t\alpha_1(t)\,dt/ (1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt)\|\boldsymbol{w}\|^2$ directly since $z^1 \in N(\boldsymbol{L})$ and the Green's function $G^a(x,y)$ satisfies homogenous conditions for the auxiliary problem. We do not use the condition (5.28) because for the consistent problem with $\varDelta = 0$ it is satisfied trivially again. Thus, we get the general least squares solution

$$u^g = cz^1 - \frac{\gamma_1 \int_0^1 t\alpha_1(t)\,dt}{(1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt)\|\boldsymbol{w}\|^2} z^2 - \frac{\gamma_1}{\|\boldsymbol{w}\|^2} \int_0^1 G^a(x,y) \int_0^1 G^a(t,y)\,dt\,dy$$

which depends on one arbitrary constant $c \in \mathbb{R}$. Then we use the formula $v^{g1}(x) = P_{N(\boldsymbol{L})^\perp} u^g$ and obtain the minimizer

$$v^{g1}(x) = c^o z^1 - \frac{\gamma_1 \int_0^1 t\alpha_1(t)\,dt}{(1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt)\|\boldsymbol{w}\|^2} z^2 - \frac{\gamma_1}{\|\boldsymbol{w}\|^2} \int_0^1 G^a(x,y) \int_0^1 G^a(t,y)\,dt\,dy$$

where c^o can always be calculated from the following formula

$$c^o = \frac{\gamma_1 \int_0^1 t\alpha_1(t)\,dt}{(1 - \gamma_2 \int_0^1 \alpha_2(t)\,dt)\|\boldsymbol{w}\|^2} \cdot \frac{(z^1, z^2)_{H^2[0,1]}}{\|z^1\|^2_{H^2[0,1]}}$$
$$+ \frac{\gamma_1}{\|\boldsymbol{w}\|^2 \cdot \|z^1\|^2_{H^2[0,1]}} \int_0^1 \big(z^1, G^a(\cdot,y)\big)_{H^1[0,1]} \int_0^1 G^a(t,y)\,dt\,dy =$$
$$\frac{3\gamma_1 \int_0^1 t\alpha_1(t)\,dt - 8\gamma_1 \int_0^1 \alpha_1(t)\,dt + 8}{3\gamma_1^2\big(\int\limits_0^1 t\alpha_1(t)\,dt\big)^2 + 4\big(1 - \gamma_1 \int\limits_0^1 \alpha_1(t)\,dt)^2 + 3\gamma_1 \int\limits_0^1 t\alpha_1(t)\,dt\big(1 - \gamma_1 \int\limits_0^1 \alpha_1(y)\,dy\big)}.$$

Now substituting obtained expressions of v^{g1}, $G^a(x,y)$ and $P_{N(\boldsymbol{L})} G^a(x,y)$ into (5.26), we know the generalized Green's function for the problem (5.24)–(5.25). Furthermore, taking the biorthogonal fundamental system of the auxiliary problem ($\widetilde{\varDelta} \neq 0$) in Corollary 1.21, we find another minimizer

$$v^{g2} = \frac{\gamma_1 \int_0^1 t\alpha_1(t)\,dt}{1 - \gamma_2 \int_0^1 t\alpha_2(t)\,dt} v^{g1} + \frac{\gamma_1 \int_0^1 t\alpha_1(t)\,dt + \big(1 - \gamma_1 \int_0^1 \alpha_1(t)\,dt\big)x}{6(1 - \gamma_2 \int_0^1 t\alpha_2(t)\,dt)} \cdot c^o.$$

Since all functions v^{g1}, v^{g2} and $G^g(x,y)$ are known, then we can also obtain the minimum norm least squares solution

$$u^o = \int_0^1 G^g(x,y)f(y)\,dy + g_1 v^{g1}(x) + g_2 v^{g2}(x)$$

for the problem (5.24)–(5.25) with every right hand side $f \in L^2[0,1]$, $g_1, g_2 \in \mathbb{R}$. This minimizer is the exact solution to the problem (5.24)–(5.25) if this problem is consistent. From Corollary 1.5, we get the solvability condition

$$g_1 = g_2 \cdot \frac{1 - \gamma_1 \int_0^1 x\alpha_1(x)\,dx}{1 - \gamma_2 \int_0^1 x\alpha_2(x)\,dx} - \gamma_1 \int_0^1 \int_0^1 \alpha_1(x) G^a(x,y) f(y)\,dx\,dy$$

for the problem (5.24)–(5.25) with $\mathit{\Delta} = 0$.

5.6 A problem with $d = 2$

Here we are going to continue the investigation of Example 1.8, where we considered the problem

$$-u'' = f(x), \quad x \in [0,1], \tag{5.30}$$

$$u(0) = -2\int_0^1 (2-3x)u(x)\,dx + g_1, \qquad u'(1) = u'(\xi) + g_2, \tag{5.31}$$

depending on the parameter $\xi \in [0,1)$.

Let us recall that $\mathit{\Delta}$ is always trivial for this problem and $d = 2$. We will find its generalized Green's function. From Lemma 1.22, we get the representation

$$G^g(x,y) = G^c(x,y) - P_{N(\boldsymbol{L})}G^c(x,y) - y(1-y)^2 v^{g,1}(x) + \mathrm{H}(y-\xi)v^{g2}(x). \tag{5.32}$$

Now we have $1, x \in N(\boldsymbol{L})$. Using the Gram–Schmidt process, we construct the orthonormal basis of the nullspace $N(\boldsymbol{L})$ in the space $H^2[0,1]$

$$z^1 = 1, \qquad z^2 = \sqrt{\frac{3}{13}}(2x-1),$$

recall the formula (4.16) from Remark 1.26 and calculate the kernel

$$P_{N(\boldsymbol{L})}G^c(x,y) = z^1\big(z^1, G^c(\cdot,y)\big)_{H^1[0,1]} + z^2\big(z^2, G^c(\cdot,y)\big)_{H^1[0,1]}$$

. Using the Green's function and its partial derivative in the weak sense

$$G^c(x,y) = \begin{cases} y-x, & y \leqslant x, \\ 0, & y > x, \end{cases} \qquad \frac{\partial}{\partial x}G^c(x,y) = \begin{cases} -1, & y \leqslant x, \\ 0, & y > x, \end{cases} = -\mathrm{H}(x-y),$$

we calculate

$$P_{N(\boldsymbol{L})}G^c(x,y) = -\frac{1}{4}(y-1)^2 + \frac{1}{26}\cdot(2x-1)(9y^4 - 9y^3 + 12y^2 + 11y - 12).$$

Further, we are going to find the function v^{g2}, which is the minimizer to the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1$. Example 1.8 gives us two vector functions $\big(-x(1-x)^2;1;0\big)^\top$ and $\big(\mathrm{H}(x-\xi);0;1\big)^\top$, those generate the nullspace $N(\boldsymbol{L}^*)$. Using the Gram–Schmidt process, we obtain the orthogonal basis

$$\boldsymbol{w}^1 = \big(-x(1-x)^2;1;0\big)^\top, \qquad \boldsymbol{w}^2 = \Big(\mathrm{H}(x-\xi) + c\cdot x(1-x)^2; -c; 1\Big)^\top,$$

where we denoted the constant $c = 35(3\xi^4 - 8\xi^3 + 6\xi^2 + 1)/424$. Now according to (4.17), we can calculate the following projection

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1 = \boldsymbol{e}^1 - \frac{\boldsymbol{w}^1}{\|\boldsymbol{w}^1\|^2}(\boldsymbol{w}^1,\boldsymbol{e}^1) - \frac{\boldsymbol{w}^2}{\|\boldsymbol{w}^2\|^2}(\boldsymbol{w}^2,\boldsymbol{e}^1),$$

where $\|\boldsymbol{w}^1\|^2 = 106/105$ and $\|\boldsymbol{w}^2\|^2 = 2-\xi-c^2/105$. Simplifying, we obtain $\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1 = (p(x);p_1;p_2)^\top$ of the form

$$p(x) = a\cdot\mathrm{H}(x-\xi) + b\cdot x(1-x)^2, \qquad p_1 = 1-b, \qquad p_2 = a.$$

For simpler further expressions, here we denoted two constants

$$a = \frac{c}{\|\boldsymbol{w}^2\|^2} = \frac{105\cdot 424(3\xi^4 - 8\xi^3 + 6\xi^2 - 1)}{3\cdot 424^2\cdot(2-\xi) - 35(3\xi^4 - 8\xi^3 + 6\xi^2 - 1)^2},$$
$$b = \frac{c^2}{\|\boldsymbol{w}^2\|^2} + \frac{105}{106} = \frac{105}{106} + \frac{35\cdot 105\cdot(3\xi^4 - 8\xi^3 + 6\xi^2 - 1)^2}{3\cdot 424^2\cdot(2-\xi) - 35(3\xi^4 - 8\xi^3 + 6\xi^2 - 1)^2}.$$

Since $d = 2$, the solution to the problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1$ is equivalent to the solution of the differential equation $-u'' = p(x)$ only. Now we take general solution to the equation $-u'' = p(x)$, that is,

$$u^g = c_1 z^1 + c_2 z^2 + \int_0^1 G^c(x,y)p(y)\,dy.$$

It describes the general least squares solution to the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^1$ or the problem $\boldsymbol{L}u = \boldsymbol{e}^1$ and depends on two arbitrary constants $c_1, c_2 \in \mathbb{R}$ ($d = 2$). Now we use the formula $v^{g1}(x) = P_{N(\boldsymbol{L})^\perp}u^g$ and find the minimizer

$$v^{g1}(x) = c_1^o z^1 + c_2^o z^2 + \int_0^1 G^c(x,y)p(y)\,dy,$$

given explicitly

$$v^{g1} = c_1^o + c_2^o(2x-1) - \frac{b}{60}x^3(3x^2 - 10x + 10) - \frac{a}{2}(x-\xi)^2\mathrm{H}(x-\xi) \quad (5.33)$$

with such values of constants

$$c_1^o = \frac{1}{60}\big(10a(1-\xi)^3 + b\big), \quad c_2^o = \frac{1}{1092}\big(21a(5\xi^4 - 4\xi^3 - 6\xi^2 - 8\xi + 13) - 23b\big).$$

Now we look for the function v^{g2}, which is the minimizer to the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2$. Its right hand side $(p(x); p_1; p_2)^\top$ is of the form

$$p(x) = \widetilde{a}\cdot \mathrm{H}(x-\xi) + \widetilde{b}\cdot x(1-x)^2, \qquad p_1 = -\widetilde{b}, \qquad p_2 = 1 + \widetilde{a},$$

where $\widetilde{a} = -1/\|\boldsymbol{w}^2\|^2$ and $\widetilde{b} = -c/\|\boldsymbol{w}^2\|^2$. Since $d = 2$, the solution to the problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^2$ is again equivalent to the solution of the differential equation $-u'' = \widetilde{a}\cdot \mathrm{H}(x-\xi) + \widetilde{b}\cdot x(1-x)^2$, which differs from the previous equation $-u'' = a\cdot \mathrm{H}(x-\xi) + b\cdot x(1-x)^2$ with constants only. Thus, the minimizer v^{g2} also has the form as (5.33) with $\widetilde{a}, \widetilde{b}$ instead of constants a, b.

Resuming, the generalized biorthogonal fundamental system can be given by

$$v^{g,k} = c_1^{k,o} + c_2^{k,o}(2x-1) - \frac{b^k}{60}x^3(3x^2-10x+10) - \frac{a^k}{2}(x-\xi)^2\mathrm{H}(x-\xi), \quad k = 1, 2,$$

with the following values of constants

$$a^1 = -b^2 = \frac{c}{\|\boldsymbol{w}^2\|^2}, \quad a^2 = -\frac{1}{\|\boldsymbol{w}^2\|^2}, \quad b^1 = \frac{c^2}{\|\boldsymbol{w}^2\|^2} + \frac{105}{106},$$

$$c_1^{k,o} = \frac{1}{60}\big(10a^k(1-\xi)^3 + b^k\big),$$

$$c_2^{k,o} = \frac{1}{1092}\big(21a^k(5\xi^4 - 4\xi^3 - 6\xi^2 - 8\xi + 13) - 23b^k\big).$$

So, we know expressions of all functions v^{g1}, v^{g2} and $P_{N(\boldsymbol{L})}G^c(x,y)$. Substituting them into (5.32), we also know the representation of the generalize Green's function for the problem (5.30)–(5.31) with $\Delta = 0$.

6 Conclusions

Below we list basic conclusions of this chapter:

1) A differential problem (1.1)–(1.2) always has the Moore–Penrose inverse $\boldsymbol{L}^\dagger$, a generalized Green's function and the unique minimum norm least squares solution.

2) For $\Delta \neq 0$, we have that $\boldsymbol{L}^\dagger = \boldsymbol{L}^{-1}$, the minimum norm least squares solution u^o is coincident with the unique solution u, the generalized Green's function $G^g(x,y)$ is coincident with the ordinary Green's function $G(x,y)$, the biorthogonal fundamental system v^1, v^2 is coincident with the generalized biorthogonal fundamental system $v^{g,1}, v^{g,2}$.

3) The minimum norm least squares solution has literally similar representations as the unique solution: it can be described by the unique solution of the Cauchy problem or the unique solution to other relative problem (the same differential equation (1.1) but different nonlocal conditions (1.2)).

4) The generalized Green's function also has representations similar to expressions of the Green's function: it can be written using the Green's function of the Cauchy problem or the Green's function to other relative problem (the same differential equation (1.1) but different nonlocal conditions (1.2)).

5) The minimum norm least squares solution $u^o \in C^2[0,1]$ if $f \in C[0,1]$ and fully nonlocal parts of conditions (1.2) are of the form (3.13).

Chapter 2

m-th order differential problems with nonlocal conditions

1 Introduction

In this chapter, we are going to generalize results of Chapter 1 to higher order differential problems. Here the investigation object is the m-th order differential equation with m nonlocal conditions

$$\mathcal{L}u := u^{(m)} + a_{m-1}(x)u^{(m-1)} + \ldots + a_1(x)u' + a_0(x)u = f(x), \; x \in [0,1], \quad (1.1)$$

$$\langle L_k, u \rangle = g_k, \quad k = \overline{1,m}, \quad (1.2)$$

defined on the real Sobolev space $H^m[0,1]$ for $m \geqslant 2$. We take real all functions $a_0, \ldots, a_{m-1} \in C[0,1]$, $f \in L^2[0,1]$, numbers $g_k \in \mathbb{R}$ and consider the operator $\mathcal{L} : H^m[0,1] \to L^2[0,1]$. Here $L_k \in \big(C^{m-1}[0,1]\big)^*$, $k = \overline{1,m}$, are continuous linear functionals. According to [2, Alt 2016], they can always be written in the following form

$$\langle L, u \rangle := \sum_{j=0}^{m-2} \gamma_j u^{(j)}(\xi_j) + \int_0^1 u^{(m-1)}(x)\, d\mu(x) \quad (1.3)$$

for some numbers $\gamma_j \in \mathbb{R}$, points $\xi_j \in [0,1]$ and a measure $\mu \in \mathrm{rca}[0,1]$. Let us note that often most nonlocal conditions (1.3) can be represented as below

$$\langle L, u \rangle := \sum_{i=1}^{\infty} \sum_{j=0}^{m-2} a_{ij} u^{(j)}(\xi_i) + \int_0^1 \sum_{j=0}^{m-1} \alpha_j(x) u^{(j)}(x)\, dx \quad (1.4)$$

for $\xi_i \in [0,1]$ (here $0 \leqslant \xi_1 < \xi_2 < \xi_3 < \ldots$), real numbers a_{ij} and integrable functions $\alpha_j \in L^1[0,1]$.

The structure of this chapter remains as previous in order that we can comfortably compare results of second order problems to higher order problems. Here many proofs will be omitted because they are absolutely analogous to corresponding proofs from Chapter 1.

2 The vectorial problem

Let us rewrite the problem (1.1)–(1.2) into the vectorial form

$$\boldsymbol{L}u = \boldsymbol{f} \tag{2.1}$$

with $\boldsymbol{L} := (\mathcal{L}, L_1, \ldots, L_m)^\top$ and the right hand side $\boldsymbol{f} = (f, g_1, \ldots, g_m)^\top \in L^2[0,1] \times \mathbb{R}^m$. Now we take the inner product

$$(\boldsymbol{f}, \tilde{\boldsymbol{f}}) = \int_0^1 f(x)\tilde{f}(x)\,dx + g_1 \cdot \tilde{g}_1 + \ldots + g_m \cdot \tilde{g}_m$$

in the Hilbert space $L^2[0,1] \times \mathbb{R}^m$ and introduce the norm

$$\|\boldsymbol{f}\| = \|\boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^m} = (\boldsymbol{f}, \boldsymbol{f})^{1/2} = \sqrt{\|f\|^2_{L^2[0,1]} + |g_1|^2 + \ldots + |g_m|^2},$$

where $\boldsymbol{f}, \tilde{\boldsymbol{f}} \in L^2[0,1] \times \mathbb{R}^m$. Here we recall the Sobolev embedding theorem [42, Evans 2010], which gives $H^m[0,1] \subset C^{m-1}[0,1]$ and the inequality

$$\|u\|_{C^{m-1}[0,1]} \leqslant C\|u\|_{H^m[0,1]}, \qquad \forall u \in H^m[0,1], \tag{2.2}$$

with a particular constant C independent on a chosen u. From here we get $(C^{m-1}[0,1])^* \subset (H^m[0,1])^*$ and, so, each functional $L_k \in \left(C^{m-1}[0,1]\right)^*$, $k = \overline{1,m}$, belongs to the dual space $(H^m[0,1])^*$. Since $\mathcal{L}$ is also defined on $H^m[0,1]$, we consider the vectorial operator $\boldsymbol{L}$ mapping one Hilbert space $H^m[0,1]$ to another Hilbert space $L^2[0,1] \times \mathbb{R}^m$. Similarly as in Chapter 1, we obtain the following properties.

Lemma 2.1. *The operator $\boldsymbol{L} : H^m[0,1] \to L^2[0,1] \times \mathbb{R}^m$ is the continuous linear operator with the domain $D(\boldsymbol{L}) = H^m[0,1]$.*

2.1 Nullspace of the operator $\boldsymbol{L}$

As in Chapter 1, we have the closed nullspace $N(\boldsymbol{L}) = \{u \in H^m[0,1] : \boldsymbol{L}u = \boldsymbol{0}\}$ and can represent the Sobolev space $H^m[0,1]$ by the direct sum of orthogonal subspaces as follows

$$H^m[0,1] = N(\boldsymbol{L}) \oplus N(\boldsymbol{L})^\perp. \tag{2.3}$$

The nullspace of the operator $\boldsymbol{L}$ is a subset of the nullspace $N(\mathcal{L})$, i.e., $N(\boldsymbol{L}) \subset N(\mathcal{L}) = \operatorname{span}\{z^1, \dots, z^m\} \subset C^m[0,1]$, where $z^1, \dots, z^m \in C^m[0,1]$ are a fundamental system of the homogenous equation $\mathcal{L}u = 0$. Thus, the nullity $d := \dim N(\boldsymbol{L}) \in \{0, 1, \dots, m\}$. Precisely, we take the general solution of $\mathcal{L}u = 0$ in the form $u = c_1 z^1 + \ldots + c_m z^m$ with arbitrary real constants and substitute it into nonlocal conditions $\langle L_k, u\rangle = 0,\ k = \overline{1,m}$. We obtain the linear system

$$c_1\langle L_1, z^1\rangle + \ldots + c_m\langle L_1, z^m\rangle = 0,$$
$$\ldots$$
$$c_1\langle L_m, z^1\rangle + \ldots + c_m\langle L_m, z^m\rangle = 0$$

with respect to constants $c_1, \dots, c_m$. Solving values of constants, we get that the nullspace $N(\mathcal{L})$ is composed of m times continuously differentiable functions.

Denoting the determinant of the previous system

$$\varDelta := \begin{vmatrix} \langle L_1, z^1\rangle & \ldots & \langle L_1, z^m\rangle \\ \ldots & \ldots & \ldots \\ \langle L_m, z^1\rangle & \ldots & \langle L_m, z^m\rangle \end{vmatrix},$$

we separate the following situations:

- $d = 0 \Leftrightarrow \varDelta \neq 0$. Then $N(\boldsymbol{L})$ is trivial.
- $d = m \Leftrightarrow \varDelta = 0$ and all $\langle L_k, z^l\rangle = 0$ for $k, l = \overline{1,m}$. Then all constants $c_1, \dots, c_m$ remain arbitrary and $N(\boldsymbol{L}) = \operatorname{span}\{z^1, \dots, z^m\}$. So, the solution to $\boldsymbol{L}u = \boldsymbol{0}$ is now equivalent to the solution of the differential equation $\mathcal{L}u = 0$ only.
- $0 < d < m \Leftrightarrow \varDelta = 0$ and $\operatorname{rank}(\langle L_k, z^l\rangle) = m - d$ (here $k, l = \overline{1,m}$). In this case, some $m - d$ constants are solved and represented by other d arbitrary constants. In other words, there exist d rows in the determinant representation of $\varDelta$ above, those are linear combinations of the rest $m - d$ linearly independent rows. Let us denote these "*dependent*" rows by $(\langle L_{k_l}, z^1\rangle, \dots, \langle L_{k_l}, z^m\rangle)$ for $k_l,\ l = \overline{1,d}$. The independent rows are also given by $(\langle L_{k_j}, z^1\rangle, \dots, \langle L_{k_j}, z^m\rangle)$ for $k_j,\ j = \overline{d+1,m}$. Thus, the solution to the problem $\boldsymbol{L}u = \boldsymbol{0}$ is now equivalent to the solution of the simplified problem: the equation $\mathcal{L}u = 0$ with conditions $\langle L_{k_j}, u\rangle = 0,\ j = \overline{d+1,m}$, representing linearly independent rows only.

2.2 Range of the operator $\boldsymbol{L}$

Let us begin this subsection with the following property.

Theorem 2.2. *The range $R(\boldsymbol{L})$ of the operator $\boldsymbol{L}$ is closed.*

This theorem is proved absolutely analogously as Theorem 1.2 for the second order differential problem. Let us note that here we take the Green's function $G^c(x, y)$ to the Cauchy problem

$$\mathcal{L}u = f, \qquad u(0) = 0, \quad \dots, \quad u^{(m-1)}(0) = 0. \tag{2.4}$$

This Green's function $G^c(x, y)$ always exists [100, Roman 2011] and is of the form

$$G^c(x, y) = \frac{1}{W(y)} \begin{cases} \widetilde{W}(x, y), & 0 \leqslant y \leqslant x, \\ 0, & x \leqslant y \leqslant 1. \end{cases}$$

Here $W(y) := W[z^1, \dots, z^m](y)$ is the Wronskian of the fundamental system $z^1, \dots, z^m$ at the point $y \in [0, 1]$, i.e.,

$$W[z^1, \dots, z^m](y) = \begin{vmatrix} z^1(y) & (z^1)'(y) & \dots & (z^1)^{(m-2)}(y) & (z^1)^{(m-1)}(y) \\ \dots & \dots & \dots & \dots & \dots \\ z^m(y) & (z^m)'(y) & \dots & (z^m)^{(m-2)}(y) & (z^m)^{(m-1)}(y) \end{vmatrix}$$

but $\widetilde{W}(x, y)$ is the determinant obtained from the Wronskian replacing the last column by $(z^1(x), \dots, z^m(x))^\top$. Moreover, this Green's function has the following properties:

a) $G^c(x, y)$ is continuous on the entire square $0 \leqslant x, y \leqslant 1$ as well as its partial derivatives $(\partial^i / \partial x^i) G^c(x, y)$, $i = \overline{1, m-2}$;

b) $G^c(x, y)$ is C^m in x except the diagonal $x = y$;

c) $(\partial^{m-1} / \partial x^{m-1}) G^c(y + 0, y) - (\partial^{m-1} / \partial x^{m-1}) G^c(y - 0, y) = 1$;

d) $\mathcal{L} G^c(\cdot, y) = 0$ except the diagonal $x = y$;

e) $(\partial^i / \partial x^i) G^c(0, y) = 0$ for $i = \overline{0, m-1}$.

We can also derive the direct representation of the range $R(\boldsymbol{L})$. Let us first consider the composition

$$\boldsymbol{f} = (f, g_1, \dots, g_m)^\top = f\boldsymbol{e}^0 + g_1\boldsymbol{e}^1 + \dots + g_m\boldsymbol{e}^m$$

for every vector valued function $\boldsymbol{f} \in L^2[0, 1] \times \mathbb{R}^m$, where we denoted unit functions $\boldsymbol{e}^0 = (1, 0, \dots, 0)^\top$, $\boldsymbol{e}^1 = (0, 1, \dots, 0)^\top, \dots,$ $\boldsymbol{e}^m = (0, 0, \dots, 1)^\top$. Now we can provide the representation of the range.

Lemma 2.3.

1) If $d = m$, then for every $f \in L^2[0,1]$ we have

$$R(\boldsymbol{L}) = \left\{ \left(f; \int\limits_0^1 \langle L_1, G^c(\cdot, y)\rangle f(y)\, dy; \ldots; \int\limits_0^1 \langle L_m, G^c(\cdot, y)\rangle f(y)\, dy \right)^\top \right\}.$$

2) If $0 < d < m$, then $R(\boldsymbol{L})$ is represented by the vector function

$$\boldsymbol{f} = f\boldsymbol{e}^0 + \sum_{l=1}^{d} \Bigg(\sum_{j=d+1}^{m} g_{k_j} \langle L_{k_l}, v^{k_j}\rangle + \int_0^1 \langle L_{k_l}, G^a(\cdot, y)\rangle f(y)\, dy \Bigg) \boldsymbol{e}^{k_l}$$
$$+ \sum_{j=d+1}^{m} g_{k_j} \boldsymbol{e}^{k_j}, \quad \textit{where} \quad f \in L^2[0,1] \textit{ and } g_{k_j} \in \mathbb{R} \textit{ for } j = \overline{d+1, m}.$$

Here $G^a(x, y)$ is the Green's function and $\{v^1, \ldots, v^m\}$ is the biorthogonal fundamental system for the problem $\mathcal{L}u = f$ with original conditions $\langle L_{k_j}, u\rangle = 0$, $j = \overline{d+1, m}$, and conditions $\langle \ell_{k_l}, u\rangle = 0$, $l = \overline{1, d}$, replacing $\langle L_{k_l}, u\rangle = 0$. Here $\langle \ell_{k_l}, u\rangle = 0$ are selected such that for this auxiliary problem $\varDelta \neq 0$. Let us note that $\langle \ell_{k_l}, u\rangle = 0$ can always be selected from independent conditions $u^{(i)}(0) = 0$ or $u^{(i)}(1) = 0$ for $i = \overline{0, m-1}$, or their combination.

Let us recall the nullspace and range theorem. Applying Lemma 2.3, we can obtain the representation of $N(\boldsymbol{L}^*) = R(\boldsymbol{L})^\perp$ that is given below.

Corollary 2.4. *The following statements are valid:*

1) if $d = m$, then $N(\boldsymbol{L}^*)$ is spanned by vector functions

$$\boldsymbol{w}^k = -\langle L_k, G^c(\cdot, x)\rangle \boldsymbol{e}^0 + \boldsymbol{e}^k, \qquad k = \overline{1, m};$$

2) if $0 < d < m$, then $N(\boldsymbol{L}^*)$ is spanned by vector functions

$$\boldsymbol{w}^l = -\langle L_{k_l}, G^a(\cdot, x)\rangle \boldsymbol{e}^0 - \sum_{j=d+1}^{m} \langle L_{k_j}, v^{k_l}\rangle \boldsymbol{e}^{k_j} + \boldsymbol{e}^{k_l}, \qquad l = \overline{1, d}.$$

This corollary gives that $d = \dim N(\boldsymbol{L})$ and $d^* := \dim N(\boldsymbol{L}^*)$ are equal. Applying the Fredholm alternative theorem, we get solvability conditions to the problem (1.1)–(1.2) without the unique solution ($\varDelta = 0$).

Corollary 2.5. *(Solvability conditions) The problem* (1.1)–(1.2) *with $\varDelta = 0$ is solvable if and only if the conditions are valid:*

1) $\int\limits_0^1 \langle L_1, G^c(\cdot,y)\rangle f(y)\,dy = g_1, \dots, \int\limits_0^1 \langle L_m, G^c(\cdot,y)\rangle f(y)\,dy = g_m$ *for* $d = m$*;*

2) $\sum\limits_{j=d+1}^{m} g_{k_j}\langle L_{k_l}, v^{k_j}\rangle + \int_0^1 \langle L_{k_l}, G^a(\cdot,y)\rangle f(y)\,dy = g_{k_l}$ *for* $l = \overline{1,d}$ *if* $0 < d < m$*.*

Example 2.6. *Let us recall Example 1.6 from Chapter 1. Now we are going to make the generalization considering the m-th order differential problem with initial conditions and one Bitsadze–Samarskii condition*

$$u^{(m)} = f(x), \quad x \in [0,1], \tag{2.5}$$

$$u(0) = g_1,\ u'(0) = g_2,\ \dots,\ u^{(m-2)}(0) = g_{m-1},\ u(1) = \gamma u(\xi) + g_m \tag{2.6}$$

for $\gamma \in \mathbb{R}$ *and a point* $\xi \in (0,1)$*.*

Taking the fundamental system $z^j = x^{j-1}/(j-1)!$ *for* $j = \overline{1,m}$*, we calculate the determinant*

$$\Delta = \begin{vmatrix} 1 & 0 & \dots & 0 & 1-\gamma \\ 0 & 1 & \dots & 0 & 1-\gamma\xi \\ & & \dots & & \dots \\ 0 & 0 & \dots & 1 & (1-\gamma\xi^{m-2})/(m-2)! \\ 0 & 0 & \dots & 0 & (1-\gamma\xi^{m-1})/(m-1)! \end{vmatrix} = \frac{1-\gamma\xi^{m-1}}{(m-1)!}.$$

From here $\Delta = 0$ *gives the condition* $\gamma\xi^{m-1} = 1$*, where the problem* (2.5)–(2.6) *does not have the unique solution. Conditions* $u^{(j)}(0) = g_j$*,* $j = \overline{1, m-1}$*, are always independent because the basic* $(m-1)$*-rst order minor of the m-th order determinant* Δ *above is nonzero. Thus, the nullity for the problem with* $\Delta = 0$ *is always equal to* $d = 1$*. From here we get* $k_1 = m$*, which represents the last condition* $u(1) = \gamma u(\xi) + g_m$*. Then other* $k_j = j-1$*,* $j = \overline{2,m}$*, are numbering initial conditions* $\langle L_{k_j}, u\rangle := u^{(j-2)}(0) = g_{j-1}$*.*

Now we formulate the auxiliary problem with classical conditions only: $u^{(m)} = f$*,* $u^{(j)}(0) = 0$*,* $j = \overline{0, m-2}$*, and* $u(1) = 0$*. We obtained it from the original problem* (2.5)–(2.6) *taking* $\gamma = 0$ *since here* $\Delta = 1/(m-1)!$ *is nonzero. The Green's function for this auxiliary problem was investigated in [100, Roman 2011], [48, Hao et al. 2007] and is of the form*

$$G^{\mathrm{cl}}(x,y) = \frac{1}{(m-1)!}\begin{cases} (x-y)^{m-1} - x^{m-1}(1-y)^{m-1}, & y \leqslant x, \\ -x^{m-1}(1-y)^{m-1}, & y \geqslant x. \end{cases}$$

Let us take the biorthogonal fundamental system to the classical problem $(\gamma = 0)$

$$v^j := z^j - \frac{x^{m-1}}{(j-1)!} = \frac{x^{j-1} - x^{m-1}}{(j-1)!},\ j = \overline{1, m-1},\ \text{and } v^m := x^{m-1}. \tag{2.7}$$

According to Lemma 2.3, the range $R(\boldsymbol{L})$ is represented by the vector function

$$\left(f; g_1; \ldots; g_{m-1}; \sum_{j=1}^{m-1} g_j \langle L_m, v^j\rangle + \int_0^1 \langle L_m, G^{\mathrm{cl}}(\cdot, y)\rangle f(y)\, dy\right)^{\top},$$

that can be rewritten into the form

$$\left(f; g_1; \ldots; g_{m-1}; \sum_{j=1}^{m-1} g_j \frac{1-\gamma\xi^{j-1}}{(j-1)!} - \gamma \int_0^1 G^{\mathrm{cl}}(\xi, y) f(y)\, dy\right)^{\top}.$$

Moreover, from Corollary 2.4, we obtain the function

$$\boldsymbol{w}(x) = \left(\gamma G^{\mathrm{cl}}(\xi, x); \gamma - 1; \gamma\xi - 1; \frac{\gamma\xi^2 - 1}{2!}; \ldots; \frac{\gamma\xi^{m-2} - 1}{(m-2)!}; 1\right)^{\top}$$

or simplifying

$$\boldsymbol{w}(x) = \left(\frac{1}{(m-1)!}\begin{cases} \gamma(\xi - x)^{m-1} - (1-x)^{m-1}, & x \leqslant \xi, \\ -(1-x)^{m-1}, & x \geqslant \xi \end{cases};\right.$$
$$\left.\gamma - 1; \gamma\xi - 1; \frac{\gamma\xi^2 - 1}{2!}; \ldots; \frac{\gamma\xi^{m-2} - 1}{(m-2)!}; 1\right)^{\top},$$

which generates the nullspace $N(\boldsymbol{L}^*)$. Lastly, the solvability condition for the problem (2.5)–(2.6) *with $\gamma = 1/\xi^{m-1}$ (that is $\Delta = 0$) is formulated below*

$$g_m = \sum_{j=1}^{m-1} g_j \frac{1-\gamma\xi^{j-1}}{(j-1)!} - \gamma \int_0^1 G^{\mathrm{cl}}(\xi, y) f(y)\, dy.$$

We present it in the explicit form

$$g_m = \sum_{j=1}^{m-1} g_j \frac{1-\gamma\xi^{j-1}}{(j-1)!} - \frac{\gamma}{(m-1)!}\int_0^{\xi} (\xi - y)^{m-1} f(y)\, dy + \frac{1}{(m-1)!}\int_0^1 (1-y)^{m-1} f(y)\, dy.$$

Let us note that for the case $m = 2$ we use $\gamma\xi = 1$ and simplifying obtain the representation

$$g_2 = (1-\gamma)g_1 + \gamma\int_0^{\xi} y f(y)\, dy + \int_{\xi}^1 f(y)\, dy - \int_0^1 y f(y)\, dy.$$

If differs from the g_2 expression (2.18) *in Example 1.6 of Chapter 1 with the minus sign by integrals. This is because now we investigate the operator u'' but there we considered $-u''$, where the Green's function differs with the minus sign.*

3 Problem with the unique solution

Started the investigation with the properties of the vectorial operator $\boldsymbol{L}$ to the problem (1.1)–(1.2), let us now look at the unique solution. Here we are going to generalize results of Chapter 1. So, this section is based on Roman's work [101, 2011] again, where the problem (1.1)–(1.2) with $f \in C[0,1]$ and the classical unique solution $u \in C^m[0,1]$ was considered. Now we take $f \in L^2[0,1]$ and apply Roman's results investigating the unique solution $u \in H^m[0,1]$ from the Sobolev space.

First, we take the general solution to the differential equation (1.1), that is

$$u = c_1 z^1 + \ldots + c_m z^m + \int_0^1 G^c(x,y) f(y)\, dy.$$

Substituting it into nonlocal conditions (1.2), we obtain the system

$$\begin{aligned} c_1\langle L_1, z^1\rangle + \ldots + c_m\langle L_1, z^m\rangle &= g_1 - \textstyle\int_0^1 \langle L_1, G^c(\cdot,y)\rangle f(y)\, dy,\\ &\ldots \\ c_1\langle L_m, z^1\rangle + \ldots + c_m\langle L_m, z^m\rangle &= g_m - \textstyle\int_0^1 \langle L_m, G^c(\cdot,y)\rangle f(y)\, dy \end{aligned} \tag{3.1}$$

and solve constants uniquely if $\Delta \neq 0$. Then we know the representation of the unique solution to the problem (1.1)–(1.2).

Our aim is to investigate the problem with $\Delta = 0$. To make the clear background for the further investigation, first we analyze aspects of the unique solution $u = \boldsymbol{L}^{-1}\boldsymbol{f}$ using the inverse operator $\boldsymbol{L}^{-1} : L^2[0,1] \times \mathbb{R}^m \to H^m[0,1]$. Let us begin our study looking at the structure of this inverse $\boldsymbol{L}^{-1}$ and its properties.

3.1 Representation of the inverse operator

As in Chapter 1, we can derive the representation of the unique solution

$$u = \int_0^1 G(x,y) f(y)\, dy + g_1 v^1(x) + \ldots + g_m v^m(x) \tag{3.2}$$

for all $f \in L^2[0,1]$ and $g_1, \ldots, g_m \in \mathbb{R}$. Here we use the *biorthogonal fundamental system* $v^1, \ldots, v^m$, where each function is the unique solution to the corresponding problem

$$\begin{aligned} \mathcal{L} v^l &= 0,\\ \langle L_k, v^l\rangle &= \delta_k^j, \quad k,l = \overline{1,m}. \end{aligned} \tag{3.3}$$

Functions v^k, $k = \overline{1,m}$, can also be directly calculated from formulas $v^k = \Delta^k/\Delta$, where Δ^k is the determinant obtained replacing the k-th column in Δ by the column $(z^1(x), \ldots, z^m(x))^\top$. Another component in the representation (3.2) is the *Green's function* to the problem (1.1)–(1.2) [100, Roman 2011], which is of the form

$$G(x,y) := G^c(x,y) - \langle L_1, G^c(\cdot,y)\rangle v^1(x) - \ldots - \langle L_m, G^c(\cdot,y)\rangle v^m(x). \quad (3.4)$$

Denoting the *Green's operator* by

$$Gf = \int_0^1 G(x,y)f(y)\,dy,$$

we describe the unique solution as below

$$u = Gf + g_1 v^1 + \ldots + g_m v^m \quad (3.5)$$

with all functions $f \in L^2[0,1]$ and numbers $g_1,\ldots,g_m \in \mathbb{R}$. From here we obtain the structure of the inverse operator

$$\boldsymbol{L}^{-1} = \big(G,\ v^1,\ldots,\ v^m\big) : L^2[0,1]\times\mathbb{R}^m \to H^m[0,1] \quad (3.6)$$

since the unique solution is $u = \boldsymbol{L}^{-1}\boldsymbol{f}$ with every $\boldsymbol{f} = (f, g_1, \ldots, g_m)^\top$. We note that $G : L^2[0,1] \to H^m[0,1]$ and $v^k \in H^m[0,1]$, $k=\overline{1,m}$, (precisely, all $v^k \in C^m[0,1]$ according to Subsection 2.1) are characterized by the inverse operator as follows

$$Gf = \boldsymbol{L}^{-1}(f,0,\ldots,0)^\top,\ v^1 = \boldsymbol{L}^{-1}(0,1,\ldots,0)^\top,\ \ldots,\ v^m = \boldsymbol{L}^{-1}(0,0,\ldots,1)^\top.$$

All here obtained representations are analogical as in Chapter 1.

3.2 Properties of the unique solution

Substituting the extended form of the Green's function (3.4) into the representation (3.2), we rewrite the unique solution into the form

$$u = u^c + (g_1 - \langle L_1, u^c\rangle)v^1 + \ldots + (g_m - \langle L_m, u^c\rangle)v^m \quad (3.7)$$

using the unique solution

$$u^c = \int_0^1 G^c(x,y)f(y)\,dy$$

to the Cauchy problem (2.4). We can also obtain the similar expression to (3.7) considering two relative problems

$$\begin{aligned} &\mathcal{L}u = f, && \mathcal{L}v = f,\\ &\langle \widetilde{L}_k, u\rangle = \widetilde{g}_k, \quad k=\overline{1,m}, && \langle L_k, v\rangle = g_k, \quad k=\overline{1,m}, \end{aligned} \quad (3.8)$$

where functionals $\widetilde{L}_k$ and L_k, $k=\overline{1,m}$, may be different. Precisely, we have the following relation.

Corollary 2.7. *For unique solutions to the problems* (3.8), *the following equality is always satisfied*

$$v = u + (g_1 - \langle L_1, u\rangle)v^1 + \ldots + (g_m - \langle L_m, u\rangle)v^m.$$

Here conditions $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$ for both problems, respectively, are valid and the biorthogonal fundamental system $v^k,\ k = \overline{1,m}$, for the second problem (3.8) only is used. On the other hand, biorthogonal fundamental systems for these two problems (3.8) are always related as given below.

Corollary 2.8. *Let* $\widetilde{\Delta} \neq 0$ *and* $\Delta \neq 0$ *for problems* (3.9). *Then their biorthogonal fundamental systems* $\widetilde{v}^k,\ k = \overline{1,m}$, *and* $v^k,\ k = \overline{1,m}$, *are linked with the equality*

$$\begin{pmatrix} \langle L_1, \widetilde{v}^1\rangle & \ldots & \langle L_m, \widetilde{v}^1\rangle \\ \ldots & \ldots & \ldots \\ \langle L_1, \widetilde{v}^m\rangle & \ldots & \langle L_m, \widetilde{v}^m\rangle \end{pmatrix} \begin{pmatrix} v^1 \\ \ldots \\ v^m \end{pmatrix} = \begin{pmatrix} \widetilde{v}^1 \\ \ldots \\ \widetilde{v}^m \end{pmatrix}.$$

3.3 Properties of a Green's function

In this section, we present properties of a Green's function for the problem with nonlocal conditions (1.1)–(1.2). Let us begin with following features.

Corollary 2.9. *For* $y \neq y_0, y_1, y_2, \ldots$ *with any* $x \in [0,1]$, *the Green's function has such properties:*

1) $(\partial^i/\partial x^i)G(x,y),\ i = \overline{0, m-2}$, *are continuous in* (x,y);

2) $G(x,y)$ *is* C^m *in* x *except the diagonal* $x = y$;

3) $(\partial^{m-1}/\partial x^{m-1})G(y+0,y) - (\partial^{m-1}/\partial x^{m-1})G(y-0,y) = 1$;

4) $\mathcal{L}G(\cdot,y) = 0$ *except the diagonal* $x = y$;

5) $\langle L_k, G(\cdot,y)\rangle = 0$ *for* $k = \overline{1,m}$.

A relation between a generalized Green's function and an ordinary Green's function for two relative problems is given below.

Proposition 2.10. *For the problems* (3.8) *with* $\widetilde{\Delta} \neq 0$ *and* $\Delta \neq 0$, *their Green's functions* $\widetilde{G}(x,y)$ *and* $G(x,y)$, *respectively, are linked with the equality*

$$G(x,y) = \widetilde{G}(x,y) - \langle L_1, \widetilde{G}(\cdot,y)\rangle v^1(x) - \ldots - \langle L_m, \widetilde{G}(\cdot,y)\rangle v^m(x),$$

for all $x \in [0,1]$ *and a.e.* $y \in [0,1]$.

3.4 Applications to nonlocal boundary conditions

Now we are interested to examine the problem with nonlocal boundary conditions

$$\mathcal{L}u := u^{(m)} + a_{m-1}(x)u^{(m-1)} + \ldots + a_1(x)u' + a_0(x)u = f(x), \ x \in [0,1], \quad (3.9)$$

$$\langle L_k, u\rangle := \langle \kappa_k, u\rangle - \gamma_k \langle \varkappa_k, u\rangle = g_k, \quad k = \overline{1,m}, \quad (3.10)$$

where functionals κ_k, $k = \overline{1,m}$, describe classical parts but functionals $\varkappa_k$, $k = \overline{1,m}$, represent fully nonlocal parts of conditions (3.10). If all parameters γ_k, $k = \overline{1,m}$, vanish, then this problem becomes classical. Let us note that initial conditions $\langle \kappa_k, u\rangle := u^{(k-1)}(0) = g_k$, $k = \overline{1,m}$, can also be considered as particular classical conditions for m-th order problems.

As in Chapter 1, we ask what is the relation between the unique solutions to the problem with nonlocal boundary conditions (3.9)–(3.10) and the problem with classical conditions (all $\gamma_k = 0$) only? Indeed, if the *classical problem* (all $\gamma_k = 0$) has the unique solution u^{cl}, then the unique solution to the nonlocal boundary value problem (3.9)–(3.10) is described by the function u^{cl} as given below

$$u = u^{cl} + \gamma_1 \langle \varkappa_1, u^{\text{cl}}\rangle v^1 + \ldots + \gamma_m \langle \varkappa_m, u^{\text{cl}}\rangle v^m.$$

Here we assumed $\Delta \neq 0$ for the problem with nonlocal boundary conditions (3.10)–(3.11) and took its biorthogonal fundamental system v^k, $k = \overline{1,m}$.

Similarly, the expression of the Green's function $G(x,y)$ for the problem (3.9)–(3.10) can be derived using the Green's function $G^{\text{cl}}(x,y)$ of the classical problem. We provide this relation below

$$G(x,y) = G^{\text{cl}}(x,y) + \sum_{k=1}^{m} \gamma_k \langle \varkappa_k, G^{\text{cl}}(\cdot,y)\rangle v^k(x) \quad (3.11)$$

for all $x \in [0,1]$ and a.e. $y \in [0,1]$.

Moreover, following continuity properties for the Green's function are valid.

Corollary 2.11. *If fully nonlocal conditions for the problem* (3.9)–(3.10) *are of the form*

$$\langle \varkappa, u\rangle := \sum_{i=1}^{\infty} \sum_{j=0}^{m-2} \gamma_{ij} u^{(j)}(\xi_i) + \int_0^1 \sum_{j=0}^{m-2} \alpha_j(x) u^{(j)}(x)\, dx \quad (3.12)$$

with $\gamma_{ij} \in \mathbb{R}$, $\xi_i \in (0,1)$ (here $0 < \xi_1 < \xi_2 < \ldots$) and $\alpha_j \in L^1[0,1]$, then the Green's function $G(x,y)$ is continuous on the entire unit square $0 \leqslant x,y \leqslant 1$

as well as its partial derivatives $(\partial^i/\partial x^i)G(x,y)$, $i=\overline{1,m-2}$. *Moreover,* $(\partial^{m-1}/\partial x^{m-1})G(x,y)$ *and* $(\partial^m/\partial x^m)G(x,y)$ *are also continuous except the diagonal* $x=y$.

Remark 2.12. Let κ_k, $k=\overline{1,m}$, represent initial conditions instead of classical conditions, and fully nonlocal conditions for the problem (3.9)–(3.10) be of the form (3.12). Then the Green's function $G(x,y)$ is continuous on the entire unit square $0 \leqslant x,y \leqslant 1$ as well as its partial derivatives $(\partial^i/\partial x^i)G(x,y)$, $i=\overline{1,m-2}$. Partial derivatives $(\partial^{m-1}/\partial x^{m-1})G(x,y)$ and $(\partial^m/\partial x^m)G(x,y)$ are also continuous except the diagonal $x=y$. We obtain the proof analogously.

So, we have continuous partial derivatives of the Green's function

$$\frac{\partial^i}{\partial x^i}G(x,y) = \frac{\partial^i}{\partial x^i}G^{\mathrm{cl}}(x,y) + \sum_{k=1}^{m} \gamma_k \langle \varkappa_k, G^{\mathrm{cl}}(\cdot,y)\rangle (v^k)^{(i)}(x)$$

except the diagonal $x=y$, where $i=\overline{1,m}$. Further, weak derivatives of the unique solution (3.3) can be described using these classical derivatives of the Green's function in the form

$$u^{(i)} = \int_0^x \frac{\partial^i}{\partial x^i}G(x,y)f(y)\,dy + \int_x^1 \frac{\partial^i}{\partial x^i}G(x,y)f(y)\,dy + \sum_{k=1}^{m} g_k (v^k)^{(i)}(x),$$

for $i=\overline{1,m-1}$, and

$$\begin{aligned} u^{(m)} &= \int_0^x \frac{\partial^m}{\partial x^m}G(x,y)f(y)\,dy \\ &\quad + \int_x^1 \frac{\partial^m}{\partial x^m}G(x,y)f(y)\,dy + f(x) + \sum_{k=1}^{m} g_k (v^k)^{(m)}(x). \end{aligned}$$

Here derivatives $u^{(i)}$, $i=\overline{1,m-1}$, are classical because $u \in H^m[0,1] \subset C^{m-1}[0,1]$. Let us note that $u^{(m)}$ is also continuous and $u \in C^m[0,1]$ if $f \in C[0,1]$.

4 The unique minimizer

If $\varDelta = 0$, we cannot solve the problem (1.1)–(1.2) uniquely and obtain neither the representation $u = \boldsymbol{L}^{-1}\boldsymbol{f}$ nor the Green's function. As in Chapter 1, we are going to solve the problem in the least squares sense, consider properties of the unique minimizer and provide its representations, study a generalized Green's function.

4.1 The minimum norm least squares solution

Let us look for the unique function $u^o \in H^m[0,1]$, which minimizes the norm of the residual for the differential problem (1.1)–(1.2)

$$\|\boldsymbol{L}u^g - \boldsymbol{f}\| = \inf_{u \in H^m[0,1]} \|\boldsymbol{L}u - \boldsymbol{f}\| \tag{4.1}$$

and has the minimum $H^m[0,1]$ norm among all minimizers $u^g \in H^m[0,1]$ of the residual

$$\|u^o\| < \|u^g\| \qquad \forall\ u^g \neq u^o. \tag{4.2}$$

This minimizer u^o for the differential problem with nonlocal conditions (1.1)–(1.2) always exists and is unique since $\boldsymbol{L}$ is the continuous linear operator with a closed range [6, Ben-Israel and Greville 2003]. If the problem (1.1)–(1.2) has the unique solution $u = \boldsymbol{L}^{-1}\boldsymbol{f}$, it is coincident with the minimizer u^o. As in the previous chapter, we focus our study on the following representation of the minimum norm least squares solution

$$u^o = \boldsymbol{L}^{\dagger}\boldsymbol{f}, \tag{4.3}$$

where $\boldsymbol{L}^{\dagger} : L^2[0,1] \times \mathbb{R}^m \to H^m[0,1]$ is the Moore–Penrose inverse of the operator $\boldsymbol{L}$. The definition of the Moore–Penrose inverse and its properties are listed in Subsection 4.1 of Chapter 1. In example, we have the following features.

Lemma 2.13. *The Moore–Penrose inverse $\boldsymbol{L}^{\dagger}$ for the problem* (4.1)–(4.2) *is the continuous linear operator with the domain $D(\boldsymbol{L}^{\dagger}) = L^2[0,1] \times \mathbb{R}^m$ and the range $R(\boldsymbol{L}^{\dagger}) = N(\boldsymbol{L})^{\perp}$.*

We emphasize that the minimizer (4.3) to the problem $\boldsymbol{L}u = \boldsymbol{f}$ is always the minimizer to a consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{f}$ [6, Ben-Israel and Greville 2003].

4.2 The generalized Green's function

Now we rewrite the minimum norm least squares solution $u = \boldsymbol{L}^{\dagger}\boldsymbol{f}$ in the following from

$$u^o = G^g f + g_1 v^{g,1} + \ldots + g_m v^{g,m} \tag{4.4}$$

denoting $G^g f := \boldsymbol{L}^{\dagger}(f, 0, \ldots, 0)^{\top}$ and $v^{g,1} := \boldsymbol{L}^{\dagger}(0, 1, \ldots, 0)^{\top}$, …, $v^{g,m} := \boldsymbol{L}^{\dagger}(0, 0,\ \ldots, 1)^{\top}$. Here functions $v^{g,1}, \ldots, v^{g,m} \in H^m[0,1]$ because $\boldsymbol{L}^{\dagger}\boldsymbol{f} \in H^m[0,1]$ with every $\boldsymbol{f}$, but $G^g : L^2[0,1] \to H^m[0,1]$ is a continuous linear operator since $\boldsymbol{L}^\dagger$ is continuous and linear. From the representation (4.4), we obtain the desired composition of the Moore–Penrose inverse

$$\boldsymbol{L}^\dagger = (G^g,\ v^{g,1},\ \ldots,\ v^{g,m}).$$

As in Chapter 1, we concentrate our investigation on the following expression of the minimum norm least squares solution

$$u^o(x) = \int_0^1 G^g(x,y)f(y)dy + g_1 v^{g,1}(x) + \ldots + g_m v^{g,m}(x), \tag{4.5}$$

which is valid for all $f \in L^2[0,1]$, $g_1, \ldots, g_m \in \mathbb{R}$ and $x \in [0,1]$. Here the kernel $G^g(x,y)$ is $L^2[0,1]$ function in y for every fixed $x \in [0,1]$. We obtained it analogously as in the previous chapter. The representation (4.5) of the minimizer u^o resembles the representation of the unique solution

$$u(x) = \int_0^1 G(x,y)f(y)dy + g_1 v^1(x) + \ldots + g_m v^m(x), \tag{4.6}$$

for the problem (4.1)–(4.2) with $\Delta \neq 0$, where $G(x,y)$ is the Green's function and $v^1, \ldots, v^m$ are the biorthogonal fundamental system of the problem (1.1)–(1.2). According to the similarity, we call the kernel $G^g(x,y)$ – *the generalized Green's function* and the functions $v^{g,1}, \ldots, v^{g,m}$ – *the generalized biorthogonal fundamental system* for the nonlocal problem (1.1)–(1.2).

Thus, if $\Delta \neq 0$, we have that $\boldsymbol{L}^\dagger = \boldsymbol{L}^{-1}$, the minimum norm least squares solution u^o is coincident with the unique solution u, the generalized Green's function $G^g(x,y)$ is coincident with the ordinary Green's function $G(x,y)$, the generalized biorthogonal fundamental system $v^{g,k}$, $k = \overline{1,m}$, is coincident with the biorthogonal fundamental system v^k, $k = \overline{1,m}$.

Do these similarities also imply some relative properties or descriptions of these functions? We are going to provide the answer to this question below.

4.3 Properties of minimizers

First, we are going to investigate the minimum norm least solution. Here we will derive its properties, those are literally analogous to properties of the unique solution, given in Subsection 3.2. Let us begin with the generalized biorthogonal fundamental system and obtain the analogue of (3.3).

Theorem 2.14. *Every function* $v^{g,l} \in H^m[0,1]$ *is the minimum norm least squares solution to the corresponding problem*

$$\begin{aligned} \mathcal{L}v^{g,l} &= 0, \\ \langle L_k, v^{g,l}\rangle &= \delta_k^l, \quad k,l = \overline{1,m}. \end{aligned} \tag{4.7}$$

Let us now consider two relative problems (3.8). Here and further $G^g(x,y)$ is the generalized Green's function and $v^{g,l}$, $l = \overline{1,m}$, are the generalized biorthogonal fundamental system for the second problem (3.8), which may have the unique solution ($\Delta \neq 0$) or not ($\Delta = 0$).

Theorem 2.15. *If the first problem* (3.8) *has the unique solution* u ($\widetilde{\Delta} \neq 0$), *then the minimum norm least squares solution to the second problem* (3.8) *is given by*

$$u^o = u - P_{N(\boldsymbol{L})}u + (g_1 - \langle L_1, u\rangle)v^{g,1} + \ldots + (g_m - \langle L_m, u\rangle)v^{g,m}.$$

Below we present the analogue of (3.7), which is the particular case of the previous theorem.

Corollary 2.16. *The minimum norm least squares solution to the problem* (2.1)–(2.2) *can always be represented by the unique exact solution* u^c *to the Cauchy problem* (2.4) *as follows*

$$u^o = u^c - P_{N(\boldsymbol{L})}u^c + (g_1 - \langle L_1, u^c\rangle)v^{g,1} + \ldots + (g_m - \langle L_m, u^c\rangle)v^{g,m}.$$

Corollary 2.8 is generalized in the following form.

Corollary 2.17. *Let* $\widetilde{\Delta} \neq 0$ *for the first problem* (3.8). *Then the biorthogonal fundamental system* $\widetilde{v}^k$, $k = \overline{1,m}$, *of the first problem and the generalized biorthogonal fundamental system* $v^{g,k}$, $k = \overline{1,m}$, *of the second problem* (3.8) *are related by*

$$\begin{pmatrix} \langle L_1, \widetilde{v}^1\rangle & \ldots & \langle L_m, \widetilde{v}^1\rangle \\ \ldots & \ldots & \ldots \\ \langle L_1, \widetilde{v}^m\rangle & \ldots & \langle L_m, \widetilde{v}^m\rangle \end{pmatrix} \begin{pmatrix} v^{g,1} \\ \ldots \\ v^{g,m} \end{pmatrix} = \begin{pmatrix} P_{N(\boldsymbol{L})^\perp}\widetilde{v}^1 \\ \ldots \\ P_{N(\boldsymbol{L})^\perp}\widetilde{v}^m \end{pmatrix}.$$

Let us emphasize that, for $\Delta \neq 0$, the generalized biorthogonal fundamental system $v^{g,k}$, $k = \overline{1,m}$, becomes ordinary biorthogonal fundamental system v^k, $k = \overline{1,m}$. Then we also have the trivial nullspace $N(\boldsymbol{L})$, and $P_{N(\boldsymbol{L})}$ vanishes in all expressions above. Thus, we obtain that all results from this subsection are coincident with the results of Section 3.1, given for the problem (1.1)–(1.2) with the unique solution if it exists.

4.4 Properties of a generalized Green's function

From Corollary 2.4, we obtain the following representation, which is always valid since $G^c(x,y)$ always exits for the Cauchy problem (2.4).

Lemma 2.18. *The generalized Green's function of the problem* (1.1)–(1.2) *is described by the Green's function* $G^c(x,y)$ *of the Cauchy problem* (2.4), *that is,*

$$G^g(x,y) = G^c(x,y) - P_{N(\boldsymbol{L})}G^c(x,y) - \sum_{k=1}^{m} \langle L_k, G^c(\cdot,y)\rangle v^{g,k}(x).$$

Below we list other properties of a generalized Green's function.

Corollary 2.19. *For* $y \neq y_0, y_1, y_2, \ldots$ *with any* $x \in [0,1]$, *we have:*

1) $(\partial^i/\partial x^i)G^g(x,y)$, $i = \overline{0, m-2}$ *are continuous in* (x,y)*;*

2) $G^g(x,y)$ *is* H^m *in* x *except the diagonal* $x = y$*;*

3) $(\partial^{m-1}/\partial x^{m-1})G^g(y+0,y) - (\partial^{m-1}/\partial x^{m-1})G^g(y-0,y) = 1$*;*

4) $\mathcal{L}G^g(\cdot,y) = -\langle L_1, G^c(\cdot,y)\rangle \mathcal{L}v^{g,1} - \ldots - \langle L_m, G^c(\cdot,y)\rangle \mathcal{L}v^{g,m}$ *if* $x \neq y$*;*

5) $\langle L_k, G^g(\cdot,y)\rangle = \langle L_k, G^c(\cdot,y)\rangle - \sum_{l=1}^{m} \langle L_l, G^c(\cdot,y)\rangle \cdot \langle L_l, v^{g,l}\rangle$ *for* $k = \overline{1,m}$.

Moreover, a generalized Green's function may be described by an ordinary Green's function of other relative problem. We formulate this relation below.

Theorem 2.20. *If* $\widetilde{\Delta} \neq 0$ *for the first problem* (3.8), *then its Green's function* $G(x,y)$ *and the generalized Green's function* $G^g(x,y)$ *of the second problem* (3.8) *are linked as follows*

$$G^g(x,y) = G(x,y) - P_{N(\boldsymbol{L})}G(x,y) - \sum_{k=1}^{m} \langle L_k, G(\cdot,y)\rangle v^{g,k}(x)$$

for all $x \in [0,1]$ *and a.e.* $y \in [0,1]$. *Here* $P_{N(\boldsymbol{L})}G(x,y)$ *is the kernel of the operator* $P_{N(\boldsymbol{L})}G : L^2[0,1] \to H^m[0,1]$.

Remark 2.21. If $\Delta \neq 0$, then the nullspace $N(\boldsymbol{L})$ is trivial and the kernel $P_{N(\boldsymbol{L})}G(x,y)$ vanishes. Otherwise, the projection $P_{N(\boldsymbol{L})}u$ is given by

$$P_{N(\boldsymbol{L})}u = \sum_{l=1}^{d} z^l(x)(z^l, u)_{H^m[0,1]},$$

where z^l, $l = \overline{1, d}$, is an orthonormal basis of the nullspace $N(\boldsymbol{L})$ in the space $H^m[0,1]$, i.e., $(z^l, z^j)_{H^m[0,1]} = 0$ if $l \neq j$ and $(z^l, z^l)_{H^m[0,1]} = 1$. Recalling Remark 1.26 from Chapter 1, we get the formula

$$P_{N(\boldsymbol{L})} G(x,y) = \sum_{l=1}^{d} z^l(x) (z^l, G(\cdot, y))_{H^{m-1}[0,1]}$$

for the problem (3.9)–(3.10) with the operator $\mathcal{L}u := u^{(m)}$ and nonlocal boundary conditions, where fully nonlocal parts are of the form (3.12).

Let us note that here formulated properties of a generalized Green's function extend results of Subsection 3.3 to the case $\Delta = 0$. We also get the generalization of features of Subsection 4.3 from Chapter 1.

4.5 Applications to nonlocal boundary conditions

For the problem with nonlocal boundary conditions (3.9)–(3.10), we can present obtained properties in the following forms.

Corollary 2.22. *If the classical problem* (3.9)–(3.10) *(all* $\gamma_k = 0$*) has the unique solution* u^{cl}*, then the minimum norm least squares solution to the nonlocal boundary value problem* (3.9)–(3.10) *is of the form*

$$u^o = u^{\mathrm{cl}} - P_{N(\boldsymbol{L})} u^{\mathrm{cl}} + \gamma_1 \langle \varkappa_1, u^{\mathrm{cl}} \rangle v^{g,1} + \ldots + \gamma_m \langle \varkappa_m, u^{\mathrm{cl}} \rangle v^{g,m}.$$

Moreover, the generalized Green's function for the problem with nonlocal boundary conditions (3.9)–(3.10) can also be similarly described.

Corollary 2.23. *If the classical problem* (3.9)–(3.10) *(*$\gamma_k = 0$*) has the Green's function* $G^{\mathrm{cl}}(x,y)$*, then the generalized Green's function of the nonlocal problem* (3.9)–(3.10) *is given by*

$$G^g(x,y) = G^{\mathrm{cl}}(x,y) - P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x,y) + \sum_{k=1}^{m} \gamma_k \langle \varkappa_k, G^{\mathrm{cl}}(\cdot, y) \rangle v^{g,k}(x),$$

for all $x \in [0,1]$ *and a.e.* $y \in [0,1]$.

For nonlocal boundary conditions (3.11) with nonlocal parts $\varkappa_k$ of the form (3.12), we can obtain the following quality.

Proposition 2.24. *If* $f \in C[0,1]$*, then the boundary value problem* (3.9)–(3.10) *with* (3.12) *has the minimizer* $u^o \in C^m[0,1]$.

The similar property for the generalized biorthogonal fundamental system is given below.

Corollary 2.25. *For the problem* (3.9)–(3.10) *with* (3.12)*, we have* $v^{g,k} \in C^m[0,1]$ *with every* $k = \overline{1,m}$.

Finally, we get the following feature for the generalized Green's function.

Corollary 2.26. *For the problem* (3.9)–(3.10) *with* (3.12)*, the generalized Green's function* $G^g(x,y)$ *and its partial derivatives* $(\partial^i/\partial x^i)G^g(x,y)$*,* $i = \overline{1,m-2}$*, are continuous on the entire unit square* $0 \leqslant x,y \leqslant 1$*. Moreover, partial derivatives* $(\partial^{m-1}/\partial x^{m-1})G^g(x,y)$ *and* $(\partial^m/\partial x^m)G^g(x,y)$ *are also continuous except the diagonal* $x = y$*.*

Remark 2.27. Let us note that Proposition 2.24 and Corollaries 2.25–2.26 are also valid if $\langle \kappa_k, u \rangle := u^{(k)}(0)$, $k = \overline{1,m}$, are initial conditions instead of classical conditions.

As Corollary 2.26 claims, the generalized Green's function has classical partial derivatives

$$\frac{\partial^i G^g(x,y)}{\partial x^i} = \frac{\partial^i G^{\mathrm{cl}}(x,y)}{\partial x^i} - \frac{\partial^i P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x,y)}{\partial x^i} + \sum_{k=1}^{m} \gamma_k \langle \varkappa_k, G^{\mathrm{cl}}(\cdot,y) \rangle (v^{g,k})^{(i)}(x)$$

except the diagonal $x = y$, where $i = \overline{1,m-1}$. The weak derivatives of the minimizer (4.5) can be described using these classical partial derivatives of the generalized Green's function, that is,

$$(u^o)^{(i)} = \int_0^x \frac{\partial^i}{\partial x^i} G^g(x,y) f(y)\,dy + \int_x^1 \frac{\partial^i}{\partial x^i} G^g(x,y) f(y)\,dy + \sum_{k=1}^{m} g_k (v^{g,k})^{(i)}(x),$$

for $i = \overline{1,m-1}$, and

$$\begin{aligned}(u^o)^{(m)} &= \int_0^x \frac{\partial^m}{\partial x^m} G^g(x,y) f(y)\,dy \\ &\quad + \int_x^1 \frac{\partial^m}{\partial x^m} G^g(x,y) f(y)\,dy + f(x) + \sum_{k=1}^{m} g_k (v^{g,k})^{(m)}(x).\end{aligned}$$

Here derivatives $(u^o)^{(i)}$, $i = \overline{1,m-1}$, are classical since $u^o \in H^m[0,1] \subset C^{m-1}[0,1]$. Now recalling Proposition 2.24, we know that the minimizer u^o belongs to $C^m[0,1]$ if $f \in C[0,1]$.

Let us note that all properties above reduce to properties from Subsection 3.4 if $\Delta \neq 0$ for the problem with nonlocal boundary conditions (3.9)–(3.10).

Example 2.28. *Let us continue the investigation of Example 2.6. Here we are going to analyze the representation of the minimum norm least squares solution*

$$u^o(x) = \int_0^1 G^g(x,y) f(y)\,dy + g_1 v^{g1}(x) + \ldots + g_m v^{gm}(x), \tag{4.8}$$

where $f \in L^2[0,1]$ *and numbers* $g_k \in \mathbb{R}$*, for the problem* (2.5)–(2.6) *without the unique solution, that is* $\gamma = 1/\xi^{m-1}$*. Applying Corollary 2.23, we get the form of the generalized Green's function*

$$G^g(x,y) = G^{\mathrm{cl}}(x,y) - P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x,y) + \gamma v^{g,m}(x)G^{\mathrm{cl}}(\xi,y), \tag{4.9}$$

where $G^{\mathrm{cl}}(x,y)$ *is the Green's function to the classical problem* (2.5)–(2.6) *(all* $\gamma_k = 0$*) and it is presented in Example 2.6. Here we have* $d = 1$*, and* $x^{m-1} \in N(\boldsymbol{L})$ *generates the nullspace. Using Remark 2.21, we get the kernel*

$$P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x,y) = \frac{x^{m-1}}{\|t^{m-1}\|^2_{H^m[0,1]}}(t^{m-1}, G^{\mathrm{cl}}(t,y))_{H^{m-1}[0,1]}.$$

So, in the representation (4.9) *only the function* $v^{g,m}$ *remains unknown. To find* $v^{g,m}$*, we are are going to apply the similar procedure made for the second order problem in Subsection 5.1 from Chapter 1. So,* $v^{g,m}$ *is the minimizer to the problem* $\boldsymbol{L}u = \boldsymbol{e}^m$ *as well as to the consistent problem* $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^m$*. Here we recalled the notation* $\boldsymbol{e}^m = (0,0,\ldots,0,1)^\top \in L^2[0,1] \times \mathbb{R}^m$*. Let us note that we consider the problem* (2.5)–(2.6) *with* $\Delta = 0$*, i.e.,* $\gamma\xi^{m-1} = 1$*, according to Example 2.6. Then we can calculate the projection*

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^m = \boldsymbol{e}^m - \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2}(\boldsymbol{w}, \boldsymbol{e}^m)$$

using the function

$$\boldsymbol{w}(x) = \left(\gamma G^{\mathrm{cl}}(\xi,x); \gamma - 1; \frac{\gamma\xi - 1}{1!}; \frac{\gamma\xi^2 - 1}{2!}; \ldots; \frac{\gamma\xi^{m-2} - 1}{(m-2)!}; 1\right)^\top,$$

which spans the nullspace $N(\boldsymbol{L}^*)$ *and is obtained in Example 2.6. Thus, we get* $\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^m =$

$$\frac{1}{\|\boldsymbol{w}\|^2}\left(-\gamma G^{\mathrm{cl}}(\xi,x); 1-\gamma; \frac{1-\gamma\xi}{1!}; \frac{1-\gamma\xi^2}{2!}; \ldots; \frac{1-\gamma\xi^{m-2}}{(m-2)!}; \|\boldsymbol{w}\|^2 - 1\right)^\top$$

with the denominator $\|\boldsymbol{w}\|^2 = \gamma^2 \int_0^1 (G^{\mathrm{cl}}(\xi,x))^2\,dy + \sum_{j=0}^{m-2}(\gamma\xi^j - 1)^2/(j!)^2 + 1$.

Now we solve the consistent problem $\boldsymbol{L}u = \boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^m$*, given in the extended form*

$$u^{(m)} = -\gamma G^{\mathrm{cl}}(\xi,x)/\|\boldsymbol{w}\|^2, \quad x \in [0,1], \tag{4.10}$$

$$u^{(j)}(0) = (1-\gamma\xi^j)/(j! \cdot \|\boldsymbol{w}\|^2), \quad j = \overline{0, m-2}, \tag{4.11}$$

$$u(1) - \gamma u(\xi) = 1 - 1/\|\boldsymbol{w}\|^2. \tag{4.12}$$

First, we obtain the general solution to the differential equation (4.10)*, that is*

$$u = c_0 + c_1 x + \dots c_{m-1} x^{m-1} - \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\mathrm{cl}}(x,y) G^{\mathrm{cl}}(\xi,y)\, dy.$$

Substituting it into initial conditions (4.11)*, we have* $c_j = (1-\gamma\xi^j)/((j!)^2 \cdot \|\boldsymbol{w}\|^2)$*,* $j = \overline{0, m-2}$*. Since* $d = 1$ *for the consistent problem* (4.10)*–*(4.12)*, the last condition* (4.12) *is satisfied trivially. Thus, the general least squares solution*

$$u^g = \sum_{j=0}^{m-2} \frac{1-\gamma\xi^j}{(j!)^2 \cdot \|\boldsymbol{w}\|^2} x^j + c x^{m-1} - \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\mathrm{cl}}(x,y) G^{\mathrm{cl}}(\xi,y)\, dy$$

depends on one arbitrary constant $c \in \mathbb{R}$*. Since the minimizer is always of the form* $v^{g,m}(x) = P_{N(\boldsymbol{L})^\perp} u^g$*, we find its expression*

$$v^{g,m}(x) = \sum_{j=0}^{m-2} \frac{1-\gamma\xi^j}{(j!)^2 \cdot \|\boldsymbol{w}\|^2} x^j + c^o x^{m-1} - \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\mathrm{cl}}(x,y) G^{\mathrm{cl}}(\xi,y)\, dy$$

with the particular constant

$$c^o = \sum_{j=0}^{m-2} \frac{\gamma\xi^j - 1}{(j!)^2 \cdot \|\boldsymbol{w}\|^2} \cdot \frac{(x^{m-1}, x^j)_{H^m[0,1]}}{\|x^{m-1}\|^2_{H^m[0,1]}} + \frac{\gamma}{\|\boldsymbol{w}\|^2 \cdot \|x^{m-1}\|^2_{H^m[0,1]}} \int_0^1 (x^{m-1}, G^{cl}(x,y))_{H^{m-1}[0,1]} G^{\mathrm{cl}}(\xi,y)\, dy,$$

which is obtained from the equality of projections

$$c^o x^{m-1} = \sum_{j=0}^{m-2} \frac{\gamma\xi^j - 1}{(j!)^2 \cdot \|\boldsymbol{w}\|^2} P_{N(\boldsymbol{L})} x^j + \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 P_{N(\boldsymbol{L})} G^{\mathrm{cl}}(x,y) G^{\mathrm{cl}}(\xi,y)\, dy.$$

Thus, we have just found the minimizer to the problem $\boldsymbol{L}u = \boldsymbol{e}^m$*, which now gives the full representation of the generalized Green's function* (4.9)*.*

Since our goal is to get the full representation of the minimizer (4.8)*, we need to find other functions* $v^{g,j}$*,* $j = \overline{1, m-1}$*. As given in Example 2.6, the problem* (2.5)*–*(2.6) *with all* $\gamma_k = 0$ *always has the unique solution* ($\widetilde{\Delta} = 1/(m-1)! \neq 0$)*. So, we take its biorthogonal fundamental system* $\widetilde{v}^k$*,* $k = \overline{1, m}$*, given by* (2.7)*. Applying Corollary 2.17 and simplifying, we get functions*

$$v^{g,j} = \frac{\gamma\xi^{j-1} - 1}{(j!)^2 \cdot \|\boldsymbol{w}\|^2} v^{g,m} + P_{N(\boldsymbol{L})^\perp} \widetilde{v}^j, \qquad j = \overline{1, m-1}.$$

Resuming, we have the analogous situation as for the second order problem investigated in Subsection 5.1 of Chapter 1. Precisely, substituting the *obtain expression of $v^{g,m}$, we find functions $v^{g,j}$, $j = \overline{1, m-1}$, and know all representations of $G^g(x,y)$ and $v^{g,j}$, $j = \overline{1,m}$. Then we can always calculate the minimum norm least squares solution u^o with every right hand side by the formula* (4.8).

As Corollaries 2.25 and 2.26 say, here all functions $v^{g,k} \in C^m[0,1]$ and $(\partial^i/\partial x^i)G^g(x,y)$ for $i = \overline{0, m-2}$ are continuous on the entire domain $0 \leqslant x, y \leqslant 1$. Moreover, partial derivatives $(\partial^{m-1}/\partial x^{m-1})G^g(x,y)$ and $(\partial^m/\partial x^m)G^g(x,y)$ are also continuous on the unit square except the diagonal $x = y$.

5 Conclusions

In this chapter, we generalized results of the previous chapter, where a second order differential problem with nonlocal conditions was considered. Thus, basic conclusions are also similarly given:

1) A differential problem (1.1)–(1.2) always has the Moore–Penrose inverse $\boldsymbol{L}^\dagger$, a generalized Green's function and the unique minimum norm least squares solution.

2) For $\Delta \neq 0$, we have that $\boldsymbol{L}^\dagger = \boldsymbol{L}^{-1}$, the minimum norm least squares solution u^o is coincident with the unique solution u, the generalized Green's function $G^g(x,y)$ is coincident with the ordinary Green's function $G(x,y)$, the generalized biorthogonal fundamental system $v^{g,k}$, $k = \overline{1,m}$, is coincident with the biorthogonal fundamental system v^k, $k = 1, m$.

3) The minimum norm least squares solution has literally similar representations as the unique solution: it can be described by the unique solution of the Cauchy problem or the unique solution to other relative problem (the same differential equation (1.1) but different nonlocal conditions (1.2)).

4) The generalized Green's function also has representations similar to expressions of the Green's function: it can be written using the Green's function of the Cauchy problem or the Green's function to other relative problem (the same differential equation (1.1) but different nonlocal conditions (1.2)).

5) The minimum norm least squares solution $u^o \in C^m[0,1]$ if $f \in C[0,1]$ and fully nonlocal parts of conditions (1.2) are of the form (3.12).

Chapter 3

Second order discrete problems with nonlocal conditions

1 Introduction

In general, the unique solution or the minimizer of the nonlocal problem

$$\mathcal{L}u := u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad x \in [0,1], \tag{1.1}$$

$$\langle L_k, u \rangle = g_k, \quad k = 1, 2, \tag{1.2}$$

which was investigated in Chapter 1, cannot always be found analytically. Since the computer-programming science nowadays is widely developed, various numerical methods have been being investigated and applied to differential problems [51, Hernandez-Martinez *et al.* 2011], [55, Il'in and Moiseev 1987]. Then the nonlocal problem (1.1)–(1.2) is replaced by some discrete problem that, merely, is described by a linear system of equations

$$\mathbf{A}\mathbf{u} = \mathbf{b} \qquad (\mathbf{A} \in \mathbb{C}^{m\times n},\ \mathbf{u} \in \mathbb{C}^{n\times 1},\ \mathbf{b} \in \mathbb{C}^{m\times 1}). \tag{1.3}$$

Since every linear transformation from one finite-dimensional vector space to another can be represented by a matrix (uniquely described by the linear transformation and the fixed bases for the vector spaces), there is one to one correspondence between the $m \times n$ *complex matrices* $\mathbb{C}^{m\times n}$ and $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$, the *space of linear transformations mapping* $\mathbb{C}^n$ *into* $\mathbb{C}^m$. Hence, we use the same symbol A to denote both the linear transformation $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ and its matrix representation $\mathbf{A} \in \mathbb{C}^{m\times n}$. Then the discrete representation (1.3) of the differential problem (1.1)–(1.2) is equivalent to the statement that the linear transformation $\mathbf{A}$ maps $\mathbf{u}$ into $\mathbf{b}$.

In this chapter, we investigate second order discrete problems with nonlocal conditions those are analogues of second order differential nonlocal problems (1.1)–(1.2). Here considering discrete problems are not necessary discretizations of differential problems using numerical methods. Various discrete problems also arise in the theory of graphs, networks. Due to this, we are interested to consider a wider class of discrete problems, not only obtained from differential problems.

The structure of the chapter is as follows. First, we define some notation. Second, we formulate a discrete problem with nonlocal conditions and its matrix representation, discuss on properties of a discrete problem. Then already known results for the discrete problem with the unique solution are briefly presented. Further, we investigate discrete problems without the unique solution. Here we solve problems in the least squares sense and consider properties of the unique discrete minimizer. Analogous features for a generalized discrete Green's function are also derived. Afterwards, we solve discrete problems in the least squares sense introducing two finite dimensional Hilbert spaces. We apply these results to discrete problems, those approximate differential problems, and obtain sufficient convergence conditions of the discrete minimizer to the minimizer of a differential problem. Let us note that this chapter is based on papers [83,84,86,88,124, Paukštaitė and Štikonas 2012–2016].

2 Notation

First, we introduce the space of complex linear functions $F(X_n) := \{u \mid u\colon X_n \to \mathbb{C}\}$ defined on the finite set $X_n := \{0, 1, 2, \dots, n\}$, $n \geqslant 2$. We use the notation $u_i = u(i)$, $i \in X_n$, and call functions $u \in F(X_n)$ by *discrete functions* making the difference from continuous functions, investigated in previous chapters.

For the space of discrete functions $F(X_n)$, we take the standard basis of complex functions e^j, $j = \overline{0, n}$, where $e^j(i) = \delta_i^j$, $i \in X_n$, and δ_i^j is the Kronecker delta. Then we can write $u = \sum_{i=0}^n u_i e^i$. It also means that $F(X_n) \cong \mathbb{C}^{(n+1)\times 1}$, i.e., every $u \in F(X_n)$ can be uniquely described by the complex column matrix $\mathbf{u} = (u_0, u_1, \dots, u_n)^\top \in \mathbb{C}^{(n+1)\times 1}$ or $\mathbf{u} = \sum_{i=0}^n u_i \mathbf{e}^i$. Here $\mathbf{e}^0 = (1, 0, \dots, 0)^\top$, $\mathbf{e}^1 = (0, 1, \dots, 0)^\top, \dots$, $\mathbf{e}^n = (0, 0, \dots, 1)^\top$ are matrix representations of the standard basis e^j, $j = \overline{0, n}$.

Similarly, a linear functional $L \in F^*(X_n)$ can be interpreted as a complex row matrix $\mathbf{L} = (L^0, L^1, \dots, L^n) \in \mathbb{C}^{1\times(n+1)}$. For the functional $L \in F^*(X_n)$ value at the function $u \in F(X_n)$, we use the notation $\langle L, u\rangle$ or the matrix multiplication $\mathbf{Lu}$.

In analogous way, the space $F(X_m \times X_n)$ is defined [100, Roman 2011] and its elements are uniquely described by complex matrices from $\mathbb{C}^{(m+1)\times(n+1)}$. In this work, the one to one correspondence between functions $M : X_m \times X_n \to \mathbb{C}$ and matrices $\mathbf{M} = (M_{ij}) \in \mathbb{C}^{(m+1)\times(n+1)}$ is represented by $M_{ij} = M(i,j),\ i \in X_m,\ j \in X_n$. Further we use notations of the summation without the sum symbol, that is,

$$\langle L^{\cdot}, U_{\cdot j}\rangle := \sum_{l=0}^{n} L^l U_{lj}, \qquad M_{i\cdot} U_{\cdot j} := \sum_{l=0}^{n} M_{il} U_{lj}, \qquad i \in X_m,\ j \in X_k,$$

where $L \in F^*(X_n)$, $M \in F(X_m \times X_n)$, $U \in F(X_n \times X_k)$. Matrix representations of two last notations are interpreted as usual matrix multiplications $\mathbf{LU}$ and $\mathbf{MU}$, respectively. So, we understand the multiplication of two discrete functions, i.e., MU, as the discrete representation of the matrix $\mathbf{MU}$, the multiplication of two matrices. We describe the i-th row of the matrix $\mathbf{M}$ by $\mathrm{row}_i\mathbf{M} := (M_{i0}, \ldots, M_{in})$. Similarly, $\mathrm{col}_j\mathbf{M} := (M_{0j}, \ldots, M_{mj})^\top$ represents the j-th column. Their discrete representations are given by discrete functions $\mathrm{row}_i M \in F(X_n)$ and $\mathrm{col}_j M \in F(X_m)$, respectively.

Let us remark again that a discrete function u and its matrix representation $\mathbf{u}$ are always equivalent notations for the same function. Thus, the identity function $I = \mathrm{id} \in F(X_n \times X_n)$ is equivalent to the identity matrix $\mathbf{I} = \mathbf{I}_{n+1}$ of order $n+1$. We will use another notation δ_{ij} for the Kronecker delta as well.

3 Formutation of the problem

In this chapter, we investigate a second order discrete problem

$$(\mathcal{L}u)_i := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in X_{n-2}, \tag{3.1}$$

$$\langle L_k, u\rangle := \sum_{j=0}^{n} L_k^j u_j = g_k, \quad k = 1, 2, \tag{3.2}$$

with functions $a^0, a^1, a^2 \in F(X_{n-2})$, $f \in F(X_{n-2})$ and the operator $\mathcal{L} : F(X_n) \to F(X_{n-2})$. Here g_k are complex numbers but $L_k \in F^*(X_n)$ – discrete linear functionals describing nonlocal conditions.

Let us take functions $a_i^0, a_i^2 \neq 0$ for all $i \in X_{n-2}$ and consider the nonsingular second order discrete operator $\mathcal{L}$, i.e., all rows of its matrix representation

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_1^0 & a_1^1 & a_1^2 & \dots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \dots & a_{n-3}^1 & a_{n-3}^2 & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{n-2}^0 & a_{n-2}^1 & a_{n-2}^2 \end{pmatrix}$$

are linearly independent. Since discrete functionals L_k, describing nonlocal conditions (3.2), are represented by row matrices $\mathbf{L}_k = (L_k^0, L_k^1, \dots, L_k^n) \in \mathbb{C}^{1\times(n+1)}$, we rewrite the discrete problem (3.1)–(3.2) in the following matrix form

$$\mathbf{Au} = \mathbf{b}, \qquad \mathbf{A} = \begin{pmatrix} \boldsymbol{\mathcal{L}} \\ \mathbf{L}_1 \\ \mathbf{L}_2 \end{pmatrix} \tag{3.3}$$

with the right hand side $\mathbf{b} = (f_0, f_1, \dots, f_{n-2}, g_1, g_2)^\top \in \mathbb{C}^{(n+1)\times 1}$.

3.1 Nullspace of the matrix A

The problem (3.3) has the unique solution if $\det \mathbf{A} \neq 0$. The condition for the unique solvability of the problem (3.1)–(3.2) is often given by the nonzero determinant

$$\varDelta := \begin{vmatrix} \langle L_1, z^1\rangle & \langle L_1, z^2\rangle \\ \langle L_2, z^1\rangle & \langle L_2, z^2\rangle \end{vmatrix}$$

as well, where $z^1, z^2 \in F(X_n)$ is any fundamental system of the homogenous equation (3.1). We obtain it solving the homogenous problem $\mathbf{Az} = \mathbf{0}$. Precisely, we take the general solution $z = c_1 z^1 + c_2 z^2$, $c_k \in \mathbb{C}$, of the equation $\mathcal{L}z = 0$. Substituting it into homogenous conditions $\langle L_k, z\rangle = 0$, $k = 1, 2$, we get the system

$$c_1\langle L_1, z^1\rangle + c_2\langle L_1, z^2\rangle = 0,$$
$$c_1\langle L_2, z^1\rangle + c_2\langle L_2, z^2\rangle = 0$$

with the determinant $\varDelta$. Let us denote the nullity of the matrix $\mathbf{A}$ by $d := \dim N(\mathbf{A})$ and separate the following cases:

- $d = 0 \Leftrightarrow \varDelta \neq 0$. Then the nullspace $N(\mathbf{A})$ is trivial.
- $d = 2 \Leftrightarrow$ if $\varDelta = 0$ with all $\langle L_k, z^l\rangle = 0$ for $k, l = 1, 2$. Then the general solution to $\mathbf{Az} = \mathbf{0}$ depends on two arbitrary constants c_1, c_2 and $N(\mathbf{A}) = \text{span}\,\{z^1, z^2\}$. Thus, the solution to $\mathbf{Az} = \mathbf{0}$ is equivalent to the solution to the differential equation $\mathcal{L}z = 0$ only.

- $d = 1 \Leftrightarrow$ if $\Delta = 0$ and exists at least one value $\langle L_k, z^l \rangle \neq 0$. Emphasizing the number of the functional, let us say $\langle L_{k_2}, z^l \rangle \neq 0$. Then we can solve one constant c_l but other c_{3-l} remains arbitrary. Here the one condition $\langle L_{k_1}, z \rangle = 0$ ($k_1 = 3 - k_2$) is dependent because it gives no additional information how to find the arbitrary constant c_{3-l}. Thus, the solution to the problem $\mathbf{Az} = \mathbf{0}$ is equivalent to the solution of the simplified problem $\mathcal{L}z = 0$, $\langle L_{k_2}, z \rangle = 0$.

For more study on the nullspace and classifications of the nullity, we suggest to read these papers [84–87, Paukštaitė and Štikonas 2013–2015].

3.2 Range of the matrix A

For the discrete problem (3.1)–(3.2), we can obtain its range representation $R(\mathbf{A})$ that is given below.

Lemma 3.1.

1) If $d = 2$, then for all $f \in F(X_{n-2})$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \dots; f_{n-2};\ \sum_{j=0}^{n-2} \langle L_1^{\cdot}, G^c_{\cdot j} \rangle f_j;\ \sum_{j=0}^{n-2} \langle L_2^{\cdot}, G^c_{\cdot j} \rangle f_j \right)^{\top} \right\}.$$

2) If $d = 1$ and $k_1 = 1$, then for all $f \in F(X_{n-2})$ and $g_2 \in \mathbb{C}$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \dots; f_{n-2};\ g_2 \langle L_1, v^2 \rangle + \sum_{j=0}^{n-2} \langle L_1^{\cdot}, G^a_{\cdot j} \rangle f_j;\ g_2 \right)^{\top} \right\}.$$

3) If $d = 1$ and $k_1 = 2$, then for all $f \in F(X_{n-2})$ and $g_1 \in \mathbb{C}$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \dots; f_{n-2};\ g_1;\ g_1 \langle L_2, v^1 \rangle + \sum_{j=0}^{n-2} \langle L_2^{\cdot}, G^a_{\cdot j} \rangle f_j \right)^{\top} \right\}.$$

Here $G^c \in F(X_n \times X_{n-2})$ is the discrete Green's function for the discrete Cauchy problem $\mathcal{L}u = f$, $u_0 = 0$, $u_1 = 0$. Other discrete Green's function $G^a \in F(X_n \times X_{n-2})$ and the biorthogonal fundamental system $v^1, v^2 \in F(X_n)$ are taken for the problem $\mathcal{L}u = f$ with the original condition $\langle L_{3-k_1}, u \rangle = 0$ and condition $\langle \ell, u \rangle = 0$, replacing $\langle L_{k_1}, u \rangle = 0$. Here $\langle \ell, u \rangle = 0$ is selected such that for this auxiliary problem $\Delta \neq 0$.

Proof. The proof is analogous to the proof of Lemma 1.3 from Chapter 1, where we investigated the second order differential problem.

1) First, the discrete Green's function $G^c \in F(X_n \times X_{n-2})$ for the discrete Cauchy problem

$$\mathcal{L}u = f, \qquad u_0 = 0, \quad u_1 = 0, \tag{3.4}$$

always exists [100, Roman 2011] and is of the form

$$G^c_{ij} = \frac{1}{a^2_j \cdot W_{j+2}} \begin{cases} z^1_{j+1} z^2_i - z^1_i z^2_{j+1} & j < i, \\ 0, & i \leqslant j, \end{cases} \qquad i \in X_n, \; j \in X_{n-2}. \tag{3.5}$$

Here

$$W_{j+2} := W[z^1, z^2]_{j+2} = \begin{vmatrix} z^1_{j+1} & z^1_{j+2} \\ z^2_{j+1} & z^2_{j+2} \end{vmatrix}$$

denotes the Wronskian of the discrete biorthogonal fundamental system $\{z^1, z^2\}$ at a point $j+2$ for every $j \in X_{n-2}$.

Further, we take the general solution

$$u_i = c_1 z^1_i + c_2 z^2_i + \sum_{j=0}^{n-2} G^c_{ij} f_j, \qquad i \in X_n,$$

to the discrete equation (3.1). Substituting it into nonlocal conditions (3.2), we obtain the system

$$\begin{aligned} c_1 \langle L_1, z^1 \rangle + c_2 \langle L_1, z^2 \rangle &= g_1 - \textstyle\sum_{j=0}^{n-2} \langle L^{\cdot}_1, G^c_{\cdot j} \rangle f_j, \\ c_1 \langle L_2, z^1 \rangle + c_2 \langle L_2, z^2 \rangle &= g_2 - \textstyle\sum_{j=0}^{n-2} \langle L^{\cdot}_2, G^c_{\cdot j} \rangle f_j. \end{aligned}$$

Since $d = 2$, then all $\langle L_k, z^j \rangle = 0$ and we get conditions $g_1 = \sum_{j=0}^{n-2} \langle L^{\cdot}_1, G^c_{\cdot j} \rangle f_j$ and $g_2 = \sum_{j=0}^{n-2} \langle L^{\cdot}_2, G^c_{\cdot j} \rangle f_j$. Thus, the range $R(\mathbf{A})$ is composed of the following vectors $\mathbf{b} = (f_0; \ldots; f_{n-2}; g_1; g_2)^\top = \big(f_0; f_1; \ldots; f_{n-2}; \; \sum_{j=0}^{n-2} \langle L^{\cdot}_1, G^c_{\cdot j} \rangle f_j; \; \sum_{j=0}^{n-2} \langle L^{\cdot}_2, G^c_{\cdot j} \rangle f_j\big)^\top$ with an arbitrary $f \in F(X_{n-2})$.

2) Since $d = 1$, the one condition $\langle L_{k_1}, u \rangle = g_{k_1}$ (here $k_1 = 1$, $k_2 = 2$) can be omitted as dependent in the consistent problem (3.1)–(3.2). Then we choose such condition $\langle \ell, u \rangle = 0$, that the problem $\mathcal{L}u = f$, $\langle \ell, u \rangle = 0$, $\langle L_{k_2}, u \rangle = g_2$ has $\varDelta \neq 0$. According to [100, Roman 2011], his special problem has the Green's function $G^a \in F(X_n \times X_{n-2})$ and the fundamental system $v^1, v^2 \in F(X_n)$, satisfying $\langle \ell, v^k \rangle = \delta^k_1$ and $\langle L_2, v^k \rangle = \delta^k_2$ for $k = 1, 2$. In example, $\langle \ell, u \rangle = 0$ can always be one of independent conditions $u_0 = 0$, $u_1 = 0$, $u_{n-1} = 0$ or $u_n = 0$.

As in the part 1) of the proof, we take the general solution

$$u_i = c_1 v^1_i + c_2 v^2_i + \sum_{j=0}^{n-2} G^a_{ij} f_j, \qquad i \in X_n,$$

to the discrete equation (3.1). Putting it into nonlocal conditions (3.2), we analogously get the system

$$\begin{aligned} c_1\langle L_1, v^1\rangle + c_2\langle L_1, v^2\rangle &= g_1 - \textstyle\sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^a\rangle f_j, \\ c_1\langle L_2, v^1\rangle + c_2\langle L_2, v^2\rangle &= g_2 - \textstyle\sum_{j=0}^{n-2}\langle L_2^{\cdot}, G_{\cdot j}^a\rangle f_j. \end{aligned}$$

Since $\langle L_2, v^2\rangle = 1$, $\langle L_2, v^1\rangle = 0$ and $\langle L_2^{\cdot}, G_{\cdot j}^a\rangle = 0$, then $c_2 = g_2$ and $\Delta = 0$ for the problem (3.1)–(3.2) gives $\langle L_1, v^1\rangle = 0$. So, the condition $c_1\langle L_1, v^1\rangle + c_2\langle L_1, v^2\rangle = g_1 - \sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^a\rangle f_j$ can be rewritten in the form $g_1 = g_2\langle L_1, v^2\rangle + \sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^a\rangle f_j$. Finally, the range $R(\mathbf{A})$ representation is given by the vector $\mathbf{b} = (f_0; \dots; f_{n-2}; g_1; g_2)^\top = \big(f_0; \dots; f_{n-2}; g_2\langle L_1, v^2\rangle + \sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^a\rangle f_j; g_2\big)^\top$ with arbitrary $g_2 \in \mathbb{C}$ and $f \in F(X_{n-2})$.
3) The proof is obtained similarly. □

To formulate other results, it is useful to recall the notation

$$\mathbf{b} = (f_0, \dots, f_{n-2}, g_1, g_2)^\top = \sum_{j=0}^{n-2} f_i\mathbf{e}^i + g_1\mathbf{e}^{n-1} + g_2\mathbf{e}^n,$$

where $\mathbf{e}^0 = (1, 0, \dots, 0)^\top$, $\mathbf{e}^1 = (0, 1, \dots, 0)^\top$,..., $\mathbf{e}^n = (0, 0, \dots, 1)^\top$ are matrix representations of the standard basis e^j, $j = \overline{0, n}$, in the discrete space $F(X_n)$. Since the equality $R(\mathbf{A})^\perp = N(\mathbf{A}^*)$ is always valid, further we provide the representation of the nullspace $N(\mathbf{A}^*)$.

Corollary 3.2. *The following three statements are valid:*

1) if $d = 2$, then $N(\mathbf{A}^*)$ is generated by two vectors

$$\mathbf{w}^1 = -\sum_{j=0}^{n-2}\overline{\langle L_1^{\cdot}, G_{\cdot j}^c\rangle}\mathbf{e}^j + \mathbf{e}^{n-1}, \qquad \mathbf{w}^2 = -\sum_{j=0}^{n-2}\overline{\langle L_2^{\cdot}, G_{\cdot j}^c\rangle}\mathbf{e}^j + \mathbf{e}^n;$$

2) if $d = 1$ and $k_1 = 1$, then $N(\mathbf{A}^*)$ is generated by the vector

$$\mathbf{w} = -\sum_{j=0}^{n-2}\overline{\langle L_1^{\cdot}, G_{\cdot j}^a\rangle}\mathbf{e}^j + \mathbf{e}^{n-1} - \overline{\langle L_1, v^2\rangle}\mathbf{e}^n;$$

3) if $d = 1$ and $k_1 = 2$, then $N(\mathbf{A}^*)$ is generated by the vector

$$\mathbf{w} = -\sum_{j=0}^{n-2}\overline{\langle L_2^{\cdot}, G_{\cdot j}^a\rangle}\mathbf{e}^j - \overline{\langle L_2, v^1\rangle}\mathbf{e}^{n-1} + \mathbf{e}^n.$$

Proof. 1) We have the orthogonality condition $(\mathbf{b}, \tilde{\mathbf{b}}) = 0$ for all $\mathbf{b} \in R(\mathbf{A})$ and $\tilde{\mathbf{b}} = (\tilde{f}_0, \ldots, \tilde{f}_{n-2}, \tilde{g}_1, \tilde{g}_2)^\top \in R(\mathbf{A})^\perp$, i.e.,

$$\sum_{j=0}^{n-2} f_j \overline{\tilde{f}_j} + \overline{\tilde{g}_1} \sum_{j=0}^{n-2} \langle L_1^\cdot, G^c_{\cdot j}\rangle f_j + \overline{\tilde{g}_2} \sum_{j=0}^{n-2} \langle L_2^\cdot, G^c_{\cdot j}\rangle f_j = 0$$

with arbitrary $f \in F(X_{n-2})$. We rewrite it in the form

$$\sum_{l=0}^{n-2} \big(\overline{\tilde{f}_l} + \overline{\tilde{g}_1} \langle L_1^\cdot, G^c_{\cdot l}\rangle + \overline{\tilde{g}_2} \langle L_2^\cdot, G^c_{\cdot l}\rangle\big) f_l = 0.$$

Taking $f_l = \delta_{lj}$ for every fixed $j \in X_{n-2}$, we get conditions $\overline{\tilde{f}_j} + \overline{\tilde{g}_1} \langle L_1^\cdot, G^c_{\cdot j}\rangle + \overline{\tilde{g}_2} \langle L_2^\cdot, G^c_{\cdot j}\rangle = 0$ or $\tilde{f}_j = -\tilde{g}_1 \overline{\langle L_1^\cdot, G^c_{\cdot j}\rangle} - \tilde{g}_2 \overline{\langle L_2^\cdot, G^c_{\cdot j}\rangle}$ valid with every $j \in X_{n-2}$ and $\tilde{g}_1, \tilde{g}_2 \in \mathbb{C}$. Thus, the set $R(\mathbf{A})^\perp$ is composed of vectors $\tilde{\mathbf{b}} = -\sum_{j=0}^{n-2} \big(\tilde{g}_1 \overline{\langle L_1^\cdot, G^c_{\cdot j}\rangle} + \tilde{g}_2 \overline{\langle L_2^\cdot, G^c_{\cdot j}\rangle}\big)\mathbf{e}^j + \tilde{g}_1 \mathbf{e}^{n-1} + \tilde{g}_2 \mathbf{e}^n$ with all $\tilde{g}_1, \tilde{g}_2 \in \mathbb{C}$, those are generated by two linearly independent vectors $\mathbf{w}^1 = -\sum_{j=0}^{n-2} \overline{\langle L_1^\cdot, G^c_{\cdot j}\rangle} \mathbf{e}^j + \mathbf{e}^{n-1}$, $\mathbf{w}^2 = -\sum_{j=0}^{n-2} \overline{\langle L_2^\cdot, G^c_{\cdot j}\rangle} \mathbf{e}^j + \mathbf{e}^n$.

2) We write the orthogonality condition $(\mathbf{b}, \tilde{\mathbf{b}}) = 0$ in the explicit form

$$\sum_{j=0}^{n-2} \big(\overline{\tilde{f}_j} + \overline{\tilde{g}_1} \langle L_1^\cdot, G^a_{\cdot j}\rangle\big) f_j \, dx + g_2 \cdot \big(\overline{\tilde{g}_1} \langle L_1, v^2\rangle + \overline{\tilde{g}_2}\big) = 0$$

for every $g_2 \in \mathbb{C}$ and $f \in F(X_{n-2})$. Since g_2 and f obtain values independently, we take $g_2 = 0$, afterwards $f = 0$ and as in the part 1) of this proof, get conditions $\overline{\tilde{f}_j} + \overline{\tilde{g}_1} \langle L_1^\cdot, G^a_{\cdot j}\rangle = 0$, $j \in X_{n-2}$, and $\overline{\tilde{g}_1} \langle L_1, v^2\rangle + \overline{\tilde{g}_2} = 0$. Rewriting we have $\tilde{f}_j = -\tilde{g}_1 \overline{\langle L_1^\cdot, G^a_{\cdot j}\rangle}$ and $\tilde{g}_2 = -\tilde{g}_1 \overline{\langle L_1, v^2\rangle}$. Thus, the nullspace $N(\mathbf{A}^*)$ is represented by $\tilde{\mathbf{b}} = -\tilde{g}_1 \sum_{j=0}^{n-2} \overline{\langle L_1^\cdot, G^a_{\cdot j}\rangle} \mathbf{e}^j + \tilde{g}_1 \mathbf{e}^{n-1} - \tilde{g}_1 \overline{\langle L_1, v^2\rangle} \mathbf{e}^n$ with an arbitrary $\tilde{g}_1 \in \mathbb{R}$, generated by the one vector $\mathbf{w} = -\sum_{j=0}^{n-2} \overline{\langle L_1^\cdot, G^a_{\cdot j}\rangle} \mathbf{e}^j + \mathbf{e}^{n-1} - \overline{\langle L_1, v^2\rangle} \mathbf{e}^n$.
3) The proof of the last statement is analogous. □

Now recalling the Fredholm alternative theorem, we get solvability conditions for the problem (3.1)–(3.2) without the unique solution ($\mathit{\Delta} = 0$).

Corollary 3.3. *(Solvability conditions) The problem* (3.1)–(3.2) *with* $\mathit{\Delta} = 0$ *is solvable if and only if the conditions are valid:*

1) $\sum_{j=0}^{n-2} \langle L_1^\cdot, G^c_{\cdot j}\rangle f_j = g_1$, $\sum_{j=0}^{n-2} \langle L_2^\cdot, G^c_{\cdot j}\rangle f_j = g_2$ *for* $d = 2$;

2) $g_2 \langle L_1, v^2\rangle + \sum_{j=0}^{n-2} \langle L_1^\cdot, G^a_{\cdot j}\rangle f_j = g_1$ *for* $d = 1$ *and* $k_1 = 1$;

3) $g_1 \langle L_2, v^1\rangle + \sum_{j=0}^{n-2} \langle L_2^\cdot, G^a_{\cdot j}\rangle f_j = g_2$ *for* $d = 1$ *and* $k_1 = 2$.

Example 3.4. *Let us consider a differential problem*

$$-u'' = f(x), \quad x \in [0,1], \tag{3.6}$$

$$u(0) = g_1, \qquad u(1) - \gamma u(\xi) = g_2 \tag{3.7}$$

with $\xi \in (0,1)$, a real function $f \in C[0,1]$ and γ, $g_1, g_2 \in \mathbb{R}$. Let us introduce the mesh $\overline{\omega}^h := \{x_i = ih,\ i \in X_n,\ hn = 1\}$ and the submesh of inner points $\omega^h := \{x_i = ih,\ i = \overline{1, n-1},\ hn = 1\}$. We denote $f_i = f(x_{i+1}),\ i \in X_{n-2}$, and suppose ξ is coincident with a mesh point, i.e., $\xi = sh$ for some $s = \overline{1, n-2}$. Now we approximate the problem (3.6)–(3.7) *by the finite difference method and obtain the real discrete problem*

$$(\mathcal{L}u)_i := -\frac{1}{h^2}u_{i+2} + \frac{2}{h^2}u_{i+1} - \frac{1}{h^2}u_i = f_i, \quad i \in X_{n-2}, \tag{3.8}$$

$$\langle L_1, u\rangle := u_0 = g_1, \qquad \langle L_2, u\rangle := u_n - \gamma u_s = g_2. \tag{3.9}$$

Let us take the fundamental system of the homogenous equation (3.8) *as follows $z^1 = 1,\ z^2 = x,\ x \in \overline{\omega}^h$. Then the necessary and sufficient existence condition of the unique solution [100, Roman 2011] is given by*

$$\Delta = \begin{vmatrix} \langle L_1, z^1\rangle & \langle L_2, z^1\rangle \\ \langle L_1, z^2\rangle & \langle L_2, z^2\rangle \end{vmatrix} = \begin{vmatrix} 1 & 1-\gamma \\ 0 & 1-\gamma\xi \end{vmatrix} = 1 - \gamma\xi \neq 0 \quad \Leftrightarrow \quad \gamma\xi \neq 1.$$

For $\gamma\xi = 1$, we have $d = 1$ and $k_1 = 2$, $k_2 = 1$ since $\langle L_1, z^1\rangle = 1 \neq 0$. Now we formulate the auxiliary problem $\mathcal{L}u = f$, $u_0 = 0$, $u_n = 0$ with classical conditions only (take $\gamma = 0$ in (3.9)*) because here $\Delta \neq 0$. This problem has the biorthogonal fundamental system $v^1 = 1 - x$, $v^2 = x$ for $x \in \overline{\omega}^h$ and the discrete Green's function*

$$G^{\mathrm{cl}}_{ij} = h \begin{cases} x_{j+1}(1 - x_i), & x_{j+1} \leqslant x_i, \\ x_i(1 - x_{j+1}), & x_i \leqslant x_{j+1}, \end{cases} \qquad i \in X_n,\ j \in X_{n-2}. \tag{3.10}$$

So, from Lemma 3.1, we get the range representation

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-2}; g_1; (1-\gamma)g_1 - \gamma \sum_{j=0}^{n-2} G^{\mathrm{cl}}_{ij} f_j \right)^\top \right\}.$$

Moreover, Corollary 3.2 gives the vector

$$\mathbf{w} = \gamma \sum_{j=0}^{n-2} G^{\mathrm{cl}}_{sj}\mathbf{e}^j + (\gamma - 1)\mathbf{e}^{n-1} + \mathbf{e}^n,$$

which generates the nullspace $N(\mathbf{A}^*)$. Finally, we present the solvability condition

$$g_2 = (1-\gamma)g_1 - \gamma \sum_{j=0}^{n-2} G^{\mathrm{cl}}_{ij} f_j$$

for the problem (3.6)–(3.7) *with* $\Delta = 0$, *what gives* $\gamma = 1/\xi$ *in formulas above.*

4 Problem with the unique solution (case $\Delta \neq 0$)

Substituting the general solution

$$u_i = c_1 z_i^1 + c_2 z_i^2 + \sum_{j=0}^{n-2} G_{ij}^c f_j, \qquad i \in X_n,$$

of the discrete equation (3.1) into nonlocal conditions (3.2), we get the system

$$\begin{aligned} c_1\langle L_1, z^1\rangle + c_2\langle L_1, z^2\rangle &= g_1 - \textstyle\sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^c\rangle f_j, \\ c_1\langle L_2, z^1\rangle + c_2\langle L_2, z^2\rangle &= g_2 - \textstyle\sum_{j=0}^{n-2}\langle L_2^{\cdot}, G_{\cdot j}^c\rangle f_j. \end{aligned} \tag{4.1}$$

If $\Delta \neq 0$, we solve constants c_1, c_2 uniquely and obtain the representation of the unique solution to the problem (3.1)–(3.2).

On the other hand, for $\det \mathbf{A} \neq 0$, the matrix form $\mathbf{Au} = \mathbf{b}$ of the discrete problem (3.1)–(3.2) gives another representation of the unique solution $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$ with every right hand side $\mathbf{b} \in \mathbb{C}^{(n+1)\times 1}$. Now we are interested to investigate the structure of the inverse matrix $\mathbf{A}^{-1} \in \mathbb{C}^{(n+1)\times(n+1)}$. Here we recall several results from Roman's work [100, 2011] to make their generalizations in the following section for problems (3.1)–(3.2) with $\Delta = 0$, or equivalently, $\det \mathbf{A} = 0$.

4.1 Representation of the inverse matrix

Since the right hand side of the problem $\mathbf{Au} = \mathbf{b}$ has the particular form $\mathbf{b} = (f_0, f_1, \ldots, f_{n-2}, g_1, g_2)^\top$ with every $\mathbf{f} = (f_0, f_1, \ldots, f_{n-2})^\top \in \mathbb{C}^{(n+1)\times 1}$ and complex numbers g_1, g_2, the unique solution can be written in the special form

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{b} = \mathbf{Gf} + g_1\mathbf{v}^1 + g_2\mathbf{v}^2. \tag{4.2}$$

Here $\mathbf{G} \in \mathbb{C}^{(n+1)\times(n-1)}$ and $\mathbf{v}^1, \mathbf{v}^2 \in \mathbb{C}^{(n+1)\times 1}$ are submatrices of the inverse matrix

$$\mathbf{A}^{-1} = (\mathbf{G},\ \mathbf{v}^1,\ \mathbf{v}^2).$$

Let us now take the discrete representation of the solution (4.2)

$$u = Gf + g_1 v^1 + g_2 v^2,$$

which has the explicit form

$$u_i = \sum_{j=0}^{n-2} G_{ij} f_j + g_1 v_i^1 + g_2 v_i^2, \quad i \in X_n. \tag{4.3}$$

Here the kernel $G \in F(X_n \times X_{n-2})$ is also known as the *discrete Green's function* and functions $v^1, v^2 \in F(X_n)$ are called the *discrete biorthogonal fundamental system* for the problem (3.1)–(3.2) [100, Roman 2011]. Using the inverse matrix $\mathbf{B} = \mathbf{A}^{-1}$, we can always calculate the discrete Green's function as well as the discrete biorthogonal fundamental system in the following way

$$G_{ij} = B_{ij}, \quad i \in X_n, \ j \in X_{n-2}, \tag{4.4}$$

$$v_i^1 = B_{i,n-1}, \quad i \in X_n, \tag{4.5}$$

$$v_i^2 = B_{in}, \quad i \in X_n. \tag{4.6}$$

4.2 Properties of discrete Green's functions

Roman investigated discrete Green's functions and their properties in [100, 2011]. Firstly, the discrete Green's function G is the unique solution to discrete problem

$$\begin{aligned} \mathcal{L}_{i\cdot} G_{\cdot j} &= \delta_{ij}, \quad i \in X_{n-2}, \\ \langle L_k^{\cdot}, G_{\cdot j} \rangle &= 0, \quad k = 1, 2, \end{aligned} \tag{4.7}$$

for every fixed $j \in X_{n-2}$. On the other hand, discrete functions v^1 and v^2 are unique solutions to corresponding discrete problems

$$\begin{matrix} \mathcal{L}v^1 = 0, & \mathcal{L}v^2 = 0, \\ \langle L_1, v^1 \rangle = 1, \quad \langle L_2, v^1 \rangle = 0, & \langle L_1, v^2 \rangle = 0, \quad \langle L_2, v^2 \rangle = 1, \end{matrix} \tag{4.8}$$

and can always be obtained from the formulas below

$$v_i^1 = \frac{\begin{vmatrix} z_i^1 & \langle L_2, z^1 \rangle \\ z_i^2 & \langle L_2, z^2 \rangle \end{vmatrix}}{\Delta}, \qquad v_i^2 = \frac{\begin{vmatrix} \langle L_1, z^1 \rangle & z_i^1 \\ \langle L_1, z^2 \rangle & z_i^2 \end{vmatrix}}{\Delta}, \qquad i \in X_n.$$

Further we present the way to calculate the discrete Green's function.

Lemma 3.5 (Roman 2011, [100]). *If $\Delta \neq 0$, then the discrete Green's function for the problem* (3.1)–(3.2) *is given by*

$$G_{ij} = G_{ij}^c - v_i^1 \langle L_1^{\cdot}, G_{\cdot j}^c \rangle - v_i^2 \langle L_2^{\cdot}, G_{\cdot j}^c \rangle, \qquad i \in X_n, \ j \in X_{n-2}.$$

The discrete Green's function $G^c \in F(X_n \times X_{n-2})$ always exists (3.5) and describes the unique solution $u^c \in F(X_n)$ to the discrete Cauchy problem (3.4), i.e., $u_i^c = \sum_{j=0}^{n-2} G_{ij}^c f_j$, $i \in X_n$. Moreover, the unique solution u^c of the Cauchy problem always represents the unique solution to the discrete problem (3.1)–(3.2) in the following form

$$u = u^c + (g_1 - \langle L_1, u^c \rangle) v^1 + (g_2 - \langle L_2, u^c \rangle) v^2.$$

This representation also follows from the other, more general result. Precisely, unique solutions of two relative problems

$$\begin{aligned} &\mathcal{L}u = f, && \mathcal{L}v = f, \\ &\langle \widetilde{L}_k, u\rangle = \widetilde{g}_k, \quad k = 1,2, && \langle L_k, v\rangle = g_k, \quad k = 1,2, \end{aligned} \tag{4.9}$$

where functionals $\widetilde{L}_k$ and L_k, $k = 1,2$, may be different, are analogously related. We present this statement below.

Corollary 3.6 (Roman 2011, [100]). *For unique solutions of problems* (4.9)*, the following equality is always valid*

$$v = u + (g_1 - \langle L_1, u\rangle)v^1 + (g_2 - \langle L_2, u\rangle)v^2.$$

Moreover, discrete Green's functions of these problems are also similarly related.

Theorem 3.7 (Roman 2011, [100]). *Discrete Green's functions $\widetilde{G}$ and G of problems* (4.9)*, respectively, are linked with the equality*

$$G_{ij} = \widetilde{G}_{ij} - v_i^1\langle L_1^{\cdot}, \widetilde{G}_{\cdot j}\rangle - v_i^2\langle L_2^{\cdot}, \widetilde{G}_{\cdot j}\rangle, \quad i \in X_n,\ j \in X_{n-2}.$$

Let us note that here we used the biorthogonal fundamental system v^1, v^2 for the second problem (4.9) only. Furthermore, conditions $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$ for both problems, respectively, are fulfilled. Applying the previous corollary, we get the relation between biorthogonal fundamental systems for these problems (4.9) as well.

Corollary 3.8. *Let $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$ for problems* (4.9)*. Then their biorthogonal fundamental systems $\widetilde{v}^1, \widetilde{v}^2 \in F(X_n)$ and $v^1, v^2 \in F(X_n)$ are related by*

$$\begin{pmatrix} \langle L_1, \widetilde{v}^1\rangle & \langle L_2, \widetilde{v}^1\rangle \\ \langle L_1, \widetilde{v}^2\rangle & \langle L_2, \widetilde{v}^2\rangle \end{pmatrix} \begin{pmatrix} v_i^1 \\ v_i^2 \end{pmatrix} = \begin{pmatrix} \widetilde{v}_i^1 \\ \widetilde{v}_i^2 \end{pmatrix}, \qquad i \in X_n.$$

Roman applied these results to problems with nonlocal boundary conditions

$$(\mathcal{L}u)_i := a_i^2 u_{i+2} + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \qquad i \in X_{n-2}, \tag{4.10}$$

$$\langle L_k, u\rangle := \langle \kappa_k, u\rangle - \gamma_k\langle \varkappa_k, u\rangle = g_k, \quad k = 1,2, \tag{4.11}$$

where $\Delta \neq 0$. Here functionals κ_k describe classical parts but $\varkappa_k$, $k = 1,2$ represent fully nonlocal parts of conditions (4.11). For vanishing parameters $\gamma_1, \gamma_2 = 0$, the problem becomes classical. If this classical problem has the unique solution $u^{\mathrm{cl}} \in F(X_n)$, then it describes the unique solution of the problem with nonlocal boundary conditions (4.10)–(4.11) in the following form

$$u = u^{\mathrm{cl}} + \gamma_1 \langle \varkappa_1, u^{\mathrm{cl}} \rangle + \gamma_2 \langle \varkappa_2, u^{\mathrm{cl}} \rangle.$$

Analogously, the discrete Green's function for the problem with nonlocal boundary conditions (4.10)–(4.11) can also be represented

$$G_{ij} = G^{\mathrm{cl}}_{ij} + \gamma_1 v^1_i \langle \varkappa_1, G^{\mathrm{cl}}_{\cdot j} \rangle + \gamma_2 v^2_i \langle \varkappa_2, G^{\mathrm{cl}}_{\cdot j} \rangle, \qquad i \in X_n,\ j \in X_{n-2},$$

using the discrete Green's function $G^{\mathrm{cl}} \in F(X_n \times X_{n-2})$ of the classical problem $(\gamma_1, \gamma_2 = 0)$.

5 The unique discrete minimizer (case $\varDelta = 0$)

If the condition $\varDelta = 0$ or equivalent condition $\det \mathbf{A} = 0$ is satisfied, then the discrete problem (3.1)–(3.2) does not have the unique solution [100, Roman 2011]. In this case, the equivalent problem (3.3) has a singular matrix $\mathbf{A}$. So, the unique solution as well as the discrete Green's function cannot be calculated using the ordinary inverse $\mathbf{A}^{-1}$ because the representation $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$ and formulas (4.4)–(4.6) are not valid.

In this section, we are going to solve the problem (3.1)–(3.2) with $\varDelta = 0$ in the least squares sense. Thus, here we focus on the matrix representation (3.3) of the discrete problem (3.1)–(3.2). Precisely, we will look for a unique vector, which minimizes the norm of the residual $\mathbf{Au} - \mathbf{b}$ and is smallest among all minimizers. This unique minimizer as well as its representations are essential objects of our consideration in this section. Here we will derive its properties. Let us note that obtained results are analogous to all known properties from the previous section, where the discrete problem (3.1)–(3.2) with $\varDelta \neq 0$ only was considered.

5.1 The minimum norm least squares solution

The problem (3.1)–(3.2) with $\det \mathbf{A} = 0$ has a lot of solutions (consistent problem) or no solutions (inconsistent problem). Despite this, we make a following decision. Instead of the unique solution to the problem $\mathbf{Au} = \mathbf{b}$, here we look for a vector $\mathbf{u}^g \in \mathbb{C}^{(n+1)\times 1}$, which minimizes the standard Euclidean norm of the residual

$$\|\mathbf{Au}^g - \mathbf{b}\| \leqslant \|\mathbf{Au} - \mathbf{b}\|, \quad \forall\ \mathbf{u} \in \mathbb{C}^{(n+1)\times 1}. \tag{5.1}$$

Every vector, minimizing the residual (5.1), is called the *least squares solution* to the problem (3.3) [6, Ben-Israel and Greville 2003]. Among all least squares solution, we choose the unique solution $\mathbf{u}^o \in \mathbb{C}^{(n+1)\times 1}$ of the minimum norm

$$\|\mathbf{u}^o\| < \|\mathbf{u}^g\| \qquad \forall\ \mathbf{u}^g \neq \mathbf{u}^o. \tag{5.2}$$

This vector $\mathbf{u}^o$ always exists and is often called the *minimum norm least squares solution* to the problem (3.3). Considering its discrete representation $u^o \in F(X_n)$, we assign it to the equivalent discrete problem (3.1)–(3.2) as the unique minimizer as well.

Minimization steps (5.1)–(5.2) for the consistent problem means that we select the unique solution, which has the minimum standard Euclidean norm among all solutions. If the problem is inconsistent, we have the best approximate solution. As we will see, there are a lot of approximate solutions as well. Thus, here we choose the one of the minimum norm again.

Let us note that the problem (3.1)–(3.2) with $\det \mathbf{A} \neq 0$ is also involved in the minimization problem (5.1)–(5.2) because its unique solution $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$ is coincident with the unique minimizer $\mathbf{u}^o$. As the inverse matrix $\mathbf{A}^{-1}$ exists and plays the essential role for the problem (3.1)–(3.2) with $\det \mathbf{A} \neq 0$, we can also obtain an analogue for the problem with $\det \mathbf{A} = 0$. Indeed, according to Penrose [93, 1955], every matrix $\mathbf{A} \in \mathbb{C}^{(n+1)\times(n+1)}$ of the discrete problem always has the unique matrix $\mathbf{X} \in \mathbb{C}^{(n+1)\times(n+1)}$ satisfying all four Penrose equations

$$\mathbf{AXA} = \mathbf{A}, \quad \mathbf{XAX} = \mathbf{X}, \quad (\mathbf{AX})^* = \mathbf{AX}, \quad (\mathbf{XA})^* = \mathbf{XA}. \tag{5.3}$$

This matrix is often called the Moore–Penrose inverse of the matrix $\mathbf{A}$ and is denoted by $\mathbf{A}^\dagger$. Several properties of the Moore–Penrose inverse $\mathbf{A}^\dagger$ are listed below.

Lemma 3.9 (Penrose 1955, [93]; Moore and Barnard 1935, [81]; Ben-Israel and Greville 2003, [6]). *For every finite matrix $\mathbf{A} \in \mathbb{C}^{k\times m}$ and its Moore–Penrose inverse $\mathbf{A}^\dagger \in \mathbb{C}^{m\times k}$, the following conditions are valid:*

1) $\mathbf{A}^\dagger = \mathbf{A}^{-1}$ *if* $\det \mathbf{A} \neq 0$ *($\Delta \neq 0$);*

2) $(\mathbf{A}^\dagger)^\dagger = \mathbf{A}$*;*

3) $(\mathbf{A}^*)^\dagger = (\mathbf{A}^\dagger)^*$*;*

4) $N(\mathbf{A}^\dagger) = N(\mathbf{A}^*)$*;*

5) $R(\mathbf{A}^\dagger) = R(\mathbf{A}^*)$*;*

6) rank $\mathbf{A}$ = rank $\mathbf{A}^{\dagger}$ = rank $\mathbf{A}^{*}$*;*

7) $\mathbf{P}_{R(A)} = \mathbf{A}\mathbf{A}^{\dagger}$ *and* $\mathbf{P}_{R(A^*)} = \mathbf{A}^{\dagger}\mathbf{A}$.

Another property of the Moore–Penrose inverse $\mathbf{A}^{\dagger}$ says that it describes the minimum norm least squares solution

$$\mathbf{u}^o = \mathbf{A}^{\dagger}\mathbf{b} \tag{5.4}$$

of the problem (3.1)–(3.2) analogously as the unique solution is given $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$ if it exists [93, Penrose 1955], [6, Ben-Israel and Greville 2003], [79, Mayer 2004]. Using the minimum norm least squares solution $\mathbf{u}^o = \mathbf{A}^{\dagger}\mathbf{b}$, we also know the general least squares solutions u^g, that is,

$$\mathbf{u}^g = \mathbf{A}^{\dagger}\mathbf{b} + \mathbf{P}_{N(A)}\mathbf{c}, \quad \forall \mathbf{c} \in \mathbb{C}^{(n+1)\times 1}, \tag{5.5}$$

where $\mathbf{P}_{N(A)}$ denotes the orthogonal projector onto the nullspace $N(\mathbf{A})$. Let us note that the minimizer u^o is the particular least squares solution, which is uniquely characterized by the following two inequalities:

$$\|\mathbf{A}\mathbf{u}^o - \mathbf{b}\| \leqslant \|\mathbf{A}\mathbf{u} - \mathbf{b}\|, \quad \forall\, \mathbf{u} \in \mathbb{C}^{(n+1)\times 1}, \tag{5.6}$$

$$\|\mathbf{u}^o\| < \|\mathbf{u}^g\|, \quad \forall\, \mathbf{u}^g \neq \mathbf{u}^o. \tag{5.7}$$

Moreover, every least squares solution can be represented by the minimum norm least squares solution

$$u^g = u^o + P_{N(A)}c \tag{5.8}$$

with an arbitrary discrete function $c \in F(X_n)$, whereas the minimum norm least squares solution is always equal to

$$u^o = P_{N(A)^{\perp}}u^g. \tag{5.9}$$

Let us remark that the minimizer (5.4) to the problem (3.3), which may be consistent ($\mathbf{b} \in R(\mathbf{A})$) or inconsistent ($\mathbf{b} \notin R(\mathbf{A})$), is always the minimizer to a consistent problem $\mathbf{A}\mathbf{u} = \mathbf{P}_{R(A)}\mathbf{b}$ [6, Ben-Israel and Greville 2003].

5.2 Generalized discrete Green's function

Considering the form of $\mathbf{b} = (f_0, f_1, \ldots, f_{n-2}, g_1, g_2)^{\top} \in \mathbb{C}^{(n+1)\times 1}$ with every $\mathbf{f} = (f_0, f_1, \ldots, f_{n-2})^{\top} \in \mathbb{C}^{(n-1)\times 1}$ and complex numbers g_1, g_2, we write the minimum norm least squares solution (5.4) in the following special form

$$\mathbf{u}^o = \mathbf{G}^g\mathbf{f} + g_1\mathbf{v}^{g,1} + g_2\mathbf{v}^{g,2}. \tag{5.10}$$

Here $\mathbf{G}^g \in \mathbb{C}^{(n+1)\times(n-1)}$ and $\mathbf{v}^{g,1}, \mathbf{v}^{g,2} \in \mathbb{C}^{(n+1)\times 1}$ are submatrices of the Moore–Penrose inverse

$$\mathbf{A}^\dagger = (\mathbf{G}^g,\ \mathbf{v}^{g,1},\ \mathbf{v}^{g,2}).$$

We also get the discrete representation of the minimum norm least squares solution (5.9)

$$u^o = G^g f + g_1 v^{g,1} + g_2 v^{g,2},$$

which can be considered in the explicit form

$$u_i^o = \sum_{j=0}^{n-2} G_{ij}^g f_j + g_1 v_i^{g,1} + g_2 v_i^{g,2}, \quad i \in X_n. \tag{5.11}$$

This representation of the unique minimizer $u^o \in F(X_n)$ is so literally similar to the representation of the unique solution (4.3) for the particular case, investigated in Section 4. Furthermore, formulas (4.3) and (5.11) are coincident if $\Delta \neq 0$. Thus, we call the discrete kernel $G^g \in F(X_n \times X_{n-2})$ by the *generalized discrete Green's function* and functions $v^{g,1}, v^{g,2} \in F(X_n)$ – the *generalized discrete biorthogonal fundamental system* for the problem (3.1)–(3.2) [100, Roman 2011].

Moreover, we can always calculate the generalized discrete Green's function and the generalized discrete biorthogonal fundamental system using the Moore–Penrose inverse $\mathbf{B} = \mathbf{A}^\dagger$ as given below

$$G_{ij}^g = B_{ij}, \quad i \in X_n,\ j \in X_{n-2}, \tag{5.12}$$

$$v_i^{g,1} = B_{i,n-1}, \quad i \in X_n, \tag{5.13}$$

$$v_i^{g,2} = B_{in}, \quad i \in X_n. \tag{5.14}$$

Thus, for $\det \mathbf{A} \neq 0$, we have that $\mathbf{A}^\dagger = \mathbf{A}^{-1}$, the discrete minimum norm least squares solution $u^o \in F(X_n)$ is coincident with the unique discrete solution $u \in F(X_n)$, the generalized discrete Green's function G^g is coincident with the discrete Green's function G, the generalized discrete biorthogonal fundamental system $v^{g,1}, v^{g,2}$ is coincident with the discrete biorthogonal fundamental system v^1, v^2.

5.3 Properties of minimizers

In this subsection we investigate properties of minimum norm least squares solutions and generalized discrete Green's functions. Obtained results are similar to corresponding properties given in Section 4.

Lemma 3.10. *The generalized discrete Green's function $G^g \in F(X_n \times X_{n-2})$ is the minimum norm least squares solution of the following discrete problem*

$$\begin{aligned} \mathcal{L}_i . G^g_{\cdot j} &= \delta_{ij}, \quad i \in X_{n-2}, \\ \langle L^{\cdot}_k, G^g_{\cdot j}\rangle &= 0, \quad k = 1, 2, \end{aligned} \tag{5.15}$$

for every fixed $j \in X_{n-2}$.

Proof. The minimum norm least squares solution of problem (3.1)–(3.2) is described by the formula (5.11). Let us choose $j \in X_{n-2}$ and values of the right hand side $\mathbf{f} = (\delta_{0j}, \delta_{1j}, \ldots, \delta_{n-2,j})^\top$ and $g_1 = g_2 = 0$. Then for a fixed $j \in X_{n-2}$, the form of the minimum norm least squares solution (5.11) simplifies as follows

$$u^o_i = \sum_{l=0}^{n-2} G^g_{il} f_l = \sum_{l=0}^{n-2} G^g_{il} \delta_{lj} = G^g_{ij}, \qquad i \in X_n.$$

So, for each fixed $j \in X_{n-2}$ generalized Green's function $G^g_{\cdot j}$ is the minimum norm least squares solution of the problem (5.15). □

Lemma 3.11. *Discrete functions $v^{g,1}$ and $v^{g,2}$ from $F(X_n)$ are minimum norm least squares solutions of corresponding discrete problems*

$$\begin{aligned} &\mathcal{L}v^{g,1} = 0, && \mathcal{L}v^{g,2} = 0, \\ &\langle L_1, v^{g,1}\rangle = 1,\ \langle L_2, v^{g,1}\rangle = 0, \quad && \langle L_1, v^{g,2}\rangle = 0,\ \langle L_2, v^{g,2}\rangle = 1. \end{aligned} \tag{5.16}$$

Proof. The minimum norm least squares solution of the problem (3.1)–(3.2) is described by formula (5.10). For this problem, let us choose $\mathbf{f} = \mathbf{0}$ and $g_1 = 1$, $g_2 = 0$. Then from the formula (5.10) follows that $\mathbf{v}^{g,1}$ is the minimum norm least squares solution of the first problem (5.12). Afterwards choosing $\mathbf{f} = \mathbf{0}$ and $g_1 = 0$, $g_2 = 1$, we obtain similarly that $\mathbf{v}^{g,2}$ is the minimum norm least squares solution of the other problem (5.16). □

Let us now investigate two relative problems (4.9), where the first problem has the unique solution, i.e., the condition $\widetilde{\Delta} \neq 0$ is valid. Here and further $G^g \in F(X_n \times X_{n-2})$ is the generalized discrete Green's function and $v^{g,1}, v^{g,2} \in F(X_n)$ are the generalized biorthogonal fundamental system to the second problem (4.9), which may have the unique solution ($\Delta \neq 0$) or not ($\Delta = 0$).

Theorem 3.12. *If the first discrete problem* (4.9) *has the unique exact solution $u \in F(X_n)$, then the minimum norm least squares solution $u^o \in F(X_n)$ of the other problem* (4.9) *is given by*

$$u^o = u - P_{N(A)}u + v^{g,1}(g_1 - \langle L_1, u\rangle) + v^{g,2}(g_2 - \langle L_2, u\rangle).$$

Proof. Let us take the difference $w = u^o - u$ between the minimum norm least squares solution $u^o \in F(X_n)$ of the second problem (4.9) and the unique exact solution $u \in F(X_n)$ of the first problem of (4.9). Now we will show that w is the least squares solution to the problem

$$\mathcal{L}w = 0, \quad \langle L_k, w\rangle = g_k - \langle L_k, u\rangle, \quad k = 1, 2, \tag{5.17}$$

which can be represented in the equivalent matrix form $\mathbf{A}\mathbf{w} = \widetilde{\mathbf{b}}$ with the right hand side $\widetilde{\mathbf{b}} = (0, 0, \ldots, 0, g_1 - \mathbf{L}_1\mathbf{u}, g_2 - \mathbf{L}_2\mathbf{u})^\top$. Since $\mathbf{u}^o$ is the minimum norm least squares solution to the system (3.3) with the right hand side $\mathbf{b} = (f_0, f_1, \ldots, f_{n-2}, g_1, g_2)^\top$, then the inequality (5.6) is always valid, i.e.,

$$\|\mathbf{A}\mathbf{u}^o - \mathbf{b}\| \leqslant \|\mathbf{A}\mathbf{v} - \mathbf{b}\| \tag{5.18}$$

for every $\mathbf{v} \in \mathbb{C}^{(n+1)\times 1}$. Now we rewrite the norm as follows $\|\mathbf{A}\mathbf{v} - \mathbf{b}\| = \|\mathbf{A}\mathbf{v} + (\mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{u}) - \mathbf{b}\| = \|\mathbf{A}(\mathbf{v} - \mathbf{u}) - (-\mathbf{A}\mathbf{u} + \mathbf{b})\| = \|\mathbf{A}(\mathbf{v} - \mathbf{u}) - \widetilde{\mathbf{b}}\|$ for all $\mathbf{v} \in \mathbb{C}^{(n+1)\times 1}$. From here we have

$$\|\mathbf{A}(\mathbf{u}^o - \mathbf{u}) - \widetilde{\mathbf{b}}\| \leqslant \|\mathbf{A}(\mathbf{v} - \mathbf{u}) - \widetilde{\mathbf{b}}\|, \qquad \forall\, \mathbf{v} \in \mathbb{C}^{(n+1)\times 1}$$

or, denoting $\mathbf{z} = \mathbf{v} - \mathbf{u}$, obtain

$$\|\mathbf{A}\mathbf{w} - \widetilde{\mathbf{b}}\| \leqslant \|\mathbf{A}\mathbf{z} - \widetilde{\mathbf{b}}\|, \qquad \forall\, \mathbf{z} \in \mathbb{C}^{(n+1)\times 1}$$

So, $\mathbf{w}$ is a least squares solution of the problem $\mathbf{A}\mathbf{w} = \widetilde{\mathbf{b}}$ and, according to the formula (5.8), it describes the minimizer $\mathbf{w}^o = \mathbf{A}^\dagger\widetilde{\mathbf{b}}$ in the form $\mathbf{w}^o = \mathbf{P}_{N(A)^\perp}\mathbf{w}$. Using the composition $\mathbf{w} = \mathbf{P}_{N(A)^\perp}\mathbf{w} + \mathbf{P}_{N(A)}\mathbf{w}$, we get $\mathbf{w} = \mathbf{A}^\dagger\widetilde{\mathbf{b}} + \mathbf{P}_{N(A)}\mathbf{w}$. Now we recall the equality $\mathbf{w} = \mathbf{u}^o - \mathbf{u}$ and obtain

$$\mathbf{u}^o = \mathbf{u} + \mathbf{A}^\dagger\widetilde{\mathbf{b}} + \mathbf{P}_{N(A)}\mathbf{w}. \tag{5.19}$$

Here we take the composition $\mathbf{u} = \mathbf{P}_{N(A)^\perp}\mathbf{u} + \mathbf{P}_{N(A)}\mathbf{u}$ and rewrite the representation (5.19) into the form $\mathbf{u}^o = \mathbf{P}_{N(A)^\perp}\mathbf{u} + \mathbf{A}^\dagger\widetilde{\mathbf{b}} + \mathbf{P}_{N(A)}\mathbf{u}^o$, because $\mathbf{w} + \mathbf{u} = \mathbf{u}^o$. From (5.9), we have $\mathbf{u}^o \in N(\mathbf{A})^\perp$. Then $\mathbf{P}_{N(A)}\mathbf{u}^o = \mathbf{0}$ and the previous representation simplifies to $\mathbf{u}^o = \mathbf{P}_{N(A)^\perp}\mathbf{u} + \mathbf{A}^\dagger\widetilde{\mathbf{b}}$. Rewriting it into the extended form

$$\mathbf{u}^o = \mathbf{u} - \mathbf{P}_{N(A)}\mathbf{u} + (g_1 - \mathbf{L}_1\mathbf{u})\mathbf{v}^{g,1} + (g_2 - \mathbf{L}_2\mathbf{u})\mathbf{v}^{g,2},$$

we obtain the statement of this theorem. □

Remark 3.13. Let us note that, for $\det \mathbf{A} \neq 0$, the nullspace $N(\mathbf{A})$ is trivial and the orthogonal projector $\mathbf{P}_{N(A)} = \mathbf{O}$ is the zero matrix. If $\det \mathbf{A} = 0$, then the nullspace $N(\mathbf{A})$ has an orthonormal basis $\mathbf{z}^l$, $l = \overline{1,d}$, and the orthogonal projector is represented by the following matrix

$$\mathbf{P}_{N(A)} = \sum_{l=1}^{d} \mathbf{z}^l (\mathbf{z}^l)^*.$$

Further, we provide the representation of the minimum norm least squares solution, which is always applicable.

Corollary 3.14. *The minimum norm least squares solution u^o to the problem* (3.1)–(3.2) *can always be represented by the unique solution u^c to the Cauchy problem* (3.4) *as follows*

$$u^o = u^c - P_{N(A)}u^c + (g_1 - \langle L_1, u^c\rangle)v^{g,1} + (g_2 - \langle L_2, u^c\rangle)v^{g,2}.$$

Proof. It follows from Theorem 3.12 since the Cauchy problem (3.4) always has the unique solution u^c. □

Now we present the property of generalized discrete biorthogonal fundamental systems for problems (4.9), which is similar to Corollary 1.10.

Corollary 3.15. *Let $\widetilde{\Delta} \neq 0$ for the first problem* (4.9). *Then the discrete biorthogonal fundamental system $\widetilde{v}^1, \widetilde{v}^2 \in F(X_n)$ of the first problem and the generalized discrete biorthogonal fundamental system $v^{g,1}, v^{g,2} \in F(X_n)$ of the second problem* (4.9) *are related by*

$$\begin{pmatrix} \langle L_1, \widetilde{v}^1\rangle & \langle L_2, \widetilde{v}^1\rangle \\ \langle L_1, \widetilde{v}^2\rangle & \langle L_2, \widetilde{v}^2\rangle \end{pmatrix} \begin{pmatrix} v^{g,1} \\ v^{g,2} \end{pmatrix} = \begin{pmatrix} P_{N(A)^\perp}\widetilde{v}^1 \\ P_{N(A)^\perp}\widetilde{v}^2 \end{pmatrix}.$$

Proof. The proof is coincident with the proof of Corollary 1.21 from Chapter 1 but here we apply Theorem 3.12. □

5.4 Relations between generalized discrete Green's functions

Here we discuss on representations of a generalized discrete Green's functions.

Theorem 3.16. *If $\widetilde{\Delta} \neq 0$ for the first problem* (4.9), *then its discrete Green's function $G \in F(X_n \times X_{n-2})$ and the generalized discrete Green's function $G^g \in F(X_n \times X_{n-2})$ of the second problem* (4.9) *are related by the equality*

$$G^g_{ij} = G_{ij} - (P_{N(A)})_{i\cdot}G_{\cdot j} - v^{g,1}_i\langle L_1, G_{\cdot j}\rangle - v^{g,2}_i\langle L_2, G_{\cdot j}\rangle, \quad i \in X_n,\ j \in X_{n-2}.$$

Proof. For every fixed $j \in X_{n-2}$, let us investigate two discrete problems (4.7) and (5.15). Their solutions are $u = \mathrm{col}_j G$ and $v = \mathrm{col}_j G^g$, respectively. Then according to Theorem 3.12, they are related as given

$$\mathrm{col}_j G^g = \mathrm{col}_j G - P_{N(A)} \mathrm{col}_j G - v^{g,1} \langle L_1, \mathrm{col}_j G \rangle - v^{g,2} \langle L_2, \mathrm{col}_j G \rangle.$$

Rewriting in the extended form, we obtain the statement of this theorem. □

If the second problem (4.9) has the discrete Green's function as well, then Theorem 3.16 simplifies to the result from [100, Roman 2011], which is formulated in Theorem 3.7. The author also obtained the explicit representation of the discrete Green's function (see Lemma 3.5) using the discrete Green's function G^c of the initial problem (3.4), since it always exists. Now we present the extension of this result to the generalized discrete Green's function.

Corollary 3.17. *The generalized discrete Green's function* $G^g \in F(X_n \times X_{n-2})$ *to the problem* (3.1)–(3.2) *is always given by*

$$G^g_{ij} = G^c_{ij} - (P_{N(A)})_{i\cdot} G^c_{\cdot j} - v^{g,1}_i \langle L^{\cdot}_1, G^c_{\cdot j} \rangle - v^{g,2}_i \langle L^{\cdot}_2, G^c_{\cdot j} \rangle, \quad i \in X_n, \ j \in X_{n-2}.$$

Proof. Since every second order discrete initial problem (3.1)–(3.2) has the discrete Green's function (3.5), the statement of this corollary follows from Theorem 3.16 with $G = G^c$. □

Since the condition $\Delta \neq 0$ is equivalent to the inequality $\det \mathbf{A} \neq 0$, the discrete problem (3.3) has a nonsingular matrix and the orthogonal projector $\mathbf{P}_{N(A)} = \mathbf{O}$ is the zero matrix. So, we note that all statements, proved in this section for a generalized discrete Green's function G^g, a generalized discrete biorthogonal fundamental system $v^{g,1}, v^{g,2}$ and the minimum norm least squares solution u^o, are coincident with corresponding statements that are formulated in Section 4 for a discrete Green's function G, a discrete biorthogonal fundamental system v^1, v^2 and the unique solution u if the condition $\Delta \neq 0$ is satisfied.

5.5 Applications to nonlocal boundary conditions

For the problem with nonlocal boundary conditions (4.10)–(4.11), we obtain the following representation of the minimizer u^o using the unique solution u^{cl} to the classical problem ($\gamma_1, \gamma_2 = 0$).

Corollary 3.18. *If the classical problem* (4.10)–(4.11) *(*$\gamma_1, \gamma_2 = 0$*) has the unique solution* $u^{\mathrm{cl}} \in F(X_n)$*, then the minimizer to the nonlocal boundary value problem* (5.10)–(5.11) *is given by*

$$u^o = u^{\mathrm{cl}} - P_{N(A)} u^{\mathrm{cl}} + \gamma_1 \langle \varkappa_1, u^{\mathrm{cl}} \rangle v^{g1} + \gamma_2 \langle \varkappa_2, u^{\mathrm{cl}} \rangle v^{g2}.$$

Proof. We obtain this corollary applying Theorem 3.12 with $\langle L_k, u^{\mathrm{cl}} \rangle = g_k - \gamma_k \langle \varkappa_k, u^{\mathrm{cl}} \rangle$, since u^{cl} satisfies conditions $\langle \kappa_k, u^{\mathrm{cl}} \rangle = g_k,\ k = 1, 2.$ □

The generalized discrete Green's function for the problem with nonlocal boundary conditions (4.10)–(4.11) can also be similarly described.

Corollary 3.19. *If the classical problem* (4.10)–(4.11) *(*$\gamma_1, \gamma_2 = 0$*) has the discrete Green's function* $G^{\mathrm{cl}} \in F(X_n \times X_{n-2})$*, then the generalized Green's function of the nonlocal problem* (4.10)–(4.11) *is of the form*

$$G^g_{ij} = G^{\mathrm{cl}}_{ij} - (P_{N(A)})_{i\cdot} G^{\mathrm{cl}}_{\cdot j} + \gamma_1 v^{g,1}_i \langle \varkappa^{\cdot}_1, G^{\mathrm{cl}}_{\cdot j} \rangle + \gamma_2 v^{g,2}_i \langle \varkappa^{\cdot}_2, G^{\mathrm{cl}}_{\cdot j} \rangle,\ i \in X_n,\ j \in X_{n-2}.$$

Proof. It follows from Theorem 3.16, since $\langle \kappa^{\cdot}_k, G^{\mathrm{cl}}_{\cdot j} \rangle = 0$ and $\langle L^{\cdot}_k, G^{\mathrm{cl}}_{\cdot j} \rangle = -\gamma_k \langle \varkappa^{\cdot}_k, G^{\mathrm{cl}}_{\cdot j} \rangle$ for $k = 1, 2$. □

Example 3.20. *Let us now recall Example 3.4, where we approximated the differential problem*

$$-u'' = f(x), \quad x \in [0, 1], \tag{5.20}$$

$$u(0) = g_1, \qquad u(1) - \gamma u(\xi) = g_2 \tag{5.21}$$

by the real discrete second order problem

$$(\mathcal{L}u)_i := -\frac{1}{h^2} u_{i+2} + \frac{2}{h^2} u_{i+1} - \frac{1}{h^2} u_i = f_i, \quad i \in X_{n-2}, \tag{5.22}$$

$$\langle L_1, u \rangle := u_0 = g_1, \qquad \langle L_2, u \rangle := u_n - \gamma u_s = g_2. \tag{5.23}$$

Let us take $\gamma = 1/\xi$*. It gives* $\Delta = 0$*, where the discrete problem* (5.22)–(5.23) *has neither the unique exact solution nor the discrete Green's function. So, now we are going to consider its minimizer and obtain the representation of the generalized discrete Green's function.*

Indeed, we have the following expression of the minimum norm least squares solution

$$u^o_i = \sum_{i=0}^{n-2} G^g_{ij} f_j + g_1 v^{g,1}_i + g_2 v^{g,2}_i, \qquad i \in X_n, \tag{5.24}$$

described by the generalized discrete Green's function $G^g \in F(X_n \times X_{n-2})$*. Let us note [100, Roman 2011] that the problem* (5.22)–(5.23) *with classical conditions* ($\gamma = 0$) *only always has the discrete Green's function*

$$G^{\mathrm{cl}}_{ij} = h\begin{cases} x_{j+1}(1-x_i), & x_{j+1} \leqslant x_i, \\ x_i(1-x_{j+1}), & x_i \leqslant x_{j+1}, \end{cases} \qquad i \in X_n,\ j \in X_{n-2}. \tag{5.25}$$

Thus, according to Corollary 3.19, we get the following expression of the generalized discrete Green's function

$$G^g_{ij} = G^{\mathrm{cl}}_{ij} - (P_{N(A)})_{i\cdot}G^{\mathrm{cl}}_{\cdot j} + \gamma v^{g,2}_i G^{\mathrm{cl}}_{sj}, \qquad i \in X_n,\ j \in X_{n-2}. \tag{5.26}$$

If $\Delta \neq 0$, *then this representation simplifies to the description of the discrete Green's function*

$$G_{ij} = G^{\mathrm{cl}}_{ij} + \gamma v^2_i G^{\mathrm{cl}}_{sj}, \qquad i \in X_n,\ j \in X_{n-2}.$$

Indeed, here $\mathbf{P}_{N(A)} = \mathbf{O}$ *is the zero matrix and* $v^{g,2} = x/(1-\gamma\xi),\ x \in \overline{\omega}^h$, *is the unique exact solution to the problem*

$$\mathcal{L}u = 0, \qquad \langle L_1, u\rangle = 0, \qquad \langle L_2, u\rangle = 1.$$

Another situation $\Delta = 0$ *is different. Here the nullity* $d = \dim N(\mathbf{A}) = 1$ *(recall Example 3.4) and* $\mathbf{x} \in N(\mathbf{A})$ *generates the nullspace. According to Remark 3.13, we know the orthogonal projector* $\mathbf{P}_{N(A)} = \mathbf{x}\mathbf{x}^\top/\|\mathbf{x}\|^2$ *and the projection of the discrete Green's function* $\mathbf{P}_{N(A)}\mathbf{G}^{\mathrm{cl}} = \mathbf{x}\mathbf{x}^\top\mathbf{G}^{\mathrm{cl}}/\|\mathbf{x}\|^2$, *that simplifies to*

$$(P_{N(A)})_{i\cdot}G^{\mathrm{cl}}_{\cdot j} = \frac{x_i}{\|\mathbf{x}\|^2}\sum_{l=0}^{n} x_l G^{\mathrm{cl}}_{lj} = \frac{h}{(1+h)(2+h)} \cdot x_i x_{j+1}(1-x^2_{j+1}).$$

To obtain the full representation of the generalized discrete Green's function (5.26), *we need to find the discrete function* $v^{g,2}$. *It is the unique minimizer to the problem* $\mathbf{A}\mathbf{u} = \mathbf{e}^n$ *as well to the consistent problem* $\mathbf{A}\mathbf{u} = \mathbf{P}_{R(A)}\mathbf{e}^n$. *Indeed, taking the matrix representation of the minimizer* $\mathbf{v}^{g,2} = \mathbf{A}^\dagger\mathbf{e}^n$, *we get* $\mathbf{A}\mathbf{v}^{g,2} = \mathbf{A}\mathbf{A}^\dagger\mathbf{e}^n = \mathbf{P}_{R(A)}\mathbf{e}^n$. *Here we used the property* $\mathbf{A}\mathbf{A}^\dagger = \mathbf{P}_{R(A)}$ *[6, Ben-Israel and Greville 2003].*

According to Example 3.4, the nullspace $N(\mathbf{A}^*)$ *is generated by the vector*

$$\mathbf{w} = \gamma\sum_{j=0}^{n-2} G^{\mathrm{cl}}_{sj}\mathbf{e}^j + (\gamma-1)\mathbf{e}^{n-1} + \mathbf{e}^n. \tag{5.27}$$

Since $\mathbf{P}_{R(A)} = \mathbf{P}_{N(A^*)^\perp}$, *then* $\mathbf{P}_{R(A)}\mathbf{e}^n = \mathbf{e}^n - \mathbf{P}_{N(A^*)}\mathbf{e}^n = \mathbf{e}^n - \mathbf{w}/\|\mathbf{w}\|^2$ *or*

$$\mathbf{P}_{R(A)}\mathbf{e}^n = \frac{1}{\|\mathbf{w}\|^2}\Big(-\gamma\sum_{j=0}^{n-2} G^{\mathrm{cl}}_{sj}\mathbf{e}^j + (1-\gamma)\mathbf{e}^{n-1} + (\|\mathbf{w}\|^2-1)\mathbf{e}^n\Big)$$

with the denominator $\|\mathbf{w}\|^2 = \gamma^2 \sum_{j=0}^{n-2} (G_{sj}^{\mathrm{cl}})^2 + (1-\gamma)^2 + 1 = 1 + (\gamma - 1)^2(h\xi^2+3)/3 + (\gamma-1)^2 h^3/6$ *converging to* $(\gamma-1)^2 = (\xi-1)^2/\xi^2$ *if* $h \to 0$.

Now we can solve the consistent problem $\mathbf{Au} = \mathbf{P}_{R(A)}\mathbf{e}^n$, *that is,*

$$(\mathcal{L}u)_i = -\gamma G_{si}^{\mathrm{cl}}/\|\mathbf{w}\|^2, \quad i \in X_{n-2}, \tag{5.28}$$

$$u_0 = (1-\gamma)/\|\mathbf{w}\|^2, \tag{5.29}$$

$$u_n - \gamma u_s = 1 - 1/\|\mathbf{w}\|^2. \tag{5.30}$$

First, we obtain a general solution of the discrete equation (5.28), *i.e.,*

$$u_i = c_1 + c_2 x_i - \frac{\gamma}{\|\mathbf{w}\|^2} \sum_{j=0}^{n-2} G_{ij}^{\mathrm{cl}} G_{sj}^{\mathrm{cl}}, \qquad i \in X_n.$$

Substituting it into the equation (5.29), *we use conditions* $\langle L_1, G_{\cdot j}^{\mathrm{cl}} \rangle := G_{0j}^{\mathrm{cl}} = 0$ *and find the general least squares solution*

$$u_i^g = \frac{1-\gamma}{\|\mathbf{w}\|^2} + c x_i - \frac{\gamma}{\|\mathbf{w}\|^2} \sum_{j=0}^{n-2} G_{ij}^{\mathrm{cl}} G_{sj}^{\mathrm{cl}}, \quad i \in X_n, \quad c \in \mathbb{R}.$$

Here the equation (5.30) *is not used to find the constant* c *because* $d = 1$ *and this equation is a linear combination of previous equations. This condition* (5.30) *for the obtained* u^g *expression is satisfied trivially. Since the minimum norm least squares solution* $v^{g,2}$ *is the unique function from* u^g, *we have*

$$v_i^{g,2} = \frac{1-\gamma}{\|\mathbf{w}\|^2} + c^o x_i - \frac{\gamma}{\|\mathbf{w}\|^2} \sum_{j=0}^{n-2} G_{ij}^{\mathrm{cl}} G_{sj}^{\mathrm{cl}}, \quad i \in X_n,$$

with the particular constant

$$\begin{aligned} c^o &= -\frac{1}{\|\mathbf{x}\|^2}\mathbf{x}^\top \mathbf{u}^g = \frac{3}{(2+h)\|\mathbf{w}\|^2} \cdot \left(\gamma - 1 + \gamma \sum_{i=0}^{n}\sum_{j=0}^{n-2} x_i G_{ij}^{\mathrm{cl}} G_{sj}^{\mathrm{cl}}\right) \\ &= \frac{1}{40(2+h)} \cdot \frac{1}{6+(\gamma-1)^2(h^2+2h\xi^2+6)} \Big(-12\xi^4 + (7-18h)\xi^3 \\ &\quad + 2(5h^2 - 9h - 10)\xi^2 + 3h^2\xi + 30h^3 - 5h^2 + 30h - 338 + 360\gamma \Big). \end{aligned}$$

We found it calculating the projection $\mathbf{v}^{g,2} = \mathbf{P}_{N(A)^\perp}\mathbf{u}^g = \mathbf{u}^g - \mathbf{P}_{N(A)}\mathbf{u}^g = \mathbf{u}^g - \mathbf{x}\mathbf{x}^\top \mathbf{u}^g/\|\mathbf{x}\|^2$. *Simplifying we obtain*

$$\begin{aligned} v_i^{g,2} = &\frac{\xi}{6\xi^2 + (\xi-1)^2(2h\xi^2+6+h^2)} \Bigg(6\xi \cdot c^o \cdot \|\mathbf{w}\|^2 x \\ &+ 6(1-\xi) - h \begin{cases} \xi(1-\xi)(2-\xi-2h)x_i + (\xi-1)x_i^3, & x_i < \xi, \\ \xi h^2 - \xi^3 + \xi(2+\xi^2-h^2)x_i - 3\xi x_i^2 + \xi x_i^3, & x_i \geqslant \xi \end{cases} \Bigg). \end{aligned}$$

Substituting the obtained v^{g2} expression and representations of the discrete Green's function G^{cl} with $(P_{N(A)})_{i\cdot}G^{cl}_{\cdot j}$ into (5.26)*, we know the full description of the generalized discrete Green's function G^g, that is given below*

$$G^g_{ij} = h\begin{cases} x_{j+1}(1-x_i) & x_{j+1} \leqslant x_i, \\ x_i(1-x_{i+1}), & x_{j+1} \geqslant x_i, \end{cases} - \frac{h}{(1+h)(2+h)} \cdot x_i x_{j+1}(1-x^2_{j+1})$$
$$+ \frac{h}{6\xi^2+(\xi-1)^2(2h\xi^2+6+h^2)}\Bigg(6\xi\cdot c^o\cdot\|\mathbf{w}\|^2 x + 6(\xi-1)$$
$$-h\begin{cases}\xi(1-\xi)(2-\xi-2h)x_i+(\xi-1)x_i^3, & x_i<\xi, \\ \xi h^2-\xi^3+\xi(2+\xi^2-h^2)x_i-3\xi x_i^2+\xi x_i^3, & x_i \geqslant \xi\end{cases}\Bigg)\times$$
$$\times\begin{cases} x_{j+1}(1-\xi), & x_{j+1}\leqslant \xi, \\ \xi(1-x_{j+1}), & x_{j+1}\geqslant\xi.\end{cases}$$

To get the full representation of the minimizer (5.24)*, we still need to find the discrete function $v^{g,1}$. We take the biorthogonal fundamental system $\widetilde{v}^1 = 1-x$, $\widetilde{v}^2 = x$, $x \in \overline{\omega}^h$, of the classical problem ($\gamma = 0$) and apply Corollary 3.15. From there we get the equality $v^{g1} + (1-\gamma)v^{g2} = P_{N(A)^\perp}\widetilde{v}^1$. Since $P_{N(A)^\perp}\widetilde{v}^1 = \widetilde{v}^1 - P_{N(A)}\widetilde{v}^1 = 1 - 3x/(2+h)$, we have the following expression*

$$v^{g1} = (\gamma-1)v^{g2} + 1 - \frac{3}{2+h}x.$$

Substituting the obtain representation of v^{g2}, we find v^{g1} and know all representations of G^g, $v^{g,1}$ and $v^{g,2}$. So, now we can always calculate the minimum norm least squares solution u^o with every right hand side by the formula (5.24)*.*

Let us now recall the minimum norm least squares solution for the differential problem (5.20)–(5.21)*, investigated in Subsection 5.1 from Chapter 1. However, comparing corresponding expressions, we obtain no convergence. Indeed, the representation $v^{g,2}_i$ above does not converge to the function $v^{g,2}(x)$. Another discrete functions $v^{g,1}_i$ and G^g_{ij} do not converge to functions $v^{g,1}(x)$ and $G^g(x,y)$, too. Why there is no convergence of the discrete minimizer* (5.24) *to the minimizer of the differential problem* (5.20)–(5.21)*?*

The answer is as follows. First, the form of the minimum norm least squares solution (5.24) *as well as the unique solution*

$$u_i = \sum_{i=0}^{n-2} G_{ij}f_j + g_1 v^1_i + g_2 v^2_i, \qquad i \in X_n, \tag{5.31}$$

if it exists, is not applied to investigate the convergence. Since the solution of the differential problem (1.1)–(1.2) *is given by*

$$u(x) = \int_0^1 G(x,y)f(y)\,dy + g_1 v^1(x) + g_2 v^2(x), \qquad x \in [0,1], \tag{5.32}$$

(consider the analogical representation for the minimizer!), we need to take the representation of the solution (as well as the minimizer) in the form, which is compatible with the representation (5.32)*. Rewriting the representation* $G_{ij}f_j = G_{ij}f(x_{j+1}) = G_{ij}^h f(x_{j+1})h$ *with the modified discrete Green's function* $G_{ij}^h = h^{-1} \cdot G_{ij}$*, we obtain the correct form*

$$u_i = \sum_{i=0}^{n-2} G_{ij}^h f(x_{j+1})\, h + g_1 v_i^1 + g_2 v_i^2, \qquad i \in X_n.$$

Similarly, we consider the minimizer

$$u_i^o = \sum_{i=0}^{n-2} G_{ij}^{g,h} f(x_{j+1})\, h + g_1 v_i^{g,1} + g_2 v_i^{g,2}, \qquad i \in X_n. \tag{5.33}$$

Second, we have just solved the minimization problem (5.6)–(5.7) *using the standard Euclidean norm in both minimization steps. However, the minimum norm least squares solution of the differential problem* (1.1)–(1.2) *has the smallest* $H^2[0,1]$ *norm among all functions minimizing the* $L^2[0,1] \times \mathbb{R}^2$ *norm of the residual* $\boldsymbol{L}u - \boldsymbol{f}$*. Thus, the standard Euclidean norm was not compatible to investigate the convergence.*

So, now we introduce the two different norms

$$\|\mathbf{u}\|_{H^2(\overline{\omega}^h)} = \left(\sum_{i=0}^{n} u_i^2 h + \sum_{i=1}^{n} \left(\frac{u_i - u_{i-1}}{h}\right)^2 h + \sum_{i=1}^{n-1} \left(\frac{u_{i+1} - u_i + u_{i-1}}{h^2}\right)^2 h\right)^{1/2},$$

$$\|\mathbf{b}\|_{L^2(\omega^h)\times\mathbb{R}^2} = \left(\sum_{i=1}^{n-1} f^2(x_i)h + g_1^2 + g_2^2\right)^{1/2},$$

for every $\mathbf{u},\ \mathbf{b} \in \mathbb{R}^{(n+1)\times 1}$*, and the corresponding inner products those hold the equality* $\|\mathbf{u}\| = (\mathbf{u},\mathbf{u})^{1/2}$*. Let us note that here* $\mathbf{b}$ *is not necessary of the special form* $\mathbf{b} = \big(f(x_1), f(x_2), \ldots, f(x_{n-1}), g_1, g_2\big)^\top \in \mathbb{R}^{(n+1)\times 1}$*, which only helps to illustrate the similarity to the inner product in the space* $L^2[0,1]\times\mathbb{R}^2$*.*

Thus, instead of the minimization problem (5.6)–(5.7)*, now we minimize the* $L^2(\omega^h) \times \mathbb{R}^2$ *norm of the residual*

$$\|\mathbf{A}\mathbf{u}^g - \mathbf{b}\|_{L^2(\omega^h)\times\mathbb{R}^2} \leqslant \|\mathbf{A}\mathbf{u} - \mathbf{b}\|_{L^2(\omega^h)\times\mathbb{R}^2}, \quad \forall\ \mathbf{u} \in \mathbb{R}^{(n+1)\times 1}, \tag{5.34}$$

and look for the unique minimizer $\mathbf{u}^o \in \mathbb{R}^{(n+1)\times 1}$*, which has the smallest norm*

$$\|\mathbf{u}^o\|_{H^2(\overline{\omega}^h)} < \|\mathbf{u}^g\|_{H^2(\overline{\omega}^h)}, \qquad \forall\ \mathbf{u}^g \neq \mathbf{u}^o. \tag{5.35}$$

Let us note that the new minimization problem indeed has the unique minimizer u^o *since it can be reduced to the previous minimization problem*

(5.6)–(5.7). *Precisely, we need to minimize the norm*

$$\|\mathbf{A}\mathbf{u}-\mathbf{b}\|^2_{L^2(\omega^h)\times\mathbb{R}^2} = \sum_{j=0}^{n-2}\big((\mathcal{L}u)_i - f(x_{i+1})\big)^2 h + \sum_{k=1}^{2}(g_k - \langle L_k, u\rangle)^2$$

$$= \sum_{j=0}^{n-2}\big(h^{1/2}(\mathcal{L}u)_i - h^{1/2} f(x_{i+1})\big)^2 + (g_1 - \langle L_1, u\rangle)^2 + (g_2 - \langle L_2, u\rangle)^2$$

$$= \sum_{j=0}^{n-2}\big((\widetilde{\mathcal{L}}u)_i - \widetilde{f}_i\big)^2 + (g_1 - \langle L_1, u\rangle)^2 + (g_2 - \langle L_2, u\rangle)^2 = \|\widetilde{\mathbf{A}}\mathbf{u} - \widetilde{\mathbf{b}}\|^2,$$

which represents the standard Euclidean norm of the residual of the second order discrete problem

$$(\widetilde{\mathcal{L}}u)_i := h^{1/2}(\mathcal{L}u)_i = \widetilde{f}_i,\ i\in X_{n-2}, \quad \langle L_1, u\rangle = g_1, \quad \langle L_1, u\rangle = g_2, \quad (5.36)$$

also given in the matrix form $\widetilde{\mathbf{A}}\mathbf{u} = \widetilde{\mathbf{b}}$. *Here we denoted the output of the discrete operator* $\widetilde{\mathcal{L}}$ *by* $\widetilde{f}_i = h^{1/2} f(x_{i+1})$, $i \in X_{n-2}$. *As discussed in Section 5.1, there exists a set of functions* $\mathbf{u}^g \in \mathbb{R}^{(n+1)\times 1}$ *those minimize the standard Euclidean norm of the residual*

$$\|\widetilde{\mathbf{A}}\mathbf{u}^g - \widetilde{\mathbf{b}}\| \leqslant \|\widetilde{\mathbf{A}}\mathbf{u} - \widetilde{\mathbf{b}}\|, \quad \forall\ \mathbf{u}\in\mathbb{R}^{(n+1)\times 1} \quad (5.37)$$

and is of the form

$$\mathbf{u}^g = \widetilde{\mathbf{A}}^{\dagger}\widetilde{\mathbf{b}} + \mathbf{P}_{N(\widetilde{A})}\mathbf{c}, \qquad \mathbf{c}\in\mathbb{R}^{(n+1)\times 1}.$$

Minimizing the $H^2(\overline{\omega}^h)$ *norm of the general least squares solution* $\mathbf{u}^g$, *we find the desired minimizer* $\mathbf{u}^o$.

Let us first obtain the discrete function $v^{g,2} \in F(X_n)$ *for the minimization problem* (5.34)–(5.35) *of* $\mathbf{A}\mathbf{u} = \mathbf{e}^n$. *The vector* $\mathbf{v}^{g,2}$ *is also the minimizer to the problem* $\widetilde{\mathbf{A}}\mathbf{u} = \widetilde{\mathbf{e}}^n$ *as well as to the consistent problem* $\widetilde{\mathbf{A}}\mathbf{u} = \mathbf{P}_{R(\widetilde{A})}\widetilde{\mathbf{e}}^n$ *in the standard least squares sense* (5.6)–(5.7). *Since here* $\widetilde{\mathbf{e}}^n = \mathbf{e}^n$, *we consider the problem* $\widetilde{\mathbf{A}}\mathbf{u} = \mathbf{P}_{R(\widetilde{A})}\mathbf{e}^n$.

For the problem (5.36), *we have* $d = 1$, $k_1 = 2$, $k_2 = 1$ *and* $\mathbf{x}\in N(\widetilde{\mathbf{A}}) = N(\mathbf{A})$ *as previous but* $N(\widetilde{\mathbf{A}}^*)$ *is different to* $N(\mathbf{A}^*) = N(\mathbf{A}^\top)$. *Let us formulate the auxiliary problem* $\widetilde{\mathcal{L}}u = f$, $u_0 = 0$, $u_n = 0$ *with classical conditions* ($\gamma = 0$). *It has the discrete Green's function* $\widetilde{G}^{\mathrm{cl}}_{ij} = h^{1/2}\cdot G^{\mathrm{cl,h}}_{ij}$, *where*

$$G^{\mathrm{cl,h}}_{ij} = \begin{cases} x_{j+1}(1-x_i), & x_{j+1}\leqslant x_i,\\ x_i(1-x_{j+1}), & x_i \leqslant x_{j+1},\end{cases} \qquad i\in X_n,\ j\in X_{n-2}.$$

Using Corollary 3.2, we find the vector

$$\widetilde{\mathbf{w}} = \gamma\sum_{j=0}^{n-2}\widetilde{G}^{\mathrm{cl}}_{sj}\mathbf{e}^j + (\gamma-1)\mathbf{e}^{n-1} + \mathbf{e}^n$$

generating the nullspace $N(\widetilde{\mathbf{A}}^*)$. Now we calculate the projection

$$\mathbf{P}_{R(\widetilde{A})}\mathbf{e}^n = \mathbf{e}^n - \mathbf{P}_{N(\widetilde{A}^*)}\mathbf{e}^n = \mathbf{e}^n - \frac{(\widetilde{\mathbf{w}}, \mathbf{e}^n)}{\|\widetilde{\mathbf{w}}\|^2}\widetilde{\mathbf{w}},$$

that can be rewritten in the form

$$\mathbf{P}_{R(\widetilde{A})}\mathbf{e}^n = \frac{1}{\|\widetilde{\mathbf{w}}\|^2}\Big(-\gamma \sum_{j=0}^{n-2} \widetilde{G}^{\mathrm{cl}}_{sj}\mathbf{e}^j + (1-\gamma)\mathbf{e}^{n-1} + (1 - \|\widetilde{\mathbf{w}}\|^2)\mathbf{e}^n \Big).$$

Let us note that the denominator

$$\|\widetilde{\mathbf{w}}\|^2 = \gamma^2 \sum_{j=0}^{n-2} (\widetilde{G}^{\mathrm{cl}}_{sj})^2 + (1-\gamma)^2 + 1 = \gamma^2 \sum_{j=0}^{n-2} (G^{\mathrm{cl,h}}_{sj})^2 h + (1-\gamma)^2 + 1,$$

converges to the norm of the vector valued function $\boldsymbol{w} \in N(\boldsymbol{L}^*)$ obtained in Subsection 5.1 from Chapter 1, that is

$$\|\boldsymbol{w}\|^2 = \gamma^2 \int_0^1 \big(G^{\mathrm{cl}}(\xi, y)\big)^2\, dy + (1-\gamma)^2 + 1.$$

It is described by the Green's function

$$G^{\mathrm{cl}}(x,y) = \begin{cases} y(1-x), & y \leqslant x, \\ x(1-y), & y \leqslant x, \end{cases}$$

for the differential problem (5.20)–(5.21) *with classical conditions ($\gamma = 0$).*

Let us now solve the consistent problem $\widetilde{\mathbf{A}}\mathbf{u} = \mathbf{P}_{R(\widetilde{A})}\mathbf{e}^n$, which is given by

$$(\widetilde{\mathcal{L}}u)_i = -\gamma \widetilde{G}^{\mathrm{cl}}_{si}/\|\widetilde{\mathbf{w}}\|^2, \quad i \in X_{n-2}, \tag{5.38}$$

$$u_0 = (1-\gamma)/\|\widetilde{\mathbf{w}}\|^2, \tag{5.39}$$

$$u_n - \gamma u_s = 1 - 1/\|\widetilde{\mathbf{w}}\|^2. \tag{5.40}$$

First, we take a general solution of the discrete equation (5.38)

$$u_i = c_1 + c_2 x_i - \frac{\gamma}{\|\widetilde{\mathbf{w}}\|^2} \sum_{j=0}^{n-2} \widetilde{G}^{\mathrm{cl}}_{ij}\widetilde{G}^{\mathrm{cl}}_{sj} = c_1 + c_2 x_i - \frac{\gamma}{\|\widetilde{\mathbf{w}}\|^2} \sum_{j=0}^{n-2} G^{\mathrm{cl,h}}_{ij} G^{\mathrm{cl,h}}_{sj}\, h, \qquad i \in X_n.$$

Substituting it into the condition (5.39)*, we find the general least squares solution*

$$u^g_i = \frac{1-\gamma}{\|\widetilde{\mathbf{w}}\|^2} + c x_i - \frac{\gamma}{\|\widetilde{\mathbf{w}}\|^2} \sum_{j=0}^{n-2} G^{\mathrm{cl,h}}_{ij} G^{\mathrm{cl,h}}_{sj}\, h, \quad i \in X_n, \quad c \in \mathbb{R}.$$

However, the condition (5.40) *is not used to find the constant* c *since it is satisfied trivially for this* u^g *(here* $d = 1$*).*

Let us note that here obtained u^g *is the general least squares solution to* $\widetilde{\mathbf{A}}\mathbf{u} = \widetilde{\mathbf{e}}^n$*, i.e., it minimizes the standard Euclidean norm of the residual* $\widetilde{\mathbf{A}}\mathbf{u} - \widetilde{\mathbf{e}}^n$*. Together, it is the general least solution of the minimization problem* (5.34)*, i.e., it minimizes the* $L^2(\omega^h) \times \mathbb{R}^2$ *norm of the residual of the problem* $\mathbf{A}\mathbf{u} = \mathbf{e}^n$*. Our aim is to find the function* $v^{g,2}$*, which is the unique function from* u^g *of the minimum* $H^2(\overline{\omega}^h)$ *norm. So, minimizing the* $H^2(\overline{\omega}^h)$ *norm of* u^g *above, we find*

$$v_i^{g,2} = \frac{1-\gamma}{\|\widetilde{\mathbf{w}}\|^2} + c^{h,o} x_i - \frac{\gamma}{\|\widetilde{\mathbf{w}}\|^2} \sum_{j=0}^{n-2} G_{ij}^{\mathrm{cl,h}} G_{sj}^{\mathrm{cl,h}}\, h, \quad i \in X_n,$$

with the particular constant $c^{h,o} = c^o + O(h)$*, where* c^o *is the constant obtained to represent the minimizer*

$$v^{g2}(x) = \frac{1-\gamma}{\|\boldsymbol{w}\|^2} + c^o x - \frac{\gamma}{\|\boldsymbol{w}\|^2} \int_0^1 G^{\mathrm{cl}}(x,y) G^{\mathrm{cl}}(\xi, y)\, dy$$

of the differential problem (5.20)–(5.21)*, considered in Subsection 5.1 from Chapter 1. Letting* $h \to 0$*, we obtain that the discrete function* $v_i^{g,2}$ *converges to* $v^{g,2}(x_i)$*.*

Let us now find the discrete function $v^{g,1} \in F(X_n)$*, which is the minimizer to the problem* $\widetilde{\mathbf{A}}\mathbf{u} = \widetilde{\mathbf{e}}^{n-1}$ *as well as to the consistent problem* $\widetilde{\mathbf{A}}\mathbf{u} = \mathbf{P}_{R(\widetilde{A})} \widetilde{\mathbf{e}}^{n-1}$ *in the standard least squares sense* (5.6)–(5.6)*. Making similar calculations, we obtain the following representation*

$$v^{g,1} = (\gamma - 1) v^{g,2} - \frac{3(1+h)}{8 + 3h + h^2} x, \qquad x \in \overline{\omega}^h.$$

Here we again obtain the convergence of the discrete function $v^{g,1}$ *to the minimizer*

$$v^{g,1}(x) = (\gamma - 1) v^{g,2}(x) - \frac{3}{8} x, \qquad x \in [0,1],$$

of the differential problem (5.20)–(5.21)*.*

Now we present the generalized discrete Green's function

$$\widetilde{G}_{ij}^g = \widetilde{G}_{ij}^{\mathrm{cl}} - (P_{N(\widetilde{A})})_{i\cdot} \widetilde{G}_{\cdot j}^{\mathrm{cl}} + \gamma v_i^{g,2} \widetilde{G}_{sj}^{\mathrm{cl}}, \qquad i \in X_n,\ j \in X_{n-2},$$

for the problem (5.36)*, simply given by* $\widetilde{\mathbf{G}}^g = \widetilde{\mathbf{G}}^{\mathrm{cl}} - \mathbf{P}_{N(\widetilde{A})} \widetilde{\mathbf{G}}^{\mathrm{cl}} + \gamma \mathbf{v}^{g,2} \mathrm{row}_s \widetilde{\mathbf{G}}^{\mathrm{cl}}$*. This discrete Green's function represents the solution of the least squares problem* (5.37) *(that is equal to* (5.34)*), which has the minimum standard Euclidean norm. However, we our aim is to find the discrete Green's function*

$G^g \in F(X_n \times X_{n-2})$ describing the minimizer of the problem (5.37) as well but of the minimum $H^2(\overline{\omega}^h)$ norm.

According to Lemma 3.10, for every fixed $j \in X_{n-2}$, the discrete Green's function $\widetilde{G}^g$ is the minimum norm least squares solutions to the problem

$$(\widetilde{\mathcal{L}}u)_i = \delta_{ij}h^{1/2}, \quad i \in X_{n-2}, \qquad \langle L_k, u\rangle = 0, \quad k = 1, 2.$$

The general least squares solution to this problem is given by

$$U^g_{ij} = \widetilde{G}^g_{ij} + (P_{N(\widetilde{A})})_{i\cdot}C_{\cdot j}, \qquad i \in X_n,$$

with an arbitrary matrix $\mathbf{C} \in \mathbb{R}^{(n+1)\times(n+1)}$. It can be rewritten as below

$$U^g_{ij} = \widetilde{G}^g_{ij} + c_j x_i, \qquad i \in X_n,$$

with an arbitrary constant $c_j \in \mathbb{R}$ for every fixed $j \in X_{n-2}$. On the other hand, the discrete Green's function G^g represents the least squares solution to this problem as well but of the minimum $H^2(\overline{\omega}^h)$ norm. So, G^g is also the least squares solution to the same problem. Then these discrete Green's functions differ from each other with the particular constant c^o_j only, that is,

$$G^g_{ij} = \widetilde{G}^g_{ij} + c^o_j x_i, \qquad i \in X_n,$$

Now we are going to find these constants c^o_j minimizing the $H^2(\overline{\omega}^h)$ norm of the general least squares solution, which is squared below

$$\|\mathrm{col}_j\mathbf{U}^g\|^2_{H^2(\overline{\omega}^h)} = c_j^2\|\mathbf{x}\|^2_{H^2(\overline{\omega}^h)} + 2c_j(\mathbf{x}, \mathrm{col}_j\widetilde{\mathbf{G}}^g)_{H^2(\overline{\omega}^h)} + \|\mathrm{col}_j\widetilde{\mathbf{G}}^g\|^2_{H^2(\overline{\omega}^h)}.$$

Differentiating with respect to c_j, we get the following system $2c_j\|\mathbf{x}\|^2_{H^2(\overline{\omega}^h)} + 2(\mathbf{x}, \mathrm{col}_j\widetilde{\mathbf{G}}^g)_{H^2(\overline{\omega}^h)} = 0$ for every $j \in X_{n-2}$. From here we solve the constants

$$c^o_j = -\frac{(\mathbf{x}, \mathrm{col}_j\widetilde{\mathbf{G}}^g)_{H^2(\overline{\omega}^h)}}{\|\mathbf{x}\|^2_{H^2(\overline{\omega}^h)}}, \qquad j \in X_{n-2}$$

and observe that $c^o_j\mathbf{x} = -\mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\mathrm{col}_j\widetilde{\mathbf{G}}^g$. Substituting obtained constant values c^o_j, we get the representation

$$\mathbf{G}^g = \widetilde{\mathbf{G}}^g - \mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\widetilde{\mathbf{G}}^g = \widetilde{\mathbf{G}}^{\mathrm{cl}} - \mathbf{P}_{N(\widetilde{A})}\widetilde{\mathbf{G}}^{\mathrm{cl}} + \gamma\mathbf{v}^{g,2}\,\mathrm{row}_s\widetilde{\mathbf{G}}^{\mathrm{cl}} - \mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\widetilde{\mathbf{G}}^g.$$

Here $\mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\widetilde{\mathbf{G}}^g = \mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\widetilde{\mathbf{G}}^{\mathrm{cl}} - \mathbf{P}_{N(\widetilde{A})}\widetilde{\mathbf{G}}^{\mathrm{cl}}$ since $\mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\widetilde{\mathbf{G}}^g \in N(\widetilde{\mathbf{A}})$ and (see the formula (5.9)) $\mathbf{v}^{g,2} \in N(\mathbf{A})^\top = N(\widetilde{\mathbf{A}})^\top$ gives vanishing $\mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\mathbf{v}^{g,2}$. So, we have

$$\mathbf{G}^g = \widetilde{\mathbf{G}}^{\mathrm{cl}} - \mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\widetilde{\mathbf{G}}^{\mathrm{cl}} + \gamma\mathbf{v}^{g,2}\,\mathrm{row}_s\widetilde{\mathbf{G}}^{\mathrm{cl}}.$$

This discrete Green's function describes the minimizer of the problem (5.36)*, which has the minimum* $H^2(\overline{\omega}^h)$ *norm, i.e., the unique solution to the least squares problem* (5.34)–(5.35)*, that is,*

$$u_i^o = \sum_{j=0}^{n-2} G_{ij}^g f(x_{j+1}) h^{1/2} + g_1 v_i^{g,1} + g_2 v_i^{g,2}, \quad i \in X_n.$$

Recalling the desired form (5.33) *for the representation and the equality* $\widetilde{\mathbf{G}}^{\mathrm{cl}} = h^{1/2}\mathbf{G}^{\mathrm{cl,h}}$*, we rewrite the minimizer as below*

$$u_i^o = \sum_{j=0}^{n-2} G_{ij}^{g,h} f(x_{j+1}) h + g_1 v_i^{g,1} + g_2 v_i^{g,2}, \quad i \in X_n,$$

with the kernel

$$\mathbf{G}^{g,h} = \mathbf{G}^{\mathrm{cl,h}} - \mathbf{P}_{H^2(\overline{\omega}^h),N(\widetilde{A})}\mathbf{G}^{\mathrm{cl,h}} + \gamma \mathbf{v}^{g,2}\,\mathrm{row}_s \mathbf{G}^{\mathrm{cl,h}}.$$

Let us rewrite it into the discrete form

$$G_{ij}^{g,h} = G_{ij}^{\mathrm{cl,h}} - (P_{H^2(\overline{\omega}^h)N(A)}G^{\mathrm{cl,h}})_{ij} + \gamma v_i^{g,2} G_{sj}^{\mathrm{cl,h}}, \; i \in X_n, \; j \in X_{n-2}. \quad (5.41)$$

Letting $h \to 0$*, we have that discrete functions* $v_i^{g,2}$*,* $G_{ij}^{\mathrm{cl,h}}$ *converge to continuous functions* $v^{g,2}(x)$*,* $G^{\mathrm{cl}}(x,y)$*, respectively, as well as the projection*

$$(P_{H^2(\overline{\omega}^h)N(A)}G^{\mathrm{cl,h}})_{ij} = \frac{x_i}{\|\mathbf{x}\|^2_{H^2(\overline{\omega}^h)}}(\mathbf{x}, \mathrm{col}_j \mathbf{G}^{\mathrm{cl,h}})_{H^1(\overline{\omega}^h)}, \quad i \in X_n, \; j \in X_{n-2},$$

converges to the projection $P_{N(\mathbf{L})}G^{\mathrm{cl}}(x,y) = x(t, G^{\mathrm{cl}}(t,y))_{H^1[0,1]}/\|t\|^2_{H^2[0,1]}$. *Here we denoted the inner product* $(\mathbf{u}, \widetilde{\mathbf{u}})_{H^1(\overline{\omega}^h)}$*, which defines the norm*

$$\|\mathbf{u}\|_{H^1(\overline{\omega}^h)} = (\mathbf{u},\mathbf{u})^{1/2}_{H^1(\overline{\omega}^h)} = \left(\sum_{i=0}^{n} u_i^2 h + \sum_{i=0}^{n-1}\left(\frac{u_{i+1}-u_i}{h}\right)^2 h\right)^{1/2}$$

for every $\mathbf{u} \in \mathbb{R}^{(n+1)\times 1}$*, and used the following equality* $(\mathbf{x}, \mathrm{col}_j \mathbf{G}^{\mathrm{cl,h}})_{H^1(\overline{\omega}^h)} = (\mathbf{x}, \mathrm{col}_j \mathbf{G}^{\mathrm{cl,h}})_{H^2(\overline{\omega}^h)}$ *because* $x_{i+1} - 2x_i + x_{i-1} \equiv 0$ *for every* $i \in 1, 2, \ldots, n-2$*. So, the representation* (5.41) *converges to the Green's function*

$$G^g(x,y) = G^{\mathrm{cl}}(x,y) - P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x,y) + \gamma v^{g,2}(x) G^{\mathrm{cl}}(\xi, y)$$

of the differential problem (5.20)–(5.21) *pointwise as well.*

6 Minimization problem in different spaces

In the previous section, we investigated the discrete minimum norm least squares solution and its properties. This solution has the smallest Euclidean norm among all discrete functions, minimizing the same standard Euclidean norm of the residual of the discrete problem (3.3)–(3.4). However, the previous example leads us to extend the interpretation of the minimization problem (5.1)–(5.2) introducing two different finite dimensional Hilbert spaces instead of one standard Euclidean space.

Here we are going to investigate the *generalized minimum norm least squares solution*, that minimizes the residual of the discrete problem in the one norm and has the smallest second norm among all minimizers of the residual in the first norm. The introduction of two different norms allows us to analyze the convergence of the generalized minimum norm least squares solution to the minimum norm least squares solution of the differential problem (1.1)–(1.2). We also obtain properties of the generalized minimum norm least squares solution as well as its generalized discrete Green's function.

6.1 The $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm

Let us introduce two inner products $(\mathbf{u}, \mathbf{v})_{\mathcal{H}_k}$, $k = 1, 2$, for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{(n+1)\times 1}$ and denote by $\mathcal{H}_k$ the space $\mathbb{C}^{(n+1)\times 1}$ with the norm $\|\mathbf{u}\|_{\mathcal{H}_k} = (\mathbf{u}, \mathbf{u})^{1/2}_{\mathcal{H}_k}$, respectively. Now we are going to generalize results of Section 5, minimizing the $\mathcal{H}_1$ norm of the residual

$$\|\mathbf{A}\mathbf{u}^g - \mathbf{b}\|_{\mathcal{H}_1} \leqslant \|\mathbf{A}\mathbf{u} - \mathbf{b}\|_{\mathcal{H}_1}, \qquad \forall\ \mathbf{u} \in \mathbb{C}^{(n+1)\times 1}, \tag{6.1}$$

by a vector $\mathbf{u}^g$ that, in general, may be not unique. We choose the one solution $\mathbf{u}^o$ from all minimizers $\mathbf{u}^g$, for which the $\mathcal{H}_2$ norm is smallest:

$$\|\mathbf{u}^o\|_{\mathcal{H}_2} < \|\mathbf{u}^g\|_{\mathcal{H}_2}, \qquad \forall\ \mathbf{u}^g \neq \mathbf{u}^o. \tag{6.2}$$

This solution $\mathbf{u}^o$, which can be called the *generalized minimum norm least squares solution* or the $\mathcal{H}_1$ *least squares solution of the minimum* $\mathcal{H}_2$ *norm*, was also investigated by other authors [6, Ben-Israel and Greville 2003], and is of the form

$$\mathbf{u}^o = \mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2}\mathbf{b} \tag{6.3}$$

with

$$\mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2} = \mathbf{P}_{\mathcal{H}_2,N(A)^{\perp}}\mathbf{A}^{\dagger}\mathbf{P}_{\mathcal{H}_1,N(A^*)^{\perp}}.$$

Here $\mathbf{A}^{\dagger} \in \mathbb{C}^{(n+1)\times(n+1)}$ denotes the standard Moore–Penrose inverse of the matrix $\mathbf{A} \in \mathbb{C}^{(n+1)\times(n+1)}$, which was investigated in the previous section.

Moreover, $\mathbf{P}_{\mathcal{H}_2,N(A)^\perp}$ is the orthogonal projector onto the orthogonal complement $N(A)^\perp$ of the nullspace $N(A)$ in the space $\mathcal{H}_2$. Here $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ denotes the adjoint operator of the discrete operator $A = A_{\mathcal{H}_1,\mathcal{H}_2} : \mathcal{H}_2 \to \mathcal{H}_1$.

Since $A_{\mathcal{H}_1,\mathcal{H}_2}$ is the continuous linear operator with the closed range [6, Ben-Israel and Greville 2003], the discrete operator $A^\dagger_{\mathcal{H}_1,\mathcal{H}_2} : \mathcal{H}_1 \to \mathcal{H}_2$ is the Moore-Penrose inverse of $A_{\mathcal{H}_1,\mathcal{H}_2}$ and describes the unique minimizer (6.3) of the problem (6.1)–(6.2) for fixed finite dimension Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, $\mathbf{A}^\dagger_{\mathcal{H}_1,\mathcal{H}_2}$ is its matrix representation.

According to [6, Ben-Israel and Greville 2003], the general $\mathcal{H}_1$ least squares solution of (6.1) is represented by the unique minimizer (6.3) as below

$$\mathbf{u}^g = \mathbf{u}^o + \mathbf{P}_{\mathcal{H}_2,N(A)}\mathbf{c}, \qquad \forall \mathbf{c} \in \mathbb{C}^{(n+1)\times 1},$$

and, conversely, the minimizer (6.3) is always equal to

$$\mathbf{u}^o = \mathbf{P}_{\mathcal{H}_2,N(A)^\perp}\mathbf{u}^g. \tag{6.4}$$

Let us note that the minimizer (6.3) to the problem (3.3), which may be consistent ($\mathbf{b} \in R(\mathbf{A})$) or inconsistent ($\mathbf{b} \in R(\mathbf{A})^\perp$), is always the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm to a consistent problem $\mathbf{Au} = \mathbf{P}_{\mathcal{H}_1,R(A)}\mathbf{b}$ [6, Ben-Israel and Greville 2003].

6.2 Generalized discrete Green's function

Let us consider the form of $\mathbf{b} = (f_0, f_1, \ldots, f_{n-2}, g_1, g_2)^\top \in \mathbb{C}^{(n+1)\times 1}$ with every $\mathbf{f} = (f_0, f_1, \ldots,\ f_{n-2})^\top \in \mathbb{C}^{(n-1)\times 1}$ and complex numbers g_1, g_2. As in the previous section, we write the minimizer (6.3) in the following special form

$$\mathbf{u}^o = \mathbf{G}^g\mathbf{f} + g_1\mathbf{v}^{g,1} + g_2\mathbf{v}^{g,2}. \tag{6.5}$$

Here $\mathbf{G}^g \in \mathbb{C}^{(n+1)\times(n-1)}$ and $\mathbf{v}^{g,1}, \mathbf{v}^{g,2} \in \mathbb{C}^{(n+1)\times 1}$ are submatrices of the matrix representation of the Moore–Penrose inverse

$$\mathbf{A}^\dagger_{\mathcal{H}_1,\mathcal{H}_2} = (\mathbf{G}^g,\ \mathbf{v}^{g,1},\ \mathbf{v}^{g,2}).$$

The minimizer (6.5), given in explicit form

$$u_i^o = \sum_{j=0}^{n-2} G_{ij}^g f_j + g_1 v_i^{g,1} + g_2 v_i^{g,2}, \quad i \in X_n, \tag{6.6}$$

is coincident with the representation of the unique solution (4.3) for the particular case $\det \mathbf{A} \neq 0$, investigated in Section 4. Moreover, it is also describes the minimizer (5.11) taking the standard Euclidean space instead of two different spaces. Hence, we call $G^g \in F(X_n \times X_{n-2})$ by the *generalized discrete Green's function* and functions $v^{g,1}, v^{g,2} \in F(X_n)$ – the *generalized discrete biorthogonal fundamental system* describing the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm for the problem (3.1)–(3.2).

This generalized discrete Green's function and the generalized discrete biorthogonal fundamental system can always be obtained using the Moore–Penrose inverse $\mathbf{B} = \mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2}$ in the following equalities

$$
\begin{aligned}
G^g_{ij} &= B_{ij}, \quad i \in X_n, \; j \in X_{n-2}, \\
v^{g,1}_i &= B_{i,n-1}, \quad i \in X_n, \\
v^{g,2}_i &= B_{in}, \quad i \in X_n.
\end{aligned}
$$

Here again, for $\det \mathbf{A} \neq 0$, we have that $\mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2} = \mathbf{A}^{-1}$, the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm $u^o \in F(X_n)$ is coincident with the unique discrete solution $u \in F(X_n)$, its generalized discrete Green's function $G^g \in F(X_n \times X_{n-2})$ is coincident with the discrete Green's function $G \in F(X_n \times X_{n-2})$, the generalized discrete biorthogonal fundamental system $v^{g,1}, v^{g,2} \in F(X_n)$ is coincident with the discrete biorthogonal fundamental system $v^1, v^2 \in F(X_n)$.

6.3 Properties

In this subsection, we present properties of the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm and its generalized discrete Green's function, those literally resemble properties obtained in Subsection 5.3 for the standard Euclidean space. We omit proofs since they are analogous to corresponding proofs from Section 5.3.

Lemma 3.21. *For every fixed $j \in X_{n-2}$, generalized discrete Green's function $G^g \in F(X_n \times X_{n-2})$ is the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm to the discrete problem*

$$
\begin{aligned}
\mathcal{L}_{i\cdot} G^g_{\cdot j} &= \delta_{ij}, \quad i \in X_{n-2}, \\
\langle L^{\cdot}_k, G^g_{\cdot j} \rangle &= 0, \quad k = 1, 2.
\end{aligned}
$$

Lemma 3.22. *Every discrete function $v^{g,l} \in F(X_n)$, $l = 1, 2$, is the $\mathcal{H}_1$ norm least squares solution of the minimum $\mathcal{H}_2$ norm to the corresponding discrete problem*

$$
\begin{aligned}
\mathcal{L} v^{g,l} &= 0, \\
\langle L_k, v^{g,l} \rangle &= \delta^l_k, \quad k = 1, 2.
\end{aligned}
$$

Let us now investigate two discrete problems (4.9), where the second problem has generalized Green's function G^g and generalized biorthogonal system $v^{g,k}$, $k = 1, 2$, describing the generalized minimum norm least squares solution (6.6).

Theorem 3.23. *The $\mathcal{H}_1$ least squares solutions of the minimum $\mathcal{H}_2$ norm u^o and v^o to problems* (4.9)*, respectively, are linked by the equality*

$$v^o = u^o - P_{\mathcal{H}_2,N(A)}u^o + G^g(f - \mathcal{L}u^o) + v^{g,1}(g_1 - \langle L_1, u^o\rangle) + v^{g,2}(g_2 - \langle L_2, u^o\rangle).$$

Proof. Let us denote $w = v^o - u^o$ (the difference between the minimizers). Our aim is to show that w is a least squares solution to the discrete problem

$$\mathcal{L}w = f - \mathcal{L}u^o, \quad \langle L_k, w\rangle = g_k - \langle L_k, u^o\rangle, \quad k = 1, 2. \tag{6.7}$$

We rewrite this problem in the equivalent matrix form $\mathbf{Aw} = \widetilde{\mathbf{b}}$ with the right hand side $\widetilde{\mathbf{b}} = (\widetilde{f}_0, \widetilde{f}_1, \ldots, \widetilde{f}_{n-2}, g_1 - \mathbf{L}_1\mathbf{u}^o, g_2 - \mathbf{L}_2\mathbf{u}^o)^\top$, where $\widetilde{f}_i = f_i - (\mathcal{L}u^o)_i$, $i \in X_{n-2}$. Since $\mathbf{v}^o$ is the minimum norm least squares solution to the system (3.3) with the right hand side $\mathbf{b} = (f_0, f_1, \ldots, f_{n-2}, g_1, g_2)^\top$, then the inequality (5.6) is always valid, i.e., $\|\mathbf{Av}^o - \mathbf{b}\|_{\mathcal{H}^1} \leqslant \|\mathbf{Av} - \mathbf{b}\|_{\mathcal{H}^1}$ for every $\mathbf{v} \in \mathbb{C}^{(n+1)\times 1}$. Now we rewrite the norm as follows $\|\mathbf{Av} - \mathbf{b}\|_{\mathcal{H}^1} = \|\mathbf{Av} + (\mathbf{Au}^o - \mathbf{Au}^o) - \mathbf{b}\|_{\mathcal{H}^1} = \|\mathbf{A}(\mathbf{v} - \mathbf{u}^o) - (-\mathbf{Au}^o + \mathbf{b})\|_{\mathcal{H}^1} = \|\mathbf{A}(\mathbf{v} - \mathbf{u}^o) - \widetilde{\mathbf{b}}\|_{\mathcal{H}^1}$ for all $\mathbf{v} \in \mathbb{C}^{(n+1)\times 1}$. From here we have

$$\|\mathbf{A}(\mathbf{v}^o - \mathbf{u}^o) - \widetilde{\mathbf{b}}\|_{\mathcal{H}^1} \leqslant \|\mathbf{A}(\mathbf{v} - \mathbf{u}^o) - \widetilde{\mathbf{b}}\|_{\mathcal{H}^1}, \qquad \forall\, \mathbf{v} \in \mathbb{C}^{(n+1)\times 1}$$

or, denoting $\mathbf{z} = \mathbf{v} - \mathbf{u}^o$, obtain $\|\mathbf{Aw} - \widetilde{\mathbf{b}}\|_{\mathcal{H}^1} \leqslant \|\mathbf{Az} - \widetilde{\mathbf{b}}\|_{\mathcal{H}^1}$ for all $\mathbf{z} \in \mathbb{C}^{(n+1)\times 1}$.

So, $\mathbf{w}$ is a least squares solution of the problem $\mathbf{Aw} = \widetilde{\mathbf{b}}$ and, according to the formula (6.4), it describes the minimizer $\mathbf{w}^o = \mathbf{A}^\dagger_{\mathcal{H}_1,\mathcal{H}_2}\widetilde{\mathbf{b}}$ in the form $\mathbf{w}^o = \mathbf{P}_{\mathcal{H}_2,N(A)^\perp}\mathbf{w}$. Using the composition $\mathbf{w} = \mathbf{P}_{\mathcal{H}_2,N(A)^\perp}\mathbf{w} + \mathbf{P}_{\mathcal{H}_2,N(A)}\mathbf{w}$, we get $\mathbf{w} = \mathbf{A}^\dagger_{\mathcal{H}_1,\mathcal{H}_2}\widetilde{\mathbf{b}} + \mathbf{P}_{\mathcal{H}_2,N(A)}\mathbf{w}$. Now we recall the equality $\mathbf{w} = \mathbf{v}^o - \mathbf{u}^o$ and obtain

$$\mathbf{v}^o = \mathbf{u}^o + \mathbf{A}^\dagger_{\mathcal{H}_1,\mathcal{H}_2}\widetilde{\mathbf{b}} + \mathbf{P}_{\mathcal{H}_2,N(A)}\mathbf{w}. \tag{6.8}$$

Here we take the composition $\mathbf{u}^o = \mathbf{P}_{\mathcal{H}_2,N(A)^\perp}\mathbf{u}^o + \mathbf{P}_{\mathcal{H}_2,N(A)}\mathbf{u}^o$ and rewrite the representation (6.8) into the form $\mathbf{v}^o = \mathbf{P}_{\mathcal{H}_2,N(A)^\perp}\mathbf{u}^o + \mathbf{A}^\dagger_{\mathcal{H}_1,\mathcal{H}_2}\widetilde{\mathbf{b}} + \mathbf{P}_{\mathcal{H}_2,N(A)}\mathbf{v}^o$, because $\mathbf{w} + \mathbf{u}^o = \mathbf{v}^o$. From (6.4), we have $\mathbf{v}^o \in N(\mathbf{A})^\perp$. Then $\mathbf{P}_{\mathcal{H}_2,N(A)}\mathbf{v}^o = \mathbf{0}$ and the previous representation simplifies to $\mathbf{v}^o = \mathbf{P}_{\mathcal{H}_2,N(A)^\perp}\mathbf{u}^o + \mathbf{A}^\dagger_{\mathcal{H}_1,\mathcal{H}_2}\widetilde{\mathbf{b}}$. Rewriting it into the extended form

$$\mathbf{v}^o = \mathbf{u}^o - \mathbf{P}_{\mathcal{H}_2,N(A)}\mathbf{u}^o + \mathbf{G}^g(\mathbf{f} - \boldsymbol{\mathcal{L}}\mathbf{u}^o) + (g_1 - \mathbf{L}_1\mathbf{u}^o)\mathbf{v}^{g,1} + (g_2 - \mathbf{L}_2\mathbf{u}^o)\mathbf{v}^{g,2},$$

we obtain the statement of this theorem. □

Corollary 3.24. *Let the first problem* (4.9) *be (uniquely) solvable. Then the* $\mathcal{H}_1$ *least squares solutions of the minimum* $\mathcal{H}_2$ *norm* u^o *and* v^o *to problems* (4.9), *respectively, are related as follows*

$$v^o = u^o - P_{\mathcal{H}_2,N(A)}u^o + v^{g,1}(g_1 - \langle L_1, u^o\rangle) + v^{g,2}(g_2 - \langle L_2, u^o\rangle).$$

Proof. It follows from Theorem 3.23 because $\mathcal{L}u^o = f$ for the consistent problem. □

Since the discrete Cauchy problem (3.4) always has the unique solution $u^c \in F(X_n)$ (for instance, see [100, Roman 2011]), we can obtain the representation of the minimizer (6.3) as follows.

Corollary 3.25. *The* $\mathcal{H}_1$ *least squares solution of the minimum* $\mathcal{H}_2$ *norm to the problem* (3.1)–(3.2) *can always be described by the unique solution* u^c *to the Cauchy problem, that is,*

$$u^o = u^c - P_{\mathcal{H}_2,N(A)}u^c + v^{g,1}(g_1 - \langle L_1, u^c\rangle) + v^{g,2}(g_2 - \langle L_2, u^c\rangle).$$

For the problem with nonlocal boundary conditions (4.10)–(4.11), we obtain the following representation.

Corollary 3.26. *If the classical problem* (4.10)–(4.11) $(\gamma_1, \gamma_2 = 0)$ *has the unique solution* $u^{\mathrm{cl}} \in F(X_n)$, *then the* $\mathcal{H}_1$ *least squares solution of the minimum* $\mathcal{H}_2$ *norm to the nonlocal boundary value problem* (4.10)–(4.11) *is*

$$u^o = u^{\mathrm{cl}} - P_{\mathcal{H}_2,N(A)}u^{\mathrm{cl}} + v^{g,1}(g_1 - \langle L_1, u^{\mathrm{cl}}\rangle) + v^{g,2}(g_2 - \langle L_2, u^{\mathrm{cl}}\rangle).$$

6.4 Relations between generalized discrete Green's functions

We are also interested to know a representation of the generalized discrete Green's function, which describes the minimizer (6.3). Here we use a discrete Green's function, investigated in Section 4, as given below.

Theorem 3.27. *If there exists the discrete Green's function* $G \in F(X_n \in X_{n-2})$ *for the first problem* (4.9), *then the generalized discrete Green's function* $G^g \in F(X_n \times X_{n-2})$ *of the second problem is given by*

$$G^g_{ij} = G_{ij} - (P_{\mathcal{H}_2,N(A)}G)_{ij} - v_i^{g,1}\langle L_1^{\cdot}, G_{\cdot j}\rangle - v_i^{g,2}\langle L_2^{\cdot}, G_{\cdot j}\rangle, \quad i \in X_n,\ j \in X_{n-2}.$$

Here $(P_{\mathcal{H}_2,N(A)}G)_{ij}$ is trivial if $\det \mathbf{A} \neq 0$. Otherwise, it denotes the kernel of the orthogonal projection onto the nullspace $N(\mathbf{A})$ in the space

$\mathcal{H}_2$:

$$(P_{\mathcal{H}_2,N(A)}Gf)_i = \sum_{l=1}^{d} z_i^l(\mathbf{z}^l, \mathbf{Gf})_{\mathcal{H}_2} = \sum_{j=1}^{n-2}(P_{\mathcal{H}_2,N(A)}G)_{ij}f_j, \qquad i \in X_n,$$

where $\mathbf{z}^l$, $l = \overline{1,d}$, is a basis of the nullspace $N(\mathbf{A})$, orthonormal with respect to the inner product in $\mathcal{H}_2$.

Corollary 3.28. *The generalized discrete Green's function for the problem* (3.1)–(3.2) *is described by the discrete Green's function* $G^c \in F(X_n \times X_{n-2})$ *of the Cauchy problem* (3.4) *as below*

$$G_{ij}^g = G_{ij}^c - (P_{\mathcal{H}_2,N(A)}G^c)_{ij} - v_i^{g,1}\langle L_1^{\cdot}, G_{\cdot j}^c\rangle - v_i^{g,2}\langle L_2^{\cdot}, G_{\cdot j}^c\rangle, \quad i \in X_n,\ j \in X_{n-2}.$$

For the problem (4.10)–(4.11) with nonlocal boundary conditions, we obtain the following relation.

Corollary 3.29. *If there exists the discrete Green's function* $G^{\mathrm{cl}} \in F(X_n \times X_{n-2})$ *of the classical problem* (4.10)–(4.11) *(*$\gamma_1, \gamma_2 = 0$*), then the generalized discrete Green's function, describing the minimizer* (6.3) *to the problem* (4.10)–(4.11) *with nonlocal boundary conditions, is given by*

$$G_{ij}^g = G_{ij}^{\mathrm{cl}} - (P_{\mathcal{H}_2,N(A)}G^{\mathrm{cl}})_{ij} + v_i^{g,1}\langle \varkappa_1^{\cdot}, G_{\cdot j}^{\mathrm{cl}}\rangle + v_i^{g,2}\langle \varkappa_2^{\cdot}, G_{\cdot j}^{\mathrm{cl}}\rangle, \quad i \in X_n,\ j \in X_{n-2}.$$

6.5 Applications to differential problems

In this section, we discuss on real discrete problems approximating second order differential problems (1.1)–(1.2). Recalling Example 3.20, we consider the real $L^2(\omega^h) \times \mathbb{R}^2$ least squares solution of the minimum $H^2(\overline{\omega}^h)$ norm for the discrete problem (3.1)–(3.2). In other words, we minimize the $L^2(\omega^h) \times \mathbb{R}^2$ norm of the residual

$$\|\mathbf{A}\mathbf{u}^g - \mathbf{b}\|_{L^2(\omega^h)\times\mathbb{R}^2} \leqslant \|\mathbf{A}\mathbf{u} - \mathbf{b}\|_{L^2(\omega^h)\times\mathbb{R}^2}, \qquad \forall\ \mathbf{u} \in \mathbb{R}^{(n+1)\times 1}, \quad (6.9)$$

by a vector $\mathbf{u}^g$ that, in general, may be not unique. Then we select the one solution $\mathbf{u}^o$ from all minimizers $\mathbf{u}^g$, for which the $H^2(\overline{\omega}^h)$ norm is smallest:

$$\|\mathbf{u}^o\|_{H^2(\overline{\omega}^h)} < \|\mathbf{u}^g\|_{H^2(\overline{\omega}^h)}, \qquad \forall\ \mathbf{u}^g \neq \mathbf{u}^o. \quad (6.10)$$

To fulfill our plan, first, we need to represent the unique discrete solution as well as the discrete minimizer u^o in correct forms. We rewrite the unique discrete solution (4.3) as given below

$$u_i = \sum_{j=0}^{n-2} G_{ij}^h f_j h + g_1 v_i^1 + g_2 v_i^2, \qquad i \in X_n,$$

what is compatible to investigate the convergence to the unique solution

$$u = \int_0^1 G(x,y) f(y)\,dy + g_1 v^1(x) + g_2 v^2(x), \qquad x \in [0,1],$$

of the differential problem (1.1)–(1.2). Thus, now we get the composition of the inverse matrix $\mathbf{A}^{-1} = (h\mathbf{G}^h\, \mathbf{v}^1\, \mathbf{v}^2)$ with the modified discrete Green's function $\mathbf{G}^h = h^{-1}\mathbf{G}$.

Analogously we partition the Moore–Penrose inverse

$$\mathbf{A}^{\dagger}_{L^2(\omega^h)\times\mathbb{R}^2, H^2(\overline{\omega}^h)} = (h\mathbf{G}^{g,h}\,, \mathbf{v}^{g,1}\,, \mathbf{v}^{g,2})$$

introducing the modified generalized discrete Green's function $\mathbf{G}^{g,h} = h^{-1}\mathbf{G}^g$. It describes the $L^2(\omega^h)\times\mathbb{R}^2$ least squares solution of the minimum $H^2(\overline{\omega}^h)$ norm

$$\mathbf{u}^o = \mathbf{A}^{\dagger}_{L^2(\omega^h)\times\mathbb{R}^2, H^2(\overline{\omega}^h)}\mathbf{b}$$

in the desired form

$$u_i^o = \sum_{j=0}^{n-2} G_{ij}^{g,h} f_j h + g_1 v_i^{g,1} + g_2 v_i^{g,2}, \qquad i \in X_n,$$

(compare it with the representation (6.6)).

According to new partitioning of inverses, we rewrite $R(\mathbf{A})$ and $N(\mathbf{A}^*)$ representations. Since proofs are so similar, we present only the final versions.

Lemma 3.30. *1) If $d = 2$, then for all $f \in F(X_{n-2})$ we have*

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-2};\ \sum_{j=0}^{n-2} \langle L_1^{\cdot}, G_{\cdot j}^{c,h}\rangle f_j\, h;\ \sum_{j=0}^{n-2} \langle L_2^{\cdot}, G_{\cdot j}^{c,h}\rangle f_j\, h \right)^{\top} \right\}.$$

2) If $d = 1$ and $k_1 = 1$, then for all $f \in F(X_{n-2})$ and $g_2 \in \mathbb{R}$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-2};\ g_2\langle L_1, v^2\rangle + \sum_{j=0}^{n-2} \langle L_1^{\cdot}, G_{\cdot j}^{a,h}\rangle f_j\, h;\ g_2 \right)^{\top} \right\}.$$

3) If $d = 1$ and $k_1 = 2$, then for all $f \in F(X_{n-2})$ and $g_1 \in \mathbb{R}$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-2};\ g_1;\ g_1\langle L_2, v^1\rangle + \sum_{j=0}^{n-2} \langle L_2^{\cdot}, G_{\cdot j}^{a,h}\rangle f_j h \right)^{\top} \right\}.$$

Here $G^{c,h} \in F(X_n \times X_{n-2})$ is the discrete Green's function for the discrete Cauchy problem (3.4)*. Other discrete Green's function $G^{a,h} \in F(X_n \times X_{n-2})$*

and the biorthogonal fundamental system $v^1, v^2 \in F(X_n)$ are taken for the problem $\mathcal{L}u = f$ with the original condition $\langle L_{3-k_1}, u\rangle = 0$ and condition $\langle \ell, u\rangle = 0$, replacing $\langle L_{k_1}, u\rangle = 0$. Here $\langle \ell, u\rangle = 0$ is selected such that for this auxiliary problem $\Delta \neq 0$.

Further, the composition of the nullspace $N(\mathbf{A}^*)$ is presented.

Corollary 3.31. *The following three statements are valid:*

1) if $d = 2$, then $N(\mathbf{A}^*)$ is generated by two vectors

$$\mathbf{w}^1 = -\sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^{c,h}\rangle\mathbf{e}^j + \mathbf{e}^{n-1}, \qquad \mathbf{w}^2 = -\sum_{j=0}^{n-2}\langle L_2^{\cdot}, G_{\cdot j}^{c,h}\rangle\mathbf{e}^j + \mathbf{e}^{n};$$

2) if $d = 1$ and $k_1 = 1$, then $N(\mathbf{A}^*)$ is generated by the vector

$$\mathbf{w} = -\sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^{a,h}\rangle\mathbf{e}^j + \mathbf{e}^{n-1} - \langle L_1, v^2\rangle\mathbf{e}^n;$$

3) if $d = 1$ and $k_1 = 2$, then $N(\mathbf{A}^*)$ is generated by the vector

$$\mathbf{w} = -\sum_{j=0}^{n-2}\langle L_2^{\cdot}, G_{\cdot j}^{a,h}\rangle\mathbf{e}^j - \langle L_2, v^1\rangle\mathbf{e}^{n-1} + \mathbf{e}^n.$$

Now recalling the Fredholm alternative theorem, we get the solvability conditions for the problem (3.1)–(3.2) without the unique solution ($\Delta = 0$).

Corollary 3.32. *(Solvability conditions) The problem* (3.1)–(3.2) *with $\Delta = 0$ is solvable if and only if the conditions are valid:*

1) $\sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^{c,h}\rangle f_j h = g_1$, $\sum_{j=0}^{n-2}\langle L_2^{\cdot}, G_{\cdot j}^{c,h}\rangle f_j h = g_2$ for $d = 2$;

2) $g_2\langle L_1, v^2\rangle + \sum_{j=0}^{n-2}\langle L_1^{\cdot}, G_{\cdot j}^{h}\rangle f_j h = g_1$ for $d = 1$ and $k_1 = 1$;

3) $g_1\langle L_2, v^1\rangle + \sum_{j=0}^{n-2}\langle L_2^{\cdot}, G_{\cdot j}^{h}\rangle f_j h = g_2$ for $d = 1$ and $k_1 = 2$.

We note that the nullspace $N(\mathbf{A})$ and its classification remains as previous (see Subsection 3.1). Here are no changes.

Now we are going to answer what conditions guarantee a convergence of the discrete minimizer to the minimizer of the differential problem. Thus, here we recall several ideas from the theory of the discrete convergence.

First, we introduce the projection operator $\boldsymbol{\pi}_1 : H^2[0,1] \to H^2(\overline{\omega}^h)$, which projects a function $u \in H^2[0,1]$ on the mesh $\overline{\omega}^h$ by the formula $\boldsymbol{\pi}_1 u = (u(x_0), u(x_1), \ldots, u(x_n))^\top$. This poitwise definition is correct since every function from $H^2[0,1]$ belongs to $C^1[0,1]$. Second, for every $\boldsymbol{f} = (f, g_1, g_2)^\top \in L^2[0,1] \times \mathbb{R}^2$ we also take a projector $\boldsymbol{\pi}_2 : L^2[0,1] \times \mathbb{R}^2 \to L^2(\omega^h) \times \mathbb{R}^2$. If $f \in C[0,1]$, the projector may be given by the formula $\boldsymbol{\pi}_2 \boldsymbol{f} = (f(x_0), f(x_1), \ldots, f(x_{n-2}), g_1, g_2)^\top$. In general, $f \in L^2[0,1]$ and we can use the projector

$$\boldsymbol{\pi}_2 \boldsymbol{f} = \sum_{i=0}^{n-2} \left(\frac{1}{h} \int_{x_i}^{x_{i+1}} f(x)\,dx \right) \mathbf{e}^i + g_1 \mathbf{e}^{n-1} + g_2 \mathbf{e}^n.$$

Let us denote $\boldsymbol{\mathcal{O}}(h^\alpha) := \big(\mathcal{O}(h^\alpha), \ldots, \mathcal{O}(h^\alpha)\big)^\top \in \mathbb{R}^{(n+1)\times 1}$ and formulate the following statement.

Theorem 3.33. *(Sufficient convergence conditions) Let the following approximations*

$$\mathbf{A}(\boldsymbol{\pi}_1 u) = \boldsymbol{\pi}_2 \boldsymbol{L} u + \boldsymbol{\mathcal{O}}(h^\alpha), \qquad \mathbf{P}_{H^2(\overline{\omega}^h), N(A)}(\boldsymbol{\pi}_1 u) = \boldsymbol{\pi}_1(\boldsymbol{P}_{N(\boldsymbol{L})} u) + \boldsymbol{\mathcal{O}}(h^\alpha),$$
$$\mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, R(A)} \mathbf{b} = \boldsymbol{\pi}_2(\boldsymbol{P}_{R(\boldsymbol{L})} \boldsymbol{f}) + \boldsymbol{\mathcal{O}}(h^\alpha)$$

be valid for some $\alpha > 0$. If $\sup_{n\in\mathbb{N}} \|\mathbf{A}^\dagger\|_{H^2(\overline{\omega}^h),\, L^2(\omega^h)\times\mathbb{R}^2} < +\infty$, then the minimizer $\mathbf{u}^o \in H^2(\overline{\omega}^h)$ of the discrete problem (3.1)–(3.2) *converges to the minimizer $u^o \in H^2[0,1]$ of the differential problem* (1.1)–(1.2)*, i.e.,*

$$\|\mathbf{u}^o - \boldsymbol{\pi}_1 u^o\|_{C(\overline{\omega}^h)} = \max_{x_i \in \overline{\omega}^h} |u_i^o - u^o(x_i)| \to 0 \qquad \text{if} \quad n \to \infty.$$

Proof. The minimizers $\mathbf{u}^o \in \mathbb{R}^{(n+1)\times 1}$ and $u^o \in H^2[0,1]$ are solutions to consistent problems $\mathbf{A}\mathbf{u}^o = \mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, R(A)} \mathbf{b}$ and $\boldsymbol{L} u^o = \boldsymbol{P}_{R(\boldsymbol{L})} \boldsymbol{f}$, respectively. Thus, we obtain the equality $\mathbf{A}\mathbf{u}^o - \boldsymbol{\pi}_2 \boldsymbol{L} u^o = \mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, R(A)} \mathbf{b} - \boldsymbol{\pi}_2 \boldsymbol{P}_{R(\boldsymbol{L})} \boldsymbol{f} = \boldsymbol{\mathcal{O}}(h^\alpha)$. From here we have $\boldsymbol{\mathcal{O}}(h^\alpha) = \mathbf{A}\mathbf{u}^o - \boldsymbol{\pi}_2 \boldsymbol{L} u^o = \mathbf{A}\mathbf{u}^o - \mathbf{A}(\boldsymbol{\pi}_1 u^o) + \boldsymbol{\mathcal{O}}(h^\alpha)$, what provides a consistent linear system $\mathbf{A}(\mathbf{u}^o - \boldsymbol{\pi}_1 u^o) = \boldsymbol{\mathcal{O}}(h^\alpha)$. The difference $\mathbf{u}^o - \boldsymbol{\pi}_1 u^o$ is a particular solution to this system and can be obtained from the general solution

$$\mathbf{u}^o - \boldsymbol{\pi}_1 u^o = \mathbf{A}^\dagger \boldsymbol{\mathcal{O}}(h^\alpha) + \mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, N(A)} \mathbf{c}$$

with a particular vector $\mathbf{c} \in \mathbb{R}^{(n+1)\times 1}$. Here $\mathbf{A}^\dagger \boldsymbol{\mathcal{O}}(h^\alpha) = \boldsymbol{\mathcal{O}}(h^\alpha)$ because $\mathbf{A}^\dagger$ is uniformly bounded. Below we prove that $\mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, N(A)} \mathbf{c} = \boldsymbol{\mathcal{O}}(h^\alpha)$ as well, what gives $\mathbf{u}^o - \boldsymbol{\pi}_1 u^o = \boldsymbol{\mathcal{O}}(h^\alpha)$.

Since $u^o \in N(\boldsymbol{L})^\perp$ (see the formula (4.4) in Chapter 1), we have $P_{N(\boldsymbol{L})} u^o = 0$. Then $\mathbf{0} = \boldsymbol{\pi}_1 P_{N(\boldsymbol{L})} u^o = \mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, N(A)}(\boldsymbol{\pi}_1 u^o) + \boldsymbol{\mathcal{O}}(h^\alpha)$ provides the equality $\mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, N(A)}(\boldsymbol{\pi}_1 u^o) = \boldsymbol{\mathcal{O}}(h^\alpha)$. Moreover, $\mathbf{u}^o \in N(\mathbf{A})^\perp$. Then considering the orthogonal subspace $N(\mathbf{A})$ in the equality $\mathbf{u}^o - \boldsymbol{\pi}_1 u^o =$

$\boldsymbol{\mathcal{O}}(h^\alpha) + \mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, N(A)}\mathbf{c}$, we get $\mathbf{0} - \boldsymbol{\mathcal{O}}(h^\alpha) = \boldsymbol{\mathcal{O}}(h^\alpha) + \mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, N(A)}\mathbf{c}$. It means $\mathbf{P}_{L^2(\omega^h)\times\mathbb{R}^2, N(A)}\mathbf{c} = \boldsymbol{\mathcal{O}}(h^\alpha)$. Substituting this expression above, we have $\mathbf{u}^o - \boldsymbol{\pi}_1 u^o = \boldsymbol{\mathcal{O}}(h^\alpha)$. From here we obtain the desired convergence.

□

7 Conclusions

Principal conclusions of this chapter are formulated below:

1) A discrete problem (3.1)–(3.2) always has the Moore–Penrose inverse $\mathbf{A}^\dagger$, a generalized discrete Green's function and the unique minimum norm least squares solution.

2) For $\varDelta \neq 0$, we have that $\mathbf{A}^\dagger = \mathbf{A}^{-1}$, the minimum norm least squares solution u^o is coincident with the unique solution u, the generalized Green's function G^g_{ij} is coincident with the ordinary Green's function G_{ij}, the biorthogonal fundamental system v^1, v^2 is coincident with the generalized biorthogonal fundamental system $v^{g,1}, v^{g,2}$.

3) The minimum norm least squares solution has literally similar representations as the unique discrete solution: it can be described by the unique solution of the discrete Cauchy problem or the unique solution to other relative problem (the same discrete equation (3.1) but different nonlocal conditions (3.2)).

4) A generalized discrete Green's function also has representations similar to expressions of a discrete Green's function: it can be written using the discrete Green's function of the Cauchy problem or the discrete Green's function to other relative problem (the same discrete equation (3.1) but different nonlocal conditions (3.2)).

5) Obtained properties of minimizers are coincident with corresponding properties of minimizers for differential problems.

6) The discrete minimum norm least squares solution converges to the minimum norm least squares solution of the differential problem (1.1)–(1.2) if conditions of Theorem 3.33 are fulfilled.

Chapter 4

m-th order discrete problems with nonlocal conditions

1 Introduction

In this chapter, we are going to generalize results of Chapter 3. Here we consider a discrete analogue of the m-th order differential problem with nonlocal conditions

$$\mathcal{L}u := u^{(m)} + a_{m-1}(x)u^{(m-1)} + \ldots + a_1(x)u' + a_0(x)u = f(x), \ x \in [0,1], \tag{1.1}$$
$$\langle L_k, u\rangle = g_k, \quad k = \overline{1,m}, \quad m \in \mathbb{N}, \tag{1.2}$$

which was studied in Chapter 2. There we investigated various properties for the minimizer as well as its generalized Green's function of the differential problem (1.1)–(1.2). Similar results can be derived for the m-th order discrete problem as well. Here we omit many proofs because they are obtained analogously as in Chapter 3.

The structure of this chapter is as follows. First, we introduce the m-th order discrete problem and study its properties. Then known results for the unique discrete solution and its discrete Green's function are presented. Afterwards, we investigate the minimum norm least squares solution as well as the generalized discrete Green's function. Their expressions and properties will be given. Finally, we present results for the generalized minimum norm least squares solution of the discrete problem, where two different finite dimensional Hilbert spaces are introduced. Let us note that this chapter is based on papers [90, 91, Paukštaitė and Štikonas 2016,2017].

2 Formutation of the problem

In this chapter, we investigate a m-th order discrete problem

$$(\mathcal{L}u)_i := a_i^m u_{i+m} + \ldots + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad i \in X_{n-m}, \tag{2.1}$$

$$\langle L_k, u\rangle := \sum_{j=0}^{n} L_k^j u_j = g_k, \quad k = \overline{1,m}, \tag{2.2}$$

where $a^0, \ldots, a^m \in F(X_{n-m})$ but $a_i^0, a_i^m \neq 0$ with every $i \in X_{n-m}$, $f \in F(X_{n-m})$, $L_k \in F^*(X_n)$, $g_k \in \mathbb{C}$ for $k = \overline{1,m}$, and $n \geqslant m$.

Let us recall notations from Chapter 3. For the solution $u \in F(X_n)$ to the problem (2.1)–(2.2), we may obtain two equivalent notations. Precisely, the discrete complex function $u \in F(X_n)$ can be uniquely described by the complex column matrix $\mathbf{u} = (u_0, u_1, \ldots, u_n)^\top \in \mathbb{C}^{(n+1)\times 1}$. Thus, u and $\mathbf{u}$ always are two equivalent notations for the same solution. Furthermore, a discrete linear functional $L_k \in F^*(X_n)$ can be represented by a complex row matrix $\mathbf{L}_k = (L_k^0, L_k^1, \ldots, L_k^n) \in \mathbb{C}^{1\times(n+1)}$ but a discrete linear operator $\mathcal{L} : F(X_n) \to F(X_{n-m})$ is described by the matrix $\boldsymbol{\mathcal{L}} = (\mathcal{L}_{ij}) \in \mathbb{C}^{(n-m+1)\times(n+1)}$ with rows

$$\mathrm{row}_i \boldsymbol{\mathcal{L}} = (\underbrace{0, \ldots, 0}_{i}, a_i^0, a_i^1, \ldots, a_i^m, 0, \ldots, 0), \quad i \in X_{n-m}.$$

So, we can represent the discrete problem (2.1)–(2.2) by the equivalent system

$$\mathbf{Au} = \mathbf{b} \tag{2.3}$$

with the matrix $\mathbf{A} = (\boldsymbol{\mathcal{L}}; \mathbf{L}_1; \ldots; \mathbf{L}_m)^\top \in \mathbb{C}^{(n+1)\times(n+1)}$ and the right hand side $\mathbf{b} = (f_0, f_1, \ldots, f_{n-m}, g_1, g_2, \ldots, g_m)^\top \in \mathbb{C}^{(n+1)\times 1}$.

2.1 Nullspace of the matrix A

This subsection is analogous to Subsection 3.1 from the previous chapter. Indeed, the problem (2.3) has the unique solution if $\det \mathbf{A} \neq 0$, that is equivalent to the nonzero determinant

$$\Delta := \begin{vmatrix} \langle L_1, z^1\rangle & \ldots & \langle L_1, z^m\rangle \\ \ldots & \ldots & \ldots \\ \langle L_m, z^1\rangle & \ldots & \langle L_m, z^m\rangle \end{vmatrix}.$$

Here $z^1, \ldots, z^m \in F(X_n)$ are any fundamental system of the homogenous equation (2.1). Let us solve the homogenous problem $\mathbf{Az} = \mathbf{0}$. First, we take the general solution $z = c_1 z^1 + \ldots + c_2 z^m$, $c_k \in \mathbb{C}$, of the equation

$\mathcal{L}z = 0$. Substituting it into homogenous conditions $\langle L_k, z\rangle = 0,\ k = \overline{1,m}$, we get the system

$$c_1\langle L_1, z^1\rangle + \ldots + c_m\langle L_1, z^m\rangle = 0,$$
$$\ldots$$
$$c_1\langle L_m, z^1\rangle + \ldots + c_m\langle L_m, z^m\rangle = 0$$

with the determinant $\varDelta$. Let us denote the nullity of the matrix $\mathbf{A}$ by $d := \dim N(\mathbf{A})$ and separate such cases:

- $d = 0 \Leftrightarrow \varDelta \neq 0$. Then the nullspace $N(\mathbf{A})$ is trivial.
- $d = m \Leftrightarrow$ if $\varDelta = 0$ with all $\langle L_k, z^l\rangle = 0$ for $k, l = \overline{1,m}$. Then the general solution to $\mathbf{Az} = \mathbf{0}$ depends on m arbitrary constants $c_k,\ k = \overline{1,m}$, and $N(\mathbf{A}) = \operatorname{span}\{z^1, \ldots, z^m\}$. Thus, the solution to $\mathbf{Az} = \mathbf{0}$ is equivalent to the solution to the differential equation $\mathcal{L}z = 0$ only.
- $0 < d < m \Leftrightarrow \varDelta = 0$ and $\operatorname{rank}(\langle L_k, z^l\rangle) = m - d$ (here $k, l = \overline{1,m}$). In this case, some $m - d$ constants are solved and represented by other d arbitrary constants. In other words, there exist d rows in the determinant representation of $\varDelta$ above, those are linear combinations of the rest $m - d$ linearly independent rows. Let us denote these "*dependent*" rows by $(\langle L_{k_l}, z^1\rangle, \ldots, \langle L_{k_l}, z^m\rangle)$ for $k_l,\ l = \overline{1,d}$. The independent rows are also given by $(\langle L_{k_j}, z^1\rangle, \ldots, \langle L_{k_j}, z^m\rangle)$ for $k_j,\ j = \overline{d+1,m}$. Thus, the solution to the problem $\mathbf{Az} = \mathbf{0}$ is now equivalent to the solution of the simplified discrete problem: the equation $\mathcal{L}z = 0$ with conditions $\langle L_{k_j}, z\rangle = 0,\ j = \overline{d+1,m}$, representing linearly independent rows only.

For more studies on the nullspace of the m-th order discrete operator, you can also read the paper [90, Paukštaitė and Štikonas 2016].

2.2 Range of the matrix A

For the discrete problem (2.1)–(2.2), we obtain its range representation $R(\mathbf{A})$.

Lemma 4.1.

1) If $d = m$, then for all $f \in F(X_{n-m})$ we have

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-m};\ \sum_{j=0}^{n-m} \langle L_1^{\cdot}, G^c_{\cdot j}\rangle f_j; \ldots; \sum_{j=0}^{n-m} \langle L_m^{\cdot}, G^c_{\cdot j}\rangle f_j \right)^{\top} \right\}.$$

2) If $0 < d < m$, then the range $R(\mathbf{A})$ is generated by the vector

$$\mathbf{b} = \sum_{i=0}^{n-m} f_i \mathbf{e}^i + \sum_{l=1}^{d} \Bigg(\sum_{j=d+1}^{m} g_{k_j} \langle L_{k_l}, v^{k_j} \rangle + \sum_{j=0}^{n-m} \langle L_{k_l}^{\cdot}, G^a_{\cdot j} \rangle f_j \Bigg) \mathbf{e}^{n-m+k_l}$$
$$+ \sum_{j=d+1}^{m} g_{k_j} \mathbf{e}^{n-m+k_j}$$

with every $f \in F(X_{n-m})$ and $g_{k_j} \in \mathbb{C}$, $j = \overline{d+1, m}$.

Here $G^c \in F(X_n \times X_{n-m})$ is the discrete Green's function for the discrete Cauchy problem

$$\mathcal{L}u = f, \qquad u_j = 0, \ j = \overline{0, m-1}. \tag{2.4}$$

Let us note [100, Roman 2011] that this discrete Green's function $G^c \in F(X_n \times X_{n-m})$ always exists and is of the form

$$G^c_{ij} = \frac{1}{a^m_j \cdot W_{j+m}} \begin{cases} \widetilde{W}_{i,j+m} & j+m < i, \\ 0, & i \leqslant j+m, \end{cases} \qquad i \in X_n, \ j \in X_{n-m}. \tag{2.5}$$

Here

$$W_{j+m} := W[z^1, \dots, z^m]_{j+m} = \begin{vmatrix} z^1_{j+1} & z^1_{j+2} & \cdots & z^1_{j+m} \\ z^2_{j+1} & z^2_{j+2} & \cdots & z^2_{j+m} \\ \cdots & \cdots & \cdots & \cdots \\ z^m_{j+1} & z^m_{j+2} & \cdots & z^m_{j+m} \end{vmatrix}$$

is the Wronskian's determinant of the discrete biorthogonal fundamental system $\{z^1, \dots, z^m\}$ at a point $j+m$ for every $j \in X_{n-m}$. Another determinant $\widetilde{W}_{i,j+m}$ is obtained replacing the last column of the Wronskian W_{j+m} by the column $(z^1_i, \dots, z^m_i)^\top \in \mathbb{C}^{m \times 1}$ for every selected $i \in X_n$.

In Lemma 4.1, other discrete Green's function $G^a \in F(X_n \times X_{n-m})$ and the biorthogonal fundamental system $v^1, \dots, v^m \in F(X_n)$ are taken for a problem $\mathcal{L}u = f$ with original conditions $\langle L_{k_j}, u \rangle = 0$, $j = \overline{d+1, m}$, and conditions $\langle \ell_{k_l}, u \rangle = 0$, $l = \overline{1, d}$, replacing conditions $\langle L_{k_l}, u \rangle = 0$. These conditions $\langle \ell_{k_l}, u \rangle = 0$, $l = \overline{1, d}$, are selected to obtain the auxiliary problem with $\Delta \neq 0$.

Below we provide the representation of the nullspace of the adjoint operator $N(\mathbf{A}^*)$.

Corollary 4.2. *The following statements are valid:*

1) if $d = m$, then $N(\mathbf{A}^*)$ is generated by linearly independent vectors

$$\mathbf{w}^k = -\sum_{j=0}^{n-m} \overline{\langle L_k^{\cdot}, G^c_{\cdot j} \rangle} \mathbf{e}^j + \mathbf{e}^{n-m+k}, \qquad k = \overline{1, m}.$$

2) if $0 < d < m$*, then* $N(\mathbf{A}^*)$ *is generated by linearly independent vectors*

$$\mathbf{w}^l = -\sum_{j=0}^{n-m} \overline{\langle L^{\cdot}_{k_l}, G^a_{\cdot j}\rangle} \mathbf{e}^j - \sum_{j=d+1}^{m} \overline{\langle L_{k_l}, v^{k_j}\rangle} \mathbf{e}^{k_j} + \mathbf{e}^{k_l}, \qquad l = \overline{1,d}.$$

Now applying the Fredholm alternative theorem, we get the solvability conditions to the problem (2.1)–(2.2) without the unique solution ($\varDelta = 0$).

Corollary 4.3. *(Solvability conditions) The problem* (2.1)–(2.2) *with* $\varDelta = 0$ *is solvable if and only if the conditions are valid:*

1) $\sum\limits_{j=0}^{n-m} \langle L^{\cdot}_k, G^c_{\cdot j}\rangle f_j = g_k$, $k = \overline{1,m}$, *for* $d = m$*;*

2) $\sum_{j=d+1}^{m} g_{k_j} \langle L_{k_l}, v^{k_j}\rangle + \sum\limits_{j=0}^{n-m} \langle L^{\cdot}_{k_l}, G^a_{\cdot j}\rangle f_j = g_{k_l}$ *for* $l = \overline{1,d}$ *if* $0 < d < m$*.*

Example 4.4. *Let us continue the investigation of the differential problem*

$$u^{(m)} = f(x), \quad x \in [0,1], \tag{2.6}$$

$$u(0) = g_1,\ u'(0) = g_2,\ \ldots,\ u^{(m-2)}(0) = g_{m-1},\ u(1) - \gamma u(\xi) = g_m, \tag{2.7}$$

which was considered in Chapter 2. Here we take again a point $\xi \in (0,1)$*, parameters* γ, $g_k \in \mathbb{R}$, $k = \overline{1,m}$*, and a real function* $f \in C[0,1]$*.*

We suppose that ξ *is coincident with a point of the mesh* $\overline{\omega}^h$*, i.e.,* $\xi = sh$ *for some positive* $s \in X_{n-m}$*, and denote the right hand side by* $f_i = f(x_{i+1})$*,* $i \in X_{n-m}$*. Let us introduce finite differences* $\nabla^0 u_i = u_i$*,* $\nabla^1 u_i = u_{i+1} - u_i$ *and* $\nabla^{k+1} u_i = \nabla^1(\nabla^k u)_i$ *for* $k \geqslant 0$*. Then we apply the finite difference method on the uniform grid* $\overline{\omega}^h$ *and consider the following m-th order real discrete problem*

$$(\mathcal{L}u)_i := \nabla^m u_i / h^m = f_i, \quad i \in X_{n-m}, \tag{2.8}$$

$$\langle L_k, u\rangle := \nabla^{k-1} u_0 / h^{k-1} = g_k, \quad k = \overline{1, m-1}, \tag{2.9}$$

$$\langle L_m, u\rangle := u_n - \gamma u_s = g_m. \tag{2.10}$$

It can be represented by a linear system (2.3) *with the matrix* $\mathbf{A} = \mathbf{A}(\gamma)$*,* $\gamma \in \mathbb{R}$*.*

First, we consider the classical problem (2.6)–(2.7) *with* $\gamma = 0$ *and the matrix* $\mathbf{A}^{\mathrm{cl}} = \mathbf{A}(0)$*. Let us take the particular fundamental system* $z^k = x^{k-1}$*,* $k = \overline{1,m}$*, and denote the determinant* $\varDelta = \det(\langle L_k, z^l\rangle)$*, where* $k, l = \overline{1,m}$*, by* $\varDelta(L_1, \ldots, L_m)$ *as well. Then applying the additivity property of a* *column of the determinant and remembering that the determinant with two equal columns is equal to zero, we rewrite the determinant as follows*

$$\begin{aligned}\Delta^{cl} &:= \Delta|_{\gamma=0} = \Delta(L_1, L_2, \dots, L_{m-1}, \delta_n)\\ &= \Delta(\delta_0, h^{-1}(\delta_1 - \delta_0), L_3 \dots, L_{m-1}, \delta_n)\\ &= h^{-1}\Delta(\delta_0, \delta_1, L_3 \dots, L_{m-1}, \delta_n) - h^{-1}\Delta(\delta_0, \delta_0, L_3 \dots, L_{m-1}, \delta_n)\\ &= h^{-1}\Delta(\delta_0, \delta_1, L_3 \dots, L_{m-1}, \delta_n) = \dots\\ &= h^{-(m-2)(m-1)/2} \cdot \Delta(\delta_0, \delta_1, \delta_2, \dots, \delta_{m-1}, \delta_n),\end{aligned}$$

were $\langle \delta_i, u\rangle := u_i,\ i \in X_n$. *Now we observe that*

$$\begin{aligned}\Delta^{cl} &= h^{-(m-2)(m-1)/2} \cdot \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1\\ 0 & h & 2h & \dots & (m-2)h & 1\\ 0 & h^2 & (2h)^2 & \dots & \big((m-2)h\big)^2 & 1\\ & & & \dots & & \\ 0 & h^{m-1} & (2h)^{m-1} & \dots & \big((m-2)h\big)^{m-1} & 1 \end{vmatrix}\\ &= h^{-(m-2)(m-1)/2} \cdot V(x_0, x_1, \dots, x_{m-2}, 1) \neq 0\end{aligned}$$

because the Vandermonde determinant $V(a_1, \dots, a_m) = \prod_{1\leqslant i<j\leqslant m}(a_i - a_j)$ *is nonzero for every different real numbers* $a_1, \dots, a_m \in \mathbb{R}$. *From here follows that the classical problem* (2.8)–(2.10) *with* $\gamma = 0$ *always has the unique solution, since the existence condition of the unique solution is always fulfilled:*

$$\Delta^{cl} = h^{-(m-2)(m-1)/2} \prod_{0\leqslant i<j\leqslant m-2} (x_i - x_j)(x_i - 1)(x_{m-2} - 1) \neq 0.$$

Thus, $\det \mathbf{A}^{\mathrm{cl}} \neq 0$.

Now we note that the determinant Δ *for the problem* (2.8)–(2.10) *with every real* γ *is of the form*

$$\Delta := \Delta^{cl} - \gamma\Delta(L_1, \dots, L_{m-2}, \delta_s) = \Delta^{cl}(1 - \gamma v_s^m), \tag{2.11}$$

where $v_i^m := \Delta(L_1, \dots, L_{m-1}, \delta_i)/\Delta^{cl},\ i \in X_n$, *is the unique solution to the classical problem (*$\gamma = 0$ *for* (2.8)–(2.10)*), that is,*

$$\mathcal{L}u = 0, \qquad \langle L_k, u\rangle = 0,\ k = \overline{1, m-1}, \qquad u_n = 1. \tag{2.12}$$

For details see [100, Roman 2011] or the following section. So, (2.11) *is equal to zero if and only if* $\gamma v_s^m = 1$. *This condition is also equivalent to* $\det \mathbf{A} = 0$.

Let us now take the problem (2.8)–(2.10) *with the singular matrix, i.e.,* $\det \mathbf{A} = 0$ *or equivalently,* $\Delta = 0$.

First, we note that the discrete function $v^m \in F(X_n)$ belongs to $N(\boldsymbol{A})$. We can directly verify that it satisfies all homogenous equations (2.8)–(2.10). *Second, the nullity $d = \dim N(\mathbf{A}) = 1$ because all rows of the nonsingular matrix $\mathbf{A}^{\text{cl}}$ are linearly independent, and the singular $\mathbf{A}$ differs from $\mathbf{A}^{\text{cl}}$ with the last row only. From here it also follows that the functional $\mathbf{L}_m$ is a linear combination of other rows of $\mathbf{A}$, representing the operator $\boldsymbol{\mathcal{L}}$ and functionals $\mathbf{L}_k,\ k = \overline{1, m-1}$. Thus, $k_1 = m$ and $k_j = j - 1,\ j = \overline{2, m}$.*

Here we take the classical problem (2.12) *as the auxiliary problem. Since $\Delta^{cl} \neq 0$, it has the discrete Green's function $G^{\text{cl}} \in F(X_n \times X_{n-m})$ and the discrete biorthogonal fundamental system $v^k,\ k = \overline{1, m}$. Then we apply Corollary 4.2 and obtain the vector*

$$\mathbf{w} = \gamma \sum_{j=0}^{n-m} G_{sj}^{\text{cl}} \mathbf{e}^j + \gamma \sum_{j=n-m+1}^{n-1} v_s^{j-n+m} \mathbf{e}^j + \mathbf{e}^n,$$

generating the nullspace $N(\mathbf{A}^*)$. Moreover, Corollary 4.3 provides the necessary and sufficient solvability condition

$$\gamma \sum_{j=0}^{n-m} G_{sj}^{\text{cl}} f_j + \gamma \sum_{k=1}^{m-1} g_k v_s^k + g_m = 0$$

for the discrete problem (2.8)–(2.10) *with $\Delta = 0$ or equivalently* $\det \mathbf{A} = 0$, *what gives $\gamma = 1/v_s^m$ in formulas above.*

3 Problem with the unique solution

In this section, we consider the unique solution to the discrete problem (2.1)–(2.2), that is also given in the matrix form $\mathbf{A}\mathbf{u} = \mathbf{b}$. If $\det \mathbf{A} \neq 0$, we have the representation $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$. Now our aim is to analyze the structure of the inverse matrix $\mathbf{A}^{-1} \in \mathbb{C}^{(n+1)\times(n+1)}$, which was earlier investigated by Roman [100, 2011]. This information directs the way how to make generalizations for problems (2.1)–(2.2) without the unique solution ($\Delta = 0$ or, equivalently, $\det \mathbf{A} = 0$) in the following section.

3.1 Representation of the inverse matrix

Since the right hand side of the problem $\mathbf{A}\mathbf{u} = \mathbf{b}$ has the particular form $\mathbf{b} = (f_0, f_1, \ldots, f_{n-m}, g_1, g_2, \ldots; g_m)^\top$ with every $\mathbf{f} = (f_0, f_1, \ldots, f_{n-m})^\top \in \mathbb{C}^{(n+1)\times 1}$ and complex numbers $g_1, \ldots, g_m$, the unique solution can also be written in the special form

$$\mathbf{u} = \mathbf{G}\mathbf{f} + g_1\mathbf{v}^1 + \ldots + g_m\mathbf{v}^m. \tag{3.1}$$

Here $\mathbf{G} \in \mathbb{C}^{(n+1)\times(n-m+1)}$ and $\mathbf{v}^k \in \mathbb{C}^{(n+1)\times 1}$, $k = \overline{1,m}$, are submatrices of the inverse matrix

$$\mathbf{A}^{-1} = (\mathbf{G},\ \mathbf{v}^1,\ \ldots,\ \mathbf{v}^m).$$

Let us now take the discrete representation of the solution (3.1) as follows

$$u = Gf + g_1 v^1 + \ldots + g_m v^m,$$

which has the explicit form

$$u_i = \sum_{j=0}^{n-m} G_{ij} f_j + g_1 v_i^1 + \ldots + g_m v_i^m, \quad i \in X_n. \tag{3.2}$$

Here the kernel $G \in F(X_n \times X_{n-m})$ is also known as the *discrete Green's function* and functions $v^k \in F(X_n)$, $k = \overline{1,m}$, are called the *discrete biorthogonal fundamental system* for the problem (2.1)–(2.2) [100, Roman 2011]. Using the inverse matrix $\mathbf{B} = \mathbf{A}^{-1}$, we can always calculate the discrete Green's function as well as the discrete biorthogonal fundamental system in the following way

$$G_{ij} = B_{ij}, \quad i \in X_n,\ j \in X_{n-m}, \tag{3.3}$$

$$v_i^k = B_{i,n-m+k}, \quad i \in X_n,\ k = \overline{1,m}. \tag{3.4}$$

3.2 Properties of discrete Green's functions

According to Roman [100, 2011], the discrete Green's function G is the unique solution to the discrete problem

$$\begin{aligned} \mathcal{L}_{i\cdot} G_{\cdot j} &= \delta_{ij}, \quad i \in X_{n-m}, \\ \langle L_k^{\cdot}, G_{\cdot j} \rangle &= 0, \quad k = \overline{1,m}, \end{aligned} \tag{3.5}$$

for every fixed $j \in X_{n-m}$. Moreover, discrete functions v^l, $l = \overline{1,m}$, are unique solutions to corresponding discrete problems

$$\mathcal{L} v^l = 0, \qquad \langle L_k, v^l \rangle = \delta_k^l, \quad k, l = \overline{1,m}, \tag{3.6}$$

and can always be obtained from the formulas $v_i^l = \Delta_i^l / \Delta$ for $l = \overline{1,m}$. Here Δ_i^l is the determinant obtained replacing the l-th column in Δ by $(z_i^1, z_i^2, \ldots, z_i^m)^\top$. Now we can formulate the representation of the discrete Green's function.

Lemma 4.5 (Roman 2011, [100]). *If $\Delta \neq 0$, then the discrete Green's function for the problem* (2.1)–(2.2) *is given by*

$$G_{ij} = G_{ij}^c - v_i^1 \langle L_1^{\cdot}, G_{\cdot j}^c \rangle - \ldots - v_i^m \langle L_m^{\cdot}, G_{\cdot j}^c \rangle, \qquad i \in X_n,\ j \in X_{n-m}.$$

The discrete Green's function $G^c \in F(X_n \times X_{n-m})$ always exists [100, Roman 2011] and describes the unique solution $u^c \in F(X_n)$ to the discrete Cauchy problem (2.3) in the form $u^c_i = \sum_{j=0}^{n-m} G^c_{ij} f_j$, $i \in X_n$. Thus, the following representation is always valid

$$u = u^c + (g_1 - \langle L_1, u^c \rangle) v^1 + \ldots + (g_m - \langle L_m, u^c \rangle) v^m.$$

This representation follows from the other, more general result: unique solutions of two relative problems

$$\begin{aligned} &\mathcal{L}u = f, && \mathcal{L}v = f, \\ &\langle \widetilde{L}_k, u \rangle = \widetilde{g}_k, \quad k = \overline{1,m}, && \langle L_k, v \rangle = g_k, \quad k = \overline{1,m}, \end{aligned} \tag{3.7}$$

where functionals $\widetilde{L}_k$ and L_k, $k = \overline{1,m}$, may be different, are related. We formulate this statement below.

Corollary 4.6 (Roman 2011, [100]). *For unique solutions of problems* (3.7)*, the following equality is always satisfied*

$$v = u + (g_1 - \langle L_1, u \rangle) v^1 + \ldots + (g_m - \langle L_m, u \rangle) v^m.$$

The analogous relation between discrete Green's functions is also valid.

Theorem 4.7 (Roman 2011, [100]). *Discrete Green's functions $\widetilde{G}$ and G of problems* (3.7)*, respectively, are linked with the equality*

$$G_{ij} = \widetilde{G}_{ij} - v^1_i \langle L^{\cdot}_1, \widetilde{G}_{\cdot j} \rangle - \ldots - v^m_i \langle L^{\cdot}_m, \widetilde{G}_{\cdot j} \rangle, \quad i \in X_n, \ j \in X_{n-m}.$$

Let us note that here we used the biorthogonal fundamental system v^k, $k = \overline{1,m}$, for the second problem (3.7) only. Since conditions $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$ for both problems, respectively, are fulfilled, we can obtain the relation between biorthogonal fundamental systems for these problems (3.7) as well.

Corollary 4.8. *Let $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$ for problems* (3.7)*. Then their biorthogonal fundamental systems $\widetilde{v}^l \in F(X_n)$ and $v^l \in F(X_n)$ $(l = \overline{1,m})$ are related by*

$$\begin{pmatrix} \langle L_1, \widetilde{v}^1 \rangle & \ldots & \langle L_m, \widetilde{v}^1 \rangle \\ \ldots & \ldots & \ldots \\ \langle L_1, \widetilde{v}^m \rangle & \ldots & \langle L_m, \widetilde{v}^m \rangle \end{pmatrix} \begin{pmatrix} v^1 \\ \ldots \\ v^m \end{pmatrix} = \begin{pmatrix} \widetilde{v}^1 \\ \ldots \\ \widetilde{v}^m \end{pmatrix}.$$

Roman also separately investigated the unique solution ($\Delta \neq 0$) to a problem with nonlocal boundary conditions

$$(\mathcal{L}u)_i := a_i^m u_{i+m} + \ldots + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \qquad i \in X_{n-m}, \tag{3.8}$$

$$\langle L_k, u \rangle := \langle \kappa_k, u \rangle - \gamma_k \langle \varkappa_k, u \rangle = g_k, \quad k = \overline{1,m}, \tag{3.9}$$

where functionals κ_k describe classical parts but $\varkappa_k$, $k = \overline{1,m}$, represent fully nonlocal parts of conditions (3.9). If the classical problem (all $\gamma_k = 0$) has the unique solution $u^{\text{cl}} \in F(X_n)$, then the unique solution of the entire problem (4.10)–(4.11) is given by

$$u = u^{\text{cl}} + \gamma_1 \langle \varkappa_1, u^{\text{cl}} \rangle + \ldots + \gamma_m \langle \varkappa_m, u^{\text{cl}} \rangle.$$

Their discrete Green's functions $G^{\text{cl}} \in F(X_n \times X_{n-m})$ and $G \in F(X_n \times X_{n-m})$, respectively, are analogously related

$$G_{ij} = G_{ij}^{\text{cl}} + \gamma_1 v_i^1 \langle \varkappa_1^{\cdot}, G_{\cdot j}^{\text{cl}} \rangle + \ldots + \gamma_m v_i^m \langle \varkappa_m^{\cdot}, G_{\cdot j}^{\text{cl}} \rangle, \qquad i \in X_n,\ j \in X_{n-m}.$$

4 The unique discrete minimizer

If the condition $\det \mathbf{A} = 0$ or the equivalent equality $\Delta = 0$ is valid, then neither the matrix $\mathbf{A}$ has the unique inverse matrix $\mathbf{A}^{-1}$ nor the problem (2.1)–(2.2) has the unique solution $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$.

In this section, we are going to solve the problem (2.1)–(2.2) with $\Delta = 0$ in the least squares sense. Here we focus on the matrix representation (2.3) of the discrete problem (2.1)–(2.2) and look for a minimum norm least squares solution as in the previous chapter.

4.1 Generalized discrete Green's function

The representation of the minimum norm least squares solution

$$\mathbf{u}^o = \mathbf{A}^\dagger \mathbf{b} \tag{4.1}$$

to the problem (2.1)–(2.2) as well as the general least squares solution

$$\mathbf{u}^g = \mathbf{A}^\dagger \mathbf{b} + \mathbf{P}_{N(A)} \mathbf{c}, \quad \forall \mathbf{c} \in \mathbb{C}^{(n+1)\times 1}, \tag{4.2}$$

are introduced in Subsection 5.1 of Chapter 3. Properties of the Moore–Penrose inverse $\mathbf{A}^\dagger \in \mathbb{C}^{(n+1)\times(n+1)}$ are also given there. Now considering the form of $\mathbf{b} = (f_0, f_1, \ldots, f_{n-m}, g_1, \ldots, g_m)^\top \in \mathbb{C}^{(n+1)\times 1}$ with every $\mathbf{f} = (f_0, f_1, \ldots, f_{n-m})^\top \in \mathbb{C}^{(n-m+1)\times 1}$ and complex numbers g_k, $k = \overline{1,m}$, we write the minimum norm least squares solution (4.1) in the following special form

$$\mathbf{u}^o = \mathbf{G}^g\mathbf{f} + g_1\mathbf{v}^{g,1} + \ldots + g_m\mathbf{v}^{g,m}. \tag{4.3}$$

Here $\mathbf{G}^g \in \mathbb{C}^{(n+1)\times(n-m+1)}$ and $\mathbf{v}^{g,k} \in \mathbb{C}^{(n+1)\times 1}$, $k = \overline{1,m}$, are submatrices of the Moore–Penrose inverse

$$\mathbf{A}^\dagger = (\mathbf{G}^g,\ \mathbf{v}^{g,1},\ \ldots,\ \mathbf{v}^{g,m}).$$

We also get the discrete representation of the minimum norm least squares solution

$$u^o = G^g f + g_1 v^{g,1} + \ldots + g_m v^{g,m},$$

which can be considered in the explicit form

$$u_i^o = \sum_{j=0}^{n-m} G_{ij}^g f_j + g_1 v_i^{g,1} + \ldots + g_m v_i^{g,m}, \quad i \in X_n. \tag{4.4}$$

This representation of the unique minimizer $u^o \in F(X_n)$ resembles the representation of the unique solution (3.2) for the particular case, investigated in Section 3. Furthermore, formulas (3.2) and (4.4) are coincident if $\varDelta \neq 0$. Thus, the discrete kernel $G^g \in F(X_n \times X_{n-m})$ is called the *generalized discrete Green's function* and functions $v^{g,k} \in F(X_n)$, $k = \overline{1,m}$, – the *generalized discrete biorthogonal fundamental system* for the m-th order discrete problem (2.1)–(2.2).

Let us note that the generalized discrete Green's function and the generalized discrete biorthogonal fundamental system can always be calculated using the Moore–Penrose inverse $\mathbf{B} = \mathbf{A}^\dagger$ as given below

$$G_{ij}^g = B_{ij}, \quad i \in X_n,\ j \in X_{n-m}, \tag{4.5}$$

$$v_i^{g,k} = B_{i,n-m+k}, \quad i \in X_n,\ k = \overline{1,m}. \tag{4.6}$$

On the other hand, we have $\mathbf{A}^\dagger = \mathbf{A}^{-1}$ for $\det \mathbf{A} \neq 0$. Then the discrete minimum norm least squares solution $u^o \in F(X_n)$ is also coincident with the unique discrete solution $u \in F(X_n)$, the generalized discrete Green's function G^g is coincident with the discrete Green's function G, the generalized discrete biorthogonal fundamental system $v^{g,k}$, $k = \overline{1,m}$, is coincident with the discrete biorthogonal fundamental system v^k, $k = \overline{1,m}$.

4.2 Properties of minimizers

In this subsection, we investigate properties of minimum norm least squares solutions and their generalized discrete Green's functions. Obtained results extend corresponding properties from Section 3.

Lemma 4.9. *The generalized discrete Green's function $G^g \in F(X_n \times X_{n-m})$ is the minimum norm least squares solution of the following discrete problem*

$$\begin{aligned} \mathcal{L}_{i\cdot} G^g_{\cdot j} &= \delta_{ij}, \quad i \in X_{n-m}, \\ \langle L^{\cdot}_k, G^g_{\cdot j} \rangle &= 0, \quad k = \overline{1, m}, \end{aligned} \tag{4.7}$$

for every fixed $j \in X_{n-m}$.

Below the generalization of (3.6) is given, where $\varDelta = 0$ is valid.

Lemma 4.10. *Discrete functions $v^{g,l}$, $l = \overline{1, m}$, are minimum norm least squares solutions of corresponding discrete problems*

$$\begin{aligned} \mathcal{L} v^{g,l} &= 0, \\ \langle L_k, v^{g,l} \rangle &= 1, \quad k, l = \overline{1, m}. \end{aligned} \tag{4.8}$$

Let us now consider two relative problems (3.7), where the first discrete problem has the unique solution ($\widetilde{\varDelta} \neq 0$). Here and further $G^g \in F(X_n \times X_{n-m})$ is the generalized discrete Green's function and $v^{g,k} \in F(X_n)$, $k = \overline{1, m}$, are the generalized biorthogonal fundamental system to the second problem (3.7), which may have the unique solution ($\varDelta \neq 0$) or not ($\varDelta = 0$).

Theorem 4.11. *If the first discrete problem* (3.7) *has the unique exact solution $u \in F(X_n)$, then the minimum norm least squares solution $u^o \in F(X_n)$ of the other problem* (3.7) *is given by*

$$u^o = u - P_{N(A)} u + v^{g,1}(g_1 - \langle L_1, u \rangle) + \ldots + v^{g,m}(g_m - \langle L_m, u \rangle).$$

Further, we provide the representation of the minimum norm least squares solution, which is always applicable.

Corollary 4.12. *The minimum norm least squares solution u^o to the problem* (2.1)–(2.2) *can always be represented by the unique solution u^c to the Cauchy problem* (2.4) *as follows*

$$u^o = u^c - P_{N(A)} u^c + (g_1 - \langle L_1, u^c \rangle) v^{g,1} + \ldots + (g_m - \langle L_m, u^c \rangle) v^{g,m}.$$

Now we present the property of generalized discrete biorthogonal fundamental systems for problems (3.7), which is similar to Corollary 4.8.

Corollary 4.13. *Let $\widetilde{\varDelta} \neq 0$ for the first problem* (3.7)*. Then the discrete biorthogonal fundamental system $\widetilde{v}^k \in F(X_n)$, $k = \overline{1, m}$, of the first problem and the generalized discrete biorthogonal fundamental system $v^{g,k} \in F(X_n)$, $k = \overline{1, m}$, of the second problem* (3.7) *are related by*

$$\begin{pmatrix} \langle L_1, \widetilde{v}^1 \rangle & \ldots & \langle L_m, \widetilde{v}^1 \rangle \\ \ldots & \ldots & \ldots \\ \langle L_1, \widetilde{v}^m \rangle & \ldots & \langle L_m, \widetilde{v}^m \rangle \end{pmatrix} \begin{pmatrix} v^{g,1} \\ \ldots \\ v^{g,m} \end{pmatrix} = \begin{pmatrix} P_{N(A)^\perp} \widetilde{v}^1 \\ \ldots \\ P_{N(A)^\perp} \widetilde{v}^m \end{pmatrix}.$$

4.3 Relations between generalized discrete Green's functions

Below we provide the representation of a generalized discrete Green's function.

Theorem 4.14. *If $\widetilde{\Delta} \neq 0$ for the first problem* (3.7)*, then its discrete Green's function $G \in F(X_n \times X_{n-m})$ and the generalized discrete Green's function $G^g \in F(X_n \times X_{n-m})$ of the second problem* (3.7) *are related by the equality*

$$G^g_{ij} = G_{ij} - (P_{N(A)})_{i\cdot} G_{\cdot j} - v^{g,1}_i \langle L^{\cdot}_1, G_{\cdot j}\rangle - \ldots - v^{g,m}_i \langle L^{\cdot}_m, G_{\cdot j}\rangle$$

for all $i \in X_n,\ j \in X_{n-m}$.

Since the discrete Green's function G^c of the initial problem (2.4) always exists, we obtain the representation of the generalized discrete Green's function, which is always valid.

Corollary 4.15. *The generalized discrete Green's function $G^g \in F(X_n \times X_{n-m})$ to the problem* (2.1)–(2.2) *is of the form*

$$G^g_{ij} = G^c_{ij} - (P_{N(A)})_{i\cdot} G^c_{\cdot j} - v^{g,1}_i \langle L^{\cdot}_1, G^c_{\cdot j}\rangle - \ldots - v^{g,m}_i \langle L^{\cdot}_m, G^c_{\cdot j}\rangle$$

for all $i \in X_n,\ j \in X_{n-m}$.

4.4 Applications to nonlocal boundary conditions

For the problem with nonlocal boundary conditions (3.7)–(3.8), we obtain the following representation of the minimizer u^o using the unique solution u^{cl} to the classical problem ($\gamma_k = 0,\ k = \overline{1,m}$).

Corollary 4.16. *If the classical problem* (3.7)–(3.8) *($\gamma_k = 0$) has the unique solution $u^{\mathrm{cl}} \in F(X_n)$, then the minimizer to the nonlocal boundary value problem* (3.7)–(3.8) *is given by*

$$u^o = u^{\mathrm{cl}} - P_{N(A)} u^{\mathrm{cl}} + \gamma_1 \langle \varkappa_1, u^{\mathrm{cl}}\rangle v^{g,1} + \ldots + \gamma_m \langle \varkappa_m, u^{\mathrm{cl}}\rangle v^{g,m}.$$

The generalized discrete Green's function for the problem with nonlocal boundary conditions (3.7)–(3.8) is also similarly described.

Corollary 4.17. *If the classical problem* (3.7)–(3.8) *(all $\gamma_k = 0$) has the discrete Green's function $G^{\mathrm{cl}} \in F(X_n \times X_{n-m})$, then the generalized Green's function of the nonlocal problem* (3.7)–(3.8) *is of the form*

$$G^g_{ij} = G^{\mathrm{cl}}_{ij} - (P_{N(A)})_{i\cdot} G^{\mathrm{cl}}_{\cdot j} + \gamma_1 v^{g,1}_i \langle \varkappa^{\cdot}_1, G^{\mathrm{cl}}_{\cdot j}\rangle + \ldots + \gamma_m v^{g,m}_i \langle \varkappa^{\cdot}_m, G^{\mathrm{cl}}_{\cdot j}\rangle$$

for every $i \in X_n,\ j \in X_{n-m}$.

If $\Delta \neq 0$, then the inequality $\det \mathbf{A} \neq 0$ gives a nonsingular matrix for the discrete problem (2.3) and the zero orthogonal projector $\mathbf{P}_{N(A)} = \mathbf{O}$. Now all statements, proved in this section for a generalized discrete Green's function G^g, a generalized discrete biorthogonal fundamental system $v^{g,k}$, $k = \overline{1,m}$, and the minimum norm least squares solution u^o, are coincident with corresponding statements from Section 3, where a discrete Green's function G, a discrete biorthogonal fundamental system v^k, $k = \overline{1,m}$, and the unique solution u were considered.

5 Minimization problem in different spaces

In this section, we investigate the *generalized minimum norm least squares solution*, which minimizes residual $\mathbf{Au} - \mathbf{b}$ of the discrete problem (2.3) in the one norm and is of the smallest second norm among all minimizers of the residual. Here we provide properties and representations of the generalized minimum norm least squares solution and its generalized discrete Green's function.

5.1 The $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm

Let us take two inner products $(\mathbf{u}, \mathbf{v})_{\mathcal{H}_k}$, $k = 1, 2$, for $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{(n+1)\times 1}$ and denote by $\mathcal{H}_k$ the space $\mathbb{C}^{(n+1)\times 1}$ with the norm $\|\mathbf{u}\|_{\mathcal{H}_k} = (\mathbf{u}, \mathbf{u})_{\mathcal{H}_k}^{1/2}$, respectively. In this section, we minimize the $\mathcal{H}_1$ norm of the residual

$$\|\mathbf{Au}^g - \mathbf{b}\|_{\mathcal{H}_1} \leqslant \|\mathbf{Au} - \mathbf{b}\|_{\mathcal{H}_1}, \qquad \forall\, \mathbf{u} \in \mathbb{C}^{(n+1)\times 1}, \tag{5.1}$$

and investigate the minimizer $\mathbf{u}^o$ for which the $\mathcal{H}_2$ norm is smallest:

$$\|\mathbf{u}^o\|_{\mathcal{H}_2} < \|\mathbf{u}^g\|_{\mathcal{H}_2}, \qquad \forall\, \mathbf{u}^g \neq \mathbf{u}^o. \tag{5.2}$$

This $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm is introduced in Subsection 6.1 of Chapter 3 and is of the form

$$\mathbf{u}^o = \mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2}\mathbf{b}, \tag{5.3}$$

where $\mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2}$ is the matrix representation of the Moore-Penrose inverse $A^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2} : \mathcal{H}_1 \to \mathcal{H}_2$.

Considering the form of $\mathbf{b} = (f_0, f_1, \ldots, f_{n-m}, g_1, \ldots, g_m)^\top \in \mathbb{C}^{(n+1)\times 1}$ with every $\mathbf{f} = (f_0, f_1, \ldots, f_{n-m})^\top \in \mathbb{C}^{(n-m+1)\times 1}$ and complex numbers g_k, $k = \overline{1,m}$, we write the minimizer (5.3) in the following special form

$$\mathbf{u}^o = \mathbf{G}^g\mathbf{f} + g_1\mathbf{v}^{g,1} + \ldots + g_m\mathbf{v}^{g,m}. \tag{5.4}$$

Here $\mathbf{G}^g \in \mathbb{C}^{(n+1)\times(n-m+1)}$ and $\mathbf{v}^{g,k} \in \mathbb{C}^{(n+1)\times 1}$, $k = \overline{1,m}$, are submatrices of the matrix representation of the Moore–Penrose inverse

$$\mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2} = (\mathbf{G}^g,\ \mathbf{v}^{g,1},\ \ldots,\ \mathbf{v}^{g,m}).$$

The minimizer (5.4), given in the explicit form

$$u_i^o = \sum_{j=0}^{n-m} G^g_{ij} f_j + g_1 v_i^{g,1} + \ldots + g_m v_i^{g,m}, \quad i \in X_n, \tag{5.5}$$

is coincident with the representation of the unique solution (3.2) for the particular case $\det \mathbf{A} \neq 0$, investigated in Section 3. Moreover, it also describes the minimizer (4.4) taking the standard Euclidean space instead of two different spaces. Hence, we call $G^g \in F(X_n \times X_{n-m})$ by the *generalized discrete Green's function* and functions $v^{g,k} \in F(X_n)$, $k = \overline{1,m}$, – the *generalized discrete biorthogonal fundamental system* describing the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm for the problem (2.1)–(2.2).

This generalized discrete Green's function and the generalized discrete biorthogonal fundamental system can always be obtained using the Moore–Penrose inverse $\mathbf{B} = \mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2}$ in the following equalities

$$G^g_{ij} = B_{ij}, \quad i \in X_n,\ j \in X_{n-m},$$
$$v_i^{g,k} = B_{i,n-m+k}, \quad i \in X_n, \quad k = \overline{1,m}.$$

So, for $\det \mathbf{A} \neq 0$, we have the equality $\mathbf{A}^{\dagger}_{\mathcal{H}_1,\mathcal{H}_2} = \mathbf{A}^{-1}$. Then the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm $u^o \in F(X_n)$ is also coincident with the unique discrete solution $u \in F(X_n)$, the generalized discrete Green's function $G^g \in F(X_n \times X_{n-m})$ is coincident with the discrete Green's function $G \in F(X_n \times X_{n-m})$, the generalized discrete biorthogonal fundamental system $v^{g,k} \in F(X_n)$, $k = \overline{1,m}$, is coincident with the discrete biorthogonal fundamental system $v^k \in F(X_n)$, $k = \overline{1,m}$.

5.2 Properties of minimizers

In this subsection, we present properties of the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm. Let us begin with the characterization of the generalized discrete Green's function.

Lemma 4.18. *For every fixed $j \in X_{n-m}$, the generalized discrete Green's function $G^g \in F(X_n \times X_{n-m})$ is the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm to the discrete problem*

$$\mathcal{L}_i.G^g_{\cdot j} = \delta_{ij}, \quad i \in X_{n-m},$$
$$\langle L_k^{\cdot}, G^g_{\cdot j}\rangle = 0, \quad k = \overline{1,m}.$$

The generalized discrete biorthogonal fundamental system has the following property.

Lemma 4.19. *Every discrete function $v^{g,l} \in F(X_n)$, $l = \overline{1,m}$, is the $\mathcal{H}_1$ norm least squares solution of the minimum $\mathcal{H}_2$ norm to the corresponding discrete problem*

$$\mathcal{L}v^{g,l} = 0,$$
$$\langle L_k, v^{g,l} \rangle = \delta_k^l, \quad k = \overline{1,m}.$$

For two discrete problems (3.7), where the second problem has the generalized Green's function G^g and the generalized biorthogonal system $v^{g,k}$, $k = \overline{1,m}$, we obtain following relations.

Theorem 4.20. *The $\mathcal{H}_1$ least squares solutions of the minimum $\mathcal{H}_2$ norm u^o and v^o to problems* (3.7)*, respectively, are linked by the equality*

$$v^o = u^o - P_{\mathcal{H}_2,N(A)}u^o + G^g(f - \mathcal{L}u^o) + \sum_{k=1}^{m} v^{g,k}(g_k - \langle L_k, u^o \rangle).$$

Corollary 4.21. *Let the first problem* (3.7) *be (uniquely) solvable. Then the $\mathcal{H}_1$ least squares solutions of the minimum $\mathcal{H}_2$ norm (the unique solution) u^o and v^o to problems* (3.7)*, respectively, are related as follows*

$$v^o = u^o - P_{\mathcal{H}_2,N(A)}u^o + \sum_{k=1}^{m} v^{g,k}(g_k - \langle L_k, u^o \rangle).$$

Since the discrete Cauchy problem (2.4) always has the unique solution $u^c \in F(X_n)$ [100, Roman 2011], the minimizer (5.3) is always expressed as below.

Corollary 4.22. *The $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm to the problem* (2.1)–(2.2) *can always be described by the unique solution u^c to the Cauchy problem, i.e.,*

$$u^o = u^c - P_{\mathcal{H}_2,N(A)}u^c + v^{g,1}(g_1 - \langle L_1, u^c \rangle) + \ldots + v^{g,m}(g_m - \langle L_m, u^c \rangle).$$

Moreover, we obtain the following representation.

Corollary 4.23. *If the classical problem* (3.8)–(3.9) *($\gamma_k = 0$) has the unique solution $u^{\mathrm{cl}} \in F(X_n)$, then the $\mathcal{H}_1$ least squares solution of the minimum $\mathcal{H}_2$ norm to the nonlocal boundary value problem* (4.10)–(4.11) *is given by*

$$u^o = u^{\mathrm{cl}} - P_{\mathcal{H}_2,N(A)}u^{\mathrm{cl}} + v^{g,1}(g_1 - \langle L_1, u^{\mathrm{cl}} \rangle) + \ldots + v^{g,m}(g_m - \langle L_m, u^{\mathrm{cl}} \rangle).$$

5.3 Relations between generalized discrete Green's functions

Let us now provide a representation of the generalized discrete Green's function, which describes the minimizer (6.3).

Theorem 4.24. *If there exists the discrete Green's function* $G \in F(X_n \in X_{n-m})$ *for the first problem* (3.7), *then the generalized discrete Green's function* $G^g \in F(X_n \times X_{n-m})$ *of the second problem is given by*

$$G^g_{ij} = G_{ij} - (P_{\mathcal{H}_2, N(A)} G)_{ij} - v_i^{g,1} \langle L_1^{\cdot}, G_{\cdot j} \rangle - \ldots - v_i^{g,m} \langle L_m^{\cdot}, G_{\cdot j} \rangle$$

for all $i \in X_n, \ j \in X_{n-m}$.

Here $(P_{\mathcal{H}_2, N(A)} G)_{ij}$ is trivial if $\det \mathbf{A} \neq 0$. Otherwise, it denotes the kernel of the orthogonal projection onto the nullspace $N(\mathbf{A})$ in the space $\mathcal{H}_2$:

$$(P_{\mathcal{H}_2, N(A)} G f)_i = \sum_{l=1}^{d} z_i^l (\mathbf{z}^l, \mathbf{G}\mathbf{f})_{\mathcal{H}_2} = \sum_{j=1}^{n-m} (P_{\mathcal{H}_2, N(A)} G)_{ij} f_j, \qquad i \in X_n,$$

where $\mathbf{z}^l$, $l = \overline{1, d}$, is a basis of the nullspace $N(\mathbf{A})$, orthonormal with respect to the inner product in $\mathcal{H}_2$.

Corollary 4.25. *The generalized discrete Green's function for the problem* (2.1)–(3.1) *is described by the discrete Green's function* $G^c \in F(X_n \times X_{n-m})$ *of the Cauchy problem* (2.4) *in the form*

$$G^g_{ij} = G^c_{ij} - (P_{\mathcal{H}_2, N(A)} G^c)_{ij} - v_i^{g,1} \langle L_1^{\cdot}, G^c_{\cdot j} \rangle - \ldots - v_i^{g,m} \langle L_m^{\cdot}, G^c_{\cdot j} \rangle$$

for all $i \in X_n, \ j \in X_{n-m}$.

For the problem (3.8)–(3.9) with nonlocal boundary conditions, the following property is valid.

Corollary 4.26. *If there exists the discrete Green's function* $G^{\mathrm{cl}} \in F(X_n \times X_{n-m})$ *of the classical problem* (3.8)–(3.9) ($\gamma_k = 0$)*, then the generalized discrete Green's function, describing the minimizer* (5.3) *to the problem* (3.8)–(3.9) *with nonlocal boundary conditions, is given by*

$$G^g_{ij} = G^{\mathrm{cl}}_{ij} - (P_{\mathcal{H}_2, N(A)} G^{\mathrm{cl}})_{ij} + v_i^{g,1} \langle \varkappa_1^{\cdot}, G^{\mathrm{cl}}_{\cdot j} \rangle + \ldots + v_i^{g,m} \langle \varkappa_m^{\cdot}, G^{\mathrm{cl}}_{\cdot j} \rangle$$

for all $i \in X_n, \ j \in X_{n-m}$.

5.4 Applications to differential problems

In this section, our aim is to consider real discrete problems approximating m-th order differential problems (1.1)–(1.2). Let us recall the minimum norm least squares solution $u^o \in H^m[0,1]$ to the differential problem (1.1)–(1.2), which minimizes the norm of the residual

$$\|\boldsymbol{L}u^g - \boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^m} = \inf_{u\in H^m[0,1]} \|\boldsymbol{L}u - \boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^m}$$

and has the minimum $H^m[0,1]$ norm among all minimizers $u^g \in H^m[0,1]$, i.e.,

$$\|u^o\|_{H^m[0,1]} < \|u^g\|_{H^m[0,1]} \qquad \forall\ u^g \neq u^o.$$

Thus, for m-th order discrete problems, we introduce two discrete norms

$$\|\mathbf{u}\|_{H^m(\overline{\omega}^h)} = \left(\sum_{k=0}^{m}\sum_{i=0}^{n-k}\left(\frac{\nabla^k u_i}{h^k}\right)^2 h\right)^{1/2},$$

$$\|\mathbf{b}\|_{L^2(\omega^h_{n-m})\times\mathbb{R}^m} = \left(\sum_{i=0}^{n-m} f^2(x_i)h + g_1^2 + \ldots + g_m^2\right)^{1/2},$$

for every $\mathbf{u},\ \mathbf{b} \in \mathbb{R}^{(n+1)\times 1}$, and the corresponding inner products those hold the equality $\|\mathbf{u}\| = (\mathbf{u},\mathbf{u})^{1/2}$ for each norm, respectively. Here we denoted the submesh $\omega^h_{n-m} = \{x_i = ih,\ i\in X_{n-m},\ nh = 1\} \subset \overline{\omega}^h$. Let us note that here $\mathbf{b}$ is not necessary of the special form $\mathbf{b} = \big(f(x_0), f(x_1), \ldots, f(x_{n-m}), g_1, \ldots, g_m\big)^\top \in \mathbb{R}^{(n+1)\times 1}$ taking $f \in C[0,1]$, which only helps to illustrate the similarity to the inner product in the space $L^2[0,1]\times\mathbb{R}^m$.

Now we minimize the $L^2(\omega^h_{n-m})\times\mathbb{R}^m$ norm of the residual

$$\|\mathbf{A}\mathbf{u}^g - \mathbf{b}\|_{L^2(\omega^h_{n-m})\times\mathbb{R}^m} \leqslant \|\mathbf{A}\mathbf{u} - \mathbf{b}\|_{L^2(\omega^h_{n-m})\times\mathbb{R}^m}, \qquad \forall\,\mathbf{u}\in\mathbb{R}^{(n+1)\times 1}, \tag{5.6}$$

by a vector $\mathbf{u}^g$. Then we select the one solution $\mathbf{u}^o$ from all minimizers $\mathbf{u}^g$, for which the $H^m(\overline{\omega}^h)$ norm is smallest:

$$\|\mathbf{u}^o\|_{H^m(\overline{\omega}^h)} < \|\mathbf{u}^g\|_{H^m(\overline{\omega}^h)}, \qquad \forall\ \mathbf{u}^g \neq \mathbf{u}^o. \tag{5.7}$$

This minimizer is represented by the formula

$$\mathbf{u}^o = \mathbf{A}^\dagger_{L^2(\omega^h_{n-m})\times\mathbb{R}^m,H^m(\overline{\omega}^h)}\mathbf{b}$$

using the Moore–Penrose inverse $\mathbf{A}^\dagger_{L^2(\omega^h_{n-m})\times\mathbb{R}^m,H^m(\overline{\omega}^h)}$. We rewrite it in the form

$$u_i^o = \sum_{j=0}^{n-m} G^{g,h}_{ij} f_j h + g_1 v_i^{g,1} + \ldots + g_m v_i^{g,m}, \qquad i\in X_n, \tag{5.8}$$

which is a discrete analogue of the representation of minimizer

$$u^o = \int_0^1 G^g(x,y)f(y)\,dy + g_1 v^{g,1}(x) + \ldots + g_m v^{g,m}(x), \qquad x \in [0,1], \tag{5.9}$$

to the differential problem (1.1)–(1.2). Here we used the modified composition of the Moore–Penrose inverse

$$\mathbf{A}^{\dagger}_{L^2(\omega^h_{n-m})\times\mathbb{R}^m, H^m(\overline{\omega}^h)} = (h\mathbf{G}^{g,h}\ \mathbf{v}^{g,1}\ \ldots\ \mathbf{v}^{g,m})$$

taking the modified generalized discrete Green's function $\mathbf{G}^{g,h} = h^{-1}\mathbf{G}^g$. For the particular case $\det\mathbf{A} \neq 0$, it reduces to the partitioning $\mathbf{A}^{-1} = (h\mathbf{G}^h\ \mathbf{v}^1\ \ldots\ \mathbf{v}^m)$ with the modified discrete Green's function $\mathbf{G}^h = h^{-1}\mathbf{G}$.

According to this special partitioning, below we provide reformulated representations of the range $R(\mathbf{A})$ and the nullspace $N(\mathbf{A}^*)$.

Lemma 4.27. *1) If $d = m$, then for all $f \in F(X_{n-m})$ we have*

$$R(\mathbf{A}) = \left\{ \left(f_0; f_1; \ldots; f_{n-m}; \sum_{j=0}^{n-m} \langle L_1^{\cdot}, G^{c,h}_{\cdot j}\rangle f_j h; \ldots; \sum_{j=0}^{n-m} \langle L_m^{\cdot}, G^{c,h}_{\cdot j}\rangle f_j h \right)^{\top} \right\}.$$

2) If $0 < d < m$, then the range $R(\mathbf{A})$ is generated by the vector

$$\mathbf{b} = \sum_{i=0}^{n-m} f_i \mathbf{e}^i + \sum_{l=1}^{d} \left(\sum_{j=d+1}^{m} g_{k_j} \langle L_{k_l}, v^{k_j}\rangle + \sum_{j=0}^{n-m} \langle L_{k_l}^{\cdot}, G^{a,h}_{\cdot j}\rangle f_j h \right) \mathbf{e}^{n-m+k_l} + \sum_{j=d+1}^{m} g_{k_j} \mathbf{e}^{n-m+k_j}$$

with every $f \in F(X_{n-m})$ and $g_{k_j} \in \mathbb{C}$, $j = \overline{d+1, m}$.

Here $G^{c,h} \in F(X_n \times X_{n-m})$ is the modified discrete Green's function for the discrete Cauchy problem (2.4). Other modified discrete Green's function $G^{a,h} \in F(X_n \times X_{n-m})$ and the biorthogonal fundamental system $v^k \in F(X_n)$, $k = \overline{1,m}$, are taken for a problem $\mathcal{L}u = f$ with original conditions $\langle L_{k_j}, u\rangle = 0$, $j = \overline{d+1, m}$, and conditions $\langle \ell_{k_l}, u\rangle = 0$, $l = \overline{1+d}$, replacing $\langle L_{k_l}, u\rangle = 0$. Here conditions $\langle \ell_{k_l}, u\rangle = 0$ are selected to construct an auxiliary problem with $\Delta \neq 0$.

Further, the composition of the nullspace $N(\mathbf{A}^*)$ is presented.

Corollary 4.28. *The following statements are valid:*

1) if $d = m$, then $N(\mathbf{A}^*)$ is generated by linearly independent vectors

$$\mathbf{w}^k = -\sum_{j=0}^{n-m} \langle L_k^{\cdot}, G^{c,h}_{\cdot j}\rangle \mathbf{e}^j + \mathbf{e}^{n-m+k}, \qquad k = \overline{1,m}.$$

2) if $0 < d < m$*, then* $N(\mathbf{A}^*)$ *is generated by linearly independent vectors*

$$\mathbf{w}^l = -\sum_{j=0}^{n-m} \langle L_{k_l}^{\cdot}, G_{\cdot j}^{a,h} \rangle \mathbf{e}^j - \sum_{j=d+1}^{m} \langle L_{k_l}, v^{k_j} \rangle \mathbf{e}^{k_j} + \mathbf{e}^{k_l}, \qquad l = \overline{1,d}.$$

Now recalling the Fredholm alternative theorem, we get the solvability conditions for the problem (2.1)–(2.2) without the unique solution ($\varDelta = 0$).

Corollary 4.29. *(Solvability conditions) The problem* (2.1)–(2.2) *with* $\varDelta = 0$ *is solvable if and only if the conditions are valid:*

1) $\sum\limits_{j=0}^{n-m} \langle L_k^{\cdot}, G_{\cdot j}^{c,h} \rangle f_j h = g_k$*,* $k = \overline{1,m}$*, for* $d = m$*;*

2) $\sum\limits_{j=d+1}^{m} g_{k_j} \langle L_{k_l}, v^{k_j} \rangle + \sum\limits_{j=0}^{n-m} \langle L_{k_l}^{\cdot}, G_{\cdot j}^{a,h} \rangle f_j h = g_{k_l}$ *for* $l = \overline{1,d}$ *if* $0 < d < m$*.*

Let us note that the nullspace $N(\mathbf{A})$ and its classification remain as previous (see Subsection 2.1). Here are no changes.

Example 4.30. *Let us now consider the discrete problem* (2.8)–(2.10) *with* $\varDelta = 0$ *(equivalently* $\det \mathbf{A} = 0$*). It has neither the unique solution nor the discrete Green's function.*

In this case Corollary 4.26 provides the representation of the generalized discrete Green's function $G^{g,h} \in F(X_n \times X_{n-m})$*, that is,*

$$G_{ij}^{g,h} = G_{ij}^{\mathrm{cl,h}} - (P_{H^m(\overline{\omega}^h),N(A)} G^{\mathrm{cl,h}})_{ij} + \gamma v_i^{g,m} G_{sj}^{\mathrm{cl,h}} \tag{5.10}$$

for every $i \in X_n$ *and* $j \in X_{n-m}$*. Here we use the discrete Green's function* $G^{\mathrm{cl,h}} \in F(X_n \times X_{n-m})$ *for the classical problem* (2.12)*, where always* $\varDelta^{\mathrm{cl}} \neq 0$*. This inequality means the existence of the unique solution* u^{cl}*, the discrete Green's function* $G^{\mathrm{cl,h}}$ *and the discrete biorthogonal fundamental system* $v^k \in F(X_n)$*,* $k = \overline{1,m}$*, as well.*

In Example 4.4, we got $d = 1$ *and* $\mathbf{v}^m \in N(\mathbf{A})$*. According to [100, Roman 2011], we can always find this function from the formula* $v_i^m = \varDelta(L_1, \dots, L_{m-1}, \delta_i)/\varDelta^{\mathrm{cl}}$*,* $i \in X_n$*. Thus, we calculate the orthogonal projection*

$$(P_{H^m(\overline{\omega}^h),N(A)} G^{\mathrm{cl,h}})_{ij} = \frac{v_i^m}{\|\mathbf{v}^m\|^2_{H^m(\overline{\omega}^h)}} \big(\mathbf{v}^m, \mathrm{col}_j \mathbf{G}^{\mathrm{cl,h}}\big)_{H^m(\overline{\omega}^h)}.$$

To obtain the full representation of the generalized discrete Green's function (5.10)*, we still need to find the function* $v^{g,m} \in F(X_n)$*. It is the discrete* *minimizer to the consistent discrete problem* $\mathbf{A}\mathbf{u} = \mathbf{P}_{L^2(\omega^h_{n-m})\times\mathbb{R}^m, R(A)}\mathbf{e}^n$. *Let us calculate the projection* $\mathbf{P}_{L^2(\omega^h_{n-m})\times\mathbb{R}^m}\mathbf{e}^n = \mathbf{e}^n - \mathbf{P}_{L^2(\omega^h_{n-m})\times\mathbb{R}^m, N(A^*)}\mathbf{e}^n$.

First, we have $d = 1$ *and* $k_1 = m$ *(see Example 4.4) and, using Corollary 4.28, obtain the vector*

$$\mathbf{w} = \gamma \sum_{j=0}^{n-m} G^{\mathrm{cl,h}}_{sj}\mathbf{e}^j + \gamma \sum_{k=1}^{m-1} v^k_s \mathbf{e}^{n-m+k} + \mathbf{e}^n,$$

which spans the nullspace $N(\mathbf{A}^*)$. *Now we calculate*

$$\mathbf{P}_{L^2(\omega^h_{n-m})\times\mathbb{R}^m, R(A)}\mathbf{e}^n = \mathbf{e}^n - \mathbf{w}(\mathbf{w}, \mathbf{e}^n)_{L^2(\omega^h_{n-m})\times\mathbb{R}^m}/\|\mathbf{w}\|^2_{L^2(\omega^h_{n-m})\times\mathbb{R}^m}$$
$$= \frac{1}{\|\mathbf{w}\|^2}\Big(-\gamma \sum_{j=0}^{n-m} G^{\mathrm{cl,h}}_{sj}\mathbf{e}^j - \gamma \sum_{k=1}^{m-1} v^k_s \mathbf{e}^{n-m+k} + (\|\mathbf{w}\|^2 - 1)\mathbf{e}^n \Big),$$

where we denoted $\|\mathbf{w}\|^2 := \|\mathbf{w}\|^2_{L^2(\omega^h_{n-m})\times\mathbb{R}^m}$. *Hence, we solve the consistent problem* $\mathbf{A}\mathbf{u} = \mathbf{P}_{L^2(\omega^h_{n-m})\times\mathbb{R}^m, R(A)}\mathbf{e}^n$, *that is*

$$(\mathcal{L}u)_i = -\gamma G^{\mathrm{cl,h}}_{si}/\|\mathbf{w}\|^2, \quad i \in X_{n-m}, \tag{5.11}$$
$$\nabla^k u_0/h^k = -\gamma v^k_s/\|\mathbf{w}\|^2, \quad k = \overline{0, m-2}, \tag{5.12}$$
$$u_n - \gamma u_s = 1 - 1/\|\mathbf{w}\|^2. \tag{5.13}$$

First, we take the general solution

$$u_i = c_1 v^1_i + c_2 v^2_i + \dots c_m v^m_i - \frac{\gamma}{\|\mathbf{w}\|^2}\sum_{j=0}^{n-m} G^{\mathrm{cl,h}}_{ij} G^{\mathrm{cl,h}}_{sj}\, h$$

of the discrete equation (5.11). *Substituting it into conditions* (5.12), *we find constants* $c_k = -\gamma v^k_s/\|\mathbf{w}\|^2$, $k = \overline{1, m-1}$, *those represent the general least squares solution*

$$u^g_i = -\gamma \sum_{k=1}^{m-1} \frac{v^k_s}{\|\mathbf{w}\|^2} v^k_i + c v^m_i - \frac{\gamma}{\|\mathbf{w}\|^2}\sum_{j=0}^{n-m} G^{\mathrm{cl,h}}_{ij} G^{\mathrm{cl,h}}_{sj}\, h, \qquad c \in \mathbb{R}.$$

Since the minimizer is of the form $\mathbf{v}^{g,m} = \mathbf{P}_{H^2(\overline{\omega}^h), N(A)^\perp}\mathbf{u}^g$, *we calculate this projection of* $\mathbf{u}^g$ *and find the expression*

$$v^{g,m}_i = -\gamma \sum_{k=1}^{m-1} \frac{v^k_s}{\|\mathbf{w}\|^2} v^k_i + c^{h,o} v^m_i - \frac{\gamma}{\|\mathbf{w}\|^2}\sum_{j=0}^{n-m} G^{\mathrm{cl,h}}_{ij} G^{\mathrm{cl,h}}_{sj}\, h, \tag{5.14}$$

with the particular constant $c^{h,o} = c^o + \mathcal{O}(h)$. *Here* c^o *represents the minimizer*

$$v^{g,m}(x) = \sum_{k=1}^{m-1} \frac{1 - \gamma\xi^{k-1}}{\big((k-1)! \cdot \|\boldsymbol{w}\|\big)^2} x^{k-1} + c^o x^{m-1} - \frac{\gamma}{\|\boldsymbol{w}\|^2}\int_0^1 G^{\mathrm{cl}}(x, y) G^{\mathrm{cl}}(\xi, y)\, dy$$

of the differential problem (2.6)–(2.7) *(see Example 2.28 in Chapter 2, where* $\Delta = 0$ *for this differential problem, that is,* $\gamma\xi^{m-1} = 1$*). Since* $c^{h,o} \to c^o$*, we ask if the discrete minimizer* $v_i^{g,m}$ *converges to the continuous minimizer* $v^{g,m}(x)$ *letting* $h \to 0$*?*

To provide the answer, we need to discuss on several approximations. Let us take the particular fundamental system $z^k = x^k$*,* $k = \overline{1,m}$*. First, we observe that*

$$v_i^m = \frac{\Delta(L_1,\dots,L_{m-1},\delta_i)}{\Delta^{\mathrm{cl}}} = \frac{h^{-(m-2)(m-1)/2}\cdot\Delta(\delta_0,\dots,\delta_{m-2},\delta_i)}{h^{-(m-2)(m-1)/2}\cdot\Delta(\delta_0,\dots,\delta_{m-2},1)}$$

$$= \frac{V(x_0,\dots,x_{m-2},x_i)}{V(x_0,\dots,x_{m-2},1)} = \frac{\prod_{0\leqslant j<l\leqslant m-2}(x_j-x_l)(x_j-x_i)(x_{m-2}-x_i)}{\prod_{0\leqslant j<l\leqslant m-2}(x_j-x_l)(x_j-1)(x_{m-2}-1)}$$
$$= \frac{\prod_{0\leqslant j\leqslant m-2}(x_j-x_i)}{\prod_{0\leqslant j\leqslant m-2}(x_j-1)} = x_i^{m-1} + \mathcal{O}(h) = v^m(x_i) + \mathcal{O}(h).$$

Here $v^m(x) = x^{m-1}$ *is the function from the biorthogonal fundamental system of the differential problem* (2.12)*. In this representation,* $\mathcal{O}(h)$ *is obtained of the special form: it is a linear combination of* $h^\alpha x_i^j$ *for* $j = \overline{0, m-2}$ *and finite* $\alpha = 1, 2, \dots$

Second, Lemma 4.5 gives the representation $G_{ij}^{\mathrm{cl,h}} = G_{ij}^{c,h} - v_i^m G_{n,j}^{c,h}$*. Above we proved* $v_i^m = v^m(x_i) + \mathcal{O}(h)$*. Let us now recall notations for a Green's function* $G^c(x,y)$ *of a differential Cauchy problem* (2.4) *and derive the approximation* $G_{ij}^{c,h} = G^c(x_i,x_j) + \mathcal{O}(h)$*. Using properties of determinants, we obtain that* W_{j+m}*, defined in the formula* (2.5)*, is equal to*

$$W_{j+m} = \begin{vmatrix} z_{j+1}^1 & z_{j+2}^1 & \dots & z_{j+m}^1 \\ z_{j+1}^2 & z_{j+2}^2 & \dots & z_{j+m}^2 \\ \dots & \dots & \dots & \dots \\ z_{j+1}^m & z_{j+2}^m & \dots & z_{j+m}^m \end{vmatrix} = \begin{vmatrix} z_{j+1}^1 & \nabla^1 z_{j+1}^1 & \dots & \nabla^{m-1} z_{j+1}^1 \\ z_{j+1}^2 & \nabla^1 z_{j+1}^2 & \dots & \nabla^{m-1} z_{j+1}^2 \\ \dots & \dots & \dots & \dots \\ z_{j+1}^m & \nabla^1 z_{j+1}^m & \dots & \nabla^{m-1} z_{j+1}^m \end{vmatrix}$$
$$= h^{\frac{m(m-1)}{2}}\cdot \begin{vmatrix} z_{j+1}^1 & \nabla^1 z_{j+1}^1/h & \dots & \nabla^{m-1} z_{j+1}^1/h^{m-1} \\ z_{j+1}^2 & \nabla^1 z_{j+1}^2/h & \dots & \nabla^{m-1} z_{j+1}^2/h^{m-1} \\ \dots & \dots & \dots & \dots \\ z_{j+1}^m & \nabla^1 z_{j+1}^m/h & \dots & \nabla^{m-1} z_{j+1}^m/h^{m-1} \end{vmatrix}$$
$$= h^{m(m-1)/2}\big(W(x_{j+1}) + \mathcal{O}(h)\big)$$

for $j \in X_{n-m}$*. Similarly, we get* $\widetilde{W}_{i,j+m} = h^{(m-1)(m-2)/2}\big(\widetilde{W}(x_i, x_{j+1}) +$

$\mathcal{O}(h)\big)$. *Since* $a_j^m = h^{-m}$, *we calculate*

$$G_{ij}^{c,h} = h^{-1} G_{ij}^c = h^{-1} \cdot \frac{1}{a_j^m} \cdot \frac{\widetilde{W}_{i,j+m}}{W_{j+m}} = \frac{\widetilde{W}(x_i, x_{j+1}) + \mathcal{O}(h)}{W(x_{j+1}) + \mathcal{O}(h)}$$

$$= \frac{\widetilde{W}(x_i, x_{j+1})}{W(x_{j+1})} + \mathcal{O}(h) = G^c(x_i, x_{j+1}) + \mathcal{O}(h), \qquad \text{for all} \quad j + m < i.$$

Otherwise $G_{ij}^{c,h} = G_{ij}^c = 0$ *if we take* $j + m \geqslant i$. *From the last two formulas, we get the desired approximation* $G_{ij}^{c,h} = G^c(x_i, x_{j+1}) + \mathcal{O}(h)$ *for* $i \in X_n$, $j \in X_{n-m}$.

Substituting obtained approximations, we represent $G_{ij}^{\mathrm{cl,h}} = G_{ij}^{c,h} - v_i^m G_{n,j}^{c,h}$ *in the form* $G_{ij}^{\mathrm{cl,h}} = G^c(x_i, x_{j+1}) - v^m(x_i) G^c(1, x_{j+1}) + \mathcal{O}(h)$. *Since the formula* (3.4) *in Chapter 2 gives the representation* $G^{\mathrm{cl}}(x,y) = G^c(x,y) - v^m(x) G^c(1,y)$ *of the Green's function for the classical differential problem, we get* $G_{ij}^{\mathrm{cl,h}} = G^{\mathrm{cl}}(x_i, x_{j+1}) + \mathcal{O}(h)$.

Third, the discrete fundamental system v_i^k, $k = \overline{1,m}$, *is related with the biorthogonal fundamental system* $v^k(x)$, $k = \overline{1,m}$, *of the classical differential problem* (2.12) *by equalities*

$$v_i^k = \Delta_i^{cl,k} / \Delta^{cl} = v^k(x_i) + \mathcal{O}(h) = \frac{x_i^{k-1} - x_i^{m-1}}{(k-1)!} + \mathcal{O}(h) \tag{5.15}$$

for $k = \overline{1, m-1}$ *and, as proved above,* $v_i^m = v^m(x_i) + \mathcal{O}(h) = x_i^{m-1} + \mathcal{O}(h)$. *We obtain these approximations rewriting determinants* $\Delta_i^{\mathrm{cl},k} := \Delta(\delta_0, \delta_1, \ldots, \delta_{k-2}, \delta_i, \delta_k, \ldots, \delta_{m-2}, \delta_n)$ *and* Δ^{cl}, $k = \overline{1, m-1}$, *similarly as we have just did above for the modified discrete Green's function* $G_{ij}^{c,h}$.

Moreover, Example 4.4 says that $\Delta = 0$ *for discrete problem* (2.8)–(2.10) *is equivalent to the equality* $\gamma v_s^m = 1$. *Recalling the approximation* $v_i^m = x_i^{m-1} + \mathcal{O}(h)$ *for all* $i \in X_n$, *we write this equality in the form* $\gamma \xi^{m-1} = 1 + \mathcal{O}(h)$. *Letting* $h \to 0$, *we obtain the condition* $\gamma \xi^{m-1} = 1$, *where the differential problem* (2.6)–(2.7) *does not have the unique solution.*

Substituting obtained approximations in (5.14), *we get the expression*

$$v_i^{g,m} = \sum_{k=1}^{m-1} \frac{1 - \gamma \xi^{k-1}}{\big((k-1)! \cdot \|\boldsymbol{w}\|\big)^2} x^{k-1} + c^o x^{m-1}$$
$$- \frac{\gamma}{\|\boldsymbol{w}\|^2} \sum_{j=0}^{n-m} G^{\mathrm{cl}}(x_i, x_{j+1}) G^{\mathrm{cl}}(\xi, x_{j+1})\, h + \mathcal{O}(h).$$

This representation gives the approximation $v_i^{g,m} = v^{g,m}(x_i) + \mathcal{O}(h)$, *what means the convergence of that discrete minimizer to the continuous minimizer. We note that other relations* $v_i^{g,k} = v^{g,k}(x_i) + \mathcal{O}(h)$ *for* $k = \overline{1, m-1}$

are also valid. Here we find discrete functions

$$\mathbf{v}^{g,k} = \gamma v_s^k \mathbf{v}^{g,m} + \mathbf{P}_{H^m(\overline{\omega}^h),N(A)^\perp}\mathbf{v}^k, \qquad k = \overline{1, m-1},$$

from Corollary 4.13 (where we take $\widetilde{v}^k = v^k$, $k = \overline{1,m}$*) and then apply obtained approximations.*

Moreover, we rewrite the orthogonal projection in the form

$$(P_{H^m(\overline{\omega}^h),N(A)}G^{\mathrm{cl,h}})_{ij} = \frac{x_i^{m-1}}{\|t^{m-1}\|^2_{H^m[0,1]}}\big(t^{m-1}, G^{\mathrm{cl}}(t, x_{j+1})\big)_{H^m[0,1]} + \mathcal{O}(h).$$

Substituting relations $G_{ij}^{\mathrm{cl,h}} = G^{\mathrm{cl}}(x_i, x_{j+1}) + \mathrm{O}(h)$, $(P_{H^m(\overline{\omega}^h),N(A)}G^{\mathrm{cl,h}})_{ij} = P_{N(\boldsymbol{L})}G^{\mathrm{cl}}(x_i, x_{j+1}) + \mathcal{O}(h)$ *and* $v_i^{g,m} = v^{g,m}(x_i) + \mathcal{O}(h)$ *into the expression* (5.10)*, we get the approximation* $G_{ij}^{g,h} = G^g(x_i, x_{j+1}) + \mathcal{O}(h)$.

Let us now put approximations of the generalized discrete Green's function $G_{ij}^{g,h} = G^g(x_i, x_{j+1}) + \mathcal{O}(h)$ *and the generalized biorthogonal fundamental system* $v_i^{g,k} = v^{g,k}(x_i) + \mathcal{O}(h)$, $k = \overline{1,m}$*, into the representation of the discrete minimum norm least squares solution* (5.8)*. Here we observe its convergence to the minimum norm least squares solution* (5.9) *of the differential problem.*

The basic conclusion of this example in formulated below.

Corollary 4.31. *The minimizer of the discrete problem* (2.8)–(2.10) *converges to the minimizer of the differential problem* (2.6)–(2.7)*.*

Below we suggest the way how to investigate the convergence of the discrete minimizer to the minimizer of a differential problem.

To formulate the convergence conditions, we need to introduce the projection operator $\boldsymbol{\pi}_1 : H^m[0,1] \to H^m(\overline{\omega}^h)$, which projects a function $u \in H^m[0,1]$ on the mesh $\overline{\omega}^h$ by the formula $\boldsymbol{\pi}_1 u = (u(x_0), u(x_1), \ldots, u(x_n))^\top$. This poitwise definition is correct since every function from $H^m[0,1]$ belongs to $C^{m-1}[0,1]$. For every $\boldsymbol{f} = (f, g_1, \ldots, g_m)^\top \in L^2[0,1] \times \mathbb{R}^m$ we also take a projector $\boldsymbol{\pi}_2 : L^2[0,1] \times \mathbb{R}^m \to L^2(\omega^h_{n-m}) \times \mathbb{R}^m$. If $f \in C[0,1]$, the projector may be given by $\boldsymbol{\pi}_2\boldsymbol{f} = (f(x_0), f(x_1), \ldots, f(x_{n-2}), g_1, g_2, \ldots, g_m)^\top$. In general, $f \in L^2[0,1]$ and we can use the projector

$$\boldsymbol{\pi}_2\boldsymbol{f} = \sum_{i=0}^{n-m}\left(\frac{1}{h}\int_{x_i}^{x_{i+1}} f(x)\,dx\right)\mathbf{e}^i + \sum_{k=1}^{m} g_k \mathbf{e}^{n-m+k}.$$

Theorem 4.32. *(Sufficient convergence conditions) Let the following approximations*

$$\mathbf{A}(\boldsymbol{\pi}_1 u) = \boldsymbol{\pi}_2\boldsymbol{L}u + \boldsymbol{\mathcal{O}}(h^\alpha), \qquad \mathbf{P}_{H^m(\overline{\omega}^h),N(A)}(\boldsymbol{\pi}_1 u) = \boldsymbol{\pi}_1(\boldsymbol{P}_{N(\boldsymbol{L})}u) + \boldsymbol{\mathcal{O}}(h^\alpha),$$

$$\mathbf{P}_{L^2(\omega^h_{n-m})\times\mathbb{R}^m,R(A)}\mathbf{b} = \boldsymbol{\pi}_2(\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{f}) + \boldsymbol{\mathcal{O}}(h^\alpha)$$

be valid for some $\alpha > 0$. *If* $\sup_{n\in\mathbb{N}} \|\mathbf{A}^\dagger\|_{H^m(\overline{\omega}^h),\, L^2(\omega^h_{n-m})\times\mathbb{R}^m} < +\infty$, *then the minimizer* $\mathbf{u}^o \in H^m(\overline{\omega}^h)$ *of the discrete problem* (2.1)–(2.2) *converges to the minimizer* $u^o \in H^m[0,1]$ *of the differential problem* (1.1)–(1.2), *i.e.,*

$$\|\mathbf{u}^o - \boldsymbol{\pi}_1 u^o\|_{C(\overline{\omega}^h)} = \max_{x_i\in\overline{\omega}^h} |u^o_i - u^o(x_i)| \to 0 \qquad \text{if} \quad n \to \infty.$$

6 Conclusions

In this chapter, we generalized results of the previous chapter, where a second order discrete problem with nonlocal conditions was investigated. Basic conclusions of this chapter are formulated below:

1) A discrete problem (2.1)–(2.2) always has the Moore–Penrose inverse $\mathbf{A}^\dagger$, a generalized discrete Green's function and the unique minimum norm least squares solution.

2) For $\Delta \neq 0$, we have that $\mathbf{A}^\dagger = \mathbf{A}^{-1}$, the minimum norm least squares solution u^o is coincident with the unique solution u, the generalized Green's function G^g_{ij} is coincident with the ordinary Green's function G_{ij}, the biorthogonal fundamental system v^k, $k = \overline{1,m}$, is coincident with the generalized biorthogonal fundamental system $v^{g,k}$, $k = \overline{1,m}$.

3) The minimum norm least squares solution has literally similar representations as the unique discrete solution: it can be described by the unique solution of the discrete Cauchy problem or the unique solution to other relative problem (the same discrete equation (2.1) but different nonlocal conditions (2.2)).

4) A generalized discrete Green's function also has representations similar to expressions of a discrete Green's function: it can be written using the discrete Green's function of the Cauchy problem or the discrete Green's function to other relative problem (the same discrete equation (2.1) but different nonlocal conditions (2.2)).

5) Obtained properties of minimizers are coincident with corresponding properties of minimizers for differential problems.

6) The discrete minimum norm least squares solution converges to the minimum norm least squares solution of the differential problem (1.1)–(1.2) if conditions of Theorem 4.32 are satisfied.

Chapter 5

First order differential systems with nonlocal conditions

1 Introduction

In this chapter, we consider a linear system of first order differential equations with nonlocal conditions

$$\frac{d\,u^k}{dx} = \sum_{l=1}^{m} a^{kl}(x)u^l + f^k(x), \quad x \in [0,1], \tag{1.1}$$

$$\sum_{l=1}^{m} \langle L_{kl}, u^l \rangle = g_k, \quad k = \overline{1,m}, \tag{1.2}$$

where we take real numbers g_k and all functions $u^k \in H^1[0,1], f^k \in L^2[0,1]$, $a^{kl} \in C[0,1]$, $L_{kl} \in C^*[0,1]$. Similar complex system was also studied by Bryan [20, 1969]. He derived the expression of the Green's matrix and studied its properties for the system with, as he called, *general linear boundary conditions*. Generalized Green's functions for systems of ordinary differential equations and general conditions were also investigated by Boichuk and Samoilenko [13, 2004]. Authors obtained a representation of a generalized Green's matrix and solvability conditions in quite abstract form. We also derive similar results for the system (1.1)–(1.2) in the form, applied particulary for the problem with nonlocal conditions.

The structure of this chapter is as follows. First, we represent this system into the vectorial form and consider its properties. Then the case of the unique solution is investigated. We obtain several representations and properties of the unique solution and its Green's matrix. Afterwards, the problem without the unique solution is considered. Here we discuss on the unique minimizer of the residual, derive its properties and representations, study a generalized Green's matrix. Several examples are also given. Let us note that this chapter is based on the paper [89, Paukštaitė and Štikonas 2017].

2 The vectorial problem

Introducing notations $\boldsymbol{u} = (u^1, u^2, \ldots, u^m)^\top$, $\boldsymbol{u}' = \big((u^1)', (u^2)', \ldots, (u^m)'\big)^\top$, $\boldsymbol{A}(x) = (a^{kl}(x))$, $\boldsymbol{f} = (f^1, \ldots, f^m)^\top$, $\boldsymbol{L}_k = (L_{k1}, \ldots, L_{km})$, the system can be written in the equivalent form

$$\boldsymbol{\mathcal{L}u} := \boldsymbol{u}' - \boldsymbol{Au} = \boldsymbol{f}, \tag{2.1}$$

$$\langle \boldsymbol{L}_k, \boldsymbol{u}\rangle = g_k, \quad k = \overline{1, m}. \tag{2.2}$$

Here $\langle \boldsymbol{L}_k, \boldsymbol{u}\rangle$ is the usual matrix multiplication $\boldsymbol{L}_k\boldsymbol{u}$, where brackets emphasize only the nature of nonlocal conditions. Similarly, $\langle \boldsymbol{L}_k, \boldsymbol{U}\rangle = \boldsymbol{L}_k\boldsymbol{U}$ for every $m \times n$ matrix $\boldsymbol{U} = (U^{lj})$ on $[0, 1]$. We also use the two dot notation $\langle \boldsymbol{L}_{k\cdot}, \boldsymbol{U}^{\cdot j}\rangle := \sum_{l=1}^m \langle L_{kl}, U^{lj}\rangle$ for every $j = \overline{1, n}$ and denote the m-th order matrix valued identity function by $\boldsymbol{I} = \boldsymbol{I}_{m\times m}$.

Let us take the short description of the problem

$$\boldsymbol{Lu} = \boldsymbol{b} \tag{2.3}$$

with the operator $\boldsymbol{L} := (\boldsymbol{\mathcal{L}}, \boldsymbol{L}_1, \ldots, \boldsymbol{L}_m)^\top$ and the right hand $\boldsymbol{b} = (f^1, \ldots, f^m, g_1, \ldots, g_m)^\top \in (L^2[0, 1])^m \times \mathbb{R}^m$.

Since every function $u^k \in H^1[0, 1]$, then for the Hilbert space $(H^1[0, 1])^m := H^1[0, 1]\times\ldots\times H^1[0, 1]$ (here m times) we take the standard inner product and the norm

$$\|\boldsymbol{u}\| = (\boldsymbol{u}, \boldsymbol{u})^{1/2} = \left(\sum_{k=1}^m \|u^k\|^2_{H^1[0,1]}\right)^{1/2}, \qquad \forall\ \boldsymbol{u} \in (H^1[0, 1])^m.$$

Analogously, for the space $(L^2[0, 1])^m \times \mathbb{R}^m$, we introduce the norm

$$\|\boldsymbol{b}\| = (\boldsymbol{b}, \boldsymbol{b})^{1/2} = \left(\sum_{k=1}^m \|f^k\|^2_{L^2[0,1]} + g_k^2\right)^{1/2}, \qquad \forall \boldsymbol{b} \in (L^2[0, 1])^m \times \mathbb{R}^m.$$

The Sobolev embedding theorem [42, Evans 2010] says that $H^1[0, 1] \subset C[0, 1]$ and the inequality

$$\|u\|_{C[0,1]} \leqslant c\|u\|_{H^1[0,1]}, \qquad \forall u \in H^1[0, 1], \tag{2.4}$$

is valid for a particular constant c independent on a chosen u. Thus, $(C[0, 1])^* \subset (H^1[0, 1])^*$ and each functional $L_{kl} \in \big(C[0, 1]\big)^*$ belongs to the dual space $(H^1[0,1])^*$. It also means that vector valued functionals $\boldsymbol{L}_k$ belong to the dual space of $(H^1[0,1])^m$. Since $\boldsymbol{\mathcal{L}}$ is defined on $(H^1[0,1])^m$, then the vectorial operator $\boldsymbol{L}$ maps one Hilbert space $(H^1[0,1])^m$ to another Hilbert space $(L^2[0,1])^m \times \mathbb{R}^m$.

Lemma 5.1. *The operator* $\boldsymbol{L} : (H^1[0,1])^m \to (L^2[0,1])^m \times \mathbb{R}^m$ *is the continuous linear operator with the domain* $D(\boldsymbol{L}) = (H^1[0,1])^m$.

Let us note that this proof is analogous to the proof of Lemma 1.1 from Chapter 1. In this chapter, we omit analogous proves emphasizing particular technical details if needed.

2.1 Existence of the unique solution

Let us consider the condition of the existence of the unique solution to the problem (1.1)–(1.2). This condition was represented by other authors [19, Brwon and Krall 1974], [20, Bryan 1969]. However, developing the parallel to m-th order differential equations with nonlocal conditions, we present here this condition in the similar form as an existence condition for the unique solution to the m-th order equation is given in [100, Roman 2011].

So, we investigate the existence of the unique trivial solution $\boldsymbol{z} \equiv \boldsymbol{0}$ to the homogenous problem

$$\boldsymbol{z}' = \boldsymbol{A}\boldsymbol{z}, \tag{2.5}$$

$$\langle \boldsymbol{L}_k, \boldsymbol{z} \rangle = 0, \quad k = \overline{1,m}. \tag{2.6}$$

The fundamental system of the equation (2.5) is composed of m linearly independent vector functions $\boldsymbol{z}^l,\ l = \overline{1,m}$, those can be represented by the $m \times m$ order fundamental matrix $\boldsymbol{Z}(x) = (\boldsymbol{z}^1(x), \dots, \boldsymbol{z}^m(x))$. Then we put the general solution to (2.5), which is $\boldsymbol{z} = \sum_{l=1}^{n} c_l \boldsymbol{z}^l$ for $c_l \in \mathbb{R},\ l = \overline{1,m}$, into conditions (2.6) and obtain the system of m homogenous equations with unknowns $c_l,\ l = \overline{1,m}$, as follows

$$\sum_{l=1}^{m} \langle \boldsymbol{L}_k, \boldsymbol{z}^l \rangle c_l = 0, \quad k = \overline{1,m}.$$

So, the system has only the trivial solution $\boldsymbol{z} = \boldsymbol{0}$ or equivalently $c_l = 0,\ l = \overline{1,m}$, if and only if the determinant of the previous system is nonzero

$$\Delta := \begin{vmatrix} \langle \boldsymbol{L}_1, \boldsymbol{z}^1 \rangle & \langle \boldsymbol{L}_1, \boldsymbol{z}^2 \rangle & \dots & \langle \boldsymbol{L}_1, \boldsymbol{z}^m \rangle \\ \langle \boldsymbol{L}_2, \boldsymbol{z}^1 \rangle & \langle \boldsymbol{L}_2, \boldsymbol{z}^2 \rangle & \dots & \langle \boldsymbol{L}_2, \boldsymbol{z}^m \rangle \\ \dots & \dots & \dots & \dots \\ \langle \boldsymbol{L}_m, \boldsymbol{z}^1 \rangle & \langle \boldsymbol{L}_m, \boldsymbol{z}^2 \rangle & \dots & \langle \boldsymbol{L}_m, \boldsymbol{z}^m \rangle \end{vmatrix} \neq 0. \tag{2.7}$$

It literally resembles the existence condition of the unique solution to a m-th order differential equation with m nonlocal conditions, which was studied in Chapter 2.

If $\Delta = 0$, then the nullspace $N(\boldsymbol{L})$ is nontrivial. Denoting the nullity $d := \dim N(\boldsymbol{L})$, we separate the following cases:

- $d = 0 \Leftrightarrow \Delta \neq 0$. Then $N(\boldsymbol{L})$ is trivial.

- $d = m \Leftrightarrow \Delta = 0$ and all $\langle \boldsymbol{L}_k, \boldsymbol{z}^l \rangle = 0$ for $k, l = \overline{1, m}$. Then all constants $c_1, \ldots, c_m$ remain arbitrary and $N(\boldsymbol{L}) = \text{span}\{\boldsymbol{z}^1, \ldots, \boldsymbol{z}^m\}$. So, the solution to $\boldsymbol{L}\boldsymbol{z} = \boldsymbol{0}$ is now equivalent to the solution of the differential equation $\boldsymbol{z}' = \boldsymbol{A}\boldsymbol{z}$ only.

- $0 < d < m \Leftrightarrow \Delta = 0$ and $\text{rank}(\langle \boldsymbol{L}_k, \boldsymbol{z}^l \rangle) = m - d$ (here $k, l = \overline{1, m}$). Here $m - d$ constants are solved and represented by other d constants, those remain arbitrary. In other words, there exist d rows in the determinant representation of Δ above, those are linear combinations of the rest $m - d$ linearly independent rows. Let us denote these "*dependent*" rows by $(\langle \boldsymbol{L}_{k_l}, \boldsymbol{z}^1 \rangle, \ldots, \langle \boldsymbol{L}_{k_l}, \boldsymbol{z}^m \rangle)$ for $k_l,\ l = \overline{1, d}$. The independent rows are also given by $(\langle \boldsymbol{L}_{k_j}, \boldsymbol{z}^1 \rangle, \ldots, \langle \boldsymbol{L}_{k_j}, \boldsymbol{z}^m \rangle)$ for $k_j,\ j = \overline{d+1, m}$. Thus, the solution to the problem $\boldsymbol{L}\boldsymbol{z} = \boldsymbol{0}$ is now equivalent to the solution of the simplified problem: the equation $\boldsymbol{z}' = \boldsymbol{A}\boldsymbol{z}$ with conditions $\langle \boldsymbol{L}_{k_j}, \boldsymbol{z} \rangle = 0,\ j = \overline{d+1, m}$, representing linearly independent rows only.

Let us note that the nullspace $N(\boldsymbol{L})$ is closed according to Lemma 5.1 [65, Kreyszig 1978]. Then the entire space $(H^1[0,1])^m$ is represented by the direct sum of orthogonal subspaces as below

$$(H^1[0,1])^m = N(\boldsymbol{L}) \oplus N(\boldsymbol{L})^\perp. \tag{2.8}$$

Moreover, the nullspace $N(\boldsymbol{L})$ is composed of continuously differentiable functions $\boldsymbol{z}^l \in (C^1[0,1])^m$, $l = \overline{1, m}$, because all functions $a^{kl} \in C[0,1]$ [34, Coddington and Levinson 1955].

2.2 Range of the operator $\boldsymbol{L}$

We can ontain the following result.

Theorem 5.2. *The range $R(\boldsymbol{L})$ of the operator $\boldsymbol{L}$ is closed.*

This theorem is proved analogously as Theorem 1.2 for the second order differential problem in Chapter 1 is proved. We take the unique solution to the Cauchy problem

$$\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{f}, \qquad u^k(0) = 0,\ k = \overline{1,m}. \tag{2.9}$$

According to [34, Coddington and Levinson 1955], its unique solution is given by $\boldsymbol{u}^c = \int_0^x \boldsymbol{K}(x,y)\boldsymbol{f}(y)\,dy$, where $\boldsymbol{K}(x,y) = \boldsymbol{Z}(x)\boldsymbol{Z}^{-1}(y)$ is the m-th order *Cauchy matrix*. Now we introduce the Green's matrix $\boldsymbol{G}^c(x,y) = \big(G^{c,kl}(x,y)\big)$ (here $k,l = \overline{1,m}$) to the Cauchy problem as follows

$$\boldsymbol{G}^c(x,y) = \begin{cases} \boldsymbol{K}(x,y), & y < x, \\ 0, & y > x, \end{cases} \tag{2.10}$$

and get another description of the unique solution

$$\boldsymbol{u}^c = \int_0^1 \boldsymbol{G}^c(x,y)\boldsymbol{f}(y)\,dy. \tag{2.11}$$

The Green's matrix has such properties:

1) $\boldsymbol{G}^c(y+0,y) - \boldsymbol{G}(y-0,y) = \boldsymbol{I}$;

2) $\boldsymbol{G}^c(x,y)$ is C in (x,y) except the diagonal $x = y$;

3) $\boldsymbol{G}^c(x,y)$ is C^1 in x except the diagonal $x = y$;

4) $(\partial/\partial x)\boldsymbol{G}^c(x,y) - \boldsymbol{A}(x)\boldsymbol{G}(x,y) = \boldsymbol{0}$ except the diagonal $x = y$;

5) $\langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot,y)\rangle = \boldsymbol{0},\ k = \overline{1,m}.$

Applying these properties of the Green's matrix $\boldsymbol{G}^c(x,y)$, we obtain the proof of the Theorem 5.1 analogously as Theorem 1.2 in Chapter 1 is proved.

We can also obtain the direct representation of the range $R(\boldsymbol{L})$. We omit proves again because they are analogous to corresponding proves in previous chapters. First, we need to discuss on the composition

$$\boldsymbol{b} = (f^1,\dots,f^m,g_1,\dots,g_m)^\top = \sum_{k=1}^m (f^k\boldsymbol{e}^k + g_k\boldsymbol{e}^{m+k})$$

for all $\boldsymbol{b} \in (L^2[0,1])^m \times \mathbb{R}^m$. Here we denoted the unit vectorial functions $\boldsymbol{e}^1 = (1,0,\dots,0)^\top$, $\boldsymbol{e}^1 = (0,1,\dots,0)^\top$,..., $\boldsymbol{e}^{2m} = (0,0,\dots,1)^\top$. Now we can provide the representation of the range.

Lemma 5.3.

1) If $d = m$, then $R(\boldsymbol{L})$ is generated by the vector function

$$\boldsymbol{b} = \left(f^1; \dots; f^m; \int_0^1 \langle \boldsymbol{L}_1, \boldsymbol{G}^c(\cdot, y)\rangle \boldsymbol{f}(y)\, dy; \dots; \int_0^1 \langle \boldsymbol{L}_m, \boldsymbol{G}^c(\cdot, y)\rangle \boldsymbol{f}(y)\, dy\right)^\top$$

where $\boldsymbol{f} = (f^1, \dots, f^m)^\top \in (L^2[0,1])^m$.

2) If $0 < d < m$, then $R(\boldsymbol{L})$ is generated by the vector function

$$\boldsymbol{b} = \sum_{k=1}^m f^k \boldsymbol{e}^k + \sum_{l=1}^d \left(\sum_{j=d+1}^m g_{k_j} \langle \boldsymbol{L}_{k_l}, \boldsymbol{v}^{k_j} \rangle + \int_0^1 \langle \boldsymbol{L}_{k_l}, \boldsymbol{G}^a(\cdot, y)\rangle \boldsymbol{f}(y)\, dy \right) \boldsymbol{e}^{m+k_l} + \sum_{j=d+1}^m g_{k_j} \boldsymbol{e}^{m+k_j},$$

where $\boldsymbol{f} = (f^1, \dots, f^m)^\top \in (L^2[0,1])^m$ and $g_{k_j} \in \mathbb{R}$ for $j = \overline{d+1, m}$.

Here $\boldsymbol{G}^a(x, y) = (G^{a,kl}(x, y))$ is the Green's matrix and $\{\boldsymbol{v}^1, \dots, \boldsymbol{v}^m\}$ is the biorthogonal fundamental system for the problem $\boldsymbol{\mathcal{L}u} = \boldsymbol{f}$ with original conditions $\langle \boldsymbol{L}_{k_j}, \boldsymbol{u}\rangle = 0,\ j = \overline{d+1, m}$, and conditions $\langle \boldsymbol{\ell}_{k_l}, \boldsymbol{u}\rangle = 0,\ l = \overline{1, d}$, replacing $\langle \boldsymbol{L}_{k_l}, \boldsymbol{u}\rangle = 0$. Here $\langle \boldsymbol{\ell}_{k_l}, \boldsymbol{u}\rangle = 0$ are selected such that for this auxiliary problem $\varDelta \neq 0$. For details see the following section.

According to [100, Roman 2011], properties of $\boldsymbol{L}$ implies the closeness of $N(\boldsymbol{L}^*)$, where $\boldsymbol{L}^* : (L^2[0,1])^m \times \mathbb{R}^m \to (H^1[0,1])^m$ is the adjoint operator of $\boldsymbol{L}$. Then the nullspace and range theorem gives $N(\boldsymbol{L}^*) = R(\boldsymbol{L})^\perp$, which representation can also be derived in the following forms.

Corollary 5.4. *The following statements are valid:*

1) if $d = m$, then $N(\boldsymbol{L}^*)$ is generated by vector functions

$$\boldsymbol{w}^k = -\sum_{l=1}^m \langle \boldsymbol{L}_{k\cdot}, \boldsymbol{G}^{c,\cdot l}(\cdot, x)\rangle \boldsymbol{e}^l + \boldsymbol{e}^{m+k}, \qquad k = \overline{1, m}.$$

2) if $0 < d < m$, then $N(\boldsymbol{L}^*)$ is generated by vector functions

$$\boldsymbol{w}^\ell = -\sum_{l=1}^m \langle \boldsymbol{L}_{k_\ell\cdot}, \boldsymbol{G}^{a,\cdot l}(\cdot, x)\rangle \boldsymbol{e}^l - \sum_{j=d+1}^m \langle \boldsymbol{L}_{k_\ell}, \boldsymbol{v}^{k_j}\rangle \boldsymbol{e}^{m+k_j} + \boldsymbol{e}^{m+k_\ell},\ \ \ell = \overline{1, d}.$$

This corollary gives that $d = \dim N(\boldsymbol{L})$ and $d^* := \dim N(\boldsymbol{L}^*)$ are equal. Now applying the Fredholm alternative theorem, we get the solvability conditions to the problem (1.1)–(1.2) without the unique solution ($\varDelta = 0$).

Corollary 5.5. *(Solvability conditions) The problem* (1.1)–(1.2) *with $\varDelta = 0$ is solvable if and only if the conditions are valid:*

1) $\int_0^1 \langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot, y)\rangle \boldsymbol{f}(y)\, dy = g_k$*,* $k = \overline{1, m}$*, for* $d = m$*;*

2) $\sum_{j=d+1}^{m} g_{k_j} \langle \boldsymbol{L}_{k_\ell}, \boldsymbol{v}^{k_j}\rangle + \int_0^1 \langle \boldsymbol{L}_{k_\ell}, \boldsymbol{G}^a(\cdot, y)\rangle \boldsymbol{f}(y)\, dy = g_{k_l}$ *for* $\ell = \overline{1, d}$ *if* $0 < d < m$*.*

Example 5.6. *Let us consider a differential system (*$m = 2$*) with the Bitsadze–Samarkii condition*

$$\begin{gathered} (u^1)' = u^2 + f^1, \quad (u^2)' = f^2, \\ u^1(0) = g_1, \quad u^1(1) = \gamma u^1(\xi) + g_2. \end{gathered} \tag{2.12}$$

The solution to homogenous equations $(z^1)' = z^2$ *and* $(z^2)' = 0$ *(*$f^1, f^2 = 0$ *above), gives the scalar problem* $(z^1)'' = 0$*, which has the fundamental system* $\{1; x\}$ *and the general solution* $z^1 = c_1 + c_2 x$*. Then* $z^2 = (z^1)' = c_2$*. From here, we obtain the fundamental system* $\boldsymbol{z}^1 = (1, 0)^\top$*,* $\boldsymbol{z}^2 = (x, 1)^\top$ *for the differential system and calculate*

$$\Delta = \begin{vmatrix} \langle \boldsymbol{L}_1, \boldsymbol{z}^1\rangle & \langle \boldsymbol{L}_2, \boldsymbol{z}^1\rangle \\ \langle \boldsymbol{L}_1, \boldsymbol{z}^2\rangle & \langle \boldsymbol{L}_2, \boldsymbol{z}^2\rangle \end{vmatrix} = \begin{vmatrix} 1 & 1-\gamma \\ 0 & 1-\gamma\xi \end{vmatrix}.$$

If $\Delta \neq 0$*, i.e.,* $\gamma\xi \neq 1$*, the problem* (2.12) *has the unique solution.*

Let us now focus on the problem with $\gamma\xi = 1$ *(case* $\Delta = 0$*). Since* $\langle \boldsymbol{L}_1, \boldsymbol{z}^1\rangle = 1$ *does not vanish, then* $d = 1$ *and* $k_1 = 2$*,* $k_2 = 1$*. Now we formulate the auxiliary problem* $(u^1)' = u^2 + f^1$*,* $(u^2)' = f^2$*,* $u^1(0) = 0$*,* $u^1(1) = 0$*. It is obtained from the problem* (2.12) *taking* $\gamma = 0$ *and has* $\Delta\big|_{\gamma=0} = 1$*. The auxiliary problem has the biorthogonal fundamental system* $\boldsymbol{v}^1 = (1-x, -1)^\top$*,* $\boldsymbol{v}^2 = (x, 1)^\top$ *and the Green's matrix*

$$\boldsymbol{G}^{\mathrm{cl}}(x, y) = \begin{pmatrix} (\partial/\partial y) G^{\mathrm{cl}}(x, y) & -G^{\mathrm{cl}}(x, y) \\ (\partial^2/\partial x \partial y) G^{\mathrm{cl}}(x, y) & -(\partial/\partial x) G^{\mathrm{cl}}(x, y) \end{pmatrix}$$

In this representation we used the Green's function

$$G^{\mathrm{cl}}(x, y) = \begin{cases} y(1-x), & y \leqslant x, \\ x(1-y), & y > x, \end{cases}$$

for the scalar problem $-u'' = f$*,* $u(0) = 0$*,* $u(1) = 0$ *(for details, you should see Example 5.16). We can directly verify that the Green's matrix* $\boldsymbol{G}^{\mathrm{cl}}(x, y)$ *describes the unique vectorial solution to the system* (2.12)*.*

From Lemma 5.3, we get the range representation

$$\boldsymbol{b} = \left(f^1; f^2; g_1; g_1 \langle \boldsymbol{L}_2, \boldsymbol{v}^1\rangle + \int_0^1 \boldsymbol{L}_2, \boldsymbol{G}^{\mathrm{cl}}(\cdot, y)\rangle \boldsymbol{f}(y)\, dy \right)^\top,$$

which simplifies to

$$\boldsymbol{b} = \Big(f^1; f^2; g_1; g_1(1-\gamma)+\gamma\int_0^1 \big(-\frac{\partial}{\partial y}G^{\mathrm{cl}}(\xi,y)f^1(y)+G^{\mathrm{cl}}(\xi,y)f^2(y)\big)dy\Big)^{\top}.$$

Corollary 5.4 provides the function

$$\begin{aligned}\boldsymbol{w}(x) &= \Big(\gamma\frac{\partial}{\partial y}G^{\mathrm{cl}}(\xi,x); -\gamma G^{\mathrm{cl}}(\xi,x); \gamma-1; 1\Big)^{\top}\\ &= \left(\begin{cases} 1/\xi-1, & x\leqslant \xi,\\ -1, & x>\xi\end{cases}; \begin{cases} x(1/\xi-1), & x\leqslant\xi,\\ x-1, & x>\xi\end{cases}; \frac{1}{\xi}-1; 1\right)^{\top}\end{aligned}$$

generating the nullspace $N(\boldsymbol{L}^*)$. Below we formulate the solvability condition for the system (2.12) *without the unique solution ($\varDelta = 0$), that is $\gamma = 1/\xi$ below:*

$$g_2 = g_1(1-\gamma)+\gamma\int_0^1 \Big(-\frac{\partial}{\partial y}G^{\mathrm{cl}}(\xi,y)f^1(y)+G^{\mathrm{cl}}(\xi,y)f^2(y)\Big)dy.$$

Simplifying, we obtain

$$g_2 = g_1(1-\gamma)+(1-\gamma)\int_0^{\xi}\big(f^1(y)-yf^2(y)\big)\,dy-\int_{\xi}^1\big(f^1(y)+(y-1)f^2(y)\big)\,dy.$$

3 Problem with the unique solution (case $\varDelta \neq 0$)

Substituting the general solution

$$\boldsymbol{u} = c_1\boldsymbol{z}^1+\ldots+c_m\boldsymbol{z}^m+\int_0^1 \boldsymbol{G}^c(x,y)\boldsymbol{f}(y)\,dy$$

of the equation (2.1) into nonlocal conditions (2.2), we use the Fubini's theorem in measure spaces and get the system

$$\begin{aligned} c_1\langle\boldsymbol{L}_1,\boldsymbol{z}^1\rangle+\ldots+c_m\langle\boldsymbol{L}_1,\boldsymbol{z}^m\rangle &= g_1-\textstyle\int_0^1\langle\boldsymbol{L}_1,\boldsymbol{G}^c(\cdot,y)\rangle\boldsymbol{f}(y)\,dy,\\ \ldots\\ c_1\langle\boldsymbol{L}_m,\boldsymbol{z}^1\rangle+\ldots+c_m\langle\boldsymbol{L}_m,\boldsymbol{z}^m\rangle &= g_m-\textstyle\int_0^1\langle\boldsymbol{L}_m,\boldsymbol{G}^c(\cdot,y)\rangle\boldsymbol{f}(y)\,dy.\end{aligned} \tag{3.1}$$

If $\varDelta \neq 0$, we solve constants $c_1,\ldots,c_m$ uniquely and obtain the representation of the unique solution to the problem (1.1)–(1.2), simply denoted by $\boldsymbol{L}\boldsymbol{u} = \boldsymbol{b}$.

The unique solution also has the form $\boldsymbol{u} = \boldsymbol{L}^{-1}\boldsymbol{b}$, where $\boldsymbol{L}^{-1} : (L^2[0,1])^m\times\mathbb{R}^m \to (H^1[0,1])^m$ is the inverse operator of $\boldsymbol{L} : (H^1[0,1])^m \to (L^2[0,1])^m\times\mathbb{R}^m$. In this section, we are going to investigate the structure of the operator $\boldsymbol{L}^{-1}$ and its properties.

3.1 Representation of the inverse operator

First, we select the particular fundamental system $\boldsymbol{v}^l$, $l = \overline{1,m}$, satisfying the biorthogonality conditions $\langle \boldsymbol{L}_k, \boldsymbol{v}^l \rangle = \delta_k^l$ for $k, l = \overline{1,m}$. Let us call functions $\boldsymbol{v}^l$, $l = \overline{1,m}$, by the *biorthogonal fundamental system.* They are unique solutions to the problems

$$\begin{gathered} (\boldsymbol{v}^l)' = \boldsymbol{A}\boldsymbol{v}^l, \\ \langle \boldsymbol{L}_k, \boldsymbol{v}^l \rangle = \delta_k^l, \quad k = \overline{1,m}, \end{gathered} \tag{3.2}$$

respectively. Then the general solution to the problem (2.1) can also be represented by

$$\boldsymbol{u}(x) = \sum_{k=1}^{n} c_k \boldsymbol{v}^k(x) + \int_0^1 \boldsymbol{G}^c(x,y)\boldsymbol{f}(y)\,dy. \tag{3.3}$$

The biorthogonal fundamental system $\boldsymbol{v}^l$, $l = \overline{1,m}$, directly gives the constants

$$c_k = g_k - \int_0^1 \langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot,y)\rangle \boldsymbol{f}(y)\,dy, \quad k = \overline{1,m},$$

from the system (3.1). Putting these expressions into (3.3), we obtain the following representation of the solution

$$\boldsymbol{u}(x) = \int_0^1 \Big(\boldsymbol{G}^c(x,y) - \sum_{k=1}^{m} \boldsymbol{v}^k(x)\langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot,y)\rangle\Big)\boldsymbol{f}(y)\,dy + \sum_{k=1}^{m} g_k \boldsymbol{v}^k(x)$$

or simply

$$\boldsymbol{u}(x) = \int_0^1 \boldsymbol{G}(x,y)\boldsymbol{f}(y)\,dy + \sum_{k=1}^{m} g_k \boldsymbol{v}^k(x). \tag{3.4}$$

Here we denoted the kernel

$$\boldsymbol{G}(x,y) := \boldsymbol{G}^c(x,y) - \sum_{k=1}^{m} \boldsymbol{v}^k(x)\langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot,y)\rangle, \tag{3.5}$$

which is called *the Green's matrix* for the problem with nonlocal conditions (2.1)–(2.2). Let us now introduce the *Green's operator*

$$\boldsymbol{G}\boldsymbol{f} = \int_0^1 \boldsymbol{G}(x,y)\boldsymbol{f}(y)\,dy.$$

It gives the following expression of the unique solution

$$\boldsymbol{u} = \boldsymbol{G}\boldsymbol{f} + g_1\boldsymbol{v}^1 + \ldots + g_m\boldsymbol{v}^m \tag{3.6}$$

for all $\boldsymbol{f} \in (L^2[0,1])^m$ and $g_k \in \mathbb{R}$, $k = \overline{1,m}$.

Now recalling another representation of the unique solution $\boldsymbol{u} = \boldsymbol{L}^{-1}\boldsymbol{f}$, we obtain the following structure of the inverse operator

$$\boldsymbol{L}^{-1} = \big(\boldsymbol{G},\ \boldsymbol{v}^1, \dots,\ \boldsymbol{v}^m\big) : (L^2[0,1])^m \times \mathbb{R}^m \to (H^1[0,1])^m. \tag{3.7}$$

Here $\boldsymbol{G} : (L^2[0,1])^m \to (H^1[0,1])^m$ is the Green's operator and $\boldsymbol{v}^l \in (H^1[0,1])^m$ (precisely, $\boldsymbol{v}^l \in (C^1[0,1])^m$ according to Subsection 2.1) are also characterized by the inverse operator as given below

$$\boldsymbol{G}\boldsymbol{f} = \boldsymbol{L}^{-1}(\boldsymbol{f}^\top, 0, \dots, 0)^\top, \qquad \boldsymbol{v}^1 = \boldsymbol{L}^{-1}\boldsymbol{e}^m, \quad \dots, \quad \boldsymbol{v}^m = \boldsymbol{L}^{-1}\boldsymbol{e}^{2m}.$$

3.2 Properties of the unique solution

Using the formula (2.10), we rewrite constants in the form $c_k = g_k - \langle \boldsymbol{L}_k, \boldsymbol{u}^c\rangle$, $k = \overline{1,m}$. Now we put these values of constants into the formula (3.3) and get the representation of the unique solution to the problem (2.1)–(2.2) via the unique solution $\boldsymbol{u}^c$ to the Cauchy problem as below

$$\boldsymbol{u} = \sum_{k=1}^{m} (g_k - \langle \boldsymbol{L}_k, \boldsymbol{u}^c\rangle)\boldsymbol{v}^k + \boldsymbol{u}^c. \tag{3.8}$$

Let us now consider two problems with the same equation but different nonlocal conditions

$$\begin{aligned} &\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{f}, && \boldsymbol{v}' = \boldsymbol{A}\boldsymbol{v} + \boldsymbol{f}, \\ &\langle \widetilde{\boldsymbol{L}}_k, \boldsymbol{u}\rangle = \widetilde{g}_k, \quad k = \overline{1,m}, && \langle \boldsymbol{L}_k, \boldsymbol{v}\rangle = g_k, \quad k = \overline{1,m}, \end{aligned} \tag{3.9}$$

supposing these problems have unique solutions $\boldsymbol{u}$ and $\boldsymbol{v}$, i.e., both $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$, respectively. The difference $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ is the unique solution to the problem

$$\boldsymbol{w}' = \boldsymbol{A}\boldsymbol{w}, \qquad \langle \boldsymbol{L}_k, \boldsymbol{w}\rangle = g_k - \langle \boldsymbol{L}_k, \boldsymbol{u}\rangle, \quad k = \overline{1,m}.$$

For this problem, we apply the formula (3.8), which represents the unique solution to (2.1)–(2.2). Now $\boldsymbol{u}^c = \boldsymbol{0}$ and right hand sides g_k are replaced by $g_k - \langle \boldsymbol{L}_k, \boldsymbol{u}\rangle$ for $k = \overline{1,m}$. So, we obtain

$$\boldsymbol{w} = \sum_{k=1}^{m} (g_k - \langle \boldsymbol{L}_k, \boldsymbol{u}\rangle)\boldsymbol{v}^k.$$

Recalling the notation $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$, we get the following property.

Lemma 5.7. *The relation between the solutions of the problems* (3.9)

$$\boldsymbol{v} = \boldsymbol{u} + \sum_{k=1}^{m} (g_k - \langle \boldsymbol{L}_k, \boldsymbol{u}\rangle)\boldsymbol{v}^k, \tag{3.10}$$

is valid, where $\boldsymbol{v}^l,\ l=\overline{1,m}$, *are the biorthogonal fundamental system of the second problem* (3.9).

Let us take biorthogonal fundamental systems $\widetilde{\boldsymbol{v}}^l,\ l=\overline{1,m}$, and $\boldsymbol{v}^l,\ l=\overline{1,m}$, for problems (3.9), respectively. These functions are unique solutions to problems (3.9), where $\boldsymbol{f}=\boldsymbol{0}$ and $\widetilde{g}_k=g_k=\delta_k^l,\ k,l=\overline{1,m}$. For every fixed $l=\overline{1,m}$, we apply the formula (3.10) taking $\boldsymbol{v}=\boldsymbol{v}^l$ and $\boldsymbol{u}=\widetilde{\boldsymbol{v}}^l$, i.e.,

$$\boldsymbol{v}^l=\widetilde{\boldsymbol{v}}^l+\sum_{k=1}^{m}(\delta_k^l-\langle \boldsymbol{L}_k,\widetilde{\boldsymbol{v}}^l\rangle)\boldsymbol{v}^k,\qquad l=\overline{1,m}.$$

Rewriting, we obtain the linear system

$$\sum_{k=1}^{n}\langle \boldsymbol{L}_k,\widetilde{\boldsymbol{v}}^l\rangle\boldsymbol{v}^k=\widetilde{\boldsymbol{v}}^l,\qquad l=\overline{1,m},$$

with the nonsingular matrix $(\langle \boldsymbol{L}_k,\widetilde{\boldsymbol{v}}^l\rangle),\ k,l=\overline{1,m}$, since (2.7) is valid for every fundamental system and, particulary, for $\boldsymbol{z}^l=\widetilde{\boldsymbol{v}}^l,\ l=\overline{1,m}$.

Lemma 5.8. *The relation*

$$\sum_{k=1}^{m}\langle \boldsymbol{L}_k,\widetilde{\boldsymbol{v}}^l\rangle\boldsymbol{v}^k=\widetilde{\boldsymbol{v}}^l,\qquad l=\overline{1,m},\tag{3.11}$$

between the fundamental systems for the problems (3.11) *is valid.*

The obtained relation between two biorthogonal fundamental systems allows us to find the biorthogonal fundamental system if the biorthogonal fundamental system for other relative problem is known.

3.3 Properties of a Green's matrix

First, every condition (2.2) can also be represented by

$$\langle \boldsymbol{L}_k,\boldsymbol{u}\rangle=\int_0^1 d\boldsymbol{\mu}_k\,\boldsymbol{u},\quad k=\overline{1,m},$$

where $\boldsymbol{\mu}_k$ is a $1\times m$ row matrix of bounded variation elementwise. Thus, $\boldsymbol{\mu}_k$ can have at most countably many discontinuity points $y_l,\ l=1,2,3,\dots$, and need only be differentiable almost everywhere, i.e., $\boldsymbol{\mu}_k'=\boldsymbol{L}_k$ almost everywhere. Thus, for every fixed $x\in[0,1]$, the Green's matrix (3.5) may have at most countably many discontinuities at $y_l,\ l=1,2,3,\dots$, as well [20, Bryan 1969]. In other words, the square $[0,1]\times[0,1]$ may be divided into $N\in\overline{\mathbb{N}}$ rectangular domains (each rectangle $x\in[0,1],\ y_{l-1}<y<y_l,\ l=\overline{1,N}$, $y_0=0,\ y_N=1$), where the Green's matrix (3.5) may satisfy the analogue

of classical properties of the Green's matrix (2.10) or the Green's function from Chapter 2. Indeed, properties of the Green's matrix for the problem with nonlocal conditions, except the discontinuities, were obtained by other authors [19, Brown and Krall 1974], [20, Bryan 1969], where nonlocal conditions where often called *general conditions*. Below we list several properties, those resemble the classical results for the Green's function, or can be obtained applying the properties of (2.10) to (3.5). So, except the discontinuities at lines $y = y_0,\ y_1,\ y_2, \ldots,$ for $x \in [0,1]$, we have the following properties of the Green's matrix (3.5):

1) $\boldsymbol{G}(y+0,y) = \boldsymbol{I} - \sum_{k=1}^{m} \boldsymbol{v}^k(y)\langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot,y)\rangle$,
   $\boldsymbol{G}(y-0,y) = -\sum_{k=1}^{m} \boldsymbol{v}^k(y)\langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot,y)\rangle$;
2) $\boldsymbol{G}(y+0,y) - \boldsymbol{G}(y-0,y) = \boldsymbol{I}$;
3) $\boldsymbol{G}(x,y)$ is C in (x,y) except the diagonal $x = y$;
4) $\boldsymbol{G}(x,y)$ is C^1 in x except the diagonal $x = y$;
5) $(\partial/\partial x)\boldsymbol{G}(x,y) - \boldsymbol{A}(x)\boldsymbol{G}(x,y) = \boldsymbol{0}$ except the diagonal $x = y$;
6) $\langle \boldsymbol{L}_k, \boldsymbol{G}(\cdot,y)\rangle = 0,\ k = \overline{1,m}$.

Let us write solutions to problems (3.8) with $\widetilde{g}_k = g_k = 0,\ k = \overline{1,m}$, via their Green's matrices

$$\boldsymbol{u}(x) = \int_0^1 \widetilde{\boldsymbol{G}}(x,y)\boldsymbol{f}(y)\,dy, \qquad \boldsymbol{v}(x) = \int_0^1 \boldsymbol{G}(x,y)\boldsymbol{f}(y)\,dy,$$

respectively. Putting them into the formula (3.10), we can get

$$\int_0^1 \boldsymbol{G}(x,y)\boldsymbol{f}(y)\,dy = \int_0^1 \widetilde{\boldsymbol{G}}(x,y)\boldsymbol{f}(y)\,dy - \sum_{k=1}^{m}\langle \boldsymbol{L}_k, \int_0^1 \widetilde{\boldsymbol{G}}(\cdot,y)\boldsymbol{f}(y)\,dy\,\rangle \boldsymbol{v}^k(x)$$

or

$$\int_0^1 \boldsymbol{G}(x,y)\boldsymbol{f}(y)\,dy = \int_0^1 \big(\widetilde{\boldsymbol{G}}(x,y) - \sum_{k=1}^{m} \boldsymbol{v}^k(x)\langle \boldsymbol{L}_k, \widetilde{\boldsymbol{G}}(\cdot,y)\rangle\big)\boldsymbol{f}(y)\,dy.$$

From here we obtain the following formula.

Lemma 5.9. *The relation*

$$\boldsymbol{G}(x,y) = \widetilde{\boldsymbol{G}}(x,y) - \sum_{k=1}^{m} \boldsymbol{v}^k(x)\langle \boldsymbol{L}_k, \widetilde{\boldsymbol{G}}(\cdot,y)\rangle \tag{3.12}$$

between two Green's matrices for the problems (3.8) *is valid.*

3.4 Nonlocal boundary value problem

Let us investigate the unique solution $\boldsymbol{u}$ to the problem with nonlocal boundary conditions

$$\boldsymbol{u}' = \boldsymbol{A}(x)\boldsymbol{u} + \boldsymbol{f}, \quad x \in [0,1], \tag{3.13}$$

$$\langle \boldsymbol{L}_k, \boldsymbol{u}\rangle := \langle \boldsymbol{\kappa}_k, \boldsymbol{u}\rangle - \gamma_k\langle \boldsymbol{\varkappa}_k, \boldsymbol{u}\rangle = g_k, \quad k = \overline{1,m}, \tag{3.14}$$

where $\boldsymbol{\kappa}_k$ describe classical parts of conditions (3.14) but $\boldsymbol{\varkappa}_k$ represent fully nonlocal parts, $\gamma_k \in \mathbb{R},\ k = \overline{1,m}$. We suppose that the classical problem (3.13)–(3.14) ($\gamma_k = 0,\ k = \overline{1,m}$) also has the unique solution $\boldsymbol{u}^{\mathrm{cl}}$. Putting these solutions into the formula (3.12), we obtain the representation of the unique solution to the nonlocal boundary value problem (3.13)–(3.14) via the unique solution to the classical problem only as given bellow

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{cl}} + \sum_{k=1}^{m} \gamma_k\langle \boldsymbol{\varkappa}_k, \boldsymbol{u}^{\mathrm{cl}}\rangle \boldsymbol{v}^k. \tag{3.15}$$

Here $\boldsymbol{v}^k,\ k = \overline{1,m}$, is the biorthogonal fundamental system of the nonlocal boundary value problem (3.13)–(3.14). Applying (3.12) to the Green's matrix $\boldsymbol{G}(x,y)$ of the nonlocal boundary value problem and the Green's matrix $\boldsymbol{G}^{\mathrm{cl}}(x,y)$ of the classical problem, we obtain very analogous relation.

Lemma 5.10. *The Green's matrix and the Green's matrix for the classical problem are related by the equality*

$$\boldsymbol{G}(x,y) = \boldsymbol{G}^{\mathrm{cl}}(x,y) + \sum_{k=1}^{m} \gamma_k \boldsymbol{v}^k(x)\langle \boldsymbol{\varkappa}_k, \boldsymbol{G}^{\mathrm{cl}}(\cdot,y)\rangle \tag{3.16}$$

Let us note that all properties of the unique solution and representations of the Green's matrix, obtained in this section, resemble analogical results for m-th order ordinary differential equation with nonlocal conditions. Comparing properties from Chapter 2, we see the similarity and ask the following question. What is the relation between the Green's matrix for the system and the Green's function for the m-th order differential problem with nonlocal conditions?

The answer to this question is given in the following section.

4 m-th order ordinary differential equations with nonlocal conditions

Now we are going to apply obtained results to the m-th order ordinary differential equation with nonlocal conditions

$$u^{(m)} + a_{m-1}(x)u^{(m-1)} + \ldots + a_1(x)u' + a_0(x)u = f(x), \; x \in [0,1], \quad (4.1)$$

$$\langle L_k, u\rangle := \sum_{l=1}^{m} \langle L_{kl}, u^{(l-1)}\rangle = g_k, \quad k = \overline{1,m}, \quad (4.2)$$

where $a_j \in C[0,1]$, $j = \overline{0,m-1}$, $f \in L^2[0,1]$ and $L_{kl} \in \big(C[0,1]\big)^*$ with $g_k \in \mathbb{R}$, $k,l = \overline{1,m}$. First, introducing notation $u^k = u^{(k-1)}$, $k = \overline{1,m}$, we rewrite the problem (4.1)–(4.2) into the equivalent first order system for m equations

$$\begin{aligned} &(u^k)' = u^{k+1}, \quad k = \overline{1,m-1}, \\ &(u^m)' = f - a_0u^1 - a_1u^2 - \ldots - a_{m-1}u^m \end{aligned} \quad (4.3)$$

with nonlocal conditions

$$\sum_{l=1}^{m} \langle L_{kl}, u^l\rangle = g_k, \quad k = \overline{1,m}. \quad (4.4)$$

Denoting the vector of unknown functions $\boldsymbol{u} = (u^1, u^2, \ldots, u^m)^\top$, this system can also be written in the vectorial form

$$\begin{aligned} &\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{f}, \\ &\langle \boldsymbol{L}_k, \boldsymbol{u}\rangle := \textstyle\sum_{l=1}^{m} \langle L_{kl}, u^l\rangle = g_k, \quad k = \overline{1,m}, \end{aligned} \quad (4.5)$$

with the m-th order square matrix and the right hand side

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{m-2} & -a_{m-1} \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ f \end{pmatrix}.$$

4.1 Green's function via Green's matrix

Since (4.1)–(4.2) and (4.5) describe the same problem but in different forms, the problem (4.1)–(4.2) has the unique solution $u \in H^m[0,1]$ if and only if the system (4.3) has the unique vectorial solution $\boldsymbol{u}$, where $u^k = u^{(k-1)} \in H^1[0,1]$, $k = \overline{1,m}$, and vice versa. Let us say $g_k = 0$, $k = \overline{1,m}$, and take their solutions, i.e., the unique solution to (4.1)–(4.2)

$$u(x) = \int_0^1 G(x,y)f(y)\,dy, \qquad x \in [0,1], \quad (4.6)$$

described by the Green's function $G(x, y)$ of the problem (4.1)–(4.2), and the unique solution to the system (4.5)

$$\boldsymbol{u}(x) = \int_0^1 \boldsymbol{G}(x, y)\boldsymbol{f}(y)\,dy, \qquad x \in [0, 1], \tag{4.7}$$

represented by the Green' s matrix $\boldsymbol{G}(x, y)$ of the system (4.5). Simplifying (4.7), we get

$$u^k(x) = \int_0^1 G^{km}(x, y)f(y)\,dy, \qquad k = \overline{1, m}. \tag{4.8}$$

Since $u^k = u^{(k-1)},\ k = \overline{1, m}$, we differentiate (4.6) and obtain the following relation.

Lemma 5.11. *The last column elements of the Green's matrix for the system* (4.4) *are represented by the Green's function of the scalar problem* (4.1)–(4.2) *in the form*

$$G^{km}(x, y) = \frac{\partial^{k-1}}{\partial x^{k-1}} G(x, y), \qquad k = \overline{1, m}. \tag{4.9}$$

Corollary 5.12. *The Green's function for the problem* (4.1)–(4.2) *can be represented by the function from the Green's matrix of the system* (4.4) *by*

$$G(x, y) = G^{1m}(x, y).$$

Remark 5.13. Thus, their properties (discontinuity points, jumps, properties of derivatives) have to be the same. Indeed, according to Roman [100, 2011], on particular rectangles $x \in [0, 1],\ y_{l-1} < y < y_l$ for $l = \overline{1, N}$ (they depend on nonlocal conditions (4.2)), the Green's function $G(x, y)$ and its partial derivatives from the first to $(m-2)$-th order in x, i.e., $(\partial^j/\partial x^j)G(x, y),\ j = \overline{0, m-1}$, are continuous but the $(m-1)$-th order partial derivative is also continuous except the diagonal $x = y$, where it has the jump

$$\frac{\partial^{m-1}}{\partial x^{m-1}} G(y + 0, y) - \frac{\partial^{m-1}}{\partial x^{n-1}} G(y - 0, y) = 1.$$

All these properties of the Green's function confirm the properties, given in Subsection 3.3, for the elements $G_{lm}(x, y)$ of the Green's matrix. Moreover, here we obtain additional smoothness properties $G^{lm} \in C^{m-1}[0, 1]$ except the diagonal $x = y$ and discontinuity points $y_0, y_1, y_2, \ldots$.

4.2 Green's matrix via Green's function

Formulas (4.9) represent the last column of the Green's matrix $\boldsymbol{G}(x, y)$. To find the next to last column, we consider the system (4.5) with the different right hand side

$$\boldsymbol{u}' = \boldsymbol{A}\boldsymbol{u} + \widetilde{\boldsymbol{f}}, \quad \langle \boldsymbol{L}_k, \boldsymbol{u}\rangle = 0, \quad k = \overline{1, m}, \tag{4.10}$$

where $\boldsymbol{u} = (u^1, u^2, \ldots, u^m)^\top$ and $\widetilde{\boldsymbol{f}} = (0, \ldots, 0, f^{m-1}, 0)^\top$. The solution to this system is represented only by the next to last column of the Green's matrix

$$u^k(x) = \int_0^1 G^{k,m-1}(x,y) f^{m-1}(y)\,dy, \qquad k = \overline{1, m}. \tag{4.11}$$

From the structure of equations (4.10), we get $u^k = (u^1)^{(k-1)}$, $k = \overline{1, m-1}$, and $u^m = (u^1)^{(m-1)} - f^{m-1}$. Let us temporarily suppose $f^{m-1} \in C^1[0,1]$ and observe that the system (4.10) is equivalent to the problem (4.1)–(4.2) for the function u^1 with $f = (f^{m-1})' - a_{m-1} f^{m-1}$, and $g_k = \langle L_{km}, f^{m-1}\rangle$, $k = \overline{1, m}$. By [100, Roman 2011], it has the solution $u^1(x) = \sum_{k=1}^m g_k v^k(x) + \int_0^1 G(x,y) f(y)\,dy$, that is,

$$\begin{aligned} u^1(x) = &\sum_{k=1}^m \langle L_{km}, f^{m-1}\rangle v^k(x) \\ &+ \int_0^1 G(x,y)\big((f^{m-1})'(y) - a_{m-1}(y) f^{m-1}(y)\big)\,dy. \end{aligned} \tag{4.12}$$

Since the Green's function is given by the equality $G(x,y) = G^c(x,y) - \sum_{k=1}^m v^k(x)\langle L_k, G^c(\cdot, y)\rangle$, we are going to rewrite the integral

$$\int_0^1 G(x,y)(f^{m-1})'(y)\,dy \tag{4.13}$$

$$= \int_0^1 \big(G^c(x,y) - \sum_{k=1}^m v^k(x)\langle L_k, G^c(\cdot, y)\rangle\big)(f^{m-1})'(y)\,dy \tag{4.14}$$

in another form. First, applying the integration by parts formula, we have

$$\int_0^1 G^c(x,y)(f^{m-1})'(y)\,dy = -G^c(x,0) f^{m-1}(0) - \int_0^1 \frac{\partial}{\partial y} G^c(x,y) f^{m-1}(y)\,dy,$$

since $G^c(x,1) = 0$. Second, using the Fubinni's theorem in measure spaces, the integration by parts again and properties of the Green's function $G^c(x,y)$ for the Cauchy problem (see Subsection 2.2 in Chapter 2), we get

$$\int_0^1 \langle L_k, G^c(\cdot, y)\rangle (f^{m-1})'(y)\,dy = \int_0^1 \sum_{l=1}^m \langle L_{kl}, \frac{\partial^{l-1}}{\partial x^{l-1}} G^c(\cdot, y)\rangle (f^{m-1})'(y)\,dy$$

$$= \sum_{l=1}^m \int_0^1 \int_0^1 \frac{\partial^{l-1}}{\partial x^{l-1}} G^c(x,y)\,d\mu_{kl}(x)\,(f^{m-1})'(y)\,dy$$

$$= \sum_{l=1}^{m} \int_0^1 \int_0^1 \frac{\partial^{l-1}}{\partial x^{l-1}} G^c(x,y)(f^{m-1})'(y)\, dy\, d\mu_{kl}(x)$$

$$= \sum_{l=1}^{m} \int_0^1 \Bigg(\frac{\partial^{l-1}}{\partial x^{l-1}} G^c(x,y) f^{m-1}(y) \Big|_{y=0}^{y=x-0} + \frac{\partial^{l-1}}{\partial x^{l-1}} G^c(x,y) f^{m-1}(y) \Big|_{y=x+0}^{y=1}$$

$$- \int_0^1 \frac{\partial^l}{\partial x^{l-1}\partial y} G^c(x,y) f^{m-1}(y)\, dy \Bigg) d\mu_{kl}(x) = \sum_{l=1}^{m} \int_0^1 \Bigg(- \frac{\partial^{l-1}}{\partial x^{l-1}} G^c(x,0) f^{m-1}(0)$$

$$+ \delta_l^m f^{m-1}(x) - \int_0^1 \frac{\partial^l}{\partial x^{l-1}\partial y} G^c(x,y) f^{m-1}(y)\, dy \Bigg) d\mu_{kl}(x)$$

$$= -\langle L_k, G^c(\cdot,0)\rangle f^{m-1}(0) + \langle L_{km}, f^{m-1}\rangle - \int_0^1 \langle L_k, \frac{\partial}{\partial y} G^c(\cdot,y)\rangle f^{m-1}(y)\, dy.$$

Now the integral (4.14) is given by

$$\int_0^1 G(x,y)(f^{m-1})'(y)\, dy = G^c(x,0) f^{m-1}(0) - \int_0^1 \frac{\partial}{\partial y} G^c(x,y) f^{m-1}(y)\, dy$$

$$- \sum_{k=1}^{m} v^k(x) \Bigg(\langle L_k, G^c(\cdot,0)\rangle f^{m-1}(0) + \langle L_{km}, f^{m-1}\rangle$$

$$- \int_0^1 \langle L_k, \frac{\partial}{\partial y} G^c(\cdot,y)\rangle f^{m-1}(y)\, dy \Bigg)$$

$$= G(x,0) f^{m-1}(0) - \sum_{k=1}^{m} \langle L_{km}, f^{m-1}\rangle v^k(x) - \int_0^1 \langle L_k, \frac{\partial}{\partial y} G(\cdot,y)\rangle f^{m-1}(y)\, dy$$

$$= -\sum_{k=1}^{m} \langle L_{km}, f^{m-1}\rangle v^k(x) - \int_0^1 \langle L_k, \frac{\partial}{\partial y} G(\cdot,y)\rangle f^{m-1}(y)\, dy.$$

Here we used the equality $G(x,0) = 0$, because the Green's function satisfies the homogenous equation (4.1) for $x \geqslant 0$ and homogenous nonlocal conditions $\langle L_k, G(\cdot,0)\rangle = 0$. Thus, $G(x,0)$ vanish as the unique solution to the homogenous problem with $\Delta = 0$.

Now, the representation of the function (4.12) simplifies to

$$u^1(x) = -\int_0^1 \Bigg(\frac{\partial}{\partial y} G(x,y) + a_{m-1}(y) G(x,y) \Bigg) f^{m-1}(y)\, dy \tag{4.15}$$

$$= -\sum_{j=1,\, j\neq M}^{N} \int_{y_{j-1}}^{y_j} + \int_{y_{M-1}}^{x} + \int_x^{y_M} \Bigg(\frac{\partial}{\partial y} G(x,y) + a_{m-1}(y) G(x,y) \Bigg) f^{m-1}(y)\, dy,$$

where $x \in [y_{M-1}; y_M]$. Differentiating the function u^1, we find other functions

$$u^k(x) = -\left(\sum_{j=1,\, j\neq M}^{N} \int_{y_{j-1}}^{y_j} + \int_{y_{M-1}}^{x} + \int_{x}^{y_M}\right) \left(\frac{\partial^k}{\partial x^{k-1}\partial y} G(x,y) \right.$$
$$\left. + a_{m-1}(y) \frac{\partial^{k-1}}{\partial x^{k-1}} G(x,y)\right) f^{m-1}(y)\, dy,$$

for $k = \overline{2,m}$. Here we used properties of the Green's function $G(x,y)$, which is C^{m-2} in x but has the jump on the diagonal $(\partial^{m-1}/(\partial x^{m-2}\partial y))G^c(x,x-0) - (\partial^{m-1}/(\partial x^{m-2}\partial y))G^c(x,x+0) = -1$ for $y \in [y_{j-1}; y_j]$ [100, Roman 2011]. Let us note that obtained representations of functions u^k, $k = \overline{1,m}$, are valid if $f^{m-1} \in L^2[0,1]$.

Since the formula (4.11) can be rewritten in the form

$$u^k = \left(\sum_{j=1,\, j\neq M}^{N} \int_{y_{j-1}}^{y_j} + \int_{y_{M-1}}^{x} + \int_{x}^{y_M}\right) G^{k,m-1}(x,y) f^{m-1}(y)\, dy,$$

where the Green's matrix is continuous in (x,y) on each subdomain, then other representation (4.15) gives the equality

$$G^{k,m-1}(x,y) = -\frac{\partial^k}{\partial x^{k-1}\partial y} G(x,y) - a_{m-1}(y) \frac{\partial^{k-1}}{\partial x^{k-1}} G(x,y), \qquad k = \overline{1,m}.$$

Similarly, we find the l-th column of the Green's matrix investigating the system (4.10) with the right hand side $\widetilde{\boldsymbol{f}} = (0,\ldots,0,f^l,0,\ldots,0)^\top$ for $l = \overline{1,m-2}$. Now we get $u^i = (u^1)^{(i-1)}$, $i = \overline{1,l}$, and $u^{l+i} = (u^1)^{(l+i-1)} - (f^l)^{(i-1)}$, $i = \overline{1,m-l}$. Here the system is equivalent to the problem (4.1)–(4.2) for the function u^1 with $f = (f^l)^{(m-l)} - \sum_{i=0}^{m-l} a_{l+i}(f^l)^{(i)}$, if $f^l \in C^{m-l}[0,1]$, and $g_k = \sum_{i=1}^{m-l} \langle L_{l,l+i}, (f^l)^{(i-1)}\rangle$, $k = \overline{1,m}$. According to [100, Roman 2011], it has the solution

$$u^1(x) = \sum_{k=1}^{m} g_k v^k(x) + \int_0^1 G(x,y) f(y)\, dy.$$

Differentiating this function and making simplifications as previous, we write all solutions in the form

$$u^k(x) = \int_0^1 G^{kl}(x,y) f^l(y)\, dy, \qquad k = \overline{1,m},$$

with the kernel

$$G^{kl}(x,y) = -\frac{\partial^{m+k-l-1}}{\partial x^{k-1}\partial y^{m-l}} G(x,y) - \sum_{i=0}^{m-l-1} (-1)^i \frac{\partial^i}{\partial y^i}\left(a_{l+i}(y) \frac{\partial^{k-1}}{\partial x^{k-1}} G(x,y)\right)$$

for $k = \overline{1,m}$ and $l = \overline{1,m-1}$. To make sure, we can verify directly that functions u^k, $k = \overline{1,m}$, are solutions to the system (4.10) with the right hand side $\widetilde{\boldsymbol{f}} = (0, \dots, 0, f^l, 0, \dots, 0)^\top$.

Recalling Lemma 5.11, we obtain the following feature.

Lemma 5.14. *The Green's matrix can be described by the Green's function of the scalar problem as below*

$$G^{kl}(x,y) = -\frac{\partial^{m+k-l-1}}{\partial x^{k-1}\partial y^{m-l}} G(x,y) - \sum_{i=0}^{m-l-1} (-1)^i \frac{\partial^i}{\partial y^i}\left(a_{l+i}(y)\frac{\partial^{k-1}}{\partial x^{k-1}} G(x,y)\right)$$

for $k, l = \overline{1,m}$.

4.3 The second order problem

Let us now take the second order system ($m = 2$)

$$(u^1)' = u^2 + f^1, \quad (u^2)' = a_0 u^1 + a_1 u^2 + f^2,$$
$$\langle \boldsymbol{L}_k, \boldsymbol{u}\rangle := \langle L_{k1}, u^1\rangle + \langle L_{k2}, u^2\rangle = 0, \quad k = 1, 2,$$

where $f^k \in L^2[0,1]$ and $L_{kl} \in C^*[0,1]$. It has the Green's matrix

$$\boldsymbol{G}(x,y) = \begin{pmatrix} -a_1(y)G(x,y) - G'_y(x,y) & G(x,y) \\ -a_1(y)G'_x(x,y) - G''_{xy}(x,y) & G'_x(x,y) \end{pmatrix},$$

which is described by the Green's function $G(x,y)$ for the scalar problem

$$u'' + a_1(x)u' + a_0(x)u = f(x), \quad x \in [0,1],$$
$$\langle L_k, u\rangle := \langle L_{k1}, u\rangle + \langle L_{k2}, u'\rangle = 0, \quad k = 1, 2.$$

Example 5.15. *Let us consider the Cauchy system*

$$(u^1)' = u^2 + f^1, \quad (u^2)' = f^2,$$
$$u^1(0) = 0, \quad u^2(0) = 0,$$

which can also be written in the matrix form $\boldsymbol{u}' - \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$, $\boldsymbol{u}(0) = \boldsymbol{0}$. *If* $f^1 \in C^1[0,1]$, *we obtain the scalar problem*

$$(u^1)'' = f(x), \quad x \in [0,1],$$
$$u^1(0) = 0, \quad (u^1)'(0) = f^1(0)$$

with the right hand side $f = (f^1)' + f^2$. *The differential equation* $-u'' = f$ *has the solution*

$$u^1 = c_1 + c_2 x + \int_0^1 G^c(x,y) f(y)\, dy,$$

represented by the Green's function $G^c(x,y) = (x-y)\cdot \mathrm{H}(x-y)$*. Substituting the general solution into initial conditions, we get* $c_1 = 0$ *and* $c_2 = f^1(0)$*. Below we make simplifications*

$$\begin{aligned} u^1 =& f^1(0)x + \int_0^x (x-y)((f^1)'(y) + f^2(y))\,dy \\ =& -\int_0^x f^1(y)\,dy + \int_0^x (x-y) f^2(y)\,dy \end{aligned}$$

and find another function $u^2 = (u^1)' - f^1 = \int_0^x f^2(y)\,dy$*. From here we know the representation of the Green's matrix*

$$\begin{aligned} \boldsymbol{G}^c(x,y) &= \begin{pmatrix} -\mathrm{H}(x-y) & (x-y)\cdot \mathrm{H}(x-y) \\ 0 & \mathrm{H}(x-y) \end{pmatrix} \\ &= \begin{pmatrix} -(G^c)'_y(x,y) & G^c(x,y) \\ -(G^c)''_{xy}(x,y) & (G^c)'_x(x,y) \end{pmatrix}, \end{aligned}$$

which represents the unique vectorial solution $\boldsymbol{u} = \int_0^1 \boldsymbol{G}^c(x,y)\boldsymbol{f}(y)\,dy$ *to the Cauchy system.*

Example 5.16. *Let us take a differential problem with the Bitsadze–Samarkii condition*

$$\begin{aligned} -u'' = f(x), \qquad x \in [0,1], \\ u(0) = 0; \quad u(1) = \gamma u(\xi), \end{aligned} \tag{4.16}$$

where $f \in L^2[0,1]$ *is a real function and* $\gamma \in \mathbb{R}$, $\xi \in (0,1)$*. According to [100, Roman 2011], it has the unique solution and the Green's function*

$$G(x,y) = \begin{cases} y(1-x), & y \leqslant x, \\ x(1-y), & x < y, \end{cases} + \frac{\gamma x}{1-\gamma\xi} \begin{cases} y(1-\xi), & y \leqslant \xi, \\ \xi(1-y), & \xi < y, \end{cases}$$

if and only if $\gamma\xi \neq 1$*. Denoting* $u_1 = u$ *and* $u_2 = u'$*, we rewrite the problem* (4.16) *into the equivalent system*

$$\begin{aligned} (u^1)' = u^2 + f^1, \quad (u^2)' = f^2, \\ u^1(0) = 0, \quad u^1(1) = \gamma u^1(\xi) \end{aligned}$$

with $f^1 = 0$, $f^2 = -f$*. This system has the Green's matrix*

$$\boldsymbol{G}(x,y) = \begin{pmatrix} G'_y(x,y) & -G(x,y) \\ G''_{xy}(x,y) & -G'_x(x,y) \end{pmatrix}.$$

Applying properties of the Green's function $G(x,y)$*, we can directly obtain properties of the Green's matrix, those are formulated in Subsection 3.3.*

Let us remark, that this Green's matrix has the minus sign since the operator $-u''$ *has the Green's function* $G(x,y)$*, given above, which differs with the minus sign from the Green's function of the operator* u''*.*

5 The unique minimizer (case $\varDelta = 0$)

For $\varDelta = 0$, the problem (2.1)–(2.2) is not uniquely solvable. Hence, we are going to solve it uniquely in the least squares sense.

Let us now emphasize the specific of our problem. Instead of the whole vectorial problem $\boldsymbol{L}\boldsymbol{u} = \boldsymbol{b}$, most authors analyzed the operator $\boldsymbol{\mathcal{L}}\boldsymbol{u} = \boldsymbol{u}' - \boldsymbol{A}\boldsymbol{u}$, whose domain is determined in part of the Stieltjes integral boundary conditions $\int_0^1 d\boldsymbol{\mu}_k\, \boldsymbol{u} = g_k$, $k = \overline{1,m}$, that in our notations correspond to $\langle \boldsymbol{L}_k, \boldsymbol{u} \rangle = g_k$, $k = \overline{1,m}$. Indeed, they solved the problem (2.1)–(2.2) in the least squares sense minimizing only the residual $\boldsymbol{u}' - \boldsymbol{A}\boldsymbol{u} - \boldsymbol{f}$ by functions $\boldsymbol{u}$, satisfying conditions $\int_0^1 d\,\boldsymbol{\mu}_k(x)\boldsymbol{u}(x) = g_k$, $k = \overline{1,m}$.

In this section, we minimize the norm of the residual of the whole vectorial problem $\boldsymbol{L}\boldsymbol{u} - \boldsymbol{b}$ by functions from the domain $D(\boldsymbol{L}) = (H^1[0,1])^m$ (see Lemma 5.1). We obtain the unique least squares solution for our considering least squares problem, present its properties and representations.

5.1 The minimum norm least squares solution

If $\varDelta = 0$, the differential system (1.1)–(1.2) may have a lot of solutions (consistent problem) or no solutions (inconsistent problem). From Corollary 5.5, we know the solvability conditions of the differential system. However, for both cases, we look for a unique vector valued function $\boldsymbol{u}^o \in (H^1[0,1])^m$, which has the minimum norm among all minimizers $\boldsymbol{u}^g \in (H^1[0,1])^m$ of the norm of the residual

$$\|\boldsymbol{L}\boldsymbol{u}^g - \boldsymbol{b}\|_{(L^2[0,1])^m \times \mathbb{R}^m} = \inf_{u \in (H^1[0,1])^m} \|\boldsymbol{L}\boldsymbol{u} - \boldsymbol{b}\|_{(L^2[0,1])^m \times \mathbb{R}^m}, \tag{5.1}$$

i.e.,

$$\|\boldsymbol{u}^o\|_{(H^1[0,1])^m} < \|\boldsymbol{u}^g\|_{(H^1[0,1])^m}, \qquad \forall\ \boldsymbol{u}^g \neq \boldsymbol{u}^o. \tag{5.2}$$

Minimization steps here are interpreted analogously as in previous chapters for differential and discrete problems.

The minimum norm least squares solution $\boldsymbol{u}^o$ always exists and is unique since $\boldsymbol{L}$ is the continuous linear operator with the closed range [6, Ben-Israel and Greville 2003]. Moreover, the operator $\boldsymbol{L} : (H^1[0,1])^m \to (L^2[0,1])^m \times \mathbb{R}^m$ has the Moore-Penrose inverse $\boldsymbol{L}^\dagger : (L^2[0,1])^m \times \mathbb{R}^m \to (H^1[0,1])^m$, which describes the minimizer

$$\boldsymbol{u}^o = \boldsymbol{L}^\dagger \boldsymbol{b} \tag{5.3}$$

similarly as the unique solution is represented by $\boldsymbol{u} = \boldsymbol{L}^{-1}\boldsymbol{b}$, if it exists. Let us note that the system (1.1)–(1.2) with $\varDelta \neq 0$ is also involved in the minimization problem (5.1)–(5.2) because its unique solution $\boldsymbol{u} = \boldsymbol{L}^{-1}\boldsymbol{b}$ is coincident with the unique minimizer $\boldsymbol{u}^o$.

Every least squares solution, which is a minimizer of (5.1), is characterized by the minimum norm least squares solution

$$\boldsymbol{u}^g = \boldsymbol{u}^o + P_{N(\boldsymbol{L})}\boldsymbol{c}$$

with an arbitrary vector valued function $\boldsymbol{c} \in (H^1[0,1])^m$. Let us note that the minimum norm least squares solution $\boldsymbol{u}^o$ is the unique function among all minimizers $\boldsymbol{u}^g$, which belongs to the orthogonal complement $N(\boldsymbol{L})^\perp$ since $\boldsymbol{u}^o = P_{N(\boldsymbol{L})^\perp}\boldsymbol{u}^g$ [6, Ben-Israel and Greville 2003].

5.2 Generalized Green's matrix

Let us develop the parallel to Subsection 3.1 studying the structure the Moore–Penrose inverse $\boldsymbol{L}^\dagger$. Since the right hand side is of the form $\boldsymbol{b} = (f^1, \ldots, f^m, g_1, \ldots, g_m)^\top = (\boldsymbol{f}^\top, g_1, \ldots, g_m)^\top$ and the operator $\boldsymbol{L}^\dagger$ is linear, we get the following composition of the minimum norm least squares solution

$$\boldsymbol{u}^o = \boldsymbol{L}^\dagger \boldsymbol{b} = \boldsymbol{G}^g\boldsymbol{f} + g_1\boldsymbol{v}^{g,1} + \ldots + g_m\boldsymbol{v}^{g,m},$$

where we introduced the operator $\boldsymbol{G} : (L^2[0,1])^m \to (H^1[0,1])^m$ and vector valued functions $\boldsymbol{v}^{g,k} \in (H^1[0,1])^m$ by formulas

$$\boldsymbol{G}^g\boldsymbol{f} := \boldsymbol{L}^\dagger(\boldsymbol{f}^\top, 0, \ldots, 0)^\top, \qquad \boldsymbol{v}^{g,k} := \boldsymbol{L}^\dagger\boldsymbol{e}^{m+k}, \quad k = \overline{1,m}.$$

So, we get the representation of the Moore–Penrose inverse

$$\boldsymbol{L}^\dagger = (\boldsymbol{G}^g,\ \boldsymbol{v}^{g,1},\ \ldots,\ \boldsymbol{v}^{g,m}).$$

Let us note that $\boldsymbol{G}^g$ is a continuous linear matrix valued operator elementwise. The linearity is obvious because $\boldsymbol{L}^\dagger$ is linear. Second, each k-th component ($k = \overline{1,m}$) of the vector valued function $\boldsymbol{G}^g\boldsymbol{f}$ can be partitioned into the composition $(\boldsymbol{G}^g\boldsymbol{f})^k = G^{g,k1}f^1 + \ldots + G^{g,km}f^m$ introducing operators $G^{g,kl} : L^2[0,1] \to H^1[0,1]$. Hence, we get the partitioning $\boldsymbol{G}^g = (G^{g,kl})$.

To proof the continuity, we observe that $G^{g,kl}f \in \mathbb{R}$ for every fixed $x \in [0,1]$ and a function $f \in L^2[0,1]$. Here we applied the Sobolev embedding theorem, which says that $G^{g,kl}f \in H^1[0,1] \subset C[0,1]$ and gives the inequality $\|G^{g,kl}f\|_{C[0,1]} \leqslant C\|G^{g,kl}f\|_{H^1[0,1]}$ with a particular finite constant C independent of f. Now for every fixed $x \in [0,1]$, we define the linear functional $F_{kl} : L^2[0,1] \to \mathbb{R}$ by

$$\langle F_{kl}, f\rangle = G^{g,kl}f(x), \qquad \forall\ f \in L^2[0,1].$$

It is continuous because it is bounded

$$
\begin{gathered}
|\langle F_{kl}, f\rangle| = |G^{g,kl} f(x)| \leqslant \sup_{x\in[0,1]} |G^{g,kl} f(x)| = \|G^{g,kl} f\|_{C[0,1]} \\
\leqslant C \cdot \|G^{g,kl} f\|_{H^1[0,1]} = C \cdot \|\big(\boldsymbol{L}^{\dagger}(f\boldsymbol{e}^l)\big)^k\|_{H^1[0,1]} \leqslant C \cdot \|\boldsymbol{L}^{\dagger}(f\boldsymbol{e}^l)\|_{(H^1[0,1])^m} \\
\leqslant C \cdot \|\boldsymbol{L}^{\dagger}\| \cdot \|f\|_{L^2[0,1]}
\end{gathered}
$$

for the finite constant $C \cdot \|\boldsymbol{L}^{\dagger}\|$ and every $f \in L^2[0,1]$. Then according to the Riesz representation theorem for continuous linear functionals in the Hilbert space [65, Kreyszig 1978], there exists the unique function $G^{g,kl}(x,\cdot) \in L^2[0,1]$ that F_{kl} can be represented by the inner product in the space $L^2[0,1]$ as follows

$$
\langle F_{kl}, f\rangle = G^{g,kl} f(x) = \int_0^1 G^{g,kl}(x,y) f(y)dy, \quad \forall f \in L^2[0,1],
$$

and this equality is valid for every $x \in [0,1]$.

Let us now denote the m-th order matrix valued function composed of these functions $G^{g,kl}(x,y)$ by $\boldsymbol{G}^g(x,y) = \big(G^{g,kl}(x,y)\big)$. Then the minimum norm least squares solution (5.3) can be given by

$$
\boldsymbol{u}^o(x) = \int_0^1 \boldsymbol{G}^g(x,y)\boldsymbol{f}(y)dy + g_1\boldsymbol{v}^{g,1}(x) + \ldots + g_m\boldsymbol{v}^{g,m}(x) \tag{5.4}
$$

for all $\boldsymbol{f} \in (L^2[0,1])^m$, numbers $g_1, \ldots, g_m \in \mathbb{R}$ and $x \in [0,1]$. For the particular case $\Delta \neq 0$ investigated in Section 3, the problem (1.1)–(1.2) has the unique solution of the form

$$
\boldsymbol{u}(x) = \int_0^1 \boldsymbol{G}(x,y)\boldsymbol{f}(y)dy + g_1\boldsymbol{v}^1(x) + \ldots + g_m\boldsymbol{v}^m(x), \tag{5.5}
$$

where $\boldsymbol{G}(x,y)$ is the Green's matrix and $\boldsymbol{v}^k$, $k = \overline{1,m}$, are the biorthogonal fundamental system of the problem (1.1)–(1.2). According to the similarity, we call the kernel $\boldsymbol{G}^g(x,y)$ – the *generalized Green's matrix* and the functions $\boldsymbol{v}^{g,k}$, $k = \overline{1,m}$ – the *generalized biorthogonal fundamental system* for the nonlocal problem (1.1)–(1.2).

For $\Delta \neq 0$, we have $\boldsymbol{L}^{\dagger} = \boldsymbol{L}^{-1}$. Here the minimum norm least squares solution $\boldsymbol{u}^o$ is coincident with the unique solution $\boldsymbol{u}$, the generalized Green's matrix $\boldsymbol{G}^g(x,y)$ is coincident with the ordinary Green's function $\boldsymbol{G}(x,y)$, the generalized biorthogonal fundamental system $\boldsymbol{v}^{g,k}$, $k = \overline{1,m}$, is coincident with the biorthogonal fundamental system $\boldsymbol{v}^k$, $k = \overline{1,m}$.

5.3 Properties of minimizers

We can derive properties and representations of minimizers those extend known results for the system with the unique solution. Since proofs are analogous as in previous chapter, we provide results without proofs. Let us begin with the following characterization of the generalized biorthogonal fundamental system that is the analogue of (3.2).

Lemma 5.17. *Every function* $\boldsymbol{v}^{g,l}$, $l = \overline{1,m}$, *is the minimum norm least squares solution to the corresponding system*

$$\mathcal{L}\boldsymbol{v}^{g,l} = \mathbf{0},$$
$$\langle \boldsymbol{L}_k, \boldsymbol{v}^{g,l}\rangle = \delta_k^l, \quad k, l = \overline{1,m}.$$

Let us now consider two relative systems (3.9), where the first system has the unique solution, i.e., the condition $\widetilde{\Delta} \neq 0$ is valid. Here and further $\boldsymbol{G}^g(x,y)$ is the generalized Green's matrix and $\boldsymbol{v}^{g,k}$, $k = \overline{1,m}$, are the generalized biorthogonal fundamental system for the second problem (3.9), which may have the unique solution ($\Delta \neq 0$) or not ($\Delta = 0$).

Theorem 5.18. *If the first problem* (3.9) *has the unique solution* $\boldsymbol{u}$, *then the minimum norm least squares solution for the second problem* (3.9) *is given by*

$$\boldsymbol{u}^o = \boldsymbol{u} - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{u} + \sum_{k=1}^{m}(g_k - \langle \boldsymbol{L}_k, \boldsymbol{u}\rangle)\boldsymbol{v}^{g,k}. \tag{5.6}$$

Below we formulate the representation of the minimizer, which is always applicable since the Cauchy problem (2.9) always has the unique solution $\boldsymbol{u}^c$.

Corollary 5.19. *The minimum norm least squares solution to the system* (1.1)–(1.2) *is of the form*

$$\boldsymbol{u}^o = \boldsymbol{u}^c - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{u}^c + (g_1 - \langle \boldsymbol{L}_1, \boldsymbol{u}^c\rangle)\boldsymbol{v}^{g,1} + \ldots + (g_m - \langle \boldsymbol{L}_m, \boldsymbol{u}^c\rangle)\boldsymbol{v}^{g,m}.$$

The generalized biorthogonal fundamental systems for problems (3.9) are also related. We present their connection below.

Corollary 5.20. *Let* $\widetilde{\Delta} \neq 0$ *for the first problem* (3.9)*. Then the biorthogonal fundamental system* $\widetilde{\boldsymbol{v}}^l$, $l = \overline{1,m}$, *of the first problem and the generalized biorthogonal fundamental system* $\boldsymbol{v}^{g,k}$, $k = \overline{1,m}$, *of the second problem* (3.9) *are related by*

$$\sum_{k=1}^{m}\langle \boldsymbol{L}_k, \widetilde{\boldsymbol{v}}^l\rangle \boldsymbol{v}^{g,k} = \boldsymbol{P}_{N(\boldsymbol{L})^\perp}\widetilde{\boldsymbol{v}}^l, \qquad l = \overline{1,m}.$$

5.4 Properties of a generalized Green's matrix

We can obtain the representation of a generalized Green's matrix, which is analogous to the definition of an ordinary Green's matrix (3.5).

Lemma 5.21. *The generalized Green's matrix for the system* (1.1)–(1.2) *is of the form*

$$\boldsymbol{G}^g(x,y) := \boldsymbol{G}^c(x,y) - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^c(x,y) - \sum_{k=1}^{m} \boldsymbol{v}^{g,k}(x)\langle \boldsymbol{L}_k, \boldsymbol{G}^c(\cdot,y)\rangle. \quad (5.7)$$

In this formula, we used the kernel $\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^c(x,y)$ of an operator $\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^c : (L^2[0,1])^m \to (H^1[0,1])^m$, which vanishes if $\Delta \neq 0$. For $\Delta = 0$, we rewrite inner products in

$$\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^c\boldsymbol{f}(x) = \sum_{l=1}^{d} \boldsymbol{z}^l(x)(\boldsymbol{z}^l, \boldsymbol{G}^c\boldsymbol{f})_{(H^1[0,1])^m} = \int_0^1 \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^c(x,y)\boldsymbol{f}(y)\,dy$$

and get the kernel $\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^c(x,y)$, which is equal to

$$\sum_{l=1}^{d} \boldsymbol{z}^l(x)\Big((\boldsymbol{z}^l, \boldsymbol{G}^c(\cdot,y))_{(H^1[0,y])^m} + (\boldsymbol{z}^l, \boldsymbol{G}^c(\cdot,y))_{(H^1[y,1])^m} + \big((\boldsymbol{z}^l)'(y)\big)^\top\Big).$$

Here $\boldsymbol{z}^l$, $l = \overline{1,d}$, is the orthonormal basis of the nullspace $N(\boldsymbol{L})$.

As in Subsection 3.3, the following properties of the generalized Green's matrix (5.7) are derived.

Lemma 5.22. *For $y \neq y_0, y_1, y_2, \ldots$ with any $x \in [0,1]$, we have:*

1) $\boldsymbol{G}^g(y+0,y) - \boldsymbol{G}^g(y-0,y) = \boldsymbol{I}$;

2) $\boldsymbol{G}^g(x,y)$ *is* C *in* (x,y) *except the diagonal* $x = y$;

3) $\boldsymbol{G}^g(x,y)$ *is* H^1 *in* x *except the diagonal* $x = y$.

Moreover, the generalized Green's matrix can be described using a Green's matrix of a relative problem as given below.

Theorem 5.23. *If the first problem* (3.9) *has the ordinary Green's matrix* $\boldsymbol{G}(x,y)$, *then the generalized Green's matrix for the second problem* (3.9) *is given by*

$$\boldsymbol{G}^g(x,y) = \boldsymbol{G}(x,y) - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}(x,y) - \sum_{k=1}^{m} \boldsymbol{v}^{g,k}(x)\langle \boldsymbol{L}_k, \boldsymbol{G}(\cdot,y)\rangle, \quad (5.8)$$

for all $x \in [0,1]$ *and a.e.* $y \in [0,1]$. *Here* $\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}(x,y)$ *denotes the kernel of the operator* $\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G} : (L^2[0,1])^m \to (H^1[0,1])^m$.

5.5 Nonlocal boundary value problem

Below we apply obtained results to the problem with nonlocal boundary conditions (3.13)–(3.14). We suppose that the classical problem (3.13)–(3.14) ($\gamma_k = 0,\ k = \overline{1,m}$) has the unique solution $\boldsymbol{u}^{\text{cl}}$. Then the minimizer of the full problem (3.13)–(3.14) is given by

$$\boldsymbol{u}^o = \boldsymbol{u}^{\text{cl}} - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{u}^{\text{cl}} + \sum_{k=1}^{m} \gamma_k \langle \boldsymbol{\varkappa}_k, \boldsymbol{u}^{\text{cl}} \rangle \boldsymbol{v}^{g,k}.$$

Here $\boldsymbol{v}^{g,k},\ k = \overline{1,m}$, is the generalized biorthogonal fundamental system of the nonlocal boundary value problem (3.13)–(3.14). Applying (5.8) to the Green's matrix $\boldsymbol{G}(x,y)$ of the nonlocal boundary value problem and the Green's matrix $\boldsymbol{G}^{\text{cl}}(x,y)$ of the classical problem, we obtain very analogous relation.

Lemma 5.24. *The relation*

$$\boldsymbol{G}^g(x,y) = \boldsymbol{G}^{\text{cl}}(x,y) - \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^{\text{cl}}(x,y) + \sum_{k=1}^{m} \gamma_k \boldsymbol{v}^{g,k}(x) \langle \boldsymbol{\varkappa}_k, \boldsymbol{G}^{\text{cl}}(\cdot,y) \rangle$$

between the generalized Green's matrix and the ordinary Green's matrix for the classical problem is valid a.e.

Let us note that $\boldsymbol{P}_{N(\boldsymbol{L})}$ is the zero operator if $\Delta \neq 0$. Then the biorthogonal fundamental system $\boldsymbol{v}^{g,k}$, $k = 1, m$, is equal to the biorthogonal fundamental system $\boldsymbol{v}^k$, $k = \overline{1,m}$, and all results from this section are coincident with analogous properties for the unique solution and the Green's matrix, given in Section 3.

Example 5.25. *Let us recall Example 5.6 and continue the investigation of the differential system*

$$\begin{gathered} (u^1)' = u^2 + f^1, \quad (u^2)' = f^2, \\ u^1(0) = g_1, \quad u^1(1) = \gamma u^1(\xi) + g_2. \end{gathered} \tag{5.9}$$

It has the unique solution if the inequality $\Delta \neq 0$ is valid, i.e., $\gamma\xi \neq 1$. The Green's matrix for this problem is presented in Example 5.16. Let us find the generalized Green's matrix for the case $\Delta = 0$, that is $\gamma\xi = 1$.

First, the problem with classical conditions ($\gamma = 0$ above) has $\Delta \neq 0$ and, according to subsection 4.3, has the Green's matrix

$$\boldsymbol{G}^{\text{cl}}(x,y) = \begin{pmatrix} (\partial/\partial y)G^{\text{cl}}(x,y) & -G^{\text{cl}}(x,y) \\ (\partial^2/\partial x \partial y)G^{\text{cl}}(x,y) & -(\partial/\partial x)G^{\text{cl}}(x,y) \end{pmatrix}, \qquad x \neq y,$$

which is represented by the Green's function

$$G^{\mathrm{cl}}(x,y)=\begin{cases} y(1-x), & y\leqslant x,\\ x(1-y), & y\geqslant x,\end{cases}$$

for the scalar problem $-u''=f$*,* $u(0)=0$*,* $u(1)=0$ *with* $f=-f^2-(f^1)'$*. So, we can obtain the generalized Green's matrix from the following formula*

$$\boldsymbol{G}^g(x,y)=\boldsymbol{G}^{\mathrm{cl}}(x,y)-\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^{\mathrm{cl}}(x,y)-\gamma\boldsymbol{v}^{g,2}(x)\mathrm{row}_1\,\boldsymbol{G}^{\mathrm{cl}}(\xi,y). \tag{5.10}$$

Since $d=1$ *(see Example 5.6) and* $\boldsymbol{z}^2=(x,1)^\top\in N(\boldsymbol{L})$*, we calculate the projection*

$$\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{G}^{\mathrm{cl}}(x,y)=\Big(\big(\boldsymbol{z}^2,\boldsymbol{G}^{\mathrm{cl}}(\cdot,y)\big)_{(H^1[0,y])^2}+\big(\boldsymbol{z}^2,\boldsymbol{G}^{\mathrm{cl}}(\cdot,y)\big)_{(H^1[y,1])^2}$$
$$+\big((\boldsymbol{z}^2)'(y)\big)^\top\Big)\cdot\frac{\boldsymbol{z}^2}{\|\boldsymbol{z}^2\|^2_{(H^1[0,1])^2}}=-\frac{1}{14}\begin{pmatrix} x(5+3y^2) & x(y-y^3)\\ 5+3y^2 & y-y^3\end{pmatrix}.$$

Now we are going to find another unknown function $\boldsymbol{v}^{g,2}=(v^{g,2;1},v^{g,2;2})^\top$ *in the expression* (5.10)*. It is the minimizer to the problem* $\boldsymbol{L}\boldsymbol{u}=\boldsymbol{e}^4$*. According to properties of minimizers [100, Roman 2011], it is also the minimum norm least squares solution to the consistent problem* $\boldsymbol{L}\boldsymbol{u}=\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^4$*.*

Thus, we calculate the projection

$$\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^4=\boldsymbol{e}^4-\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|^2_{(L^2[0,1])^2\times\mathbb{R}^2}}(\boldsymbol{w},\boldsymbol{e}^4)_{(L^2[0,1])^2\times\mathbb{R}^2}.$$

Here we took the function $\boldsymbol{w}=\big(\gamma(\partial/\partial y)G^{\mathrm{cl}}(\xi,x);-\gamma G^{\mathrm{cl}}(\xi,x);\gamma-1;1\big)^\top$*, which generates the nullspace* $N(\boldsymbol{L}^*)=R(\boldsymbol{L})^\perp$*. Hence, we obtain the following representation of the problem* $\boldsymbol{L}\boldsymbol{u}=\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{e}^4$*:*

$$(u^1)'-u^2=\gamma(\partial/\partial y)G^{\mathrm{cl}}(\xi,x)/\|\boldsymbol{w}\|^2, \tag{5.11}$$
$$(u^2)'=-\gamma G^{\mathrm{cl}}(\xi,x)/\|\boldsymbol{w}\|^2, \tag{5.12}$$
$$u^1(0)=(\gamma-1)/\|\boldsymbol{w}\|^2, \tag{5.13}$$
$$u^1(1)-\gamma u_1(\xi)=1-1/\|\boldsymbol{w}\|^2, \tag{5.14}$$

where $\|\boldsymbol{w}\|=\|\boldsymbol{w}\|_{(L^2[0,1])^2\times\mathbb{R}^2}=2-\xi+(1-\gamma)^2(3\xi^2+2\xi+1)/2$*. We take the fundamental system* $\boldsymbol{z}^1=(1,0)^\top$*,* $\boldsymbol{z}^2=(x,1)^\top$ *(see Example 5.6) and get the general solution*

$$\boldsymbol{u}=c_1\boldsymbol{z}^1+c_2\boldsymbol{z}^2+\frac{\gamma}{\|\boldsymbol{w}\|^2}\int_0^1\boldsymbol{G}^{\mathrm{cl}}(x,y)\begin{pmatrix}(\partial/\partial y)G^{\mathrm{cl}}(\xi,y)\\ -G^{\mathrm{cl}}(\xi,y)\end{pmatrix}dy,$$

for $c_1, c_2 \in \mathbb{R}$. *Substituting it into conditions, we find* $c_1 = (1-\gamma)/\|\boldsymbol{w}\|^2$ *and know the general least squares solution*

$$\boldsymbol{u}^g = \frac{\gamma - 1}{\|\boldsymbol{w}\|^2}\boldsymbol{z}^1 + c\boldsymbol{z}^2 + \frac{\gamma}{\|\boldsymbol{w}\|^2}\int_0^1 \boldsymbol{G}^{\mathrm{cl}}(x,y)\begin{pmatrix} (\partial/\partial y)G^{\mathrm{cl}}(\xi,y) \\ -G^{\mathrm{cl}}(\xi,y) \end{pmatrix} dy,$$

to the problem (5.11)–(5.14)*, depending on one arbitrary constant* $c \in \mathbb{R}$. *Taking* $c^o = (5\xi^6 + 70\xi^4 + 187\xi^2 - 120\xi)/280(\xi^5 - \xi^4 - \xi^3 + 2\xi^2 - \xi + 1)$*, we obtain the desired minimizer* $\boldsymbol{v}^{g,2}$. *We found this constant value applying the property* $\boldsymbol{v}^{g,2} = \boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{u}^g$. *Below we present the explicit form of the vector valued minimizer*

$$v^{g,2;1} = \frac{\gamma-1}{\|\boldsymbol{w}\|^2} + c^o x - \frac{\gamma}{\|\boldsymbol{w}\|^2}\int_0^1 (G^{\mathrm{cl}})'_y(x,y)(G^{\mathrm{cl}})'_y(\xi,y) + G^{\mathrm{cl}}(x,y)G^{\mathrm{cl}}(\xi,y)\,dy,$$

$$v^{g,2;2} = c^o - \frac{\gamma}{\|\boldsymbol{w}\|^2}\left[\int_0^x + \int_x^1\right](G^{\mathrm{cl}})''_{xy}(x,y)(G^{\mathrm{cl}})'_y(\xi,y) + (G^{\mathrm{cl}})'_x(x,y)G^{\mathrm{cl}}(\xi,y)\,dy.$$

Finally, we substitute obtained expressions into (5.10) *and have that the generalized Green's function* $\boldsymbol{G}^g(x,y)$ *is equal to*

$$\begin{pmatrix} (G^{\mathrm{cl}})'_y + \frac{x(5+3y^2)}{14} - \gamma v^{g,2;1}(x)(G^{\mathrm{cl}})'_y(\xi,y) & -G^{\mathrm{cl}} + \frac{x(y-y^3)}{14} + \gamma v^{g,2;1}(x)G^{\mathrm{cl}}(\xi,y) \\ (G^{\mathrm{cl}})''_{xy} + \frac{5+3y^2}{14} - \gamma v^{g,2;2}(x)(G^{\mathrm{cl}})'_y(\xi,y) & -(G^{\mathrm{cl}})'_x + \frac{y-y^3}{14} + \gamma v^{g,2;2}(x)G^{\mathrm{cl}}(\xi,y) \end{pmatrix}.$$

Let us note that another minimizer $\boldsymbol{v}^{g,1}$ *can be found from Corollary 5.20, which gives the equality* $\langle \boldsymbol{L}_1, \widetilde{\boldsymbol{v}}^1\rangle\boldsymbol{v}^{g,1} + \langle \boldsymbol{L}_2, \widetilde{\boldsymbol{v}}^1\rangle\boldsymbol{v}^{g,2} = \boldsymbol{P}_{N(\boldsymbol{L})^\perp}\widetilde{\boldsymbol{v}}^1$. *Here we take the biorthogonal fundamental system* $\widetilde{\boldsymbol{v}}^1 = (1-x,-1)^\top$, $\boldsymbol{v}^2 = (x,1)^\top$ *for the problem* (5.9) *with classical conditions* ($\gamma = 0$) *and, simplifying, obtain*

$$\boldsymbol{v}^{g,1} = (\gamma - 1)\boldsymbol{v}^{g,2} + (1,0)^\top - \frac{3}{14}(x,1)^\top.$$

Since we know the generalized biorthogonal fundamental system $\boldsymbol{v}^{g,1}, \boldsymbol{v}^{g,2}$ *and the generalized Green's matrix* $\boldsymbol{G}^g(x,y)$*, we can also calculate the minimizer using the representation*

$$\boldsymbol{u}^o = \int_0^1 \boldsymbol{G}^g(x,y)\boldsymbol{f}(y)\,dy + g_1\boldsymbol{v}^{g,1}(x) + g_2\boldsymbol{v}^{g,2}(x).$$

6 Conclusions

Basic conclusions of this chapter are formulated below:

1) A differential problem (1.1)–(1.2) always has the Moore–Penrose inverse $\boldsymbol{L}^\dagger$, a generalized Green's matrix and the unique minimum norm least squares solution.

2) For $\varDelta \neq 0$, we have that $\boldsymbol{L}^{\dagger} = \boldsymbol{L}^{-1}$, the minimum norm least squares solution $\boldsymbol{u}^{o}$ is coincident with the unique solution $\boldsymbol{u}$, the generalized Green's matrix $\boldsymbol{G}^{g}$ is coincident with the ordinary Green's matrix $\boldsymbol{G}$, the biorthogonal fundamental system $\boldsymbol{v}^{k}$, $k = \overline{1, m}$, is coincident with the generalized biorthogonal fundamental system $\boldsymbol{v}^{g,k}$, $k = \overline{1, m}$.

3) The minimum norm least squares solution has literally similar representations as the unique solution: it can be described by the unique solution of the Cauchy problem or the unique solution to other relative problem (same differential equations (1.1) but different nonlocal conditions (1.2)).

4) A generalized Green's matrix also has representations similar to expressions of a Green's matrix: it can be written using the Green's matrix of the Cauchy problem or a Green's matrix to other relative problem (same differential equations (1.1) but different nonlocal conditions (1.2)).

5) A Green's function of a scalar problem describes a Green's matrix of a differential system, which is obtained representing a scalar problem in a system form, and vice versa.

Chapter 6

First order discrete systems with nonlocal conditions

1 Introduction

In this chapter, a linear system of first order discrete equations with nonlocal conditions is considered. We are going to develop an analogy to Chapter 5, where first order differential systems were investigated.

The structure of this chapter is as follows. First, we introduce some notation. Then we represent a discrete system into the equivalent "matrix" form and consider its properties. Afterwards a problem with the unique solution is investigated. Here we obtain several representations and properties of the unique solution. A discrete Green's matrix is also studied. Later, the problem without the unique solution is considered in the least squares sense. Here we discuss on the unique discrete minimizer of the residual, derive its properties and representations, study a generalized discrete Green's matrix. An example is also given.

2 Notation

In Chapter 3, we introduced a discrete function $u \in F(X_n)$, which can be uniquely described by the complex column matrix $\mathbf{u} = (u_0, u_1, \ldots, u_n)^\top \in \mathbb{C}^{(n+1)\times 1}$. Let us now denote the collection of functions $u^k \in F(X_n)$, $k = \overline{1,m}$, by $\boldsymbol{U} = (u^1, u^2, \ldots, u^m)^\top \in F^m(X_n) := F(X_n) \times \ldots \times F(X_n)$. We also take its matrix representation $\mathbf{U} = (\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^m)^\top \in \mathbb{C}^{m\times(n+1)}$.

Similarly, a collection of linear functionals $L_l \in F^*(X_n)$, $l = \overline{1,m}$, can be given by $\boldsymbol{L} = (L_1, L_2, \ldots, L_m) \in \big(F^m(X_n)\big)^*$ but its matrix matrix representation is of the form $\mathbf{L} = (\mathbf{L}_1^\top, \mathbf{L}_2^\top, \ldots, \mathbf{L}_m^\top) \in \mathbb{C}^{(n+1)\times m}$. For a functional $\boldsymbol{L} \in \big(F^m(X_n)\big)^*$ value at $\boldsymbol{U} \in F^m(X_n)$, we use the notation $\langle \boldsymbol{L}, \boldsymbol{U}\rangle := \sum_{l=1}^m \langle L_l, u^l\rangle = \sum_{l=1}^m \sum_{i=0}^n L_l^i u_i^l = \sum_{l=1}^m \mathbf{L}_l \mathbf{u}^l$, where $\mathbf{L}_l \mathbf{u}^l$ are usual multiplications of matrices $\mathbf{L}_l \in \mathbb{C}^{1\times(n+1)}$ and $\mathbf{u}^l \in \mathbb{C}^{(n+1)\times 1}$. Let us denote its matrix representation by $\langle \mathbf{L}, \mathbf{U}\rangle$, where brackets emphasize the nature of nonlocal conditions.

Analogously, a space $F^{m_1\times m_2}(X_{n_1} \times X_{n_2})$ is defined. It is composed of $m_1 \times m_2$ order matrix valued functions $\boldsymbol{M} = (M^{kl})$ for $k = \overline{1, m_1}$ and $l = \overline{1, m_2}$, those elements $M^{kl} \in F(X_{n_1} \times X_{n_2})$ are uniquely represented by complex matrices from $\mathbb{C}^{(n_1+1)\times(n_2+1)}$. In this work, the one to one correspondence among discrete functions $M^{kl} : X_{n_1} \times X_{n_2} \to \mathbb{C}$ and matrices $\mathbf{M}^{kl} = (M_{ij}^{kl}) \in \mathbb{C}^{(n_1+1)\times(n_2+1)}$ is represented by $M_{ij}^{kl} = M^{kl}(i, j)$, $i \in X_{n_1}$, $j \in X_{n_2}$. In other words, an element $\boldsymbol{M} \in F^{m_1\times m_2}(X_{n_1} \times X_{n_2})$ can be understood as a "*matrix of matrices*" since $\boldsymbol{M} \cong (\mathbf{M}^{kl})$.

We also use the notation of the double summation without the sum symbol as

$$\langle \boldsymbol{L}, \boldsymbol{M}_{\cdot j}^{\cdot l}\rangle = \sum_{j=0}^{n_1} \sum_{k=1}^{m_1} L_k^i M_{ij}^{kl}, \qquad l = \overline{1, m_2},\ j \in X_{n_2},$$

where $\boldsymbol{L} \in \big(F^{m_1\times m_2}(X_{n_1} \times X_{n_2})\big)^*$, $\boldsymbol{M} \in F^{m_1\times m_2}(X_{n_1} \times X_{n_2})$. Let us denote matrix representations of the last notation by $\langle \mathbf{L}, \mathbf{M}\rangle \in \mathbb{C}^{m_2\times(n_2+1)}$. In this chapter, we will use the matrix valued zero function $\boldsymbol{O}$ of any desired dimensions as well.

3 Formulation of the problem

Let us investigate a first order discrete system with nonlocal conditions

$$(\mathcal{L}U)_i^k := u_{i+1}^k - \sum_{l=1}^m a_i^{kl} u_i^l = f_i^k, \quad i \in X_{n-1}, \tag{3.1}$$

$$\sum_{l=1}^m \langle L_{kl}, u^l\rangle = g_k, \quad k = \overline{1, m}, \tag{3.2}$$

where functions $\boldsymbol{U} = (u^1, \ldots, u^m)^\top \in F^m(X_n)$, $\boldsymbol{F} = (f^1, \ldots, f^m)^\top \in F^m(X_{n-1})$ and $a^{kl} \in F(X_{n-1})$, $g_k \in \mathbb{C}$. Here $\boldsymbol{\mathcal{L}} : F^m(X_n) \to F^m(X_{n-1})$ is the discrete operator and $L_{kl} \in F^*(X_n)$ are discrete linear functionals. Denoting collections of functionals by $\boldsymbol{L}_k = (L_{k1}, L_{k2}, \ldots, L_{km}) \in \big(F^m(X_n)\big)^*$ for every $k = \overline{1, m}$, this problem is given by

$$\boldsymbol{\mathcal{L}U} = \boldsymbol{F}, \qquad \langle \boldsymbol{L}_k, \boldsymbol{U}\rangle = g_k, \quad k = \overline{1, m}.$$

We also use the following description

$$\boldsymbol{A}\boldsymbol{U} = \boldsymbol{B} \tag{3.3}$$

of the problem (3.1)–(3.2) with an operator $\boldsymbol{A} := (\boldsymbol{\mathcal{L}}, \boldsymbol{L}_1, \ldots, \boldsymbol{L}_m) : F^m(X_n) \to F^m(X_{n-1}) \times \mathbb{C}^m$ and $\boldsymbol{B} = (\boldsymbol{F}, g_1, \ldots, g_m)^\top \in F^m(X_{n-1}) \times \mathbb{C}^m$.

3.1 The discrete Cauchy problem

Let us now consider a discrete Cauchy problem

$$u_{i+1}^k = \sum_{l=1}^{m} a_i^{kl} u_i^l + f_i^k, \quad i \in X_{n-1}, \tag{3.4}$$

$$u_0^k = 0, \quad k = \overline{1, m}. \tag{3.5}$$

It always has the unique solution $\boldsymbol{U}^c \in F^m(X_n)$. To prove the existence of the unique solution, we take the homogenous Cauchy problem

$$z_{i+1}^k = \sum_{l=1}^{m} a_i^{kl} z_i^l, \quad i \in X_{n-1}, \tag{3.6}$$

$$z_0^k = 0, \quad k = \overline{1, m}. \tag{3.7}$$

Substituting conditions (3.7) into the equation (3.6) with $i = 1$, we find all $z_1^k = 0$. Applying the mathematical induction, we find all trivial values $z_i^k = 0$ uniquely for every $k = \overline{1, m}$ and $i \in X_{n-1}$. Thus, the homogenous Cauchy problem has only the trivial solution, what means that the Cauchy problem (3.4)–(3.5) with every right hand side always has the unique solution $\boldsymbol{U}^c$. We are going to obtain its representation.

To find the unique solution, we will use the method of variation of parameters. Here we need to take the fundamental system of the homogenous equation (3.6). First, the problem (3.6)–(3.7) has $m(n+1)$ equations and $m(n+1)$ unknowns, where all $m(n+1)$ equations are linearly independent since this problem always has the unique solution. Second, the subsystem (3.6) has mn linearly independent equations with $m(n+1)$ unknowns. Thus, we can solve exactly mn unknowns via other m unknowns, those remain arbitrarily, what gives the nullity $\dim N(\boldsymbol{\mathcal{L}}) = m$. It also means that a fundamental system of the equation (3.6) is composed of m linearly independent solutions $\boldsymbol{Z}^l \in F^m(X_n)$, $l = \overline{1, m}$, those matrix representations are $\mathbf{Z}^l = (z_i^{l,k}) \in \mathbb{C}^{m \times (n+1)}$, where $k = \overline{1, m}$ and $i \in X_n$. Let us denote the discrete fundamental matrix at a point $i \in X_n$ by

$$\mathbf{Z}_i = \begin{pmatrix} z_i^{1,1} & z_i^{2,1} & \dots & z_i^{m,1} \\ z_i^{1,2} & z_i^{2,2} & \dots & z_i^{m,2} \\ \dots & \dots & \dots & \dots \\ z_i^{1,m} & z_i^{2,m} & \dots & z_i^{m,m} \end{pmatrix}.$$

So, the general solution to the homogenous equation (3.6) is given by $z_i^k = c^1 z_i^{1,k} + c^2 z_i^{2,k} + \ldots + c^m z_i^{m,k}$, where $k = \overline{1,m}$ and $i \in X_n$. Then we look for the unique solution to the problem (3.4)–(3.5) in the form (method of variation of parameters)

$$u_i^k = c_i^1 z_i^{1,k} + c_i^2 z_i^{2,k} + \ldots + c_i^m z_i^{m,k} = \sum_{j=1}^{m} c_i^j z_i^{j,k} \tag{3.8}$$

with unknown functions $c^l \in F(X_n)$, $l = \overline{1,m}$. Substituting this representation into the problem (3.4), we rewrite

$$\begin{aligned} f_i^k = (\mathcal{L}U)_i^k &= u_{i+1}^k - \sum_{l=1}^{m} a_i^{kl} u_i^l = \sum_{j=1}^{m} c_{i+1}^j z_{i+1}^{j,k} - \sum_{l=1}^{m} a_i^{kl} \sum_{j=1}^{m} c_i^j z_i^{j,l} \\ &= \sum_{j=1}^{m} c_i^j \left(z_{i+1}^{j,k} - \sum_{l=1}^{m} a_i^{kl} z_i^{j,l} \right) + \sum_{j=1}^{m} (c_{i+1}^j - c_i^j) z_{i+1}^{j,k} = \sum_{j=1}^{m} (c_{i+1}^j - c_i^j) z_{i+1}^{j,k}. \end{aligned}$$

Here we used trivial equalities $z_{i+1}^{j,k} - \sum_{l=1}^{m} a_i^{kl} z_i^{j,l} = 0$ since (3.6) are valid for a fundamental system $\boldsymbol{Z}^l \in F^m(X_n)$, $l = \overline{1,m}$.

Let us denote the difference $\tilde{c}_{i+1}^j = c_{i+1}^j - c_i^j$ and obtain the system

$$\sum_{j=1}^{m} \tilde{c}_{i+1}^j z_{i+1}^{j,k} = f_i^k, \qquad k = \overline{1,m},\ i \in X_{n-1},$$

with unknowns $\tilde{c}_{i+1}^j$. Since the discrete fundamental matrix $\boldsymbol{Z}_{i+1}$ is nonsingular at every point $i+1$, we find $\tilde{c}_{i+1}^k = \sum_{l=1}^{m} (Z_{i+1})^{-1,kl} f_i^l$. Recalling notation $\tilde{c}_{i+1}^k = c_{i+1}^k - c_i^k$, we solve

$$c_i^k = \sum_{j=0}^{i-1} \left(c_{i+1}^k - c_i^k \right) + c_0^k = \sum_{j=0}^{i-1} (Z_{j+1})^{-1,kl} f_j^l + c_0^k.$$

Now we substitute these values in the formula (3.8) and obtain the representation

$$u_i^k = \sum_{j=0}^{i-1} \sum_{l=1}^{m} K_{ij}^{kl} f_j^l + \sum_{l=1}^{m} c_0^l z_i^{l,k}$$

of the general solution to the problem (3.4) with m arbitrary constants c_0^l. Here we introduce the notation the *discrete Cauchy matrix* at a point (i, j) by $\mathbf{K}_{ij} = \mathbf{Z}_i(\mathbf{Z}_{j+1})^{-1} \in \mathbb{C}^{m\times m}$ for every fixed $i \in X_n$, $j \in X_{n-1}$.

Using initial conditions (3.5), we find trivial constant values $c_0^l = 0$ since functions $z^{l,k} \in F(X_n)$ for fixed k, l are linearly independent as a fundamental system at the point $i = 0$. Thus, we find the unique solution $u_i^k = \sum_{j=0}^{n-1}\sum_{l=1}^{m} K_{ij}^{kl} f_j^l$ to the problem (3.4)–(3.5). Now we introduce the *discrete Green's matrix* $\boldsymbol{G}^c \in F^{m\times m}(X_n \times X_{n-1})$ for the Cauchy problem:

$$G_{ij}^{c,kl} = \begin{cases} K_{ij}^{kl}, & j < i, \\ 0, & j \geqslant i, \end{cases} \qquad k, l = \overline{1, m}, \quad i \in X_n,\ j \in X_{n-1}, \tag{3.9}$$

and have the unique solution of the form

$$u_i^{c,k} = \sum_{j=0}^{n-1}\sum_{l=1}^{m} G_{ij}^{c,kl} f_j^l, \qquad k = \overline{1, m},\ i \in X_n. \tag{3.10}$$

3.2 Existence of the unique solution

Let us now obtain a condition of the existence of the unique solution to the problem (3.1)–(3.2). It is equivalent to investigate the existence of the unique trivial solution $\boldsymbol{Z} \equiv \boldsymbol{O}$ to the homogenous problem $\boldsymbol{A}\boldsymbol{Z} = \boldsymbol{O}$, that is

$$\boldsymbol{\mathcal{L}}\boldsymbol{Z} = \boldsymbol{O}, \tag{3.11}$$

$$\langle \boldsymbol{L}_k, \boldsymbol{Z}\rangle = 0, \quad k = \overline{1, m}. \tag{3.12}$$

We put the general solution of the probem (3.11), which is $\mathbf{Z} = \sum_{l=1}^{n} c_l \boldsymbol{Z}^l$ for $c_l \in \mathbb{C}$, $l = \overline{1, m}$, into conditions (3.12) and obtain the system of m homogenous equations with unknowns c_l, $l = \overline{1, m}$, as follows

$$\sum_{l=1}^{m} \langle \boldsymbol{L}_k, \boldsymbol{Z}^l\rangle c_l = 0, \quad k = \overline{1, m}.$$

So, the problem has only the trivial solution $\mathbf{Z} = \mathbf{0}$ or equivalently $c_l = 0$, $l = \overline{1, m}$, if and only if the determinant of the previous system is nonzero

$$\mathit{\Delta} := \begin{vmatrix} \langle \boldsymbol{L}_1, \boldsymbol{Z}^1\rangle & \langle \boldsymbol{L}_1, \boldsymbol{Z}^2\rangle & \dots & \langle \boldsymbol{L}_1, \boldsymbol{Z}^m\rangle \\ \langle \boldsymbol{L}_2, \boldsymbol{Z}^1\rangle & \langle \boldsymbol{L}_2, \boldsymbol{Z}^2\rangle & \dots & \langle \boldsymbol{L}_2, \boldsymbol{Z}^m\rangle \\ \dots & \dots & \dots & \dots \\ \langle \boldsymbol{L}_m, \boldsymbol{Z}^1\rangle & \langle \boldsymbol{L}_m, \boldsymbol{Z}^2\rangle & \dots & \langle \boldsymbol{L}_m, \boldsymbol{Z}^m\rangle \end{vmatrix} \neq 0. \tag{3.13}$$

If $\mathit{\Delta} = 0$, then the nullspace $N(\boldsymbol{A})$ is nontrivial. Denoting the nullity $d := \dim N(\boldsymbol{A})$, we separate the following cases:

- $d = 0 \Leftrightarrow \varDelta \neq 0$. Then $N(\boldsymbol{A})$ is trivial.
- $d = m \Leftrightarrow \varDelta = 0$ and all $\langle \boldsymbol{L}_k, \boldsymbol{Z}^l \rangle = 0$ for $k, l = \overline{1, m}$. Then all constants $c_1, \dots, c_m$ remain arbitrary and $N(\boldsymbol{A}) = \operatorname{span}\{\boldsymbol{Z}^1, \dots, \boldsymbol{Z}^m\}$. So, the solution to $\boldsymbol{AZ} = \boldsymbol{O}$ is now equivalent to the solution of the differential equation $\boldsymbol{\mathcal{L}Z} = \boldsymbol{O}$ only.
- $0 < d < m \Leftrightarrow \varDelta = 0$ and $\operatorname{rank}(\langle \boldsymbol{L}_k, \boldsymbol{Z}^l \rangle) = m - d$ (here $k, l = \overline{1, m}$). Here $m - d$ constants are solved and represented by other d constants, those remain arbitrary. In other words, there exist d rows in the determinant representation of $\varDelta$ above, those are linear combinations of the rest $m - d$ linearly independent rows. Let us denote these "*dependent*" rows by $(\langle \boldsymbol{L}_{k_l}, \boldsymbol{Z}^1 \rangle, \dots, \langle \boldsymbol{L}_{k_l}, \boldsymbol{Z}^m \rangle)$ for $k_l,\ l = \overline{1, d}$. The independent rows are also given by $(\langle \boldsymbol{L}_{k_j}, \boldsymbol{Z}^1 \rangle, \dots, \langle \boldsymbol{L}_{k_j}, \boldsymbol{Z}^m \rangle)$ for $k_j,\ j = \overline{d+1, m}$. Thus, the solution to the problem $\boldsymbol{AZ} = \boldsymbol{O}$ is now equivalent to the solution of the simplified problem: the equation $\boldsymbol{\mathcal{L}Z} = \boldsymbol{O}$ with conditions $\langle \boldsymbol{L}_{k_j}, \boldsymbol{Z} \rangle = \boldsymbol{O}$, $j = \overline{d+1, m}$, representing linearly independent rows only.

3.3 Range of the operator $\boldsymbol{A}$

We are going to obtain the representation of the range $R(\boldsymbol{A})$. We omit proofs again because they are analogous to corresponding proofs in previous chapters. First, we need to discuss on the composition

$$\boldsymbol{B} = (f^1, \dots, f^m, g_1, \dots, g_m)^\top = \sum_{k=1}^{m} (f^k \boldsymbol{E}^k + g_k \boldsymbol{E}^{m+k})$$

for all $\boldsymbol{B} \in F^m(X_{n-1}) \times \mathbb{C}^m$, where we took discrete matrix valued functions $\boldsymbol{E}^k \in F^m(X_{n-1}) \times \mathbb{C}^m$. Now we can provide the representation of the range.

Lemma 6.1.

1) If $d = m$, then $R(\boldsymbol{A})$ is generated by the matrix valued function

$$\boldsymbol{B} = \sum_{k=1}^{m} \left(f^k \boldsymbol{E}^k + \sum_{j=0}^{n-1} \sum_{l=1}^{m} \langle \boldsymbol{L}_k, \boldsymbol{G}^{c,\cdot l}_{\cdot j} \rangle f^l_j \boldsymbol{E}^{m+k} \right),$$

where $\boldsymbol{F} = (f^1, \dots, f^m)^\top \in F^m(X_{n-1})$.

2) If $0 < d < m$, then for $\boldsymbol{F} = (f^1, \dots, f^m)^\top \in F^m(X_{n-1})$ and $g_{k_i} \in \mathbb{R}$, $i = \overline{d+1, m}$, $R(\boldsymbol{A})$ is generated by the matrix valued function

$$\boldsymbol{B} = \sum_{k=1}^{m} f^k \boldsymbol{E}^k + \sum_{\ell=1}^{d} \Bigg(\sum_{i=d+1}^{m} g_{k_i} \langle \boldsymbol{L}_{k_\ell}, \boldsymbol{V}^{k_i} \rangle + \sum_{j=0}^{n-1} \sum_{l=1}^{m} \langle \boldsymbol{L}_{k_\ell}, \boldsymbol{G}^{a,\cdot l}_{\cdot j} \rangle f^l_j \Bigg) \boldsymbol{E}^{m+k_\ell}$$
$$+ \sum_{i=d+1}^{m} g_{k_i} \boldsymbol{E}^{m+k_i}.$$

Here $\boldsymbol{G}^a \in F^{m\times m}(X_n \times X_{n-1})$ is a *discrete Green's matrix* and $\boldsymbol{V}^l$, $l = \overline{1,m}$, is the biorthogonal fundamental system for the problem $\boldsymbol{\mathcal{L}U} = \boldsymbol{F}$ with original conditions $\langle \boldsymbol{L}_{k_j}, \boldsymbol{U} \rangle = 0$, $j = \overline{d+1,m}$, and conditions $\langle \boldsymbol{\ell}_{k_l}, \boldsymbol{U} \rangle = 0$, $l = \overline{1,d}$, replacing $\langle \boldsymbol{L}_{k_l}, \boldsymbol{U} \rangle = 0$. Here $\langle \boldsymbol{\ell}_{k_l}, \boldsymbol{U} \rangle = 0$ are selected such that for this auxiliary problem $\mathit{\Delta} \neq 0$. For details see the following section.

Since the nullspace and range theorem gives the equality $N(\boldsymbol{A}^*) = R(\boldsymbol{A})^\perp$, we can provide the following representation.

Corollary 6.2. *The following statements are valid:*

1) *if $d = m$, then $N(\boldsymbol{A}^*)$ is generated by functions $\boldsymbol{W}^k$, those at a point $i \in X_n$ are equal to*

$$\boldsymbol{W}^k_i = -\sum_{l=1}^{m} \overline{\langle \boldsymbol{L}_k, \boldsymbol{G}^{c,\cdot l}_{\cdot i} \rangle} \boldsymbol{E}^l + \boldsymbol{E}^{m+k}, \qquad k = \overline{1,m}.$$

2) *if $0 < d < m$, then $N(\boldsymbol{A}^*)$ is generated by functions $\boldsymbol{W}^\ell$, those at a point $i \in X_n$ are equal to*

$$\boldsymbol{W}^\ell_i = -\sum_{l=1}^{m} \overline{\langle \boldsymbol{L}_{k_\ell}, \boldsymbol{G}^{a,\cdot l}_{\cdot i} \rangle} \boldsymbol{E}^l - \sum_{j=d+1}^{m} \overline{\langle \boldsymbol{L}_{k_\ell}, \boldsymbol{V}^{k_j} \rangle} \boldsymbol{E}^{m+k_j} + \boldsymbol{E}^{m+k_\ell}, \qquad \ell = \overline{1,d}.$$

This corollary gives that $d = \dim N(\boldsymbol{A})$ and $d^* := \dim N(\boldsymbol{A}^*)$ are equal. Now applying the Fredholm alternative theorem, we get the solvability conditions to the problem (3.1)–(3.2) without the unique solution ($\mathit{\Delta} = 0$).

Corollary 6.3. *(Solvability conditions) The problem* (3.1)–(3.2) *with* $\mathit{\Delta} = 0$ *is solvable if and only if the conditions are valid:*

1) $\sum_{j=0}^{n-1} \sum_{l=1}^{m} \langle \boldsymbol{L}_k, \boldsymbol{G}^{c,\cdot l}_{\cdot j} \rangle f^l_j = g_k$, $k = \overline{1,m}$, *for $d = m$;*

2) $\sum_{i=d+1}^{m} g_{k_i} \langle \boldsymbol{L}_{k_\ell}, \boldsymbol{V}^{k_i} \rangle + \sum_{j=0}^{n-1} \sum_{l=1}^{m} \langle \boldsymbol{L}_{k_\ell}, \boldsymbol{G}^{a,\cdot l}_{\cdot j} \rangle f^l_j = g_{k_\ell}$ *for $\ell = \overline{1,d}$ if $0 < d < m$.*

4 Problem with the unique solution (case $\varDelta \neq 0$)

Substituting the general solution

$$u_i^k = c_1 z_i^{1,k} + \ldots + c_m z_i^{m,k} + \sum_{j=0}^{n-1} \sum_{l=1}^{m} G_{ij}^{c,kl} f_j^l$$

of the equation (3.1) into nonlocal conditions (3.2), we get the system

$$\begin{aligned} &c_1 \langle \boldsymbol{L}_1, \boldsymbol{Z}^1 \rangle + \ldots + c_m \langle \boldsymbol{L}_1, \boldsymbol{Z}^m \rangle = g_1 - \textstyle\sum_{j=1}^{n-1} \sum_{l=1}^{m} \langle \boldsymbol{L}_1, \boldsymbol{G}_{\cdot j}^{c,\cdot l} \rangle f_j^l, \\ &\qquad \ldots \\ &c_1 \langle \boldsymbol{L}_m, \boldsymbol{Z}^1 \rangle + \ldots + c_m \langle \boldsymbol{L}_m, \boldsymbol{Z}^m \rangle = g_m - \textstyle\sum_{j=1}^{n-1} \sum_{l=1}^{m} \langle \boldsymbol{L}_m, \boldsymbol{G}_{\cdot j}^{c,\cdot l} \rangle f_j^l. \end{aligned} \tag{4.1}$$

If $\varDelta \neq 0$, we solve constants $c_1, \ldots, c_m$ uniquely and obtain the representation of the unique solution to the problem (3.1)–(3.2), simply denoted by $\boldsymbol{AU} = \boldsymbol{B}$.

On the other hand, the unique solution is also described by the inverse operator $\boldsymbol{A}^{-1} : F^m(X_{n-1}) \times \mathbb{C}^m \to F^m(X_n)$ in the form $\boldsymbol{U} = \boldsymbol{A}^{-1}\boldsymbol{B}$. Thus, first, we are going to investigate the structure of the operator $\boldsymbol{A}^{-1}$ and its properties.

4.1 Representation of the inverse operator

Let us take the particular fundamental system $\boldsymbol{V}^l = (v_i^{l,k})$, $l = \overline{1,m}$ (here $k = \overline{1,m}$ and $i \in X_n$), which satisfies the biorthogonality conditions $\langle \boldsymbol{L}_k, \boldsymbol{V}^l \rangle = \delta_k^l$ for $k, l = \overline{1,m}$. We call these functions $\boldsymbol{V}^l$, $l = \overline{1,m}$, by the *biorthogonal fundamental system*. They are unique solutions to following problems

$$\begin{aligned} \boldsymbol{\mathcal{L}} \boldsymbol{V}^l &= \boldsymbol{O}, \\ \langle \boldsymbol{L}_k, \boldsymbol{V}^l \rangle &= \delta_k^l, \quad k = \overline{1,m}, \end{aligned} \tag{4.2}$$

respectively. So, the general solution to the problem (3.1) is represented by

$$u_i^k = c_1 v_i^{1,k} + \ldots + c_m v_i^{m,k} + \sum_{j=0}^{n-1} \sum_{l=1}^{m} G_{ij}^{c,kl} f_j^l \tag{4.3}$$

The biorthogonal fundamental system $\boldsymbol{V}^l$, $l = \overline{1,m}$, directly gives us the constants

$$c_k = g_k - \sum_{j=1}^{n-1} \sum_{l=1}^{m} \langle \boldsymbol{L}_k, \boldsymbol{G}_{\cdot j}^{c,\cdot l} \rangle f_j^l, \quad k = \overline{1,m},$$

from the system (4.1). Putting these expressions into (4.3), we obtain the following representation of the solution

$$u_i^k = \sum_{j=0}^{n-1} \sum_{l=1}^{m} \big(G_{ij}^{c,kl} - \sum_{\ell=1}^{m} v_i^{\ell,k} \langle \boldsymbol{L}_\ell, \boldsymbol{G}_{\cdot j}^{c,\cdot l} \rangle \big) f_j^l + g_1 v_i^{1,k} + \ldots + g_m v_i^{m,k}.$$

Let us denote the kernel

$$G_{ij}^{kl} := G_{ij}^{c,kl} - \sum_{\ell=1}^{m} v_i^{\ell,k} \langle \boldsymbol{L}_\ell, \boldsymbol{G}_{\cdot j}^{c,\cdot l} \rangle, \tag{4.4}$$

where $i \in X_n$, $j \in X_{n-1}$, $k, l = \overline{1,m}$, and obtain the representation

$$u_i^k = \sum_{j=0}^{n-1} \sum_{l=1}^{m} G_{ij}^{kl} f_j^l + g_1 v_i^{1,k} + \ldots + g_m v_i^{m,k}, \tag{4.5}$$

simply given by

$$\boldsymbol{U} = \boldsymbol{G}\boldsymbol{F} + g_1 \boldsymbol{V}^1 + \ldots + g_m \boldsymbol{V}^m. \tag{4.6}$$

We call the kernel $\boldsymbol{G} = (G_{ij}^{kl}) \in F^{m \times m}(X_n \times X_{n-1})$ by the *discrete Green's matrix* for the problem with nonlocal conditions (3.1)–(3.2).

Let us recall the representation of the unique solution $\boldsymbol{U} = \boldsymbol{A}^{-1}\boldsymbol{B}$ using the linear inverse operator $\boldsymbol{A}^{-1} : F^m(X_{n-1}) \times \mathbb{C}^m \to F^m(X_n)$. Thus, we get the following composition

$$\boldsymbol{A}^{-1} = (\boldsymbol{G},\ \boldsymbol{V}^1, \ldots,\ \boldsymbol{V}^m)$$

via the discrete Green's matrix $\boldsymbol{G}$ and the biorthogonal fundamental system $\boldsymbol{V}^l$, $l = \overline{1,m}$.

4.2 Properties of unique solutions

Using the formula (3.10), we rewrite constants in the form $c_k = g_k - \langle \boldsymbol{L}_k, \boldsymbol{U}^c \rangle$, $k = \overline{1,m}$. Substituting them into (4.3), we obtain the following representation of the unique solution.

Lemma 6.4. *The unique solution*

$$\boldsymbol{U} = \sum_{k=1}^{m} (g_k - \langle \boldsymbol{L}_k, \boldsymbol{U}^c \rangle) \boldsymbol{v}^k + \boldsymbol{U}^c$$

to the problem (3.1)–(3.2) *is always described by the unique solution* $\boldsymbol{U}^c$ *to the Cauchy problem* (3.10).

Let us now consider two problems with the same equation but different nonlocal conditions

$$\begin{aligned} &\boldsymbol{\mathcal{L}}\boldsymbol{U} = \boldsymbol{F}, && \boldsymbol{\mathcal{L}}\boldsymbol{V} = \boldsymbol{F}, \\ &\langle \widetilde{\boldsymbol{L}}_k, \boldsymbol{U} \rangle = \widetilde{g}_k, \quad k = \overline{1,m}, && \langle \boldsymbol{L}_k, \boldsymbol{V} \rangle = g_k, \quad k = \overline{1,m}, \end{aligned} \tag{4.7}$$

supposing these problems have unique solutions $\boldsymbol{U}$ and $\boldsymbol{V}$, i.e., both $\widetilde{\Delta} \neq 0$ and $\Delta \neq 0$, respectively.

Lemma 6.5. *The relation between the solutions of the problems* (4.7)

$$\boldsymbol{V} = \boldsymbol{U} + \sum_{k=1}^{m} (g_k - \langle \boldsymbol{L}_k, \boldsymbol{U} \rangle) \boldsymbol{V}^k,$$

is valid, where $\boldsymbol{V}^l$, $l = \overline{1,m}$, *are the biorthogonal fundamental system of the second problem* (4.7).

A similar relation is obtained for discrete Green's matrices as well. We present it below.

Lemma 6.6. *Discrete Green's matrices* $\widetilde{\boldsymbol{G}}$ *and* $\boldsymbol{G}$ *for two problems* (4.7), *respectively, are linked with the equality*

$$G_{ij}^{kl} = \widetilde{G}_{ij}^{kl} - \sum_{\ell=1}^{m} v_i^{\ell,k} \langle \boldsymbol{L}_\ell, \widetilde{\boldsymbol{G}}_{\cdot j}^{\cdot l} \rangle, \qquad i \in X_n,\ j \in X_{n-1}, \quad k, l = \overline{1,m}.$$

Another relation is gieven for biorthogonal fundamental systems.

Lemma 6.7. *The relation*

$$\sum_{k=1}^{m} \langle \boldsymbol{L}_k, \widetilde{\boldsymbol{V}}^l \rangle \boldsymbol{V}^k = \widetilde{\boldsymbol{V}}^l, \qquad l = \overline{1,m},$$

is valid between the fundamental systems $\widetilde{\boldsymbol{V}}^l$, $l = \overline{1,m}$, *and* $\boldsymbol{V}^k$, $k = \overline{1,m}$, *for the problems* (4.7), *respectively.*

4.3 Nonlocal boundary value problem

Let us now take a discrete problem with nonlocal boundary conditions

$$\boldsymbol{\mathcal{L}} \boldsymbol{U} = \boldsymbol{F}, \tag{4.8}$$

$$\langle \boldsymbol{L}_k, \boldsymbol{U} \rangle := \langle \boldsymbol{\kappa}_k, \boldsymbol{U} \rangle - \gamma_k \langle \boldsymbol{\varkappa}_k, \boldsymbol{U} \rangle = g_k, \quad k = \overline{1,m}. \tag{4.9}$$

Here $\boldsymbol{\kappa}_k$ describe classical parts but $\boldsymbol{\varkappa}_k$ represent fully nonlocal parts of conditions (4.9), $\gamma_k \in \mathbb{R}$, $k = \overline{1,m}$. If the *classical problem* (4.8)–(4.9) ($\gamma_k = 0$, $k = \overline{1,m}$) has the unique solution $\boldsymbol{U}^{\mathrm{cl}}$, then Lemma 6.5 provides the following relation.

Lemma 6.8. *The unique solution* $\boldsymbol{U}$ *to the problem* (4.8)–(4.9) *and the unique solution* $\boldsymbol{U}^{\mathrm{cl}}$ *to the classical problem (all* $\gamma_k = 0$*) are related as given below*

$$\boldsymbol{U} = \boldsymbol{U}^{\mathrm{cl}} + \sum_{k=1}^{m} \gamma_k \langle \boldsymbol{\varkappa}_k, \boldsymbol{U}^{\mathrm{cl}} \rangle \boldsymbol{V}^k.$$

Here $\boldsymbol{V}^k$, $k = \overline{1,m}$, *is the biorthogonal fundamental system of the nonlocal boundary value problem* (4.8)–(4.9).

Discrete Green's matrices are also similarly connected.

Lemma 6.9. *The discrete Green's matrix* $\boldsymbol{G}$ *for the problem* (4.8)–(4.9) *and the discrete Green's matrix* $\boldsymbol{G}^{\mathrm{cl}}$ *for the classical problem (all* $\gamma_k = 0$*) are related by the equality*

$$G_{ij}^{kl} = G_{ij}^{\mathrm{cl},kl} + \sum_{\ell=1}^{m} \gamma_\ell v_i^{\ell,k} \langle \boldsymbol{\varkappa}_\ell, \boldsymbol{G}_{\cdot j}^{\mathrm{cl},\cdot l} \rangle.$$

5 The unique minimizer (case $\varDelta = 0$)

If $\varDelta = 0$, we cannot solve the problem (3.1)–(3.2) uniquely since the problem may have a lot of solutions or no solutions. We are going to solve the problem in the least squares sense, consider properties and representations of the "*best*" least squares solution.

5.1 The minimum norm least squares solution

Let us take an inner product $(\boldsymbol{B}, \widetilde{\boldsymbol{B}})_1$ and the norm $\|\boldsymbol{B}\|_1 = (\boldsymbol{B}, \boldsymbol{B})_1^{1/2}$ in the space $F^m(X_{n-1}) \times \mathbb{C}^m$ for every $\boldsymbol{B}, \widetilde{\boldsymbol{B}} \in F^m(X_{n-1}) \times \mathbb{C}^m$. In the space $F^m(X_n)$, we consider another inner product $(\boldsymbol{U}, \widetilde{\boldsymbol{U}})_2$ with the norm $\|\boldsymbol{U}\|_2 = (\boldsymbol{U}, \boldsymbol{U})_2^{1/2}$ for $\boldsymbol{U}, \widetilde{\boldsymbol{U}} \in F^m(X_n)$. For instance, we will use two inner products $(\boldsymbol{B}, \widetilde{\boldsymbol{B}})_{(L^2(\omega^h))^m \times \mathbb{R}^m}$ and $(\boldsymbol{U}, \widetilde{\boldsymbol{U}})_{(H^1(\overline{\omega}^h))^m}$ with norms

$$\begin{aligned}
\|\boldsymbol{B}\|_{(L^2(\omega^h_{n-1}))^m \times \mathbb{R}^m} &= (\boldsymbol{B}, \boldsymbol{B})^{1/2}_{(L^2(\omega^h))^m \times \mathbb{R}^m} \\
&= \left(\sum_{k=1}^{m} \|f^k\|^2_{L^2(\omega^h_{n-1})} + g_1^2 + \ldots + g_m^2 \right)^{1/2}, \\
\|\boldsymbol{U}\|_{(H^1(\overline{\omega}^h))^m} = (\boldsymbol{U}, \boldsymbol{U})^{1/2}_{(H^1(\overline{\omega}^h))^m} &= \left(\sum_{k=1}^{m} (u^k, u^k)^2_{H^1(\overline{\omega}^h)} \right)^{1/2}
\end{aligned}$$

in the example below (case $m = 2$); here all functions $\boldsymbol{U}, \widetilde{\boldsymbol{U}} \in F^m(X_n)$ and $\boldsymbol{B}, \widetilde{\boldsymbol{B}} \in F^m(X_{n-1}) \times \mathbb{R}^m$ are real, and we denoted the norm $\|f\|_{L^2(\omega^h_{n-1})} = \left(\sum_{i=0}^{n-1} f_i^2 h\right)^{1/2}$ for every $f \in F(X_{n-1})$.

Introducing two different norms, we can minimize the norm of the residual

$$\|\boldsymbol{A}\boldsymbol{U}^g - \boldsymbol{B}\|_1 = \inf_{\boldsymbol{U} \in F^m(X_n)} \|\boldsymbol{A}\boldsymbol{U} - \boldsymbol{B}\|_1 \tag{5.1}$$

and look for a unique discrete matrix valued function $\boldsymbol{U}^o \in F^m(X_n)$, which has the minimum norm

$$\|\boldsymbol{U}^o\|_2 < \|\boldsymbol{U}^g\|_2 \qquad \forall\, \boldsymbol{U}^g \neq \boldsymbol{U}^o \tag{5.2}$$

among all minimizers $\boldsymbol{U}^g \in F^m(X_n)$ of the norm of the residual.

This minimizer $\boldsymbol{U}^o$ is called the *minimum norm least squares solution* to the problem (3.1)–(3.2). It always exists and is unique since a finite dimensional discrete operator $\boldsymbol{A}$ is continuous linear with the closed range [6, Ben-Israel and Greville 2003]. Moreover, the minimum norm least squares solution

$$\boldsymbol{U}^o = \boldsymbol{A}^\dagger \boldsymbol{B} \tag{5.3}$$

is described by the Moore–Penrose inverse $\boldsymbol{A}^\dagger : F^m(X_{n-1}) \times \mathbb{C}^m \to F^m(X_n)$ of an operator $\boldsymbol{A} : F^m(X_n) \to F^m(X_{n-1}) \times \mathbb{C}^m$.

Using the minimum norm least squares solution, we can represent all least squares solutions $\boldsymbol{U}^g = \boldsymbol{U}^o + \boldsymbol{P}_{N(\boldsymbol{A})}\boldsymbol{C}$ with an arbitrary discrete matrix valued function $\boldsymbol{C} \in F^m(X_n)$. Here $\boldsymbol{P}_{N(\boldsymbol{A})}$ denotes the orthogonal projector onto $N(\boldsymbol{A})$, the nullspace of $\boldsymbol{A}$. The minimum norm least squares solution is also uniquely characterized by the equality $\boldsymbol{U}^o = \boldsymbol{P}_{N(\boldsymbol{A})^\perp}\boldsymbol{U}^g$. Moreover, the function $\boldsymbol{U}^o$ is always the minimizer to the consistent problem $\boldsymbol{A}\boldsymbol{U} = \boldsymbol{P}_{R(\boldsymbol{A})}\boldsymbol{B}$.

5.2 Generalized discrete Green's matrix

Since $\boldsymbol{B} = (\boldsymbol{F}, g_1, \ldots, g_m)^\top \in F^m(X_{n-1}) \times \mathbb{C}^m$ and the Moore–Penrose inverse $\boldsymbol{A}^\dagger$ is linear [6, Ben-Israel and Greville 2003], we rewrite the minimizer (5.3) in the special form

$$\boldsymbol{U}^o = \boldsymbol{G}^g \boldsymbol{F} + g_1 \boldsymbol{V}^{g,1} + \ldots + g_m \boldsymbol{V}^{g,m}. \tag{5.4}$$

Here we introduced an operator $\boldsymbol{G}^g : F^m(X_{n-1}) \to F^m(X_n)$ and functions $\boldsymbol{V}^l \in F^m(X_n)$, $l = \overline{1,m}$, those are parts of the Moore-Penrose inverse

$$\boldsymbol{A}^\dagger = (\boldsymbol{G}^g,\ \boldsymbol{V}^{g,1},\ \ldots,\ \boldsymbol{V}^{g,m}).$$

The minimizer (5.4) can also be given in the discrete form

$$(u^o)_i^k = \sum_{j=0}^{n-1} \sum_{l=1}^{m} G_{ij}^{g,kl} f_j^l + g_1 v_i^{g,1;k} + \ldots + g_m v_i^{g,m;k}, \tag{5.5}$$

where $k = \overline{1,m}$ and $i \in X_n$. This representation simplifies to the formula (4.5) of the unique solution if it exists ($\Delta \neq 0$). Thus, we call the kernel $\boldsymbol{G}^g = (G_{ij}^{g,kl}) \in F^{m\times m}(X_n \times X_{n-1})$ by the *generalized discrete Green's matrix* and the set of discrete matrix valued functions $\boldsymbol{V}^{g,l} = (v_i^{g,l;k}) \in F^m(X_n)$ – the *generalized biorthogonal fundamental system* for the problem with nonlocal conditions (3.1)–(3.2).

5.3 Properties of minimizers

Let us begin with the following characterization of the generalized discrete biorthogonal fundamental system that is the analogue of (4.2).

Lemma 6.10. *Every function $\boldsymbol{V}^{g,l}$, $l = \overline{1,m}$, is the minimum norm least squares solution to the corresponding problem*

$$\boldsymbol{\mathcal{L}} \boldsymbol{V}^{g,l} = \boldsymbol{O},$$
$$\langle \boldsymbol{L}_k, \boldsymbol{V}^{g,l} \rangle = \delta_k^l, \quad k,l = \overline{1,m}.$$

Since the proof is analogous as in previous chapters, we omit it. We present other results without proofs as well.

Let us now consider two relative problems (4.7), where the first problem has the unique solution, i.e., the condition $\widetilde{\Delta} \neq 0$ is valid. Here and further $\boldsymbol{G}^g$ is the generalized discrete Green's matrix and $\boldsymbol{V}^{g,k}$, $k = \overline{1,m}$, are the generalized discrete biorthogonal fundamental system for the second problem (4.7), which may have the unique solution ($\Delta \neq 0$) or not ($\Delta = 0$).

Theorem 6.11. *If the first problem* (4.7) *has the unique solution $\boldsymbol{U}$, then the minimum norm least squares solution for the second problem* (4.7) *is of the form*

$$\boldsymbol{U}^o = \boldsymbol{U} - \boldsymbol{P}_{N(\boldsymbol{A})} \boldsymbol{U} + \sum_{k=1}^{m} (g_k - \langle \boldsymbol{L}_k, \boldsymbol{U} \rangle) \boldsymbol{V}^{g,k}.$$

Since the discrete Cauchy problem (3.4)–(3.5) always has the unique solution $\boldsymbol{U}^c$, we obtain the following representation.

Corollary 6.12. *The minimizer of the problem* (3.1)–(3.2) *is always given by*

$$\boldsymbol{U}^o = \boldsymbol{U}^c - \boldsymbol{P}_{N(\boldsymbol{A})} \boldsymbol{U}^c + \sum_{k=1}^{m} (g_k - \langle \boldsymbol{L}_k, \boldsymbol{U}^c \rangle) \boldsymbol{V}^{g,k}.$$

For a problem with nonlocal boundary conditions (4.8)–(4.9), we can derive one more description of the minimizer as well.

Corollary 6.13. *Let the classical problem* (4.8)–(4.9) *(all $\gamma_k = 0$) has the unique solution $\boldsymbol{U}^{\mathrm{cl}}$. Then the minimizer of the problem* (4.8)–(4.9) *is described as below*

$$\boldsymbol{U}^o = \boldsymbol{U}^{\mathrm{cl}} - \boldsymbol{P}_{N(\boldsymbol{A})} \boldsymbol{U}^{\mathrm{cl}} + \sum_{k=1}^{m} \gamma_k \langle \boldsymbol{\varkappa}_k, \boldsymbol{U}^{\mathrm{cl}} \rangle \boldsymbol{V}^{g,k}.$$

The generalized biorthogonal fundamental systems for problems (4.7) are also related. We present their connection below.

Corollary 6.14. *Let $\widetilde{\Delta} \neq 0$ for the first problem* (4.7)*. Then the biorthogonal fundamental system $\widetilde{\boldsymbol{V}}^l$, $l = \overline{1, m}$, of the first problem and the generalized biorthogonal fundamental system $\boldsymbol{V}^{g,k}$, $k = \overline{1, m}$, of the second problem* (4.7) *are related by*

$$\sum_{k=1}^{m} \langle \boldsymbol{L}_k, \widetilde{\boldsymbol{V}}^l \rangle \boldsymbol{V}^{g,k} = \boldsymbol{P}_{N(\boldsymbol{A})^\perp} \widetilde{\boldsymbol{V}}^l, \qquad l = \overline{1, m}.$$

5.4 Properties of a generalized discrete Green's matrix

Let us provide the representation of a generalized Green's matrix, which is analogous to the definition of an discrete Green's matrix (4.4).

Lemma 6.15. *The generalized discrete Green's matrix for the problem* (3.1)–(3.2) *is of the form*

$$G_{ij}^{g,kl} = G_{ij}^{c,kl} - \left(P_{N(\boldsymbol{A})} G^c\right)_{ij}^{kl} - \sum_{\ell=1}^{m} v_i^{g,\ell;k} \langle \boldsymbol{L}_\ell, \boldsymbol{G}_{\cdot j}^{c,\cdot l} \rangle,$$

where $i \in X_n$, $j \in X_{n-1}$, $k, l = \overline{1, m}$.

Here we denoted by $\left(P_{N(\boldsymbol{A})} G\right)_{ij}^{kl}$ the kernel of the discrete operator $\boldsymbol{P}_{N(\boldsymbol{A})} \boldsymbol{G} : F^m(X_{n-1}) \to F^m(X_n)$. The following description of a generalized discrete Green's matrix is also valid.

Theorem 6.16. *If the first problem* (4.7) *has the discrete Green's matrix $\boldsymbol{G}$, then the generalized discrete Green's matrix for the second problem* (4.7) *is given by*

$$G_{ij}^{g,kl} = G_{ij}^{kl} - \left(P_{N(\boldsymbol{A})} G\right)_{ij}^{kl} - \sum_{\ell=1}^{m} v_i^{g,\ell;k} \langle \boldsymbol{L}_\ell, \boldsymbol{G}_{\cdot j}^{\cdot l} \rangle$$

for all $i \in X_n$, $j \in X_{n-1}$, $k, l = \overline{1, m}$.

For the problem with nonlocal boundary conditions (4.8)–(4.9), we obtain the following relation.

Corollary 6.17. *Let the classical problem* (4.8)–(4.9) *(all $\gamma_k = 0$) has the discrete Green's matrix $\boldsymbol{G}^{\mathrm{cl}}$. Then the generalized discrete Green's matrix for the problem* (4.8)–(4.9) *is of the form*

$$G_{ij}^{g,kl} = G_{ij}^{\mathrm{cl},kl} - \left(P_{N(\boldsymbol{A})} G^{\mathrm{cl}}\right)_{ij}^{kl} + \sum_{\ell=1}^{m} v_i^{g,\ell;k} \gamma_\ell \langle \boldsymbol{\varkappa}_\ell, \boldsymbol{G}_{\cdot j}^{\mathrm{cl},\cdot l} \rangle.$$

Example 6.18. *Let us now continue the investigation of the differential system with the Bitsadze–Samarkii condition*

$$\begin{gathered}(u^1)' = u^2 + f^1, \quad (u^2)' = f^2,\\ u^1(0) = g_1, \quad u^1(1) = \gamma u^1(\xi) + g_2,\end{gathered} \tag{5.6}$$

where functions $f^1, f^2 \in C[0,1]$, g_1, g_2, γ *are real numbers and a point* $\xi \in (0,1)$.

Let us denote $f_i^1 = f^1(x_i)$, $f_i^2 = f^2(x_i)$ *for* $i \in X_{n-1}$ *and suppose* ξ *is coincident with a point of the mesh* $\overline{\omega}^h$, *i.e.,* $\xi = sh$ *for some* $s = \overline{1, n-2}$. *Approximating the differential problem* (5.6) *by the finite difference method, we obtain a discrete problem*

$$\frac{u_{i+1}^1 - u_i^1}{h} = u_i^2 + f_i^1, \qquad \frac{u_{i+1}^2 - u_i^2}{h} = f_i^2, \qquad i \in X_{n-1}, \tag{5.7}$$

$$\langle \boldsymbol{L}_1, \boldsymbol{U} \rangle := u_0^1 = g_1, \qquad \langle \boldsymbol{L}_2, \boldsymbol{U} \rangle := u_n^1 - \gamma u_s^1 = g_2. \tag{5.8}$$

We note that the solution to the problem (5.7)–(5.8) *is equivalent to the solution of a second order discrete problem*

$$-\frac{u_{i+2}^1 - 2u_{i+1}^1 + u_i^1}{h^2} = f_i, \qquad i \in X_{n-2}, \tag{5.9}$$

$$u_0^1 = g_1, \qquad u_n^1 - \gamma u_s^1 = g_2, \tag{5.10}$$

which is obtained rewriting equations (5.7) *in the form* (5.9) *and denoting* $f_i = -f_i^2 - (f_{i+1}^1 - f_i^1)/h$. *From here we can find the solution* u_i^1, $i \in X_n$, *and then calculate another function* $u_i^2 = (u_{i+1}^1 - u_i^1)/h - f_i^1$ *for* $i \in X_{n-1}$ *but* $u_n^2 = u_{n-1}^2 + hf_{n-1}^2$.

The homogenous equation (5.9) *has the fundamental system* $z^1 = 1$ *and* $z^2 = x$, $x \in \overline{\omega}^h$. *It gives the fundamental system* $\boldsymbol{Z}^1 = (1,0)^\top$, $\boldsymbol{Z}^2 = (x,1)^\top$ *for the homogenous problem* (5.7) *as well. Now we calculate*

$$\varDelta = \begin{vmatrix} \langle \boldsymbol{L}_1, \boldsymbol{Z}^1 \rangle & \langle \boldsymbol{L}_2, \boldsymbol{Z}^1 \rangle \\ \langle \boldsymbol{L}_1, \boldsymbol{Z}^2 \rangle & \langle \boldsymbol{L}_2, \boldsymbol{Z}^2 \rangle \end{vmatrix} = \begin{vmatrix} 1 & 1-\gamma \\ 0 & 1-\gamma\xi \end{vmatrix} = 1 - \gamma\xi.$$

If $\varDelta \neq 0$, *i.e.,* $\gamma\xi \neq 1$, *then the problem* (5.7)–(5.8) *has the unique collection of solutions* $\boldsymbol{U} = (u^1, u^2) \in F^2(X_n)$.

First, we solve the problem (5.9)–(5.10) *for the discrete function* $u^1 \in F(X_n)$. *Since* $\gamma\xi \neq 1$, *the problem* (5.9)–(5.10) *has the discrete Green's function*

$$G_{ij}^h = \begin{cases} x_{j+1}(1 - x_i), & j+1 \leqslant i, \\ x_i(1 - x_{j+1}), & j+1 \geqslant i, \end{cases} + \frac{\gamma x_i}{1 - \gamma\xi} \begin{cases} x_{j+1}(1 - \xi), & j+1 \leqslant s, \\ \xi(1 - x_{j+1}), & j+1 \geqslant s, \end{cases}$$

for $i \in X_n$ *and* $j \in X_{n-2}$*, but we denote* $G^h_{i,n-1} := 0$, $G^h_{i,-1} = 0$. *It describes the general solution to the equation* (5.9) *in the form*

$$u^1_i = c_1 + c_2 x_i + \sum_{j=0}^{n-2} G^h_{ij} f_j h = c_1 + c_2 x_i + \sum_{j=0}^{n-1} G^h_{ij} f^2_j\, h - \sum_{j=0}^{n-1} \frac{G^h_{ij} - G^h_{i,j-1}}{h} f^1_j\, h$$

with two arbitrary constants $c_1, c_2 \in \mathbb{R}$. *Substituting it into nonlocal conditions* (5.10)*, we find the unique solution*

$$u^1_i = g_1 \frac{1 + (\gamma - 1)x}{1 - \gamma\xi} + g_2 \frac{x}{1 - \gamma\xi} + \sum_{j=0}^{n-1} G^h_{ij} f^2_j\, h - \sum_{j=0}^{n-1} \frac{G^h_{ij} - G^h_{i,j-1}}{h} f^1_j\, h, \ i \in X_n.$$

Using formulas $u^2_i = (u^1_{i+1} - u^1_i)/h - f^1_i$ *for* $i \in X_{n-1}$ *and* $u^2_n = u^2_{n-1} + hf^2_{n-1}$*, we get*

$$u^2_i = \frac{g_1(\gamma - 1)}{1 - \gamma\xi} + \frac{g_2}{1 - \gamma\xi} + \sum_{j=0}^{n-1} \nabla_+ G^h_{i,j} f^2_j\, h - \sum_{j=0}^{n-1} \nabla_+(\nabla_- G^h_{ij}) f^1_j\, h - f^1_i, \ i \in X_n.$$

Here we introduced two notations $\nabla_+ G^h_{ij} = (G^h_{i+1,j} - G^h_{i,j})/h$ *and* $\nabla_- G^h_{ij} = (G^h_{i,j} - G^h_{i,j-1})/h$. *Now we can write the unique solution in the form* $\boldsymbol{U} = g_1 \boldsymbol{V}^1 + g_2 \boldsymbol{V}^2 + \boldsymbol{G}^h \boldsymbol{F}$ *using the biorthogonal fundamental system*

$$\boldsymbol{V}^1 = \left(\frac{1 + (\gamma - 1)x}{1 - \gamma\xi}, \frac{\gamma - 1}{1 - \gamma\xi} \right)^\top, \qquad \boldsymbol{V}^2 = \left(\frac{x}{1 - \gamma\xi}, \frac{1}{1 - \gamma\xi} \right)^\top$$

and the discrete Green's matrix $\boldsymbol{G}^h$*, which at a point* (i, j) *is given by*

$$\boldsymbol{G}^h_{ij} = \begin{pmatrix} \nabla_- G^h_{ij} & -G^h_{ij} \\ \nabla_+(\nabla_- G^h_{ij}) - \delta_{ij}/h & -\nabla_+ G^h_{ij} \end{pmatrix}, \quad i \in X_n, \ j \in X_{n-1}. \quad (5.11)$$

Comparing this expression with the representation of the generalized Green's matrix $\boldsymbol{G}(x, y)$ *in Example 5.16, here we see additional* δ_{ij}/h. *We note that it simplifies since* $\nabla_+(\nabla_- G^h_{ij}) = -1 + \delta_{ij}/h$ *contains* δ_{ij}/h *with a plus sign. Moreover, we considered Green's matrices in the previous chapter, defined on two domains* $x < y$ *and* $x > y$ *(i.e., see the formula* (2.10) *in Chapter 5).*

If $\Delta = 0$*, the problem* (5.7)–(5.8) *does not have the unique solution nor the discrete Green's matrix. Thus, we are going to derive its generalized discrete Green's matrix* $\boldsymbol{G}^{g,h}$.

Formula (5.3) *provides the representation*

$$\boldsymbol{G}^g_{ij} = \boldsymbol{G}^{\text{cl}}_{ij} - (\boldsymbol{P}_{N(\boldsymbol{A})} \boldsymbol{G}^{\text{cl}})_{ij} + \gamma \boldsymbol{V}^{g,2}_i \boldsymbol{G}^{\text{cl},1\cdot}_{sj} \quad (5.12)$$

via the discrete Green's matrix

$$\boldsymbol{G}_{ij}^{\mathrm{cl,h}} = \begin{pmatrix} \nabla_- G_{ij}^{\mathrm{cl,h}} & -G_{ij}^{\mathrm{cl,h}} \\ \nabla_+(\nabla_- G_{ij}^{\mathrm{cl,h}}) - \delta_{ij}/h & -\nabla_+ G_{ij}^{\mathrm{cl,h}} \end{pmatrix}, \qquad i \in X_n, \quad j \in X_{n-1},$$

obtained taking $\gamma = 0$ *in* (5.11)*. Here* $\nabla_+(\nabla_- G_{ij}^{\mathrm{cl,h}}) - \delta_{ij}/h = (-1+\delta_{ij}/h) - \delta_{ij}/h = -1$ *for all* $i \in X_n$ *and* $j \in X_{n-1}$*.*

Since $\langle \boldsymbol{L}_1, \boldsymbol{Z}^1 \rangle = 1 \neq 0$*, we get* $d = 1$ *(we have* $0 < d \leqslant 2 = m$ *according to Subsection 3.2) and* $k_1 = 2$*,* $k_2 = 1$ *for the problem* (5.7)–(5.8) *with* $\Delta = 0$*. We also see that* $\boldsymbol{Z}^2 \in N(\boldsymbol{A})$ *and calculate the discrete kernel* $(\boldsymbol{P}_{N(\boldsymbol{A})}\boldsymbol{G}^{\mathrm{cl,h}})_{ij}$*, which is equal to*

$$(\boldsymbol{P}_{N(\boldsymbol{A})}\boldsymbol{G}^{\mathrm{cl,h}})_{ij} = \frac{\boldsymbol{Z}_i^2}{\|\boldsymbol{Z}^2\|_2^2}\big(\boldsymbol{Z}^2, \boldsymbol{G}_{\cdot,j}^{\mathrm{cl,h}}\big)_{(H^1(\overline{\omega}^h))^2} = -\frac{1}{(2+h)(7+h)} \times$$
$$\times \begin{pmatrix} x_i\big(5 - h + h^2 + 3x_{j+1}(x_{j+1} - h)\big), & x_i\big(x_{j+1}(1-h) - x_{j+1}^3\big) \\ 5 - h + h^2 + 3x_{j+1}(x_{j+1} - h) & x_{j+1}(1-h) - x_{j+1}^3 \end{pmatrix}.$$

Now we are going to find the function $\boldsymbol{V}^{g,2}$*. It is the minimizer to the problem* $\boldsymbol{AU} = \boldsymbol{E}^4$*. According to properties of minimizers [6, Ben-Israel and Greville 2003], it is also the minimum norm least squares solution to the consistent problem* $\boldsymbol{AU} = \boldsymbol{P}_{R(\boldsymbol{A})}\boldsymbol{E}^4$*.*

Thus, we calculate the projection

$$\boldsymbol{P}_{R(\boldsymbol{A})}\boldsymbol{E}^4 = \boldsymbol{E}^4 - \frac{\boldsymbol{W}}{\|\boldsymbol{W}\|^2}(\boldsymbol{W}, \boldsymbol{E}^4)_{(L^2(\omega_{n-1}^h))^2 \times \mathbb{R}^2}.$$

Here $\|\boldsymbol{W}\| := \|\boldsymbol{W}\|_{(L^2(\omega_{n-1}^h))^2 \times \mathbb{R}^2}$ *for the function* $\boldsymbol{W}_i = \big(\gamma\nabla_- G_{si}^{\mathrm{cl,h}}; -\gamma G_{si}^{\mathrm{cl,h}}; \gamma - 1; 1\big)^\top$*, which generates the nullspace of the adjoint operator, i.e.,* $\boldsymbol{W} \in N(\boldsymbol{A}^*) = R(\boldsymbol{A})^\perp$*. Let us note that we have an approximation* $\|\boldsymbol{W}\|^2 = \|\boldsymbol{w}\|^2_{(L^2[0,1])^2 \times \mathbb{R}^2} + \mathcal{O}(h)$*, where the vector function* $\boldsymbol{w} = \big(\gamma(\partial/\partial y)G^{\mathrm{cl}}(\xi, x); -\gamma G^{\mathrm{cl}}(\xi, x); \gamma - 1; 1\big)^\top$ *generates the nullspace* $N(\boldsymbol{L}^*) = R(\boldsymbol{L})^\perp$ *for the differential problem* (5.6)*.*

Now we write the problem $\boldsymbol{AU} = \boldsymbol{P}_{R(\boldsymbol{A})}\boldsymbol{E}^4$ *in the explicit form*

$$\frac{u_{i+1}^1 - u_i^1}{h} - u_i^2 = \gamma\nabla_- G_{si}^{\mathrm{cl,h}}/\|\boldsymbol{W}\|^2, \;\; \frac{u_{i+1}^2 - u_i^2}{h} = -\gamma G_{si}^{\mathrm{cl,h}}/\|\boldsymbol{W}\|^2, \quad (5.13)$$

$$u_0^1 = (\gamma - 1)/\|\boldsymbol{W}\|^2, \quad (5.14)$$

$$u_n^1 - \gamma u_s^1 = 1 - 1/\|\boldsymbol{W}\|^2. \quad (5.15)$$

We solve this problem analogously as the corresponding differential problem from Example 5.25 in Chapter 5. That is, we take the general solution $\boldsymbol{U}$ *to discrete equations* (5.13)*, substitute it into the condition* (5.14) *(another* *condition is satisfied trivially), find constant c_1 value and obtain the general least squares solution $\boldsymbol{U}^g$. Calculating the projection $\boldsymbol{V}^{g,2} = \boldsymbol{P}_{N(\boldsymbol{A})}\boldsymbol{U}^g$, we find the desired minimizer. Below we present it in the explicit form*

$$v_i^{g,2;1} = \frac{\gamma - 1}{\|\boldsymbol{W}\|^2} + c^{h,o}x_i - \frac{\gamma}{\|\boldsymbol{W}\|^2}\sum_{j=0}^{n-1}\left(\nabla_- G_{ij}^{\mathrm{cl,h}} \cdot \nabla_- G_{sj}^{\mathrm{cl,h}} + G_{ij}^{\mathrm{cl,h}} \cdot G_{sj}^{\mathrm{cl,h}}\right) h,$$

$$v_i^{g,2;2} = c^{h,o} - \frac{\gamma}{\|\boldsymbol{W}\|^2}\sum_{j=0,\ j\neq i}^{n-1}\left(\nabla_+\nabla_- G_{ij}^{\mathrm{cl,h}} \cdot \nabla_- G_{sj}^{\mathrm{cl,h}} + \nabla_+ G_{ij}^{\mathrm{cl,h}} \cdot G_{sj}^{\mathrm{cl,h}}\right) h + \mathcal{O}(h)$$

with the constant $c^{h,o} = c^o + \mathcal{O}(h)$, which describes the minimizer $\boldsymbol{V}^{g,2}$ of the differential problem (5.6) *(see Example 5.25 in Chapter 5). Finally, we substitute obtained expressions into* (5.10) *and have that the generalized Green's matrix $\boldsymbol{G}_{ij}^{g,h}$ is equal to*

$$\begin{pmatrix} \nabla_- G_{ij}^{\mathrm{cl,h}} + \frac{x_i(5+3x_{j+1}^2)}{14} - \gamma v_i^{g,2;1}\nabla_- G_{sj}^{\mathrm{cl,h}} & -G_{ij}^{\mathrm{cl,h}} + \frac{x_i(x_{j+1}-x_{j+1}^3)}{14} + \gamma v_i^{g,2;1} G_{sj}^{\mathrm{cl,h}} \\ \nabla_+\nabla_- G_{ij}^{\mathrm{cl,h}} + \frac{5+3x_{j+1}^2}{14} - \gamma v_i^{g,2;2}\nabla_- G_{sj}^{\mathrm{cl,h}} & -\nabla_i G_{ij}^{\mathrm{cl,h}} + \frac{x_{j+1}-x_{j+1}^3}{14} + \gamma v_i^{g,2;2} G_{sj}^{\mathrm{cl,h}} \end{pmatrix}$$

plus a term $\boldsymbol{\mathcal{O}}(h)$. This equality is valid if $i \in X_n$, $j \in X_{n-1}$ and $i \neq j+1$ (observe the analogy to a differential case, where $x \neq y$!). Recalling Example 5.25 in Chapter 5, we get the approximation $\boldsymbol{G}_{ij}^{g,h} = \boldsymbol{G}^g(x_i, x_{j+1}) + \boldsymbol{\mathcal{O}}(h)$ for all $i \in X_n$, $j \in X_{n-1}$ if $i \neq j+1$ (except $x = y$, where the Green's matrix $\boldsymbol{G}^g(x,y)$ has the jump). For $i = j+1$, that is $j = i-1$, we get the vanishing element $(G^{g,h})_{i,i-1}^{kl} f_{i-1}^l\, h = \mathcal{O}(h)$ in the sum (5.16) *below for a chosen number i.*

Applying Corollary 6.14, we find another minimizer

$$\boldsymbol{V}^{g,1} = (\gamma - 1)\boldsymbol{V}^{g,2} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{3(1+h)}{(2+h)(7+h)}\begin{pmatrix} x \\ 1 \end{pmatrix}, \qquad x \in \overline{\omega}^h.$$

Since we know the generalized biorthogonal fundamental system $\boldsymbol{V}^{g,1}, \boldsymbol{V}^{g,2}$ and the generalized discrete Green's matrix $\boldsymbol{G}^{g,h}$, we can always calculate the minimizer using the representation $\boldsymbol{U}^o = \boldsymbol{G}^{g,h}\boldsymbol{F} + g_1\boldsymbol{V}^{g,1} + g_2\boldsymbol{V}^{g,2}$, that explicitly is

$$(u^o)_i^k = \sum_{j=0}^{n-1}\sum_{l=1}^{m}(G^{g,h})_{ij}^{kl} f_j^l\, h + g_1 v_i^{g,1;k} + \ldots + g_m v_i^{g,m;k}, \tag{5.16}$$

where $k = \overline{1,2}$ and $i \in X_n$.

Let us finally note that discrete functions $\boldsymbol{V}^{g,1}, \boldsymbol{V}^{g,2}$ converge to continuous functions $\boldsymbol{v}^{g,1}, \boldsymbol{v}^{g,2}$ (representations in Example 5.25, Chapter 5) and the generalized discrete Green's matrix $\boldsymbol{G}^{g,h}$ converges to the generalized Green's matrix $\boldsymbol{G}^g$ pointwise (see the formula (5.10) *in Chapter 5) if $h \to 0$ except* *the diagonal $x = y$. From here, we also get the pointwise convergence of the discrete minimizer $\boldsymbol{U}^o$ of the minimizer $\boldsymbol{u}^o$ to the differential problem* (5.6).

Below we suggest the way how to investigate the convergence of the discrete minimizer to the minimizer of a differential problem. First, for every function $\boldsymbol{u} \in \left(H^1[0,1]\right)^m$ we take its discretization $\boldsymbol{\pi}_1\boldsymbol{u} = \left(u^k(x_i)\right) \in \mathbb{R}^{m\times(n+1)}$ on the mesh $\overline{\omega}^h$. This formula is correct since $\boldsymbol{u} \in \left(H^1[0,1]\right)^m$ is an elementwise continuous function. Second, for every $\boldsymbol{f} = (f^1, \dots, f^m, g_1, \dots, g_m)^\top \in \left(L^2[0,1]\right)^m \times \mathbb{R}^m$ we also take some projector $\boldsymbol{\pi}_2 : \left(L^2[0,1]\right)^m \times \mathbb{R}^m \to F^m(X_{n-1}) \times \mathbb{R}^m$.

Theorem 6.19. *(Sufficient convergence conditions) Let the following approximations*

$$\begin{aligned}
&\boldsymbol{A}(\boldsymbol{\pi}_1\boldsymbol{u}) = \boldsymbol{\pi}_2\boldsymbol{L}\boldsymbol{u} + \boldsymbol{\mathcal{O}}(h^\alpha),\\
&\boldsymbol{P}_{N(\boldsymbol{A})}(\boldsymbol{\pi}_1\boldsymbol{u}) = \boldsymbol{\pi}_1\boldsymbol{P}_{N(\boldsymbol{L})}\boldsymbol{u} + \boldsymbol{\mathcal{O}}(h^\alpha),\\
&\boldsymbol{P}_{R(\boldsymbol{A})}\boldsymbol{B} = \boldsymbol{\pi}_2\boldsymbol{P}_{R(\boldsymbol{L})}\boldsymbol{f} + \boldsymbol{\mathcal{O}}(h^\alpha)
\end{aligned}$$

be valid for some $\alpha > 0$. If $\sup_{n\in\mathbb{N}} \|\mathbf{A}^\dagger\|_{1,2} < +\infty$, then the minimizer $\boldsymbol{U}^o$ of the discrete problem (3.1)–(3.2) *converges to the minimizer $\boldsymbol{u}^o \in \left(H^1[0,1]\right)^m$ of the differential problem* (1.1)–(1.2) *from Chapter 5, i.e.,*

$$\|\boldsymbol{U}^o - \boldsymbol{\pi}_1\boldsymbol{u}^o\|_{(C(\overline{\omega}^h))^m} := \max_{x_i\in\overline{\omega}^h,\ k=\overline{1,m}} |u_i^{o,k} - u^{o,k}(x_i)| \to 0 \qquad \text{if} \quad n \to \infty.$$

6 Conclusions

Below we present principal conclusions of this chapter:

1) A discrete problem (3.1)–(3.2) always has the Moore–Penrose inverse $\boldsymbol{A}^\dagger$, a generalized discrete Green's matrix and the unique minimum norm least squares solution.

2) For $\Delta \neq 0$, we have that $\boldsymbol{A}^\dagger = \boldsymbol{A}^{-1}$, the minimum norm least squares solution $\boldsymbol{U}^o$ is coincident with the unique solution $\boldsymbol{U}$, the generalized discrete Green's matrix $\boldsymbol{G}^g$ is coincident with the ordinary discrete Green's matrix $\boldsymbol{G}$, the biorthogonal fundamental system $\boldsymbol{V}^k$, $k = \overline{1,m}$, is coincident with the generalized discrete biorthogonal fundamental system $\boldsymbol{V}^{g,k}$, $k = \overline{1,m}$.

3) The minimum norm least squares solution has literally similar representations as the unique solution: it can be described by the unique solution of the discrete Cauchy problem or the unique solution to other

relative problem (same discrete equations (3.1) but different nonlocal conditions (3.2)).

4) A generalized discrete Green's matrix also has representations similar to expressions of a discrete Green's matrix: it can be written using the Green's matrix of the discrete Cauchy problem or a Green's matrix to other relative problem (same discrete equations (3.1) but different nonlocal conditions (3.2)).

5) Obtained properties of minimizers are coincident with corresponding properties of minimizers for differential problems.

6) The discrete minimum norm least squares solution converges to the minimum norm least squares solution of a differential problem (3.1)–(3.2) if conditions of Theorem 6.19 are valid.

General conclusions

Basic conclusions of this thesis are formulated below:

1) Considered problems always have Moore–Penrose inverses, generalized Green's functions/matrices and unique minimum norm least squares solutions.

2) For $\Delta \neq 0$, a Moore–Penrose inverse is coincident with an ordinary inverse, a minimum norm least squares solution is coincident with a unique solution, a generalized Green's function/matrix is coincident with an ordinary Green's function/matrix, the biorthogonal fundamental system is coincident with the generalized biorthogonal fundamental system in both the differential and discrete cases.

3) In all cases, a minimum norm least squares solution has literally similar representations as a corresponding unique solution: it can be described by the unique solution of a Cauchy problem or the unique solution to other relative problem (the same differential/discrete equation but different nonlocal conditions).

4) A generalized Green's function also has representations similar to expressions of a Green's function: it can be written using the Green's function of a Cauchy problem or a Green's function to other relative problem (same differential/discrete equations but different nonlocal conditions). For generalized Green's matrices, we have the analogical statement.

5) Obtained properties of discrete minimizers are coincident with corresponding properties of minimizers for differential problems.

6) Under certain conditions (Theorems 3.33, 4.32 or 6.19), a discrete minimum norm least squares solution converges to a minimum norm least squares solution of a differential problem.

Bibliography

[1] H. Albrecher, W.J. Runggaldier and W. Schachermayer. *Advanced Financial Modelling.* Walter de Gruyter, 2009.

[2] H.W. Alt. *Linear Functional Analysis: An Application-Oriented Introduction.* Springer-Verlag London, 2016.

[3] A. Ashyralyev. A note on the Bitsadze–Samarskii type nonlocal boundary value problem in a Banach space. *J. Math. Anal. Appl.*, **344**:557–573, 2008. Available from Internet: `https://doi.org/10.1016/j.jmaa.2008.03.008`.

[4] Н.С. Бахвалов, Н.П. Жидков, Г.М. Кобелков. *Численные методы.* Бином, 2011.

[5] Z. Bai. Positive solutions of some nonlocal fourth-order boundary value problem. *Appl. Math. Comput.*, **215**:4191—-4197, 2010. Available from Internet: `https://doi.org/10.1016/j.amc.2009.12.040`.

[6] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications, Second edition.* Springer Verlag, New York, 2003.

[7] A.V. Bitsadze and A.A. Samarskii. Some elementary generalizations of linear elliptic boundary value problems. *Soviet Math. Dokl.*, **10**:398–400, 1969. (see [8])

[8] А.В. Бицадзе, А.А. Самарский. О некоторых простейших обобщениях эллиптических краевых задач. *Докл. АН СССР*, **185**:739–740, 1969. Available from Internet: `http://samarskii.ru/articles/1969/1969-002ocr.pdf`.

[9] A. Bjerhammar. Application of calculus of matrices to method of least squares with special refference to geodetic calculations. *Trans. Roy. Inst. Tech. Stockholm*, **49**:86, 1951.

[10] A. Bjerhammar. Rectangular reciprocal matrices, with special reference to geodeditic calculations. *Bull. Geodesique*, pp. 188–220, 1951.

[11] A. Bjerhammar. A generalized matrix algebra. *Trans. Roy. Inst. Tech. Stockholm*, **124**:32, 1958.

[12] J.M. Blackledge. *Digital Signal Processing: Mathematical and Computational Methods, Software Development and Applications*. Woodhead Publishing, second edition, 1983.

[13] A.A. Boichuk and A. M. Samoilenko. *Generalized inverse operators and Fredholm boundary-value problems*. VSP, Utrecht-Boston, The Netherlands, 2004. Available from Internet: https://doi.org/10.1515/9783110944679.

[14] M.G. Bostock. Green's functions, source signatures, and the normalization of teleseismic wave fields. *Jornal of Geophysical Research*, **109**, 2004. Available from Internet: https://doi.org/10.1029/2003JB002783.

[15] E. Bounitzky. Sur la fonction de Green des équations différentielles linéaires ordinaires. *J. Math. Pures Appl.*, **5**(6):65–125, 1909.

[16] J.S. Bradley. Generalized Green's functions for compatible differential systems. *Amer. J. Math.*, **51**:397–416, 1929.

[17] L.A. Brooks and P. Gerstoft. Green's function approximation from cross-correlations of 20-100 hz noise during a tropical storm. *J. Acoust. Soc. Am.*, **125**(2):723–734, 2009. Available from Internet: https://doi.org/10.1121/1.3056563.

[18] R.C. Brown. Generalized Green's functions and generalized inverses for linear differential systems with Stieltjes boundary conditions. *J. Differential Equations*, **16**:335–341, 1974.

[19] R.C. Brown and A.M. Krall. Ordinary differential operators under Stieltjes boundary conditions. *Trans. Amer. Math. Soc*, **198**:73–92, 1974. Available from Internet: https://doi.org/10.1090/S0002-9947-1974-0358436-2.

[20] R.N. Bryan. A linear differential system with general linear boundary conditions. *J. Differential Equations*, **5**(1):38–48, 1969. Available from Internet: https://doi.org/10.1016/0022-0396(69)90102-8.

[21] V. Būda, M. Sapagovas and R. Čiegis. Two-dimensional model of nonlinear diffusion. *Matem, i Mshinn. Metody v Mikroelektronike*, pp. 36–43, 1985. (in Russian)

[22] A. Cabada. *Green's Functions in the Theory of Ordinary Differential Equations*. Springer, New York, 2014. Available from Internet: https://doi.org/10.1007/978-1-4614-9506-2.

[23] B. Chanane. Computing the eigenvalues of a class of nonlocal Sturm–Liouville problems. *Math. Comput. Mod.*, **50**(1–2):225–232, 2009. Available from Internet: https://doi.org/10.1016/j.mcm.2008.10.021.

[24] Y.S. Choi and K.Y. Chan. A parabolic equation with nonlocal boundary conditions arising from electrochemistry. *Nonlinear Analysis, Theory, Method. Appli.*, **18**(4):317–331, 1992.

[25] F.R.K. Chung and S.-T. Yau. Discrete green's functions. *J. Combin. Theory Ser A*, **91**:191–214, 2000. Available from Internet: `https://doi.org/10.1006/jcta.2000.3094`.

[26] R. Ciegis. Numerical solution of a problem with small parameter for the highest derivative and a nonlocal condition. *Lith. Math. J.*, **28**(1):90–96, 1988. http://dx.doi.org/10.1007/BF00972255.

[27] R. Čiegis. Finite-difference schemes for nonlinear parabolic problem with nonlocal boundary conditions. In M. Ahues, C. Constanda and A. Largillier(Eds.), *Integral Methods in Science and Engineering: Analytic and Numerical Techniques*, p. 47–52, Birkhauser, Boston, 2004.

[28] R. Čiegis, O. Suboč and A. Bugajev. Parallel algorithms for three-dimensional parabolic and pseudoparabolic problems with different boundary conditions. *Nonlinear Anal. Model. Control*, **19**(3):382–395, 2014. http://dx.doi.org/10.15388/NA.2014.3.5.

[29] R. Čiegis and N. Tumanova. Numerical solution of parabolic problems with nonlocal boundary conditions. *Numer. Funct. Anal. Optim.*, **31**(12):1318–1329, 2010. http://dx.doi.org/10.1080/01630563.2010.526734.

[30] R. Čiegis and N. Tumanova. On construction and analysis of finite difference schemes for pseudoparabolic problems with nonlocal boundary conditions. *Math. Model. Anal.*, **19**(2):281–297, 2014. http://dx.doi.org/10.3846/13926292.2014.910562.

[31] R. Čiegis, A. Štikonas, O. Štikonienė and O. Suboč. Stationary problems with nonlocal boundary conditions. *Math. Model. Anal.*, **6**(2):178–191, 2001. http://dx.doi.org/10.1080/13926292.2001.9637157.

[32] R. Čiegis, A. Štikonas, O. Štikonienė and O. Suboč. A monotonic finite-difference scheme for a parabolic problem with nonlocal conditions. *Differ. Equ.*, **38**(7):1027–1037, 2002. http://dx.doi.org/10.1023/A:1021167932414.

[33] R. Čiupaila, Ž. Jesevičiūtė and M. Sapagovas. On the eigenvalue problem for one-dimensional differential operator with nonlocal integral condition. *Nonlinear Anal. Model. Control*, **9**(2):109–116, 2004. Available from Internet: `http://www.mii.lt/na/issues/NA_0902/NA09201.pdf`.

[34] E.A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. McGraw Hill Book Co., Inc., New York, Toronto, London, 1955.

[35] R. Cooper and G. Madden (Eds.). *Digital Signal Processing: Mathematical and Computational Methods, Software Development and Applications*. Physica-Verlag Heidelberg, first edition, 2004. Available from Internet: `https://doi.org/10.1007/978-3-7908-2676-0`.

[36] W. Day. A decreasing property of solutions of the parabolic equations with applications to thermoelasticity. *Quart. Applied Mathem.*, **41**:468–475, 1983.

[37] W.A. Day. Extensions of a property of the heat equation to linear thermoelasticity and order theories. *Quart. Appl. Math.*, **40**:319–330, 1982.

[38] G. Dorfleitner, P. Schneider, K. Hawlitschek and A. Buch. Pricing options with Green's functions when volatility, interest rate and barriers depend on time. *Quatitative Finance*, **8**(2), 2008. Available from Internet: `https://doi.org/10.1080/14697680601161480`.

[39] E.N. Economou. *Green's Functions in Quatum Physics*. Springer, Berlin, Heidelberg, 2006. Available from Internet: `https://doi.org/10.1007/3-540-28841-4`.

[40] W.W. Elliot. Green's functions for differential systems containing a parameter. *Amer. J. Math.*, **50**:243–258, 1928.

[41] W.W. Elliot. Generalized Green's functions for compatible differential systems. *Amer. J. Math.*, **51**:397–416, 1929.

[42] L.C. Evans. *Partial Differential Equations*. Amer. Math. Soc., Rrovidence, RI, second edition, 2010.

[43] D. Franco, G. Infante and F.M. Minhós. Nonlocal boundary value problems. *Boundary Value Problems*, **2012**(Article ID 23):1, 2012. Available from Internet: `https://doi.org/10.1186/1687-2770-2012-23`.

[44] I. Fredholm. Sur une classe d'*é*quations fonctionelles. *Acta Math.*, **27**:365–390, 1903.

[45] M.I. Freedman and K. Tseng. A first-order time-domain Green's function approach to supersonic unsteady flow. Technical Report NASA-CR-172208, NAS 1.26:172208, Boston Univiversity, Departament of Mathematics, Boston, United States, 1985.

[46] K. Ghanbary. Similarities of discrete and continuous Sturm Liouville problems. *Electron. J. Differential Equations*, **172**:1–8, 2007.

[47] G. Green. An essay on the application of mathematical analysis to the theories of elasticity and magnetism. *Trans. Amer. Math. Soc*, 1828.

[48] X. Hao, L. Liu and Y. Wu. Positive solutions for nonlinear nth-order singular nonlocal boundary value problems. *Bound. Value Probl.*, **2007**:1–10, 2007. Available from Internet: `https://doi.org/10.1155/2007/74517`.

[49] X. Hao, L. Liu and Y. Wu. Iterative solution to singular nth-order nonlocal boundary value problems. *Boundary Value Problems*, **2015**(125):1–10, 2015. Available from Internet: `https://doi.org/10.1186/s13661-015-0393-6`.

[50] A. Hasanov Hasanoğlu and V.G. Romanov. *Introduction to Inverse Problems for Differential Equations*. Springer International Publishing AG, Switzerland, 2017. Available from Internet: `https://doi.org/10.1007/978-3-319-62797-7`.

[51] E. Hernandez-Martinez, F.J. Valdes-Parada and L. Alvarez-Ramirez. A Green's function formulation of nonlocal finite-difference schemes for reaction-diffusion equations. *BJ. Comput. Appl. Math.*, **235**(9):3096–3103, 2011. Available from Internet: `https://doi.org/10.1016/j.cam.2010.10.015`.

[52] M.R. Hestenes. Relative self-adjoint operators in Hilbert space. *Pacific J. Math.*, **11**:1315–1357, 1961.

[53] D. Hilbert. Gundzüge einer allgemeinen Theorie der linearen Integralgleicungen. *Götingen Nachrichten*, **27**:49–51;213–259, 1904.

[54] V.A. Il'in and E.I. Moiseev. Nonlocal boundary value problem of the first kind for a Sturm–Liouville operator in its differential and finite difference aspects. *Differ. Uravn.*, **23**(7):803–810, 1987.

[55] V.A. Il'in and E.I. Moiseev. Nonlocal boundary-value problem of the second kind for a Sturm–Liouville operator. *Differ. Equ.*, **23**(8):979–987, 1987. (Translated from *Дифференц. Уравнения*, **23**(8):1422–1431, 1987, in Russian)

[56] В.А. Ильин. Необходимые и достаточные условия базисности подсистемы собственных и присоединённых функций пучка М.В.Келдыша обыкновенных дифференциальных операторов. *Докл. АН СССР*, **227**(4):796–799, 1976.

[57] G. Infante. Eigenvalues of some non-local boundary-value problems. *Proc. Edinb. Math. Soc. (Series 2)*, **46**:75–86, 2003. Available from Internet: `https://doi.org/10.1017/S0013091501001079`.

[58] G. Infante and J.R.L. Webb. Positive solutions of some nonlocal boundary value problems. *Abstr. Appl. Anal.*, **18**:1047–1060, 2003. Available from Internet: `https://doi.org/10.1155/S1085337503301034`.

[59] G. Infante and J.R.L. Webb. Three-point boundary value problems with solutions that change sign. *J. Integral Equations Appl.*, **15**(1):37–57, 2003. Available from Internet: `https://doi.org/10.1216/jiea/1181074944`.

[60] W.R. Jones. Differential Systems with Integral Bondary Conditions. *J. Differential Equations*, **3**:191–202, 1967.

[61] G. Kalna and S. McKee. The thermostat problem with a nonlocal nonlinear boundary conditions. *IMA J. Applied Math.*, **69**:437–462, 2004.

[62] L.I. Kamynin. A boundary value problem in the theory of heat conduction with nonclassical boundary conditions. *Comp. Math. Math. Phys.*, **6**(4):1006–1024, 1964. (in Russian)

[63] A.N. Kolmogorov and S.V. Fomin. *Elements of the Theory of Functions and Functional Analysis.* Dover Books, 1957.

[64] L. Kong and Q. Kong. Higher order boundary value problems with nonhomogeneous boundary conditions. *Nonlinear Anal.*, **72**:240–261, 2010. Available from Internet: `https://doi.org/10.1016/j.na.2009.06.050`.

[65] E. Kreyszig. *Introductory Functional Analysis with Applications.* John Willey & Sons Inc., 1 edition, 1978.

[66] K.Q. Lan. Positive characteristic values and optimal constants for three-point boundary value problems. In *Differential & Difference Equations and Applications*, pp. 623–633, New York, 2006. Hindawi Publ. Corp.

[67] E.M. Landesman. Hilbert–space methods in elliptic partial differential equations. *Pacific J. Math.*, **21**:113–131, 1967.

[68] J. Liu, B. Jia and Q. Zhu. An estimate for the three-dimensional discrete Green's function and applications. *J. Math. Anal.*, **370**(2):350–363, 2010. Available from Internet: `https://doi.org/10.1016/j.jmaa.2010.05.002`.

[69] J. Locker. On constructing least squares solutions to two-point boundary value problems. *Trans. Amer. Math. Soc.*, **203**:175–183, 1975.

[70] J. Locker. The generalized Green's function for an n-th order linear differential operator. *Trans. Amer. Math. Soc.*, **228**:243–268, 1977.

[71] W.S. Loud. Generalized inverses and generalized Green's functions. *SIAM J. Appl. Math.*, **14**:342–369, 1966.

[72] W.S. Loud. Some examples of generalized Green's functions and generalized Green's matrices. *SIAM Rev.*, **12**:194–210, 1970.

[73] R.Y. Ma. Positive solutions of a nonlinear three-point boundary-value problem. *Electron. J. Differential Equations*, **1998**(34):1–8, 1998.

[74] R.Y. Ma. A survey on nonlocal boundary value problems. *Appl. Math. E-Notes*, **7**:257–279, 2007.

[75] R. Marczuk and L. Majkut. Modelling of Green function in a rectangular room based upon the geometrical-filtration model. *Archives of Acoustics*, **31**(2):213–230, 2006. Available from Internet: `https://doi.org/10.1029/2003JB002783`.

[76] D. Maroncelli and J. Rodríguez. A least squares solution to linear boundary value problems with impulses. *Differential Equations and Applications*, **5**(4):519–525, 2013. Available from Internet: `https://doi.org/10.7153/dea-05-30`.

[77] I. Matsionis. A generalized Green's function. *Lith. Math. J.*, **13**(1):76–80, 1973.

[78] Y.A. Melnikov and M.Y. Melnikov. *Green's Functions. Constructions and Applications.* De Gruyter, 2012.

[79] C.D. Meyer. *Matrix Analysis and Applied Linear Algebra.* SIAM, 2004.

[80] E.H. Moore. On the reciprocal of the general algebraic matrix. *Bull. Am. Math. Soc.*, **26**:394–395, 1920.

[81] E.H. Moore and R.W. Barnard. General Analysis. Memoirs of the Americal Philosophycal Society, vol. i. *Am. Philos. Soc.*, 1935.

[82] А.М. Нахушев. *Уравнения математической биологии.* Нальчик: Высшая школа, 1995. (in Russian)

[83] G. Paukštaitė and A. Štikonas. Generalized Green's functions for second-order discrete boundary-value problems with nonlocal boundary conditions. *Liet. matem. rink. Proc. LMS,* Ser. A, **53**:96–101, 2012. Available from Internet: `ftp://ftp.science.mii.lt/pub/Publications/LMR/53(2012)/Series_A/MATHEMATICS/Paukstaite_Stikonas.pdf`.

[84] G. Paukštaitė and A. Štikonas. Investigation of matrix nullity for the second order discrete nonlocal boundary value problem. *Liet. matem. rink. Proc. LMS,* Ser. A, **54**:49–54, 2013. Available from Internet: ftp://ftp.science.mii.lt/pub/Publications/LMR/54(2013)/Series_A/MATHEMATICS/Paukstaite_Stikonas.pdf.

[85] G. Paukštaitė and A. Štikonas. Classification of the nullity for the second order discrete nonlocal problems. *Liet. matem. rink. Proc. LMS,* Ser. A, **55**:40–45, 2014. Available from Internet: http://www.mii.lt/LMR/A/2014/08.htm.

[86] G. Paukštaitė and A. Štikonas. Generalized Green's functions for the second order discrete problems with nonlocal conditions. *Lith. Math. J.*, **54**(2):203–219, 2014. Available from Internet: https://doi.org/10.1007/s10986-014-9238-8.

[87] G. Paukštaitė and A. Štikonas. Nullity of the second order discrete problem with nonlocal multipoint boundary conditions. *Liet. matem. rink. Proc. LMS,* Ser. A, **56**:72–77, 2015. Available from Internet: http://www.mii.lt/LMR/A/2015/13.htm.

[88] G. Paukštaitė and A. Štikonas. Ordinary and generalized Green's functions for the second order discrete nonlocal problems. *Bound. Value Probl.*, **2015**:207.1–19, 2015. Available from Internet: https://doi.org/10.1186/s13661-015-0474-6.

[89] G. Paukštaitė and A. Štikonas. Green's Matrices for First Order Differential Systems with Nonlocal Conditions. *Math. Model. Anal.*, **22**:213–227, 2016. Available from Internet: https://doi.org/10.3846/13926292.2017.1291456.

[90] G. Paukštaitė and A. Štikonas. Nullspace of the m-th order discrete problem with nonlocal conditions. *Liet. matem. rink. Proc. LMS,* Ser. A, **57**:59–64, 2016. Available from Internet: https://www.mii.lt/LMR/A/2016/11.htm.

[91] G. Paukštaitė and A. Štikonas. Generalized Green's functions for mth-order discrete nonlocal problems. *Lith. Math. J.*, **57**(1):109–127, 2017. Available from Internet: https://doi.org/10.1007/s10986-017-9346-3.

[92] G. Paukštaitė and A. Štikonas. The minimizer for the second order differential problem with one nonlocal condition. *Liet. matem. rink. Proc. LMS,* Ser. A, **58**:28–33, 2017. Available from Internet: https://doi.org/10.15388/LMR.A.2017.06.

[93] R. Penrose. A generalized inverse for matrices. *Proc. Cambridge Philos. Soc.*, **51**:406–413, 1955. Available from Internet: https://doi.org/10.1017/S0305004100030401.

[94] R. Penrose. On the best approximate solution of linear matrix equations. *Proc. Cambridge Philos. Soc.*, **52**:17–19, 1956.

[95] S. Pečiulytė. *Investigation and numerical analysis of boundary problems with various types nonlocal conditions.* PhD thesis, Institute of Mathematics and Informatics, 2007. Available from Internet: `http://www.mii.lt/files/mii_dis_2007_peciulyte.pdf`.

[96] W.A. Porter and J.P. Williams. Extension of the minimum effort control problem. *J. Math. Anal. Appl*, **13**:536–549, 1966.

[97] W.T. Reid. Generalized Green's matrices for compatible of systems differential equations. *Michigan J. Math.*, **13**:97–108, 1966.

[98] W.T. Reid. Generalized Green's matrices for two-point boundary problems. *SIAM J. Appl. Math.*, **15**:856–870, 1967.

[99] B. Riemann. Ueber die Fortpflanzug ebener Lufwelen von endlicher Schwingugsweite. *Abhandlungen der Königlichen Gesellshaft der Wissenschapten zu Göttingen*, **8**, 1860.

[100] S. Roman. *Green's functions for boundary-value problems with nonlocal boundary conditions.* PhD thesis, Vilnius University, 2011. Available from Internet: `http://www.mii.lt/files/s_roman_mii_disertacija.pdf`.

[101] S. Roman. Linear differential equation with additional conditions and formulae for Green's function. *Math. Model. Anal.*, **16**(3):401–417, 2011. Available from Internet: `https://doi.org/10.3846/13926292.2011.602125`.

[102] S. Roman and A. Štikonas. Green's function for discrete problems with nonlocal boundary conditions. *Liet. matem. rink. LMD darbai*, **52**:291–296, 2011. Available from Internet: `ftp://ftp.science.mii.lt/pub/Publications/52_TOMAS(2011)/SKAICIAVIMO_MATEMATIKA/sm_Roman_Stik.pdf`.

[103] S. Roman and A. Štikonas. Green's function for discrete second-order problems with nonlocal boundary conditions. *Bound. Value Probl.*, **2011**(Article ID 767024):1–23, 2011. Available from Internet: `https://doi.org/10.1155/2011/767024`.

[104] A.A. Samarskii. *The Theory of Difference Schemes.* Marcel Dekker, Inc., New York, Basel, 2001. (see [108])

[105] A.A. Samarskii and E.S Nikolaev. *Numerical Methods for Grid Equations.* Birkhäuser, 1978. (Vol. I Direct Methods, Vol.II Iterative Methods, see [107])

[106] А.А. Самарский, А.В. Гулин. *Численные методы.* Наука, Москва, 1989.

[107] А.А. Самарский, Е.С. Николаев. *Методы решения сеточных уравнений.* Наука, Москва, 1978.

[108] А.А. Самарский. *Теория разностных схем.* Наука, Москва, 1983.

[109] M. Sapagovas. The difference method for the solution of the problem on the equilibrium of a drop of liquid. *Different. Equat. and their Applic.*, **31**:63–72, 1982. (М. Сапаговас. Разностный метод решения задачи о равновесии капли жидкости, *Диффернциальные уравнения и их применения*, in Russian)

[110] M. Sapagovas. Numerical methods for the solution of the equation of surface with prescribed mean curvature. *Lith. Math. J.*, **23**(3):321–326, 1983. Available from Internet: `https://doi.org/10.1007/BF00966474`.

[111] M. Sapagovas. On the stability of a finite–difference scheme for nonlocal parabolic boundary–value problems. *Lith. Math. J.*, **48**(3):339–356, 2008. http://dx.doi.org/10.1007/s10986-008-9017-5.

[112] M. Sapagovas. On the spectral properties of three-layer difference schemes for parabolic equations with nonlocal conditions. *Differ. Equ.*, **48**(7):1018–1027, 2012. http://dx.doi.org/10.1134/S0012266112070129.

[113] M. Sapagovas, T. Meškauskas and F. Ivanauskas. Numerical spectral analysis of a difference operator with non-local boundary conditions. *Appl. Math. Comput.*, **218**(14):7515–7527, 2012. http://dx.doi.org/10.1134/S0012266112070129.

[114] M.P. Sapagovas. The numerical method for the solution of the problem on the equilibrium of a drop of liquid. *Comput. Math.,* Banach Center Publ., **13**:45–59, 1984. (in Russian)

[115] M.P. Sapagovas. A boundary value problem with a nonlocal condition for a system of ordinary differential equations. *Differ. Equ.*, **36**(7):1078–1085, 2000. Available from Internet: `https://doi.org/10.1007/BF02754510`.

[116] M.P. Sapagovas. Difference method of increased order of accuracy for the Poisson equation with nonlocal conditions. *Differ. Equ.*, **44**(7):1018–1028, 2008. http://dx.doi.org/10.1134/S0012266108070148.

[117] М.П. Сапаговас, Р.Ю. Чегис. О некоторых краевых задачах с нелокальным условием. *Дифференц. Уравнения*, **23**(7):1268–1274, 1987. (M.P. Sapagovas and R.Yu. Čiegis. On some boundary problems with nonlocal conditions, *Differ. Uravn.*, in Russian)

[118] K. Schuegerl. *Bioreaction Engineering.* Vol. I. Fundamentals, Thermodinamics, Formal Kinetics, Idealized Reactor Types and Operation Modes. John Wiley and Sons Ltd., Chichester, 1987.

[119] L. Schwartz. Théorie des noyaux. In *Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., August 30–September 6*, volume 1, pp. 220–230, 1950.

[120] C.J.R. Sheppard, S.S. Kou and J. Lin. The Green-function transform and wave propagation. In *Front. Phys.*, 2014.

[121] P. Slavík. The Least Extremal Solution of the Operator Equation $\mathbf{AXB} = \mathbf{C}$. *J. Math. Anal. Appl.*, **148**:251–262, 1990.

[122] S.L. Sobolev. Méthode nouvelle à résoudre le problème de Cauchy pour les équations linéaires hyperboliques normales. *Mat. Sbornik*, **1**:39–72, 1936.

[123] A. Štikonas. A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions. *Nonlinear Anal. Model. Control*, **19**(3):301–334, 2014. Available from Internet: `https://doi.org/10.15388/NA.2014.3.1`.

[124] A. Štikonas and G. Paukštaitė. The minimum norm least squares solution to the discrete nonlocal problems. In *IAIP Conf. Proc.*, 2016. Available from Internet: `https://doi.org/10.1063/1.4965403`.

[125] A. Štikonas and S. Roman. Stationary problems with two additional conditions and formulae for Green's functions. *Numer. Funct. Anal. Optim.*, **30**(9):1125–1144, 2009. Available from Internet: `https://doi.org/10.1080/01630560903420932`.

[126] A. Štikonas and O. Štikonienė. Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions. *Math. Model. Anal.*, **14**(2):229–246, 2009. Available from Internet: `https://doi.org/10.3846/1392-6292.2009.14.229-246`.

[127] J.-P. Sun and H.-E. Zhang. Existence of solutions to third-order m-point boundary-value problems. *Electron. J. Differential Equations*, **2008**(125):1–9, 2008.

[128] Y. Sun. Eigenvalues and symmetric positive solutions for a three-point boundary-value problem. *Electron. J. Differential Equations*, **2005**(127):1–7, 2005. Available from Internet: `http://ejde.math.txstate.edu/Volumes/2005/127/sun.pdf`.

[129] L.X. Truong, L.T.P. Ngoc and N.T. Long. Positive solutions for an m-point boundary-value problem. *Electron. J. Differential Equations*, **2008**(111):1–11, 2008. Available from Internet: `http://ejde.math.txstate.edu/Volumes/2008/111/truong.pdf`.

[130] Y.-Y. Tseng. Virtual solutions and general inversions. *Uspehi Mat. Nauk (N.S.)*, **11**:213–215, 1956.

[131] W.D.A. Westfall. Existence of the Generalized Green's Function. *The Annals of Mthematics, second series*, **10**(4):177–180, 1909 Jul.

[132] W.M. Whyburn. On the missing eigenvalue problem for an inverse Sturm–Liouville problem. *J. Math. Pures Appl.*, **48**:692–704, 1942.

[133] O. Wyler. On two-point boundary value problems. *Ann. Mat. Pura Appl.*, **6**:127–142, 1965.

[134] D. Xie, Y. Liu and C. Bai. Green's function and positive solutions of a singular nth-order three-point boundary value problem on time scales. *Electron. J. Qual. Theory Differ. Equ.*, **2009**(38):1–14, 2009. Available from Internet: `http://www.math.u-szeged.hu/ejqtde/p392.pdf`.

[135] V.A. Yurko and C. Yang. Recovering differential operators with nonlocal boundary conditions. *arXiv.org*, 2014. Available from arXiv:1410.2017.

[136] Z. Zhao. Positive solutions for singular three-point boundary-value problems. *Electron. J. Differential Equations*, **2007**(156):1–8, 2007. Available from Internet: `http://ejde.math.txstate.edu/Volumes/2007/156/zhao.pdf`.

FOR NOTES

Vilniaus universiteto leidykla
Universiteto g. 1, LT-01513 Vilnius
El. p. info@leidykla.vu.lt,
www.leidykla.vu.lt
Tiražas 20 egz.