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Recursive calculation of risk measures in discrete time risk model with inhomogeneous claims

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Contents

Chapter 1

Introduction

1.1 Scope and relevance of the Problem

The problems considered in the thesis belong to the area of applied probability known as Risk theory. The roots of Risk theory lie in Cramér-Lundberg model (or classical risk model), which was introduced in 1903 by Swedish actuary Filip Lundberg [48]. Lundberg's work was republished by Harald Cramér in 1930 [18]. Cramér-Lundberg model is a continuous-time real-valued stochastic process modelling the change of insurer's capital in time. This model has tree components: insurer's intial capital, constant premium inflow, and randomly stopped sum of i.i.d. claims. The cumulative number of claims is modelled by Poisson process. A more general model (known as renewal risk model) was proposed in 1957 by Sparre Andersen [65]. In this model the cumulative number of claims is modelled by renewal process. In 1988, Gerber [28] introduced discrete time risk model, which is a discrete-time and integer-valued version of the aforementioned models.

The main quantity of interest in Risk theory is ruin probability, which is a probability that insurer's capital will be zero or negative at some time in the future. Ruin probability is defined as a function of insurer's initial capital. In continuous time risk models, it is not easy to obtain analytical formulas for the computation of ruin probability, and in many cases only asymptotics and upper estimates of ruin probability can be realistically computed. By contrary, in discrete time risk models the computation of ruin probabilities is a lot easier task. This is usually done by obtaining recursive relations between consecutive values of ruin probability. Recursive calculation of ruin probabilities was first considered in [20] and [25].

Recent developments in Risk theory gave birth to more complex models and risk measures. For instance, the inhomogeneous discrete time risk models (i.e. models with not necessarily identically distributed claims) have been extensively analyzed. However, to obtain formulas for the computation of ruin probabilities some claims' non-homogeneity structure has to be assumed. For example, models with cyclically distributed claims with an arbitrary cycle length could be considered. Such models are also called seasonal discrete time risk models. There exists practical motivation for seasonal risk models in different spheres of insurance risks. In [7] the effect of seasonality on fracture risk is found to be statistically significant. Another example of risk influenced by seasonality is dairy production loss risk, as found in [22]. In this thesis, discrete time risk model with two seasons, also called bi-seasonal model, is considered. Furthermore, it is also realistic to consider models with claims' independence assumption relaxed. Here we introduce bi-seasonal discrete time risk model with dependent claims. The algorithm for computing ruin probabilities is obtained for this model.

As for risk measures, in 1998 a new risk measure called Gerber-Shiu expected discounted penalty function was introduced [29]. This function captures the economic costs to the insurer at the time of ruin, accounting for the force of interest as well. In this thesis, we derived an algorithm for computing the values of Gerber-Shiu function in bi-seasonal discrete time risk model.

1.2 Aim and objectives of research

The aim of this work is to create an algorithm for the calculation of risk measures in inhomogeneous discrete time risk models with independent and dependent claims. To do this, the following objectives are:

(i) to derive an algorithm for calculating the values of the particular case of the Gerber-Shiu discounted penalty function in the bi-seasonal discrete time risk model;

(ii) to create an algorithm for computing the values of the ruin probability in the bi-seasonal discrete time risk model with dependent claims;

(iii) to describe the case when net profit condition is not satisfied in the bi-seasonal discrete time risk model with dependent claims;

(iv) to illustrate the applicability and investigate the computational properties of the algorithms with numerical examples;

(v) to create methods for measuring approximation errors of the algorithms.

1.3 Methodology of research

The main results of the thesis are proved using the classical methods of probability theory and mathematical analysis, with an emphasis on discrete differentiation.

1.4 Scientific novelty

- The results of the thesis extend the results obtained by Damarackas and Šiaulys [19]. In this paper the calculation of ruin probability in the bi-seasonal discrete time risk model was considered. Both more general risk measure (Gerber-Shiu function) and more general model are considered in the thesis. Bi-seasonal model with dependent claims is introduced for the first time here in the dissertation.
- The recursive calculation of ruin probability in any kind of discrete time risk model with dependent and differently distributed claims was not considered in the scientific literature before.
- The new algorithm is derived for calculating Gerber-Shiu function values which is both more computationally feasible and less prone to numerical errors.

1.5 Approbation of dissertation results

The results of the thesis are presented in one international conference, one national conference and seminar:

- **Navickienė, Olga**; Šiaulys, Jonas. Gerber-Shiu Discounted Penalty Function for the Bi-seasonal Discrete Time Risk Model. The 10th Tartu conference on Multivariate Statistics, 28 June - 1 July 2016, Tartu, Estonia: abstracts. Tartu: University of Tartu Press, 2016. ISBN: 9789949771530. p. 40.
- **Navickienė, Olga**; Šiaulys, Jonas. Gerber-Shiu Discounted Penalty Function for the Bi-seasonal Discrete Time Risk Model with Indepen-

dent Claims. The LVI Conference of Lithuanian Mathematical Society, 20 June - 21 June 2016, Vilnius, Lithuania, at Vilnius Gediminas Technical University.

- **Navickienė, Olga**. Recursive calculation of risk measures in discrete time risk model with inhomogeneous claims. 25 September 2018. Scientific seminar of Finance and Insurance Mathematics of Institute of Mathematics of Faculty of Mathematics and Informatics of Vilnius University.

1.6 Main publications

The thesis is prepared based on two publications, which are to be published in journals indexed in Clarivate Analytics Web of Science in December 2018 and January 2019, respectively:

- **Olga Navickienė**, Jonas Sprindys, Jonas Šiaulys. Gerber-Shiu discounted penalty function for the bi-seasonal discrete time risk model. *Informatica* (accepted).
- **Olga Navickienė**, Jonas Sprindys, Jonas Šiaulys. Ruin probability for the bi-seasonal discrete time risk model with dependent claims. *Modern Stochastics: Theory and Applications*, https://doi.org/10.15559/18-VMSTA118.

1.7 Other publications

During the preparation of the thesis some other publications were issued as well:

- Ewart, Jacqui; Leichteris, Edgaras; Mačiulis, Algimantas; McLean, Hamish; Mikulskienė, Birutė; Paražinskaitė, Gintarė; Paunksnienė, Žaneta; Pitrėnaitė-Žilėnienė, Birutė; Stasiukynas, Andrius; Žalėnienė, Inga; de Lange, Michiel; Brunalas, Benas; Gudelytė, Laura; Kalinauskas, Marius; Mačiulienė, Monika; **Navickienė, Olga**; Skaržauskienė, Aelita; Stokaitė, Viktorija; Tamošiūnaitė, Rūta; Tvaronavičienė, Agnė; Valys, Taurimas. Social Technologies and Collective Intelligence: monograph. Vilnius: Mykolas Romeris University, 2015. 520 p.:. ISBN: 9789955197089.

- Gudelytė, Laura; **Navickienė, Olga**; Valentinaitė, Aistė. Overview of Features and Issues in Designing Evaluation Indices for Social Phenomena. Social technologies: research papers. Vilnius: Mykolas Romeris University. ISSN: 2029-7564. 2014, 4(2), p. 401-413. [DOAJ; Academic Search Research and Development (EBSCO); IndexCopernicus]

- Gudelytė, Laura; **Navickienė, Olga**. Modelling of Systemic Risk of Banking Sector. Social technologies: research papers. Vilnius: Mykolas Romeris University. ISSN: 2029-7564. 2013, 3(2), p. 359-371. [DOAJ; Academic Search Research and Developement (EBSCO); IndexCopernicus] [M.kr.: 04S, 01P]

- Kosareva, Natalja; Krylovas, Aleksandras; **Navickienė, Olga**. Economic and Social Phenomena Indicators Design Methodology Based on Averaging Values of Dichotomous Operators. Whither our economies – 2013: 3nd international scientific conference: conference proceedings. Vilnius: Mykolas Romeris University. ISSN: 2029-8501. 2013, p. 35-42. [Business Source Corporate Plus]

- **Navickienė, Olga**; Krylovas, Aleksandras; Kosareva, Natalja. The Construction of Test Reliability Statistical Criteria by a Computer Simulation. Lietuvos matematikos rinkinys: Proceedings of the Lithuanian Mathematical Society, Series A. Vilnius: Vilnius University Institute of Mathematics and Informatics. ISSN: 0132-2818. 2013, T. 54, p. 37-42. [MathSciNet; MLA]

In addition, two more earlier issued publications are listed below:

- Markšaitis, Hamletas Vladislavas; **Navickienė, Olga**. Probability Theory and Mathematical Statistics: educational publication. Vilnius: Mykolas Romeris University, 2012. 184 p.:. ISBN: 9789955193920 (e-book).

- Markšaitis, Hamletas Vladislavas; **Sajadian, Olga**. Linear Algebra and Mathematical Analysis Basics: educational publication. Vilnius: Mykolas Romeris University, 2010. 147 p.:. ISBN: 9789955191889.

1.8 Thesis structure

The scientific relevance, the aim and main objectives, the scientific novelty and approved results of the thesis are presented in the Introduction.

In Chapter 2, a brief review of important results from the discrete Risk theory is provided. Chapters 3 and 4 deal with the main results of the thesis.

In Chapter 3, discrete time risk model with two seasons is considered. In such model, the claims repeat with time periods of two units, i.e. claim distributions coincide at all even instants and at all odd instants. We derive a recursive algorithm for calculating the values of the particular case of the Gerber-Shiu discounted penalty function. Theoretical results are illustrated with the numerical examples.

In Chapter 4, we introduce discrete time risk model with two seasons and dependent claims. A recursive algorithm is created for computing the values of ruin probability. Theoretical results are illustrated with the numerical examples as well.

Finally, in Chapter 5 a short summary of the results obtained is provided. In the Appendices, the algorithms' code in R language is provided which was constructed for experimental needs.

Chapter 2

Review on risk models

Risk theory is the area of applied probability which uses mathematical models to describe an insurer's vulnerability to ruin. In such models the most interesting quantities are risk measures, such as the probability of ruin.

The roots of Risk theory lie in Cramér-Lundberg model (or classical risk model), which was introduced in 1903 by Swedish actuary Filip Lundberg [48]. Lundberg's work was republished by Harald Cramér in 1930 [18]. Cramér-Lundberg model is a continuous-time real-valued stochastic process modelling the change of insurer's capital in time. This model has tree components: insurer's intial capital, constant premium inflow, and randomly stopped sum of i.i.d. claims. The cumulative number of claims is modelled by Poisson process. A more general model (known as renewal risk model) was proposed in 1957 by Sparre Andersen [65]. In this model the cumulative number of claims is modelled by renewal process.

2.1 Renewal risk model

In this section, the components of renewal risk model are presented.

Definition 2.1.1. *Let $\theta_1, \theta_2, \ldots$ be independent copies of nonnegative random variable θ. Then the random walk*

$$T_0 = 0,\ T_n = \theta_1 + \ldots + \theta_n,\ n \in \mathbb{N} = \{1, 2, \ldots\}$$

is said to be a renewal sequence and the counting process

$$\Theta(t) = \#\{n \geqslant 1 : T_n \leqslant t\},\ t \geqslant 0,$$

is the corresponding renewal counting process.

Definition 2.1.2. *The aggregate claim amount process is a process defined by*

$$S(t) = \sum_{i=1}^{\Theta(t)} Z_i, \ t \geqslant 0,$$

where $Z_1, Z_2, \ldots$ are independent copies of nonnegative random variable Z, and sequences $\{Z_i, i \in \mathbb{N}\}$ and $\{\theta_i, i \in \mathbb{N}\}$ are mutually independent.

Definition 2.1.3. *The surplus process is a process defined by*

$$W_u(t) = u + ct - S(t), \ t \geqslant 0,$$

where $u = W_u(0)$ is the initial surplus, c is premium payment rate, and $S(t)$ is the aggregate claim amount process.

In Figure 2.1 sample trajectory of the surplus process $W_u(t)$ is depicted.

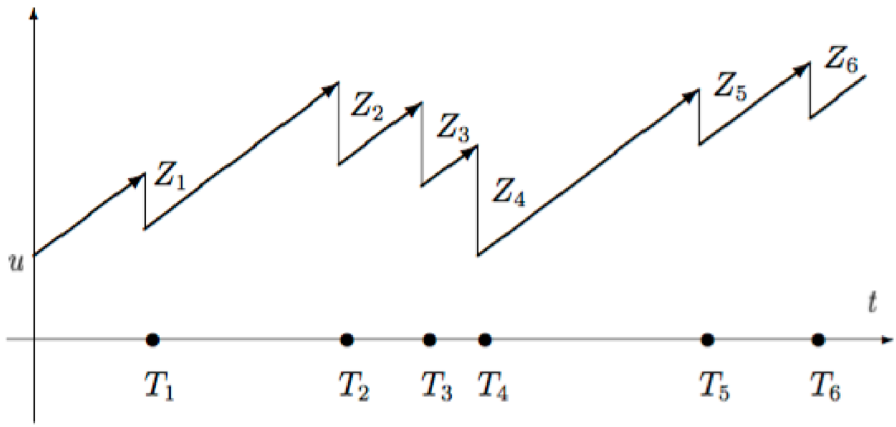

Figure 2.1: Behaviour of the surplus process $W_u(t)$

If the interclaim durations $\theta_1, \theta_2, \ldots$ are independent copies of an exponentially distributed random variable θ, then the general renewal risk model reduces to classical risk model. In this model, the renewal process $\Theta(t)$ is Poisson process with parameter λ. This parameter is the same as the exponential distribution parameter of θ. In other words, in the classical risk renewal model

$$\mathbb{P}\Big(\Theta(t) = k\Big) = \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t}, \ k \in \mathbb{N}_0, \ t > 0.$$

If θ is a degenerate random variable with $\mathbb{P}(\theta = 1) = 1,\ u \in \mathbb{N}_0,\ c = 1,\ Z \in \mathbb{N}_0$, then renewal risk model reduces to the so-called discrete-time risk model. In this model $\Theta(t) = \lfloor t \rfloor, t > 0$. Discrete-time risk model was first considered in 1988 by Gerber [28].

2.2 Discrete-time risk model

Definition 2.2.1. *We say that the insurer's surplus W_u varies according to a (classical) discrete time risk model if*

$$W_u(n) = u + n - \sum_{i=1}^{n} Z_i$$

for each $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and the following assumptions hold:

- *the initial insurer's surplus $u \in \mathbb{N}_0$,*
- *claim amounts $\{Z_1, Z_2, \ldots\}$ are independent copies of a nonnegative integer-valued random variable Z.*

In Figure 2.2 sample trajectory of the surplus process $W_u(n)$ is depicted.

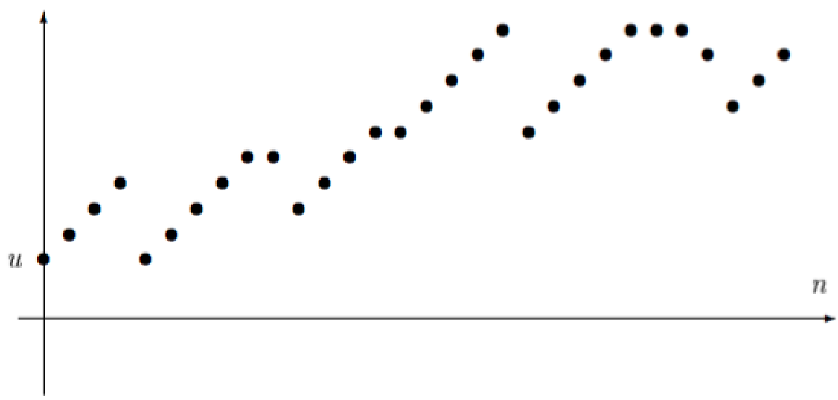

Figure 2.2: Behaviour of the surplus process $W_u(n)$

At each time moment $n \in \mathbb{N}$, the trajectory of the surplus process $W_u(n)$ can either increase by one (if $Z_n = 0$), stay at the same level (if $Z_n = 1$), or decrease by $a \in \mathbb{N}$ (if $Z_n = a + 1$). If the trajectory of the surplus process $W_u(n)$ is above zero, then insurer works successfully, because at each time moment there is enough capital to pay claims. However, sometimes the trajectory of $W_u(n)$ becomes such that $W_u(T_u) \leqslant 0$ at certain time moment T_u. In this case, it is said that at the time moment T_u the ruin has occurred, because insurer does not have enough capital to cover claims.

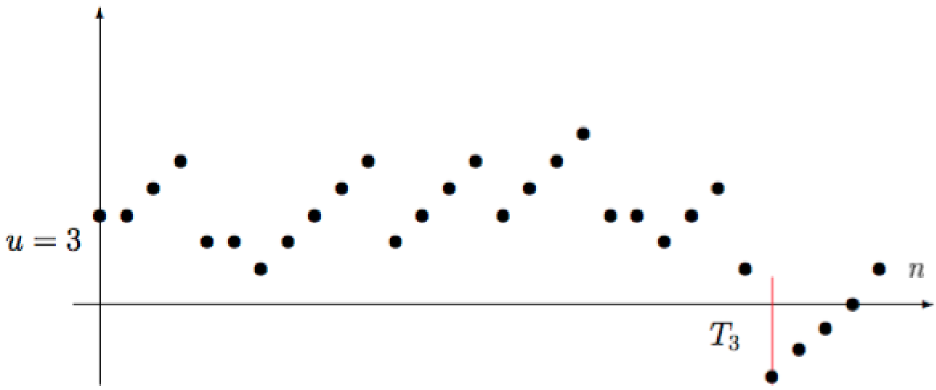

Figure 2.3: Sample trajectory of $W_3(n)$ with ruin occurring at time moment T_3

Time of ruin T_u and related critical characteristics are defined in the next section. We remark only that these characteristics are defined for more general renewal risk models as well. However, in this thesis we will restrict ourselves only to the analysis of discrete time risk models, so the definitions are provided only for such models.

2.3 Main characteristics

The main quantity of interest in Risk theory is ruin probability, which is a probability that insurer's capital will be zero or negative at some time in the future.

Definition 2.3.1. *Time of ruin is an extended random variable defined for the discrete time risk model by*

$$T_u = \begin{cases} \min\{n \geqslant 1 : W_u(n) \leqslant 0\}, \\ \infty, \text{ if } W_u(n) > 0 \text{ for all } \ n \in \mathbb{N}. \end{cases}$$

Definition 2.3.2. *Ruin probability is defined by*

$$\psi(u) = \mathbb{P}(T_u < \infty).$$

In continuous time risk models, it is not easy to obtain analytical formulas for the computation of ruin probability, and in many cases only asymptotics and upper estimates of ruin probability can be realistically computed.

By contrary, in discrete time risk models the computation of ruin probabilities is a lot easier task. This is usually done by obtaining recursive relations between consecutive values of ruin probability. Recursive calculation of ruin probabilities was first considered in [20] and [25]. Since then, many different methods for the calculation of ruin probabilities were developed. For instance, in [23] the following recursive formulas for the classical discrete time risk model are presented.

Theorem 2.3.1. *Let the classical discrete time risk model be generated by a nonnegative and integer valued random variable Z. If $\mathbb{E}Z < 1$, then the following recursive relations hold:*

$$\begin{aligned} \psi(0) &= \mathbb{E}Z, \\ \psi(u) &= \sum_{j=1}^{u-1}(1-F_Z(j))\psi(u-j) + \sum_{j=u}^{\infty}(1-F_Z(j)),\ u = 1,2,\ldots, \end{aligned}$$

where F_Z denotes the cumulative distribution function of Z.

If the expectation of claim random variable Z in classical discrete time risk model is large (in such situation, it is said that Z does not satisfy the net profit condition), then the ruin probability has a very simple behaviour, as described in [54].

Theorem 2.3.2. *Let the classical discrete time risk model be generated by a nonnegative and integer valued random variable Z. If $\mathbb{E}Z \geqslant 1$, then $\psi(u) = 1$ for all $u \in \mathbb{N}_0$.*

If we denote $Z_i = I_i X_i,\ i \in \mathbb{N}$ in discrete time risk model, where I_i is an indicator of claim occurrence and $X_i \in \mathbb{N}$ is the amount of claim, then the discrete time risk model is equivalent to the following model

$$W_u(t) = u + t - \sum_{i=1}^{\Theta(t)} X_i,\ t \in \mathbb{N},$$

where $\Theta(t) = I_1 + I_2 + \ldots + I_t$. Here $I_1, I_2, \ldots$ is a sequence of i.i.d. Bernoulli random variables.

By expressing model in such form, analytical formulas can be derived for the calculation of ruin probability and related quantities. Examples of such formulas are provided below. The first formula provided was derived in Gerber [28].

Theorem 2.3.3. *Let us denote* $q = \mathbb{P}(I_i = 1)$ *and* $\mu = \mathbb{E}X$. *Also, denote* $S_k = X_1 + \ldots + X_k$, $k \in \mathbb{N}$, $S_0 = 0$. *Furthermore, the notation* $a^{(k)} = k!\binom{a}{k}$ *for the factorial powers of* a *is used. The ultimate ruin probability for the discrete time risk model in the case* $q\mu < 1$ *can be expressed as*

$$\begin{aligned}
\psi(0) &= q\mu, \\
\psi(u) &= (1-q\mu)\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{q}{1-q}\right)^k \mathbb{E}\Big[(S_k-u)_+^{(k)}(1-q)^{S_k-u}\Big], u \in \mathbb{N}.
\end{aligned}$$

Let $\varphi(u) = 1 - \psi(u)$ be the corresponding survival probability, with initial surplus $u \in \mathbb{N}_0$. Shiu [64] derived following formulas for $\varphi(u)$, corresponding to those in Theorem 2.3.3, by alternative methods. Also, Shiu [64] defines ruin as the event that the surplus $W_u(n)$ becomes strictly negative, whereas in Gerber [28] the ruin is defined when the surplus $W_u(n)$ being non-positive for some $n \in \mathbb{N}$.

Theorem 2.3.4. *The survival probability for the discrete time risk model can be expressed as*

$$\begin{aligned}
\varphi(0) &= \frac{1-q\mu}{1-q}, \\
\varphi(u) &= \varphi(0)\sum_{k=0}^{\infty}\left(\frac{-q}{1-q}\right)^k \mathbb{E}\left[\binom{u+k-S_k}{k}(1-q)^{S_k-u}1_+(u-S_k)\right], \ u \in \mathbb{N},
\end{aligned}$$

where $1_+(k) = 1$ *for* $k \in \mathbb{N}_0$, *and* 0 *otherwise.*

2.4 Gerber-Shiu expected discounted penalty function

Recent developments in Risk theory gave birth to more complex risk measures. In 1998, a risk measure called Gerber-Shiu expected discounted penalty function (or, shorter, Gerber-Shiu function) was introduced in [29]. This function captures the economic costs to the insurer at the time of ruin, accounting for the force of interest as well.

Definition 2.4.1. *Gerber-Shiu function for the discrete time risk model is defined by*

$$\Psi_{\delta,w}(u) = \mathbb{E}\Big(\mathrm{e}^{-\delta T_u}\, w\big(W_u(T_u-1), |W_u(T_u)|\big)\,\mathbb{1}_{\{T_u<\infty\}}\Big),$$

where force of interest $\delta \geqslant 0$, $w(x, y)$ *is an arbitrary function of two nonnegative arguments, and* T_u *denotes the time of ruin.*

In the particular case when $\delta = 0$ and $w(x, y) = 1$ for all nonnegative x and y, the discounted penalty function is equal to the ruin probability.

As with the ruin probability, recursive methods are widely used to calculate the values of Gerber-Shiu function in discrete time risk models. Below we provide the formulas derived by Li and Garrido [37].

Theorem 2.4.1. *Let the classical discrete time risk model be generated by a nonnegative and integer-valued random variable* Z *with* $\mathbb{E}Z < 1$. *Then the values of function* $\Psi_{\delta,w}$ *can be calculated using the following formulas*

$$\begin{aligned}
\Psi_{\delta,w}(0) &= \mathrm{e}^{-\delta} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \varrho^k w(k, l) \mathbb{P}(Z = k + l + 1), \\
\Psi_{\delta,w}(u) &= \mathrm{e}^{-\delta} \sum_{k=0}^{u-1} \Psi_{\delta,w}(u - k) \sum_{l=0}^{\infty} \varrho^l \, \mathbb{P}(Z = k + l + 1) \\
&+ \mathrm{e}^{-\delta} \varrho^{-u} \sum_{k=u}^{\infty} \varrho^k \sum_{l=0}^{\infty} w(k, l) \mathbb{P}(Z = k + l + 1),
\end{aligned}$$

where $\varrho \in (0, 1)$ *is the root of equation*

$$s\mathrm{e}^{\delta} = \sum_{k=0}^{\infty} s^k \, \mathbb{P}(Z = k).$$

By arguments provided in [37], the solution to the equation above exists and is unique for $\delta > 0$.

Furthermore, denote by $\psi_\delta(u) = \mathbb{E}\big(\mathrm{e}^{-\delta T_u} \mathbb{1}_{\{T_u < \infty\}}\big)$. Clearly, $\psi_\delta(u)$ is a special case of Gerber-Shiu function when $w(x, y) = 1$. An application of Theorem 2.4.1 gives a recursive formula for $\psi_\delta(u)$ in the corollary below.

Theorem 2.4.2. *The function* $\psi_\delta(u)$ *in classical discrete time risk model can be expressed as*

$$\psi_\delta(u) = \sum_{z=0}^{u-1} \psi_\delta(u - z) g(z|0) + H(u), \; u \in \mathbb{N}, \tag{2.4.1}$$

where $g(z|0) = \sum_{x=0}^{\infty} \mathrm{e}^{-\delta} \rho^x \mathbb{P}(Z = x + z + 1)$, *and* $H(u) = \psi_\delta(0) - \sum_{z=0}^{u-1} g(z|0)$ *with*
$\psi_\delta(0) = H(0) = \frac{\mathrm{e}^{-\delta} - \rho}{1 - \rho}$.

Li and Garrido [37] further show in the Theorem below that $\psi_\delta(u)$ can be expressed as a compound geometric tail.

Theorem 2.4.3. *The solution to equation (2.4.1) can be expressed as a compound geometric sum*

$$\psi_\delta(u) = \frac{\beta}{1+\beta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\beta} \right)^n \bar{L}^{*n}(u-1), \ u \in \mathbb{N}_0.$$

Here β *is defined as* $1/(1+\beta) := \sum_{z=0}^{\infty} g(z|0) = \frac{\mathrm{e}^{-\delta}-\rho}{1-\rho}$. *Furthermore,* $l(z) = (1+\beta)g(z|0)$, *which is a proper probability mass function on* $\mathbb{N}_0$. *Also,* $\bar{L}(u) = \sum_{z=u+1}^{\infty} l(z)$ *is the tail probability of* l, *while* $\bar{L}^{*n}$ *is the* n*-th convolution of* $\bar{L}$, *with* $\bar{L}(-1) = \bar{L}^{*n}(-1) = 1$.

Chapter 3

Gerber-Shiu discounted penalty function for the bi-seasonal discrete time risk model

In this chapter, we consider so called bi-seasonal discrete time risk model, which is the direct generalization of the classical discrete time risk model described in the Chapter 2. The rest part of the chapter is organized in the following way: Section 3.2 deals with proofs of the main results; in Section 3.3 we describe an algorithm for calculating values of Gerber-Shiu function; next, in Section 3.4 we present a few numerical examples which illustrate the applicability of our algorithm; in Section 3.5, some concluding remarks and directions for future work are provided; finally, in Appendix A the algorithm code in R language is provided.

3.1 Definitions and main results

Definition 3.1.1. *We say that the insurer's surplus W_u varies according to the bi-seasonal risk model if*

$$W_u(n) = u + n - \sum_{i=1}^{n} Z_i$$

for each $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and the following assumptions hold:

- *the initial insurer's surplus $u \in \mathbb{N}_0$,*

- *the random claim amounts* $\{Z_1, Z_2, \ldots\}$ *are nonnegative integer-valued independent r.v.s,*
- *there exist r.v.s* X *and* Y *such that* $Z_{2k+1} \stackrel{d}{=} X$, $k \in \mathbb{N}_0$, *and* $Z_{2k} \stackrel{d}{=} Y$, $k \in \mathbb{N}$.

If $X \stackrel{d}{=} Y$, then the bi-seasonal discrete time risk model becomes the classical discrete time risk model.

There exists practical motivation for the seasonal risk models in the different spheres of insurance risks. In Bischoff-Ferrari et al. [7] the effect of seasonality on a fracture risk is found to be statistically significant. Another example of risk influenced by seasonality is dairy production loss risk, as found by Deng et al. [22].

The Gerber-Shiu discounted penalty function $\Psi_{\delta,w}$ is one of the main critical characteristics for the risk models of any types. According to the definition presented in Gerber and Shiu [29] for the discrete time risk model

$$\Psi_{\delta,w}(u) = \mathbb{E}\Big(\mathrm{e}^{-\delta T_u}\, w\big(W_u(T_u - 1), |W_u(T_u)|\big)\, \mathbb{I}_{\{T_u < \infty\}}\Big),$$

where force of interest $\delta \geqslant 0$, $w(x, y)$ is an arbitrary function of two nonnegative arguments, and T_u denotes the time of ruin, i.e.

$$T_u = \begin{cases} \min\{n \geqslant 1 : W_u(n) \leqslant 0\}, \\ \infty, \text{ if } W_u(n) > 0 \text{ for all } \ n \in \mathbb{N}. \end{cases}$$

Function w has practical interpretations. For example, if w was interpreted as the benefit amount of reinsurance payable at the time of ruin, then $\Psi_{\delta,w}(u)$ is the single premium of the reinsurance.

In the particular case considered in this paper when $w(x, y) = 1$ for all nonnegative x and y, the discounted penalty function is equal to the following expression

$$\psi_\delta(u) = \Psi_{\delta,1}(u) = \mathbb{E}\big(\mathrm{e}^{-\delta T_u}\, \mathbb{I}_{\{T_u < \infty\}}\big).$$

If, in addition, force of interest $\delta = 0$, then the Gerber-Shiu discounted penalty function is equal to the ruin probability

$$\psi(u) = \psi_0(u) = \Psi_{0,1}(u) = \mathbb{P}(T_u < \infty).$$

After Gerber and Shiu [29] presented the concept of function named on their behalf various properties of this function were considered by many authors. The main part of the known results on the Gerber-Shiu function is related with the Sparre Andersen model and various generalizations of this model. For instance, several cases of the Sparre Andersen model were considered by Dickson and Qazvini [26], Landriault and Willmot [34], Li and Garrido [38], Li and Sendova [41], Lin et al. [44], Schmidli [60], Willmot and Dickson [69]. Properties of the Gerber-Shiu function in the risk renewal models perturbed by diffusion were investigated by Chi et al. [15], Tsai and Willmot [66], Tsai [67], Xu et al. [71], Zhang and Cheung [72], Zhang et al. [75], Zhang et al. [76], Zhang et al. [77]. The Gerber-Shiu function of the risk models with various special strategies were considered by Avram et al. [3], Bratiichuk [9], Cheung and Liu [13], Cheung et al. [14], Dong et al. [27], Lin and Pavlova [42], Lin and Sendova [43], Liu et al. [47], Marciniak and Palmowski [51], Shi et al. [62], Shiraishi [63], Woo et al. [70], Zhang et al. [73], Zhou et al. [78]. This function for the risk models with various dependence structures or for risk models with investment strategies was considered by Cheung et al. [12], Cossette et al. [17], Li and Lu [40], Mihályko and Mihályko [53], Schmidli [61] among others.

In the above articles, the general risk renewal models of continuous time were considered. In such a case, the defective renewal equation is the main tool to obtain a suitable information about the exact values or the asymptotic behaviour of the Gerber-Shiu function. If we consider the discrete time risk model, then the recursive relations between values of the Gerber-Shiu function play role of the defective renewal equation. Various properties of the Gerber-Shiu function in the discrete time risk models were considered by Bao and Liu [4], Cheng et al. [11], Li and Wu [35], Li [36], Li and Garrido [37], Li et al. [39], Liu et al. [45], Liu and Guo [46], Marceau [50], Pavlova and Willmot [56]. For instance, in Li and Garrido [37], it is shown that values of function $\Psi_{\delta,w}$ of the homogeneous discrete time risk model can be calculated using the formulas provided in Chapter 2.

In this chapter, we consider the behaviour of the special case of Gerber-Shiu penalty function for the bi-seasonal discrete time risk model which is a particular case of nonhomogeneous discrete time risk models. We observe that the definitions of T_u and $\Psi_{\delta,w}(u)$ are the same for both classical and bi-seasonal discrete time risk models. However, the methods of $\Psi_{\delta,w}(u)$ calculation in the bi-seasonal discrete time risk model are much more complicated, since the claims are no longer identically distributed. For the continuous risk renewal models T_u and $\Psi_{\delta,w}(u)$ are defined similarly as in discrete models. We do not provide exact definitions here in the continuous case, since we consider only discrete time risk models.

Our results supplement the results of Castañer et al. [10], Răducan et al. [57] and Răducan et al. [58]. We derive the specific recursive equality for function ψ_δ. Using the derived formula we construct an algorithm to calculate approximate values of this function. The running of the algorithm is illustrated by several examples. The ideas from Bieliauskienė and Šiaulys [6], Damarackas and Šiaulys [19], De Vylder and Goovaerts [20], Dickson and Waters [25] were used to get the main results of this chapter.

We consider the bi-seasonal discrete time risk model generated by two nonnegative, independent and integer valued random variables X and Y. By

$$x_k = \mathbb{P}(X = k),\ y_k = \mathbb{P}(Y = k),\ q_k = \mathbb{P}(Q = k),\ k \in \mathbb{N}_0,$$

we denote the local probabilities of random variables X, Y and $Q = X + Y$ respectively. Distribution functions of these random variables we denote by F_X, F_Y and F_Q, i.e.

$$\begin{aligned} F_X(u) &= \mathbb{P}(X \leqslant u) = \sum_{k=0}^{\lfloor u \rfloor} x_k, \\ F_Y(u) &= \mathbb{P}(Y \leqslant u) = \sum_{k=0}^{\lfloor u \rfloor} y_k, \\ F_Q(u) &= \mathbb{P}(Q \leqslant u) = \sum_{k=0}^{\lfloor u \rfloor} q_k, \end{aligned}$$

for each real u. The notation $\overline{F}$ is used for the tail of an arbitrary distribution function F, i.e. $\overline{F}(u) = 1 - F(u)$ for each $u \in \mathbb{R}$.

The following two assertions enable us to construct an algorithm for calculating values of function $\psi_\delta(u)$ in the bi-seasonal discrete time risk model.

Theorem 3.1.1. *Let the bi-seasonal discrete time risk model be generated by two nonnegative, independent and integer valued random variables X and Y. If $\mathbb{E}X + \mathbb{E}Y < 2$, then $\lim_{u\to\infty} \psi_\delta(u) = 0$ for an arbitrary fixed $\delta \geqslant 0$. In addition, if $\max\{\mathbb{E}e^{hX}, \mathbb{E}e^{hY}\} < \infty$ for some positive h, then $\sum_{l=0}^{\infty} \psi_\delta(l) < \infty$ for each fixed $\delta \geqslant 0$.*

Theorem 3.1.2. *Let all the conditions of the Theorem 3.1.1 be satisfied. Furthermore, let $\delta > 0$, and ψ_δ denote the Gerber-Shiu function with $w(x, y) = 1$ for all nonnegative x and y. Also denote $\mathcal{S}_\delta := \sum_{l=0}^{\infty} \psi_\delta(l)$.*

- *If $q_0 = \mathbb{P}(X + Y = 0) > 0$, then*

$$\psi_\delta(n) = a_n \psi_\delta(0) + b_n \mathcal{S}_\delta + d_n \tag{3.1.1}$$

for each $n \in \mathbb{N}_0$, where a_n, b_n, d_n are three sequences of real numbers defined recursively by the following equations:

$$a_0 = 1,\ a_1 = -\frac{1}{y_0},\ a_n = \frac{1}{q_0}\Big(\mathrm{e}^{2\delta} a_{n-2} - \sum_{i=1}^{n-1} q_i a_{n-i} - x_{n-1}\Big), n \in \{2, 3, \ldots\};$$

$$b_0 = 0,\ b_1 = -\frac{\mathrm{e}^{2\delta} - 1}{y_0},$$

$$b_n = \frac{1}{q_0}\Big(\mathrm{e}^{2\delta} b_{n-2} - \sum_{i=1}^{n-1} q_i b_{n-i} - x_{n-1}(\mathrm{e}^{2\delta} - 1)\Big), n \in \{2, 3, \ldots\};$$

$$d_0 = 0,\ d_1 = \frac{\mathrm{e}^{\delta}\mathbb{E}X + y_0 + \mathbb{E}Y - 1}{y_0},$$

$$d_n = \frac{1}{q_0}\Big(\mathrm{e}^{2\delta} d_{n-2} - \sum_{i=1}^{n-1} q_i d_{n-i} + x_{n-1} y_0 d_1 - \mathrm{e}^{\delta}\overline{F}_X(n-2)$$
$$- \sum_{i=0}^{n-2} x_i \overline{F}_Y(n-1-i)\Big), n \in \{2, 3, \ldots\}.$$

- *If $x_0 = \mathbb{P}(X = 0) = 0$ and $y_0 = \mathbb{P}(Y = 0) \neq 0$, then*

$$\psi_\delta(n) = \tilde{a}_n \psi_\delta(0) + \tilde{b}_n \mathcal{S}_\delta + \tilde{d}_n \tag{3.1.2}$$

for each $n \in \mathbb{N}_0$, where $\tilde{a}_n$, $\tilde{b}_n$, $\tilde{d}_n$ are three sequences of real numbers defined *recursively by the following equations:*

$$\tilde{a}_0 = 1,\ \tilde{a}_1 = -\frac{1}{y_0},\ \tilde{a}_n = \frac{1}{q_1}\Big(e^{2\delta}\tilde{a}_{n-1} - \sum_{i=1}^{n-1} q_{i+1}\tilde{a}_{n-i} - x_n\Big), n \in \{2,3,\ldots\};$$

$$\tilde{b}_0 = 0,\ \tilde{b}_1 = -\frac{e^{2\delta}-1}{y_0},$$

$$\tilde{b}_n = \frac{1}{q_1}\Big(e^{2\delta}\tilde{b}_{n-1} - \sum_{i=1}^{n-1} q_{i+1}\tilde{b}_{n-i} - x_n(e^{2\delta}-1)\Big), n \in \{2,3,\ldots\};$$

$$\tilde{d}_0 = 0,\ \tilde{d}_1 = \frac{e^{\delta}\mathbb{E}X + y_0 + \mathbb{E}Y - 1}{y_0},$$

$$\tilde{d}_n = \frac{1}{q_1}\Big(e^{2\delta}\tilde{d}_{n-1} - \sum_{i=1}^{n-1} q_{i+1}\tilde{d}_{n-i} + x_n y_0 \tilde{d}_1 - e^{\delta}\overline{F}_X(n-1) - \sum_{i=0}^{n-2} x_{i+1}\overline{F}_Y(n-1-i)\Big), n \in \{2,3,\ldots\}.$$

- *If* $x_0 \neq 0$ *and* $y_0 = 0$, *then*

$$\psi_\delta(n) = \hat{b}_n\mathcal{S}_\delta + \hat{d}_n \tag{3.1.3}$$

for each $n \in \mathbb{N}_0$, *where* $\hat{b}_n$, $\hat{d}_n$ *are two sequences of real numbers defined recursively by the following equations:*

$$\hat{b}_0 = -(e^{2\delta}-1),\ \hat{b}_n = \frac{1}{q_1}\Big(e^{2\delta}\hat{b}_{n-1} - \sum_{i=1}^{n-1} q_{i+1}\hat{b}_{n-i}\Big), n \in \mathbb{N};$$

$$\hat{d}_0 = e^{\delta}\mathbb{E}X + \mathbb{E}Y - 1,$$

$$\hat{d}_n = \frac{1}{q_1}\Big(e^{2\delta}\hat{d}_{n-1} - \sum_{i=1}^{n-1} q_{i+1}\hat{d}_{n-i} - e^{\delta}\overline{F}_X(n-1) - \sum_{i=0}^{n-1} x_i\overline{F}_Y(n-i)\Big),$$
$$n \in \mathbb{N}.$$

Remark 3.1.1. *We observe that case* $x_0 = y_0 = 0$ *is impossible due to the requirement* $\mathbb{E}(X+Y) < 2$. *This observations shows that all possible cases of the discrete r.v.s* X *and* Y *are considered in the Theorem 3.1.2.*

3.2 Proofs of the main results

Proof of Theorem 3.1.1. To prove the first proposition of the Theorem, the Lemma below is used.

Lemma 3.2.1. *(see Theorem 2.3 of Damarackas and Šiaulys [19])*

Let the bi-seasonal discrete time risk model be generated by two non-negative, independent and integer valued random variables X and Y. If $\mathbb{E}X + \mathbb{E}Y < 2$, then $\lim_{u\to\infty} \psi(u) = 0$.

The statement of the Lemma 3.2.1 implies that $\lim_{u\to\infty} \psi_\delta(u) = 0$ for an arbitrary fixed $\delta \geqslant 0$, because $0 \leqslant \psi_\delta(u) \leqslant \psi(u)$ for all $\delta, u \geqslant 0$.

To show that Gerber-Shiu function series converges, we will prove that this function has an exponential upper bound under conditions of this Theorem. To prove this, the Lemma below will be used.

Lemma 3.2.2. *(see Lemma 1 by Andrulytė et al. [2])*

Let $\eta_1, \eta_2, \ldots$ be independent r.v.s such that

$$\begin{aligned}
&\sup_{i\in\mathbb{N}} \mathbb{E}\big(\mathrm{e}^{h\eta_i}\big) < \infty \; for\ some\ h > 0, \\
&\lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| 1\!\!1_{\{\eta_i \leqslant -u\}}\big) = 0, \\
&\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\eta_i < 0.
\end{aligned}$$

Then, there exist positive constants c_1 and c_2 such that

$$\mathbb{P}\left(\sup_{k\geqslant 1} \sum_{i=1}^{k} \eta_i > u\right) \leqslant c_1 \mathrm{e}^{-c_2 u},\ u \geqslant 0.$$

Let us denote $\eta_i = Z_i - 1$ for $i \in \mathbb{N}$. The conditions of the Theorem imply that all the requirements of the Lemma 3.2.2 are satisfied. Firstly,

$$\sup_{i\in\mathbb{N}} \mathbb{E}\big(\mathrm{e}^{h\eta_i}\big) = \max\Big\{\mathbb{E}\big(\mathrm{e}^{h(X-1)}\big), \mathbb{E}\big(\mathrm{e}^{h(Y-1)}\big)\Big\} < \infty.$$

Secondly,

$$\begin{aligned}
&\lim_{u\to\infty} \sup_{i\in\mathbb{N}} \mathbb{E}\big(|\eta_i| 1\!\!1_{\{\eta_i \leqslant -u\}}\big) \\
= \quad &\lim_{u\to\infty} \max\Big\{\mathbb{E}\big((1-X) 1\!\!1_{\{X \leqslant 1-u\}}\big), \mathbb{E}\big((1-Y) 1\!\!1_{\{Y \leqslant 1-u\}}\big)\Big\} = 0,
\end{aligned}$$

because for $u > 1$ events $\{X \leqslant 1-u\}$ and $\{Y \leqslant 1-u\}$ have zero probability.

Lastly, we can show that for $n = 2k,\ k \in \mathbb{N}$,

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\eta_i = \frac{n\mathbb{E}X + n\mathbb{E}Y - 2n}{2n}.$$

For $n = 2k+1,\ k \in \mathbb{N}_0$, we have that

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\eta_i = \frac{(n+1)\mathbb{E}X + n\mathbb{E}Y - (2n+1)}{2n+1}.$$

Therefore, it follows that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\eta_i = \frac{\mathbb{E}X + \mathbb{E}Y - 2}{2} < 0,$$

because $\mathbb{E}X + \mathbb{E}Y < 2$ by the conditions of the Theorem.

Hence, according to the Lemma 3.2.2, we have

$$\psi_\delta(u) \leqslant \psi(u) = \mathbb{P}\left(\sup_{k\geqslant 1}\sum_{i=1}^{k} \eta_i > u\right) \leqslant c_1 \mathrm{e}^{-c_2 u},\ u \geqslant 0,$$

for some positive constants c_1, c_2.

Therefore, it follows immediately that

$$\sum_{l=0}^{\infty} \psi_\delta(l) \leqslant c_1 \sum_{l=0}^{\infty} \mathrm{e}^{-c_2 l} = c_1 \sum_{l=0}^{\infty} \left(\frac{1}{\mathrm{e}^{c_2}}\right)^l = c_1 \frac{1}{1-\mathrm{e}^{-c_2}} < \infty$$

for each fixed $\delta \geqslant 0$. □

Proof of Theorem 3.1.2. Suppose that $\delta > 0$ and $u \in \mathbb{N}_0$. According to the definition of function ψ_δ we have

$$\begin{aligned}
\psi_\delta(u) &= \sum_{m=1}^{\infty} \mathbb{E}\big(\mathrm{e}^{-\delta m} \mathbb{1}_{\{T_u = m\}}\big) \\
&= \sum_{m=1}^{\infty} \mathrm{e}^{-\delta m}\, \mathbb{P}\Bigg(\sum_{i=1}^{j} Z_i < j + u \text{ for } j \in \{1, 2, \ldots, m-1\} \\
&\qquad \text{and} \sum_{i=1}^{m} Z_i \geqslant m + u\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-\delta}\,\mathbb{P}(Z_1 \geqslant 1+u) + \mathrm{e}^{-2\delta}\,\mathbb{P}(Z_1 < 1+u, Z_1+Z_2 \geqslant 2+u) \\
&+ \sum_{m=3}^{\infty} \mathrm{e}^{-\delta m}\,\mathbb{P}\Big(\sum_{i=1}^{j} Z_i < j+u \text{ for } j \in \{1,2,\ldots,m-1\} \\
&\qquad \text{and } \sum_{i=1}^{m} Z_i \geqslant m+u\Big).
\end{aligned}
$$

Since $X \overset{d}{=} Z_1 \overset{d}{=} Z_3 \overset{d}{=} Z_5 \overset{d}{=} \ldots$ and $Y \overset{d}{=} Z_2 \overset{d}{=} Z_4 \overset{d}{=} Z_6 \overset{d}{=} \ldots$ we get that

$$
\begin{aligned}
\psi_\delta(u) &= \mathrm{e}^{-\delta} \sum_{l \geqslant 1+u} x_l + \mathrm{e}^{-2\delta} \sum_{l \leqslant u} \sum_{k \geqslant 2+u-l} x_l y_k \\
&+ \sum_{m=3}^{\infty} \mathrm{e}^{-\delta m}\,\mathbb{P}\Big(Z_1 \leqslant u, Z_1+Z_2 \leqslant 1+u, Z_1+Z_2+\sum_{i=3}^{j} Z_i < j+u \\
&\qquad \text{for } j \in \{3,\ldots,m-1\} \text{ and } Z_1+Z_2+\sum_{i=3}^{m} Z_i \geqslant m+u\Big) \\
&= \mathrm{e}^{-\delta}\overline{F}_X(u) + \mathrm{e}^{-2\delta} \sum_{l=0}^{u} x_l \overline{F}_Y(1+u-l) \\
&+ \sum_{l=0}^{u} \sum_{k=0}^{1+u-l} x_l y_k \sum_{m=3}^{\infty} \mathrm{e}^{-\delta m}\,\mathbb{P}\Big(\sum_{i=3}^{j} Z_i < j+u-l-k \\
&\qquad \text{for } j \in \{3,\ldots,m-1\} \text{ and } \sum_{i=3}^{m} Z_i \geqslant m+u-k-l\Big) \\
&= \mathrm{e}^{-\delta}\overline{F}_X(u) + \mathrm{e}^{-2\delta} \sum_{l=0}^{u} x_l \overline{F}_Y(1+u-l) \\
&+ \mathrm{e}^{-2\delta} \sum_{l=0}^{u} \sum_{k=0}^{1+u-l} x_l y_k \sum_{m=3}^{\infty} \mathrm{e}^{-\delta(m-2)}\,\mathbb{P}\Big(\sum_{i=1}^{j} Z_i < j+u-l-k \\
&\qquad \text{for } j \in \{1,\ldots,m-3\} \text{ and } \sum_{i=1}^{m-2} Z_i \geqslant m+u-k-l\Big) \\
&= \mathrm{e}^{-\delta}\overline{F}_X(u) + \mathrm{e}^{-2\delta} \sum_{l=0}^{u} x_l \overline{F}_Y(1+u-l) \\
&+ \mathrm{e}^{-2\delta} \sum_{l=0}^{u} \sum_{k=0}^{1+u-l} x_l y_k\, \psi_\delta(u+2-k-l). \qquad (3.2.1)
\end{aligned}
$$

For each $m \in \mathbb{N}_0$

$$
q_m = \mathbb{P}(Q=m) = \sum_{k=0}^{m} x_k y_{m-k}.
$$

Therefore the last sum in the equality (3.2.1) can be expressed by

$$\begin{aligned}
&\sum_{l=0}^{1+u}\sum_{k=0}^{1+u-l} x_l y_k \,\psi_\delta(u+2-(k+l)) - x_{u+1}y_0\psi_\delta(1) \\
=& \sum_{l=0}^{1+u} q_l\psi_\delta(u+2-l) - x_{u+1}y_0\psi_\delta(1) \\
=& \sum_{l=0}^{u} q_l\psi_\delta(u+2-l) + (q_{u+1}-x_{u+1}y_0)\psi_\delta(1).
\end{aligned}$$

Substituting this expression into the equality (3.2.1) we obtain that

$$\begin{aligned}
\psi_\delta(u) &= \mathrm{e}^{-\delta}\overline{F}_X(u) + \mathrm{e}^{-2\delta}\sum_{l=0}^{u} x_l\overline{F}_Y(1+u-l) \\
&+ \mathrm{e}^{-2\delta}\left(\sum_{l=0}^{u} q_l\psi_\delta(u+2-l) + (q_{u+1}-x_{u+1}y_0)\psi_\delta(1)\right)
\end{aligned} \tag{3.2.2}$$

for each $u \in \mathbb{N}_0$.

By summing these last equalities from $u = 0$ to $u = N \in \mathbb{N}$ we get that

$$\begin{aligned}
\sum_{u=0}^{N}\psi_\delta(u) &= \mathrm{e}^{-\delta}\sum_{u=0}^{N}\overline{F}_X(u) + \mathrm{e}^{-2\delta}\sum_{u=0}^{N}\sum_{l=0}^{u} x_l\overline{F}_Y(1+u-l) \\
&+ \mathrm{e}^{-2\delta}\left(\sum_{u=0}^{N}\sum_{l=0}^{u} q_l\psi_\delta(u+2-l)\right. \\
&+ \left.\psi_\delta(1)\sum_{u=0}^{N}(q_{u+1}-x_{u+1}y_0)\right)
\end{aligned} \tag{3.2.3}$$

We observe that

$$\sum_{u=0}^{N}\sum_{l=0}^{u} x_l\overline{F}_Y(1+u-l) = \sum_{u=1}^{N+1}\overline{F}_Y(u)F_X(N+1-u)$$

and, similarly,

$$\sum_{u=0}^{N}\sum_{l=0}^{u} q_l\psi_\delta(u+2-l) = \sum_{u=2}^{N+2}\psi_\delta(u)F_Q(N+2-u).$$

Hence, it follows from the equality (3.2.3) that

$$\sum_{u=0}^{N+2}\psi_\delta(u)\big(1-\mathrm{e}^{-2\delta}F_Q(N+2-u)\big) = \mathrm{e}^{-\delta}\sum_{u=0}^{N}\overline{F}_X(u)$$
$$+\,\mathrm{e}^{-2\delta}\sum_{u=1}^{N+1}\overline{F}_Y(u)F_X(N+1-u)$$
$$+\,\psi_\delta(N+1)+\psi_\delta(N+2)$$
$$+\,\mathrm{e}^{-2\delta}\psi_\delta(1)\sum_{u=0}^{N}(q_{u+1}-x_{u+1}y_0)$$
$$-\,\mathrm{e}^{-2\delta}\big(\psi_\delta(0)F_Q(N+2)+\psi_\delta(1)F_Q(N+1)\big) \quad (3.2.4)$$

for each $N\in\mathbb{N}$.

Now we are in a position to let $N\to\infty$. It is obvious that:

$$\lim_{N\to\infty}\sum_{u=0}^{N}\overline{F}_X(u)=\mathbb{E}X,\quad \lim_{N\to\infty}F_Q(N+1)=\lim_{N\to\infty}F_Q(N+2)=1, \quad (3.2.5)$$

$$\lim_{N\to\infty}\sum_{u=0}^{N}q_{u+1}=1-q_0,\quad \lim_{N\to\infty}\sum_{u=0}^{N+1}x_{u+1}=1-x_0. \quad (3.2.6)$$

The Theorem 3.1.1 implies that

$$\lim_{N\to\infty}\psi_\delta(N+1)=\lim_{N\to\infty}\psi_\delta(N+2)=0. \quad (3.2.7)$$

Consider the second term in the right side of the equality (3.2.4). Obviously

$$\lim_{N\to\infty}\sum_{u=1}^{N+1}\overline{F}_Y(u)F_X(N+1-u)\leqslant\sum_{u=1}^{\infty}\overline{F}_Y(u).$$

On the other hand, for an arbitrary $M\in\mathbb{N}$

$$\lim_{N\to\infty}\sum_{u=1}^{N+1}\overline{F}_Y(u)F_X(N+1-u) \;\geqslant\; \lim_{N\to\infty}F_X(N+1-M)\sum_{u=1}^{M}\overline{F}_Y(u)$$
$$=\;\sum_{u=1}^{M}\overline{F}_Y(u). \quad (3.2.8)$$

Consequently,

$$\begin{aligned}\lim_{N\to\infty}\sum_{u=1}^{N+1}\overline{F}_Y(u)F_X(N+1-u) &= \sum_{u=1}^{\infty}\overline{F}_Y(u)\\ &= y_2+2y_3+3y_4+\dots\\ &= y_0+\mathbb{E}Y-1. \qquad (3.2.9)\end{aligned}$$

Only the left side of equality (3.2.4) is left for consideration. Due to the Theorem 3.1.1

$$\lim_{N\to\infty}\sum_{u=0}^{N+2}\psi_\delta(u)=\mathcal{S}_\delta<\infty.$$

In addition,

$$\lim_{N\to\infty}\sum_{u=0}^{N+2}\psi_\delta(u)F_Q(N+2-u)\leqslant\sum_{u=0}^{\infty}\psi_\delta(u)=\mathcal{S}_\delta,$$

and, for an arbitrary chosen $M\in\mathbb{N}$,

$$\lim_{N\to\infty}\sum_{u=0}^{N+2}\psi_\delta(u)F_Q(N+2-u)\geqslant\lim_{N\to\infty}F_Q(N+2-M)\sum_{u=0}^{M}\psi_\delta(u).$$

Hence,

$$\lim_{N\to\infty}\sum_{u=0}^{N+2}\psi_\delta(u)\big(1-\mathrm{e}^{-2\delta}F_Q(N+2-u)\big)=\big(1-\mathrm{e}^{-2\delta}\big)\mathcal{S}_\delta. \qquad (3.2.10)$$

Substituting all limiting relations (3.2.5)–(3.2.10) into the equality (3.2.4) we get

$$\big(1-\mathrm{e}^{-2\delta}\big)\mathcal{S}_\delta=\mathrm{e}^{-\delta}\mathbb{E}X+\mathrm{e}^{-2\delta}(y_0+\mathbb{E}Y-1)-\mathrm{e}^{-2\delta}\psi_\delta(1)y_0-\mathrm{e}^{-2\delta}\psi_\delta(0). \qquad (3.2.11)$$

From this point we consider the three cases described in the formulation of Theorem separately.

(I) If $q_0>0$ then the equality (3.2.11) implies that

$$\psi_\delta(1)=a_1\psi_\delta(0)+b_1\mathcal{S}_\delta+d_1$$

where a_1, b_1 and d_1 are as defined in the formulation of the Theorem. So, we have that the main equality (3.1.1) holds if $n\in\{0,1\}$.

Now we need to prove this equality for all $n\in\mathbb{N}$. For this we use an induction. Suppose that the equality (3.1.1) holds for all $n \in \{0, 1, \ldots, K\}$ for the defined sequences a_n, b_n and d_n.

The induction hypothesis and equality (3.2.2) with $u = K - 1$ imply that

$$\begin{aligned} \mathrm{e}^{2\delta}\psi_\delta(K-1) &= \mathrm{e}^{2\delta}\big(a_{K-1}\,\psi_\delta(0) + b_{K-1}\,\mathcal{S}_\delta + d_{K-1}\big) \\ &= \mathrm{e}^{\delta}\overline{F}_X(K-1) + \sum_{l=0}^{K-1} x_l \overline{F}_Y(K-l) + q_0\,\psi_\delta(K+1) \\ &+ \sum_{l=1}^{K-1} q_l \big(a_{K+1-l}\,\psi_\delta(0) + b_{K+1-l}\,\mathcal{S}_\delta + d_{K+1-l}\big) \\ &+ (q_K - x_K y_0)\big(a_1\psi_\delta(0) + b_1\mathcal{S}_\delta + d_1\big). \end{aligned}$$

Therefore,

$$\begin{aligned} q_0\,\psi_\delta(K+1) &= \psi_\delta(0)\big(\mathrm{e}^{2\delta}a_{K-1} - \sum_{l=1}^{K} q_l a_{K+1-l} + x_K y_0 a_1\big) \\ &+ \mathcal{S}_\delta\big(\mathrm{e}^{2\delta}b_{K-1} - \sum_{l=1}^{K} q_l b_{K+1-l} + x_K y_0 b_1\big) \\ &+ \Bigg(\mathrm{e}^{2\delta}d_{K-1} - \sum_{l=1}^{K} q_l d_{K+1-l} + x_K y_0 d_1 \\ &- \mathrm{e}^{\delta}\overline{F}_X(K-1) - \sum_{l=0}^{K-1} x_l \overline{F}_Y(K-l)\Bigg), \end{aligned}$$

or

$$\psi_\delta(K+1) = a_{K+1}\psi_\delta(0) + b_{K+1}\mathcal{S}_\delta + d_{K+1}$$

due to the definition of sequences a_n, b_n and d_n.

The induction principle implies that the equality (3.1.1) holds for all $n \in \mathbb{N}_0$. The first part of the Theorem 3.1.2 is proved.

(II) If $x_0 = 0$, $y_0 \neq 0$, then the equality (3.2.11) implies that

$$\psi_\delta(1) = \tilde{a}_1\psi_\delta(0) + \tilde{b}_1\mathcal{S}_\delta + \tilde{d}_1$$

where $\tilde{a}_1$, $\tilde{b}_1$ and $\tilde{d}_1$ are as defined in the formulation of the Theorem. So, we have that the main equality (3.1.2) holds if $n \in \{0, 1\}$. Similarly as in the case (I), we finish the proof using the induction method and equality (3.2.2).

(III) If $x_0 \neq 0$, $y_0 = 0$, then equality (3.2.11) implies that

$$\psi_\delta(0) = \hat{b}_0 \mathcal{S}_\delta + \hat{d}_0$$

where $\hat{b}_0$ and $\hat{d}_0$ are as defined in the formulation of the Theorem. So, we have that the main equality (3.1.3) holds if $n = 0$. Similarly as in the case (I), we finish the proof using the induction method and equality (3.2.2).

Now Theorem 3.1.2 is proved. □

3.3 Algorithm for finding the values of function ψ_δ

In this section, we describe an algorithm for calculating values of $\psi_\delta(u)$ in the case of the bi-seasonal risk model. The algorithm was implemented with R language, using increased numerical precision package *Rmpfr*. Our algorithm is based on the formula (3.1.1) from the Theorem 3.1.2 and the results of the Theorem 3.1.1. As usual, it is assumed that we have a positive force of interest δ, and the bi-seasonal discrete time risk model is generated by two nonnegative, integer-valued and differently distributed r.v.s X, Y with local probabilities $x_k = \mathbb{P}(X = k)$, $y_k = \mathbb{P}(Y = k)$, $k \in \mathbb{N}_0$. Of course, these two r.v.s should satisfy all the requirements of the Theorem 3.1.2. Below we present the detailed, step by step algorithm for calculating $\psi_\delta(u)$, $u \in \mathbb{N}_0$ in the case when $x_0 y_0 > 0$. The other possible cases:$\{x_0 = 0, y_0 > 0\}$, and $\{x_0 > 0, y_0 = 0\}$, which were described in the Theorem 3.1.2, can be considered similarly.

Step 1: Select $N \in \{10, 20, 30, ..., 100\}$ and $K \in \{1, ..., 5\}$.

Step 2: Calculate coefficients a_n, b_n, d_n for all $n \in \{0, 1, ..., N\}$ using the formulas from the Theorem 3.1.2.

Step 3: Find $\hat{\psi}_\delta(0)$ and $\hat{\mathcal{S}}_\delta$ satisfying the following system of linear equations

$$\begin{cases} a_{N-K}\hat{\psi}_\delta(0) + b_{N-K}\hat{\mathcal{S}}_\delta + d_{N-K} = 0, \\ a_N \hat{\psi}_\delta(0) + b_N \hat{\mathcal{S}}_\delta + d_N = 0. \end{cases} \tag{3.3.1}$$

- *Due to the main formula* (3.1.1) *of the Theorem 3.1.2 the desired* *quantity* $\psi_\delta(0)$ *together with sum* $\mathcal{S}_\delta$ *satisfy the following system*

$$\begin{cases} a_{N-K}\psi_\delta(0) + b_{N-K}\mathcal{S}_\delta + d_{N-K} = \psi_\delta(N-K), \\ a_N\psi_\delta(0) + b_N\mathcal{S}_\delta + d_N = \psi_\delta(N). \end{cases} \tag{3.3.2}$$

- *However, according to the Theorem 3.1.1* $\psi_\delta(N-K)$ *and* $\psi_\delta(N)$ *are close to the zero for sufficiently large* N*. We get system* (3.3.1) *from* (3.3.2) *by changing values of* $\psi_\delta(N-K)$ *and* $\psi_\delta(N)$ *to zeroes.*

Step 4: Test the error $|\psi_\delta(0) - \hat{\psi}_\delta(0)|$.

- *Using the Cramer's rule for both systems of linear equations* (3.3.1)*,* (3.3.2) *and the trivial estimate* $|\psi_\delta(n)| \leqslant 1$*,* $n \in \mathbb{N}_0$*, we derive that*

$$|\psi_\delta(0) - \hat{\psi}_\delta(0)| \leqslant \frac{e^{-\delta}\big(|b_{N-K}| + |b_N|\big)}{|a_{N-K}b_N - b_{N-K}a_N|}.$$

- *Numerical simulations have showed that the upper estimate of* $\psi_\delta(0)$ *approximation error tends to 0 as* N *grows. This is consistent with the behaviour of the approximation error itself. As for the parameter* K*, its choice does not have the clear effect on the upper estimate of* $\psi_\delta(0)$ *approximation error.*

Step 5: If the size of error in the Step 4 is suitable, then pass to the Step 6. If the size of error is not suitable, then return to the Step 1 choosing different parameters N and K.

- *We remark only that the sets provided in the Step 1 for choosing these parameters are not strictly defined, and different sets can be used successfully. However, choosing* N *much larger than 100 would result in very large coefficients* a_N*,* b_N *and* d_N*, and owing to that some computational difficulties may arise. Besides that, in this case computational speed would be reduced. And conversely, choosing* N *too small would result in a big approximation error of* $\psi_\delta(0)$ *when changing the system* (3.3.2) *to the* (3.3.1)*, since* $\psi_\delta(N)$ *does not converge to zero so quickly. As for the parameter* K*, it should be chosen to minimize the upper estimate of* $\psi_\delta(0)$ *approximation error.*

Step 6: Calculate $\psi_\delta(1)$ according to the formula (3.1.1) by supposing that $\psi_\delta(0) = \hat{\psi}_\delta(0)$ and $\mathcal{S}_\delta = \hat{\mathcal{S}}_\delta$.

Step 7: Calculate values of $\psi_\delta(u)$ for $u \geqslant 2$ while the algorithm works correctly, applying either formula (3.1.1) from Theorem 3.1.2 or the main recursive formula (3.2.2) from the proof of Theorem 3.1.2.

- *By saying that the algorithm works correctly, we mean that its results do not conflict with mathematical properties. Namely, $\psi_\delta(u)$ is a function taking values between 0 and 1, nonincreasing with respect to u and decreasing with respect to δ. However, sometimes algorithm produces results that are not compatible with these properties. This could happen due to the following reasons:*
  (i) In some particular cases of X, Y and δ, coefficients a_n, b_n, d_n, $n \in \mathbb{N}_0$ in the main equality of the Theorem 3.1.2 are rapidly growing and fluctuating. Consequently, it is quite difficult to get precise values of these coefficients.
  (ii) Also, computational errors could arise because by using the formula (3.1.1) *from the Theorem 3.1.2, we are calculating a "small" quantity $\psi_\delta(u)$ as a sum containing "large" in absolute value summands.*

Remark 3.3.1. Many ideas for constructing a recursive algorithm were taken from Damarackas and Šiaulys [19]. In this chapter the infinite time ruin probability, which is a special case of Gerber-Shiu function with $\delta = 0$ and $w(x, y) = 1$, was considered. In this chapter we have extended the results to the case $\delta > 0$.

Remark 3.3.2. In Bieliauskienė and Šiaulys [6], analogous problem to ours is considered. While we analyze a less general model than the one provided in this paper, there are some advantages in our algorithm. Namely, our approach of finding $\psi_\delta(0)$ is more efficient. The formula provided in the Theorem 3 of [6] is applicable to all numerical examples of the Section 3.4 except the last one, which deals with random variables having infinite support. But the problem with this formula is its combinatorial form, and even for relatively simple distributions it is not easy to implement. The computational speed is also reduced for the same reason. Furthermore, our proposed algorithm is less prone to computational errors, because we do not use multiple way recursion.

3.4 Numerical examples

In this section, we present four numerical examples for calculating the values of $\psi_\delta(u)$, $u \in \mathbb{N}_0$, in the bi-seasonal discrete time risk model. In all examples we consider function ψ_δ with three different values of the interest force $\delta \in \{0; 0.01; 0.1\}$. Our algorithm does not allow to compute function values for case $\delta = 0$, so the algorithm and its $\psi_\delta(0)$ approximation error upper estimate provided in Damarackas and Šiaulys [19] were used for this case. Since the function $\psi_\delta(u)$ seems to decay exponentially, all the figures are plotted in log scale (with base 10).

Example 3.4.1. *Let us assume that the bi-seasonal discrete time risk model is generated by the following independent random claim amounts X and Y*

X	0	1	2
$\mathbb{P}$	0.6	0.2	0.2

;

Y	0	1	2	3
$\mathbb{P}$	0.5	0.2	0.2	0.1

.

In this example, both claim amounts are ”good” because $\max\{\mathbb{E}X, \mathbb{E}Y\} < 1$, and all conditions of the Theorem 3.1.2 are satisfied. Using the algorithm presented in the Section 3.3 we obtain values of $\psi_\delta(u)$ for $u \in \{0, 1, ..., 15\}$. These values are presented in the Table 3.1 and values of $\log \psi_\delta(u)$ are shown in the Figure 3.1. The upper estimate of $\psi_\delta(0)$ approximation error, described in the Step 4 of algorithm, is provided in the parenthesis near value of δ. Results of this example are based on the value of $\psi_\delta(0)$ which is obtained with $N = 50$ and $K = 2$.

Table 3.1: Values of $\psi_\delta(u)$ in Example 3.4.1

u	$\delta = 0$ (< 0.000000001)	$\delta = 0.01$ (0.083494161)	$\delta = 0.1$ (0.000001786)
0	0.735808540	0.715289725	0.588111815
1	0.528382921	0.505099453	0.379732449
2	0.308008652	0.283691781	0.168950439
3	0.186932507	0.166883336	0.082819297
4	0.109425467	0.094115383	0.036822099
5	0.064774209	0.053789118	0.016949434
6	0.038352631	0.030752904	0.007818717
7	0.022665488	0.017539770	0.003572849
8	0.013406572	0.010015276	0.001640920
9	0.007928948	0.005717783	0.000753055
10	0.004688946	0.003263965	0.000345342
11	0.002773172	0.001863371	0.000158466
12	0.001639884	0.001063758	0.000072701
13	0.000970174	0.000607275	0.000033353
14	0.000573054	0.000346681	0.000015302
15	0.000340345	0.000197913	0.000007020

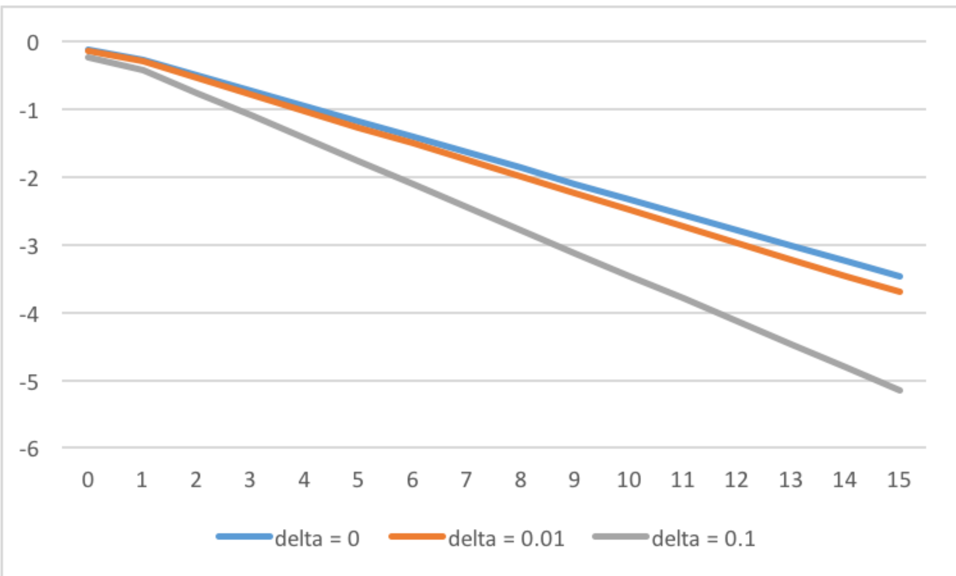

Figure 3.1: Values of $\log \psi_\delta(u)$ in Example 3.4.1

Example 3.4.2. *Suppose now that the bi-seasonal discrete time risk model is generated by r.v.s X and Y having the following distributions*

X	0	1	Y	0	1	2
$\mathbb{P}$	0.4	0.6	$\mathbb{P}$	0.1	0.6	0.3

We observe that $\mathbb{E}X < 1$, $\mathbb{E}Y \geqslant 1$, but $\mathbb{E}X + \mathbb{E}Y < 2$ in this case. Consequently, the model is "good" only on average and all conditions of the Theorem 3.1.2 are satisfied. Using the algorithm from the Section 3.3, the Table 3.2 is filled out with values of $\psi_\delta(u)$ for $u \in \{0, 1, ..., 15\}$. Results of this example are based on the value of $\psi_\delta(0)$ which is obtained with $N = 40$ and $K = 1$. The values of $\log \psi_\delta(u)$ are also shown in Figure 3.2.

Table 3.2: Values of $\psi_\delta(u)$ in Examples 3.4.2

u	$\delta = 0$ (< 0.000000001)	$\delta = 0.01$ (0.006459348)	$\delta = 0.1$ (< 0.000000001)
0	0.850000000	0.826902130	0.697524567
1	0.500000000	0.455345718	0.274354439
2	0.250000000	0.207339723	0.075270358
3	0.125000000	0.094411255	0.020650757
4	0.062500010	0.042989761	0.005665627
5	0.031250000	0.019575203	0.001554390
6	0.015625000	0.008913485	0.000426454
7	0.007812502	0.004058717	0.000116999
8	0.003906251	0.001848120	0.000032099
9	0.001953125	0.000841533	0.000008807
10	0.000976563	0.000383189	0.000002416
11	0.000488281	0.000174483	0.000000663
12	0.000244141	0.000079450	0.000000182
13	0.000122070	0.000036177	0.000000050
14	0.000061035	0.000016473	0.000000014
15	0.000030518	0.000007501	0.000000004

Example 3.4.3. *Let us consider the mirror reflection of the bi-seasonal discrete time risk model from Example 3.4.2, i.e. the order of claims appearance is reversed.*

From the obtained calculations we can easily see that when the positions of claims are changed, the values of $\psi_\delta(u)$ are also changing. The numerical values of $\psi_\delta(u)$ of this model are given in the Table 3.3 and $\log \psi_\delta(u)$ are shown in the Figure 3.2 with $N = 50$ and $K = 3$.

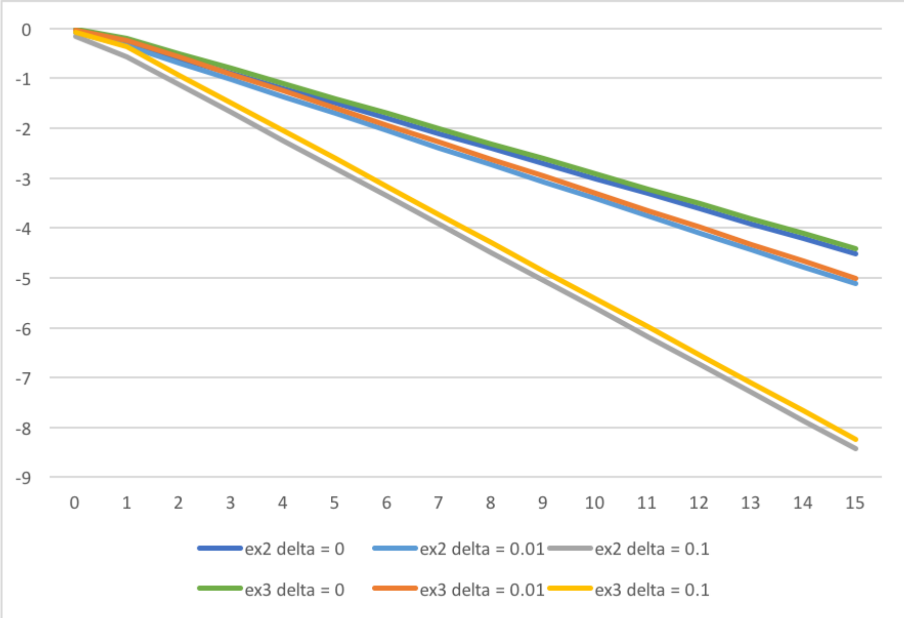

Figure 3.2: Values of $\log \psi_\delta(u)$ in Examples 3.4.2 and 3.4.3

Table 3.3: Values of $\psi_\delta(u)$ in Examples 3.4.3

u	$\delta = 0$ (< 0.000000001)	$\delta = 0.01$ (0.001104494)	$\delta = 0.1$ (< 0.000000001)
0	0.950000000	0.936126346	0.839178292
1	0.625000000	0.588031587	0.427209666
2	0.312500000	0.267757665	0.117206868
3	0.156250000	0.121922306	0.032156225
4	0.078125010	0.055516800	0.008822203
5	0.039062510	0.025279337	0.002420411
6	0.019531260	0.011510838	0.000664050
7	0.009765629	0.005241411	0.000182185
8	0.004882816	0.002386654	0.000049983
9	0.002441409	0.001086753	0.000013713
10	0.001220706	0.000494848	0.000003762
11	0.000610354	0.000225327	0.000001032
12	0.000305178	0.000102602	0.000000283
13	0.000152590	0.000046719	0.000000078
14	0.000076296	0.000021273	0.000000021
15	0.000038149	0.000009687	0.000000006

Example 3.4.4. *Suppose that the bi-seasonal discrete time risk model is generated by r.v.s X and Y, where X has Poisson distribution with*

parameter $\lambda = 0.8$ *and* Y *has geometric distribution with parameter* $p = 0.7$.

In this case, the model generators have infinite supports, but all requirements of the Theorem 3.1.2 are satisfied. So we can use the algorithm from the Section 3.3 to calculate values of $\psi_\delta(u)$. These values are given in the Table 3.4 and shown in the Figure 3.3. The results are obtained by choosing $N = 60$ and $K = 4$ in the first step of the algorithm.

Table 3.4: Values of $\psi_\delta(u)$ in Example 3.4.4

u	$\delta = 0$ (< 0.000000001)	$\delta = 0.01$ (0.089541014)	$\delta = 0.1$ (0.000002568)
0	0.678504300	0.667146224	0.582922968
1	0.357239100	0.346815995	0.278446415
2	0.170682700	0.162951735	0.116632815
3	0.080801850	0.075772347	0.047817117
4	0.038827470	0.035788750	0.020007214
5	0.018862780	0.017104346	0.008536891
6	0.009203741	0.008213946	0.003676915
7	0.004496317	0.003949953	0.001588588
8	0.002197207	0.001900018	0.000686862
9	0.001073798	0.000913991	0.000297021
10	0.000524834	0.000439670	0.000128443
11	0.000256585	0.000211501	0.000055544
12	0.000125498	0.000101741	0.000024019
13	0.000061448	0.000048942	0.000010387
14	0.000030139	0.000023543	0.000004492
15	0.000014871	0.000011325	0.000001942

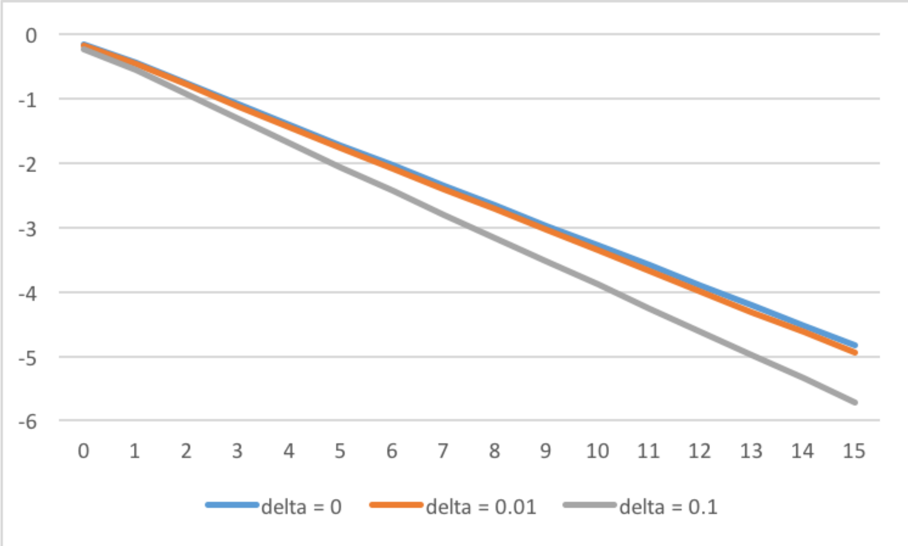

Figure 3.3: Values of $\log \psi_\delta(u)$ in Example 3.4.4

3.5 Concluding remarks

In this chapter, the bi-seasonal discrete time risk model is considered. We derived a recursive algorithm for calculating the values of a special case of Gerber-Shiu discounted penalty function. Theoretical results are illustrated by some numerical examples.

The results obtained in this chapter could be improved in the following directions:

- Instead of taking $w(x, y) = 1$ in the Gerber-Shiu function, arbitrary function $w(x, y)$ could be taken. This would allow to reflect insurer's economic costs at the time of ruin in a more realistic way.

- Our results could be generalized to the models with more complex structure of claims' non-homogeneity. For instance, models with cyclically distributed claims with an arbitrary cycle length could be considered. In this chapter, the model with cycle length equal to 2 is considered.

- In the Step 4 of our presented algorithm, more subtle estimation of $\psi_\delta(0)$ approximation error could be derived.

- In the bi-seasonal discrete time risk model, claims with distributions satisfying $\max\{\mathbb{E}e^{hX}, \mathbb{E}e^{hY}\} = \infty$ for all positive h could be considered. The difficulty arises here because one of the limiting relations in the Theorem 3.1.1 does not hold anymore. Therefore an alternate way of finding $\psi_\delta(0)$ and $\psi_\delta(1)$ should be derived.

Chapter 4

Ruin probability for the bi-seasonal discrete time risk model with dependent claims

In this chapter, we consider the bi-seasonal discrete time risk model with dependent claims. The aim of this chapter is to derive an algorithm for computing the values of the ultimate ruin probability in the bi-seasonal discrete time risk model with dependent claims. Theoretical results are illustrated with numerical examples. The rest of the chapter is organized as follows. In Section 4.1, we present our main results. In Section 4.2 the proofs of the main results are given. Finally, in Section 4.3 we present some examples, which show the applicability of our results.

4.1 Definitions and main results

Definition 4.1.1. *We say that the insurer's surplus W_u varies according to the bi-seasonal risk model with dependent claims if*

$$W_u(n) = u + n - \sum_{i=1}^{n} Z_i$$

for all $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and the following assumptions hold:

- *the initial insurer's surplus is $u \in \mathbb{N}_0$,*
- *there exists a random vector (X, Y) such that $(Z_{2k-1}, Z_{2k}) \overset{d}{=} (X, Y)$, $k \in \mathbb{N}$,*
- *the random vectors (Z_{2k-1}, Z_{2k}), $k \in \mathbb{N}$, are independent,*

• *the generating random vector* (X, Y) *has the distribution defined by the table below, where* $h_{ij} = \mathbb{P}(X = i, Y = j), i, j \in \mathbb{N}_0$*:*

	0	1	2	3	...
0	$h_{0,0}$	$h_{0,1}$	$h_{0,2}$	$h_{0,3}$	...
1	$h_{1,0}$	$h_{1,1}$	$h_{1,2}$	$h_{1,3}$	...
2	$h_{2,0}$	$h_{2,1}$	$h_{2,2}$	$h_{2,3}$	...
...	...	...	...	...	...

If X and Y are independent random variables, then the model reduces to the one considered in Chapter 3 and [19]. If, in addition, X and Y are identically distributed, then the bi-seasonal discrete time risk model with dependent claims becomes the classical discrete time risk model.

The time of ruin and the ruin probability are the main extremal characteristics of insurance risk models. In this chapter these quantities are defined identically to the analogous quantities considered in Chapters 2 and 3.

In the case of the classical discrete time risk model, recursive procedures for calculating exact values of $\psi(u)$ are well known. These procedures and related information can be found in [21], [24], [23], [25], [28], [29], [39], [64], [68] among others.

The recursive calculation of $\psi(u)$ is relatively simple in the classical discrete time risk model because of the explicit formula for $\psi(0)$. If the consecutive claim amounts $Z_1, Z_2, \ldots$ are no longer identically distributed or independent, then the classical discrete time risk model becomes the inhomogeneous discrete time risk model. For all such models, the algorithms for finding values of the ruin probabilities are much more complicated. Several results related to the calculation of the ruin probabilities for inhomogeneous renewal risk models can be found in [1], [4], [5], [6], [8], [10], [16], [19], [30], [31], [32], [58], [57], [59] and [74].

Let us introduce some notation used in our results. By

$$\begin{aligned} x_k &= \mathbb{P}(X = k) = \sum_{j=0}^{\infty} h_{k,j}, \\ y_k &= \mathbb{P}(Y = k) = \sum_{j=0}^{\infty} h_{j,k}, \end{aligned}$$

$$q_k = \mathbb{P}(Q = k) = \sum_{l=0}^{k} h_{l,k-l}, \quad k \in \mathbb{N}_0,$$

we denote the marginal distributions of the random variables X, Y and their sum $Q = X + Y$, respectively. The distribution functions of these random variables are denoted by F_X, F_Y and F_Q, i.e.

$$\begin{aligned} F_X(u) &= \mathbb{P}(X \leqslant u) = \sum_{k=0}^{\lfloor u \rfloor} x_k, \\ F_Y(u) &= \mathbb{P}(Y \leqslant u) = \sum_{k=0}^{\lfloor u \rfloor} y_k, \\ F_Q(u) &= \mathbb{P}(Q \leqslant u) = \sum_{k=0}^{\lfloor u \rfloor} q_k \end{aligned}$$

for all $u \geqslant 0$. As in Chapter 3, the notation $\overline{F}$ is used for the tail of an arbitrary distribution function F.

Furthermore, the survival probability is denoted by $\varphi(u) = 1 - \psi(u)$ for all $u \in \mathbb{N}_0$. It should be noted that our main results are formulated in terms of the survival probability.

Theorem 4.1.1. *Let the bi-seasonal discrete time risk model be generated by the random vector (X, Y), where X and Y are nonnegative and integer-valued random variables such that $\mathbb{E}Q = \mathbb{E}X + \mathbb{E}Y < 2$. In this case*

$$\lim_{u \to \infty} \varphi(u) = 1. \tag{4.1.1}$$

- *If $q_0 = h_{0,0} > 0$, then*

$$\varphi(0) = (2 - \mathbb{E}Q) \lim_{n \to \infty} \frac{b_{n+1} - b_n}{a_n - a_{n+1}}, \tag{4.1.2}$$

$$\varphi(u) = a_u \varphi(0) + b_u (2 - \mathbb{E}Q), \quad u \in \mathbb{N}_0, \tag{4.1.3}$$

where a_n and b_n are two sequences of real numbers defined recursively by *the equalities:*

$$
\begin{aligned}
&a_0 = 1,\ a_1 = -\frac{1}{y_0},\ a_n = \frac{1}{q_0}\Big(a_{n-2} - \sum_{i=1}^{n-1} q_i a_{n-i} + a_1 h_{n-1,0}\Big),\\
&n \in \{2,3,\ldots\};\\
&b_0 = 0,\ b_1 = \frac{1}{y_0},\ b_n = \frac{1}{q_0}\Big(b_{n-2} - \sum_{i=1}^{n-1} q_i b_{n-i} + b_1 h_{n-1,0}\Big),\\
&n \in \{2,3,\ldots\}.
\end{aligned}
$$

- *If* $q_0 = 0$ *with* $x_0 \neq 0$ *and* $y_0 = 0$*, then*

$$
\begin{aligned}
\varphi(0) &= 2 - \mathbb{E}Q,\\
\varphi(u) &= \frac{1}{q_1}\Big(\varphi(u-1) - \sum_{k=2}^{u} q_k \varphi(u-k+1)\Big),\ u \in \mathbb{N}.
\end{aligned}
$$

- *If* $q_0 = 0$ *with* $x_0 = 0$ *and* $y_0 \neq 0$*, then*

$$
\begin{aligned}
\varphi(0) &= 0,\\
\varphi(1) &= \frac{1}{y_0}(2 - \mathbb{E}Q),\\
\varphi(u) &= \frac{1}{q_1}\Big(\varphi(u-1) - \sum_{k=2}^{u} q_k \varphi(u-k+1) + h_{u,0}\,\varphi(1)\Big),\ u \in \{2,3,\ldots\}.
\end{aligned}
$$

Theorem 4.1.2. *Let the bi-seasonal discrete time risk model be generated by random vector* (X,Y)*, where* X *and* Y *are nonnegative and integer-valued random variables such that the net profit condition is not satisfied, i.e.* $\mathbb{E}X + \mathbb{E}Y \geqslant 2$.

If $\mathbb{E}X + \mathbb{E}Y > 2$*, then* $\varphi(u) = 0$ *for all* $u \in \mathbb{N}_0$.

If $\mathbb{E}X + \mathbb{E}Y = 2$*, then we have the following possible subcases:*

- $\varphi(u) = 0,\ u \in \mathbb{N}_0,$ *if* $q_2 = h_{0,2} + h_{1,1} + h_{2,0} < 1$;
- $\varphi(0) = 0,\ \varphi(u) = 1,\ u \in \mathbb{N},$ *if* $q_2 = 1$ *and* $h_{2,0} = 0$;
- $\varphi(0) = \varphi(1) = 0,\ \varphi(u) = 1,\ u \in \{2,3,\ldots\},$ *if* $q_2 = 1$ *and* $h_{2,0} > 0$.

4.2 Proofs of the main results

Proof of the Theorem 4.1.1. The proof partly follows the scheme of the proofs given in [19]. Therefore, some of the details provided there are omitted.

At the beginning of the proof consider the general case with $\mathbb{E}Q \geqslant 0$. By the total probability formula, we get the following basic recursive formula for all $u \in \mathbb{N}_0$

$$\begin{aligned}\varphi(u) &= \sum_{k=0}^{u+1} q_k \varphi(u+2-k) - h_{u+1,0}\varphi(1) \\ &= \sum_{k=0}^{u+1} q_{u+1-k}\varphi(k+1) - h_{u+1,0}\varphi(1). \end{aligned} \tag{4.2.1}$$

The obtained equality implies that

$$\sum_{l=0}^{u} \varphi(l) = \sum_{l=0}^{u}\sum_{k=0}^{l+1} q_{l+1-k}\varphi(k+1) - \varphi(1)\sum_{l=0}^{u} h_{l+1,0}, \; u \in \mathbb{N}_0.$$

By rearranging the terms we obtain

$$\begin{aligned}\sum_{k=0}^{u+2} \varphi(k)\overline{F}_Q(u+2-k) &= \varphi(u+1) + \varphi(u+2) \\ &\quad - \varphi(1)\sum_{l=0}^{u+1} h_{l,0} - \varphi(0)F_Q(u+2). \end{aligned} \tag{4.2.2}$$

Sequence $\{\varphi(0), \varphi(1), \varphi(2), \ldots\}$ is nondecreasing and bounded. Therefore, there exists limit $\varphi(\infty) = \lim\limits_{u\to\infty} \varphi(k)$. By Silverman–Toeplitz theorem, we obtain that

$$\lim_{u\to\infty} \sum_{k=0}^{u+2} \varphi(k)\overline{F}_Q(u+2-k) = \varphi(\infty)\mathbb{E}Q. \tag{4.2.3}$$

Below we give a detailed explanation about how the equality above is derived. First, let us formulate Silverman–Toeplitz theorem.

Theorem 4.2.1. *(Silverman–Toeplitz) Let $\{a_{uk}\}$ be a matrix of complex* *numbers such that*

$$\sup_u \sum_{k=0}^{\infty} |a_{uk}| < \infty, \ \lim_{u\to\infty} \sum_{k=0}^{\infty} a_{uk} = 1, \ \lim_{u\to\infty} a_{uk} = 0, \ k \in \mathbb{N}_0.$$

Let $\{s_k, \ k \in \mathbb{N}_0\}$ *be a convergent sequence of complex numbers. Also, denote* $\sigma_u := \sum\limits_{k=0}^{\infty} a_{uk} s_k, \ u \in \mathbb{N}_0$. *Then the following statement holds:*

$$\lim_{k\to\infty} s_k = s \Leftrightarrow \lim_{u\to\infty} \sigma_u = s.$$

Let us take

$$s_k = \varphi(k), \ a_{uk} = \frac{\overline{F}_Q(u+2-k)}{\mathbb{E}Q} \mathbb{1}(k \leqslant u+2).$$

Then we can obtain that

$$\sup_u \sum_{k=0}^{\infty} |a_{uk}| = \frac{1}{\mathbb{E}Q} \sup_u \sum_{k=0}^{u+2} \overline{F}_Q(u+2-k) = 1,$$
$$\lim_{u\to\infty} \sum_{k=0}^{\infty} a_{uk} = \frac{1}{\mathbb{E}Q} \lim_{u\to\infty} \sum_{k=0}^{u+2} \overline{F}_Q(u+2-k) = 1,$$
$$\lim_{u\to\infty} a_{uk} = \frac{1}{\mathbb{E}Q} \lim_{u\to\infty} \overline{F}_Q(u+2-k)\mathbb{1}(k \leqslant u+2) = 0, \ k \in \mathbb{N}_0.$$

Therefore, we can apply Silverman–Toeplitz theorem. Since $\lim\limits_{k\to\infty} s_k = \varphi(\infty)$, we get that the relation (4.2.3) holds.

Inserting the relation (4.2.3) into the equality (4.2.2) and passing to the limit as $u \to \infty$, we get that

$$(2 - \mathbb{E}Q)\varphi(\infty) = y_0\varphi(1) + \varphi(0). \tag{4.2.4}$$

From now on until the end of the proof, let us restrict to the case $\mathbb{E}Q < 2$. Equality (4.1.1) is proved using the strong law of large numbers, and the proof is identical to the first part of Theorem 2.3 proof in [19]. Namely, according to the alternative definition of survival probability, we get that

$$\varphi(\infty) = \lim_{u\to\infty} \mathbb{P}\Big(\sup_{n\geqslant 1} \eta_n < u\Big), \tag{4.2.5}$$

where

$$\eta_n = \sum_{i=1}^{n}(Z_i - 1).$$

It is evident that for even n ($n = 2N$)

$$\frac{\eta_n}{n} = \frac{\eta_{2N}}{2N} = \frac{1}{2}\Bigg(\frac{1}{N}\sum_{i=0}^{N-1}(Z_{2i+1} - 1) + \frac{1}{N}\sum_{i=0}^{N}(Z_{2i} - 1)\Bigg),$$

and for odd n ($n = 2N + 1$)

$$\frac{\eta_n}{n} = \frac{\eta_{2N+1}}{2N+1} = \frac{N+1}{2N+1}\frac{1}{N+1}\sum_{i=0}^{N}(Z_{2i+1} - 1) + \frac{N}{2N+1}\frac{1}{N}\sum_{i=1}^{N}(Z_{2i} - 1).$$

According to the strong law of large numbers, we have

$$\frac{\eta_n}{n} \xrightarrow[n\to\infty]{} \frac{1}{2}(\mathbb{E}X - 1 + \mathbb{E}Y - 1) = \frac{\mathbb{E}S - 2}{2}.$$

It follows that

$$\mathbb{P}\Bigg(\sup_{m\geqslant n}\left|\frac{\eta_m}{m} + \mu\right| < \frac{\mu}{2}\Bigg) \xrightarrow[n\to\infty]{} 1. \tag{4.2.6}$$

with $\mu := (2 - \mathbb{E}S)/2 > 0$.

Let now $\varepsilon \in (0, 1/2)$ be temporally fixed. Due to (4.2.6),

$$\mathbb{P}\Bigg(\sup_{m\geqslant n}\left|\frac{\eta_m}{m} + \mu\right| < \frac{\mu}{2}\Bigg) \geqslant 1 - \varepsilon$$

if $n \geqslant N = N(\varepsilon)$. Consequently, for $n \geqslant N$,

$$\mathbb{P}\Bigg(\bigcap_{m=n}^{\infty}\{\eta_m \leqslant 0\}\Bigg) \geqslant \mathbb{P}\Bigg(\bigcap_{m=n}^{\infty}\left\{\left|\frac{\eta_m}{m} + \mu\right| < \frac{\mu}{2}\right\}\Bigg) \geqslant 1 - \varepsilon.$$

For an arbitrary positive u we have that

$$\begin{aligned}
\mathbb{P}\Big(\sup_{n\geqslant 1}\eta_n < u\Big) &\geqslant \mathbb{P}\Big(\sup_{n\geqslant 1}\eta_n \leqslant \frac{u}{2}\Big) \\
&= \mathbb{P}\Bigg(\Bigg\{\bigcap_{n=1}^{N-1}\Big\{\eta_n \leqslant \frac{u}{2}\Big\}\Bigg\} \cap \Bigg\{\bigcap_{n=N}^{\infty}\Big\{\eta_n \leqslant \frac{u}{2}\Big\}\Bigg\}\Bigg) \\
&\geqslant \mathbb{P}\Bigg(\Bigg\{\bigcap_{n=1}^{N-1}\Big\{\eta_n \leqslant \frac{u}{2}\Big\}\Bigg\}\Bigg) + \mathbb{P}\Bigg(\Bigg\{\bigcap_{n=N}^{\infty}\Big\{\eta_n \leqslant 0\Big\}\Bigg\}\Bigg) - 1 \\
&\geqslant \mathbb{P}\Bigg(\Bigg\{\bigcap_{n=1}^{N-1}\Big\{\eta_n \leqslant \frac{u}{2}\Big\}\Bigg\}\Bigg) - \varepsilon,
\end{aligned}$$

where the second inequality follows from the well known identity $\mathbb{P}(A\cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A\cup B)$ which holds for arbitrary events A, B.

This estimate and equality (4.2.5) imply that

$$\liminf_{u\to\infty}\varphi(u) \geqslant 1-\varepsilon.$$

The desired equality (4.1.1) now follows due to the arbitrariness of $\varepsilon \in (0, 1/2)$ in the last estimate.

Therefore, by the equalities (4.1.1) and (4.2.4), we get that

$$2 - \mathbb{E}Q = y_0\varphi(1) + \varphi(0). \tag{4.2.7}$$

Suppose now that $q_0 = h_{0,0} \neq 0$. Then the equality (4.1.3) can be derived by induction using the main recursive relation (4.2.1), with induction basis obtained from (4.2.7).

In the next part of the proof we obtain equality (4.1.2). This equality can be derived in a way similar to the fourth and fifth part of Theorem 2.3 proof in [19].

Namely, let ε be a temporarily fixed positive number. Equality $\varphi(\infty) = \lim_{u\to\infty}\varphi(u) = 1$ implies that $|\varphi(n+1) - \varphi(n)| \leqslant \varepsilon$ if $n \geqslant N = N(\varepsilon)$. Hence, according to the recursion formula (4.1.3), we get:

$$|(a_{n+1} - a_n)\varphi(0) + (b_{n+1} - b_n)(2 - \mathbb{E}Q)| \leqslant \varepsilon,\ n \geqslant N.$$

Let us temporarily suppose that

$$|a_{n+1} - a_n| \geqslant 2, \ n \in N. \tag{4.2.8}$$

In such case,

$$\left|\varphi(0) - \frac{b_{n+1} - b_n}{a_n - a_{n+1}}(2 - \mathbb{E}Q)\right| \leqslant \frac{\varepsilon}{2}, \ n \geqslant N.$$

The arbitrariness of $\varepsilon > 0$ implies the desired equality (4.1.2). It remains to prove (4.2.8).

Observe that the desired inequality (4.2.8) follows immediately from the estimates below:

$$\begin{cases} a_{2m} \geqslant a_{2(m-1)} \geqslant 1, \ m \in \mathbb{N}, \\ a_{2m+1} \leqslant a_{2m-1} \leqslant -1/y_0, \ m \in \mathbb{N}. \end{cases} \tag{4.2.9}$$

Indeed, if n is odd then $n+1$ is even, and by (4.2.9) we get

$$|a_{n+1} - a_n| = a_{n+1} - a_n \geqslant 1 + 1/y_0 \geqslant 2.$$

If n is even then $n+1$ is odd, and similarly we have that

$$|a_{n+1} - a_n| = a_n - a_{n+1} \geqslant 1 + 1/y_0 \geqslant 2.$$

Thus, for the validity of (4.1.2) it suffices to prove inequalities (4.2.9). For this, we use induction. By the definition of sequence $\{a_n\}$, we have

$$\begin{aligned} a_2 &= \frac{1}{q_0}(a_0 - q_1 a_1 + h_{1,0} a_1) = \frac{1}{q_0}\left(1 + \frac{h_{1,0} + h_{0,1}}{y_0} - \frac{h_{1,0}}{y_0}\right) \\ &= \frac{1}{q_0}\left(1 + \frac{h_{0,1}}{y_0}\right) \geqslant \frac{1}{q_0} \geqslant 1 = a_0. \end{aligned}$$

Similarly,

$$\begin{aligned} a_3 - a_1 &= \frac{1}{q_0}(a_1 - q_1 a_2 - q_2 a_1 + h_{2,0} a_1 - q_0 a_1) \\ &= \frac{1}{q_0}\left(-\frac{1}{y_0}(1 - q_2 - q_0) - q_1 a_2 - \frac{h_{2,0}}{y_0}\right) \leqslant 0. \end{aligned}$$

Hence, estimates (4.2.9) hold for $m = 1$. Suppose that inequalities (4.2.9) hold for all $m \in \{1, 2, \ldots, l-1\}$. Let us prove that both estimates are valid if $m = l$. By the definition of sequence $\{a_n\}$ we have

$$\begin{aligned} a_{2l} - a_{2(l-1)} &= \frac{1}{q_0}\left(a_{2(l-1)} - \sum_{i=1}^{2l-1} q_i a_{2l-i} - \frac{h_{2l-1,0}}{y_0} - q_0 a_{2(l-1)}\right) \\ &= \frac{1}{q_0}\left(a_{2(l-1)} - \sum_{j=1}^{l-1} q_{2j} a_{2l-2j} - \sum_{j=1}^{l} q_{2j-1} a_{2l-2j+1}\right. \\ &- \left.\frac{h_{2l-1,0}}{y_0} - q_0 a_{2(l-1)}\right). \end{aligned}$$

The induction hypothesis implies that

$$\sum_{i=1}^{l-1} q_{2j} a_{2l-2j} \leqslant a_{2(l-1)} \sum_{j=1}^{l-1} q_{2j} \leqslant (1 - q_0) a_{2(l-1)},$$

$$\sum_{i=1}^{l} q_{2j-1} a_{2l-2j+1} \leqslant a_1 \sum_{j=1}^{l} q_{2j-1} \leqslant a_1 q_{2l-1} \leqslant a_1 h_{2l-1,0} = -\frac{h_{2l-1,0}}{y_0}.$$

Therefore,

$$a_{2l} - a_{2(l-1)} \geqslant \frac{1}{q_0}\left(a_{2(l-1)} - (1-q_0) a_{2(l-1)} + \frac{h_{2l-1,0}}{y_0} - \frac{h_{2l-1,0}}{y_0} - q_0 a_{2(l-1)}\right) = 0. \tag{4.2.10}$$

Similarly, we have

$$a_{2l+1} - a_{2l-1} = \frac{1}{q_0}\left(a_{2l-1} - \sum_{j=1}^{l-1} q_{2j} a_{2l-2j+1} - \sum_{j=1}^{l} q_{2j-1} a_{2l-2j+2} - \frac{h_{2l,0}}{y_0} - q_0 a_{2l-1}\right).$$

Using the induction hypothesis and the derived estimate (4.2.10) we get

$$\sum_{j=1}^{l-1} q_{2j} a_{2l-2j+1} \geqslant a_{2l-1} \sum_{j=1}^{l} q_{2j} \geqslant a_{2l-1}(1 - q_0),$$

$$\sum_{j=1}^{l} q_{2j-1} a_{2l-2j+2} \geqslant 0.$$

Therefore,

$$a_{2l+1} - a_{2l-1} \leqslant \frac{1}{q_0}\left(a_{2l-1} - a_{2l-1}(1 - q_0) - \frac{h_{2l,0}}{y_0} - q_0 a_{2l-1}\right) \leqslant 0. \tag{4.2.11}$$

Estimates (4.2.10) and (4.2.11) imply that inequalities (4.2.9) hold for $m =$

l. The induction principle implies relation (4.2.9) for all $m \in \mathbb{N}$. The proof of equality (4.1.2) is completed.

It remains to consider the case where $q_0 = h_{0,0} = 0$. Since $\mathbb{E}Q < 2$, it follows that $q_1 \neq 0$. Two subcases can be considered separately: $x_0 \neq 0$ and $y_0 = 0$, or $x_0 = 0$ and $y_0 \neq 0$.

In the subcase where $x_0 \neq 0$ and $y_0 = 0$, we get the formula for $\varphi(0)$ from (4.2.7). The formula for $\varphi(u)$, $u \in \mathbb{N}$, follows from (4.2.1) because

$$0 = y_0 = \sum_{k=0}^{\infty} h_{k,0}$$

in the considered case.

If $x_0 = 0$ and $y_0 \neq 0$, then we get $\varphi(0) = 0$ from (4.2.1). Then the formula for $\varphi(1)$ follows from (4.2.7), and the formula for $\varphi(u)$ in the case $u \in \{2, 3, \ldots\}$ can be derived from (4.2.1).

Theorem 4.1.1 is proved.

Proof of the Theorem 4.1.2. Let us consider the cases $\mathbb{E}Q > 2$ and $\mathbb{E}Q = 2$ separately.

If $\mathbb{E}Q > 2$, then equality (4.2.4) implies that $\varphi(\infty) = 0$, since $y_0\varphi(1) + \varphi(0) \geqslant 0$. Therefore, we obtain that $\varphi(u) = 0,\ u \in \mathbb{N}_0$.

In the case $\mathbb{E}Q = 2$, we can easily see from (4.2.4) that

$$y_0\varphi(1) + \varphi(0) = 0. \tag{4.2.12}$$

Therefore, $\varphi(0) = 0$. To calculate $\varphi(u), u \in \mathbb{N}$, the subcases $q_2 < 1$ and $q_2 = 1$ can be considered separately.

If $q_2 < 1$ and $q_0 \neq 0$, then equality (4.2.12) implies that $\varphi(1) = 0$. Further, substituting $\varphi(0) = \varphi(1) = 0$ into equality (4.2.1), we get that $q_0\varphi(u) = 0$ for each $u \in \{2, 3, \ldots\}$. Therefore, in the case ($\mathbb{E}Q = 2,\ q_2 < 1,\ q_0 \neq 0$) we have that $\varphi(u) = 0$ for each $u \in \mathbb{N}_0$.

If $q_2 < 1$, $q_1 \neq 0$ and $q_0 = 0$, then, substituting $\varphi(0) = 0$ into equality (4.2.1), we get that $q_1\varphi(u) = 0$ for each $u \in \{1, 2, \ldots\}$. Therefore, in the case ($\mathbb{E}Q = 2,\ q_2 < 1,\ q_1 \neq 0,\ q_0 = 0$) we also have that $\varphi(u) = 0$ for each $u \in \mathbb{N}_0$.

Now let us consider the subcase $q_2 = h_{0,2} + h_{1,1} + h_{2,0} = 1$. There are the following possible cases:

- If $h_{2,0} > 0$, then from the main recursive formula (4.2.1) we get $\varphi(1) = 0$.
- If $h_{2,0} = 0$, then obviously $W_1(n) \geqslant 1, n \in \mathbb{N}$, and therefore, $\varphi(1) = 1$.

For $u \in \{2, 3, \ldots\}$, it is easy to show that $W_u(n) \geqslant 1$ for $n \in \mathbb{N}$, and therefore, $\varphi(u) = 1$ for such u.

Theorem 4.1.2 is proved.

4.3 Numerical examples

In this section, five numerical examples for the calculation of ruin probability $\psi(u), u \in \mathbb{N}_0$, are given. The first case deals with simple finite-support distribution, the second case deals with the bivariate Poisson distribution, and the last three cases deal with a Clayton copula. The use of copulas is beneficial since it gives the possibility of modeling marginal distributions and dependence between them separately. Furthermore, while the bivariate Poisson distribution allows to model only positive dependence between marginals, a Clayton copula enables to model negative dependence as well.

The numerical simulation procedure goes as follows. First, we can calculate sufficiently many terms of the sequences a_u and b_u from Theorem 4.1.1. Next, we can approximate $\psi(0)$ by

$$\psi_N(0) = 1 - (2 - \mathbb{E}Q)\frac{b_{N+1} - b_N}{a_N - a_{N+1}}$$

with large enough $N \in \mathbb{N}$. In all the examples below, we take $N = 20$. Using the same arguments as in Remark 2.1 of [19], we can obtain both lower and upper bounds for $\psi(0)$ by calculating $\psi_N(0)$ and $\psi_{N+1}(0)$. Then, the upper bound for the approximation error of $\psi(0)$ can be calculated by

$$\Delta = |\psi_N(0) - \psi_{N+1}(0)|.$$

Finally, we can obtain approximations of the ruin probabilities using formula (4.1.3) from Theorem 4.1.1

$$1 - \psi(u) = a_u(1 - \psi_N(0)) + b_u(2 - \mathbb{E}Q), \quad u \in \mathbb{N}.$$

Example 4.3.1. *Assume that the joint probability mass function of* (X, Y) *is given by the following distribution:*

	0	1	2	3
0	2/3	1/45	1/45	1/45
1	1/45	1/45	1/45	1/45
2	1/45	1/45	1/45	1/45
3	1/45	1/45	1/45	1/45

The correlation between the components of random vector is 0.54. In the table and graph below, the results of simulation are given. The ruin probability is calculated for $u \in \{0, 1, \ldots, 12\}$, and the upper bound for the approximation error of $\psi(0)$ is also given.

Table 4.1: Values of $\psi(u)$ in Example 4.3.1

u	cor $= 0.54$ ($\Delta < 10^{-5}$)
0	0.5101
1	0.3953
2	0.2853
3	0.1810
4	0.1145
5	0.0682
6	0.0433
7	0.0270
8	0.0168
9	0.0104
10	0.0065
11	0.0040
12	0.0025

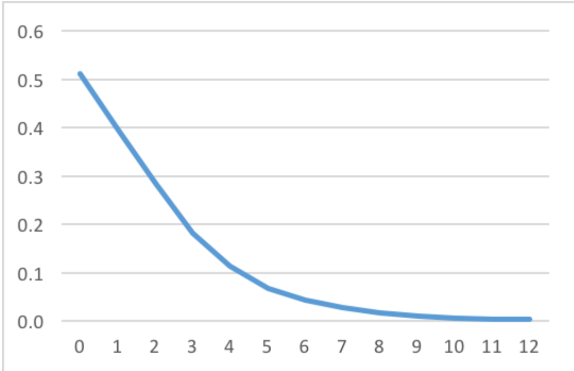

Figure 4.1: Values of $\psi(u)$ in Example 4.3.1

Example 4.3.2. *Assume that the joint probability mass function of* (X, Y) *is given by the bivariate Poisson distribution:*

$$\mathbb{P}(X = k, Y = l) = \sum_{i=0}^{\min\{k,l\}} \frac{(\lambda_1 - \lambda)^{k-i}(\lambda_2 - \lambda)^{l-i}\lambda^i}{(k-i)!(l-i)!i!} e^{-(\lambda_1+\lambda_2-\lambda)}, \ k, l \in \mathbb{N}_0$$

where $\lambda_j > 0$, $j = 1, 2$, $0 \leqslant \lambda < \min\{\lambda_1, \lambda_2\}$*. Then the marginal distribution of* X *is Poisson with parameter* λ_1*, the marginal distribution of* Y *is Poisson with parameter* λ_2*, and* $\mathrm{Cov}(X, Y) = \lambda$*. If* $\lambda = 0$*, then the two variables are independent, and the results in this case are obtained in [19].*

In this example, we take $\lambda_1 = 0.3$ and $\lambda_2 = 1.4$. We consider three possible values for the covariance parameter $\lambda = \{0.01; 0.15; 0.29\}$, and the corresponding correlations equal $\{0; 0.23; 0.46\}$.

In the table and graph below, the results of simulation are given. The ruin probability is calculated for the three values of the covariance parameter mentioned above, and the upper bounds for the approximation errors of $\psi(0)$ are also given.

From the results of simulation it could be observed, that for positively dependent claims the ruin probability is decreasing more slowly. It is also interesting to note that the value of $\psi(0)$ is largest in the case of independent claims.

Table 4.2: Values of $\psi(u)$ in Example 4.3.2

u	cor = 0 ($\Delta < 10^{-11}$)	cor = 0.23 ($\Delta < 10^{-10}$)	cor = 0.46 ($\Delta < 10^{-9}$)
0	0.7977	0.7921	0.7868
1	0.6040	0.6264	0.6480
2	0.4469	0.4875	0.5222
3	0.3269	0.3754	0.4165
4	0.2383	0.2880	0.3310
5	0.1736	0.2208	0.2628
6	0.1265	0.1692	0.2085
7	0.0921	0.1297	0.1655
8	0.0671	0.0994	0.1313
9	0.0489	0.0762	0.1042
10	0.0356	0.0584	0.0827
11	0.0260	0.0447	0.0657
12	0.0189	0.0343	0.0521

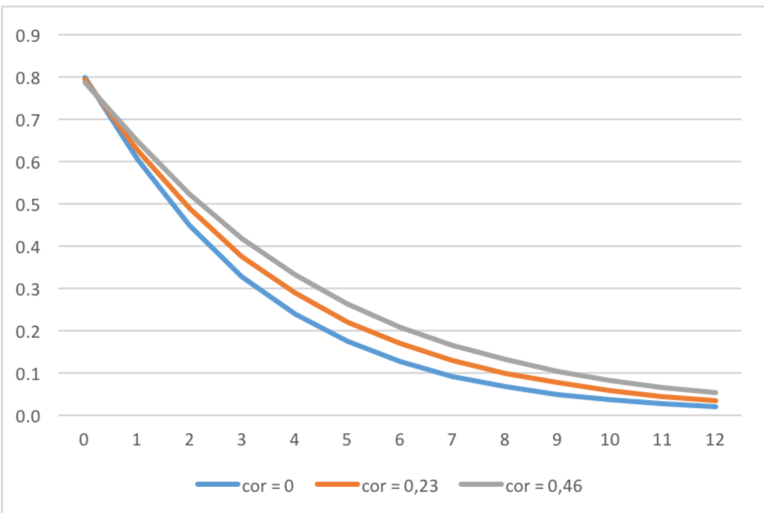

Figure 4.2: Values of $\psi(u)$ in Example 4.3.2

Example 4.3.3. *This example deals with a Clayton copula and Poisson marginals. Let us denote* $u_1 := F_X(x)$, $u_2 := F_Y(y)$. *Clayton copula is defined by*

$$C(u_1, u_2; \theta) = \max\{u_1^{-\theta} + u_2^{-\theta} - 1, 0\}^{-1\backslash\theta},\ u_1, u_2 \in [0, 1]$$

where the dependence parameter $\theta \in [-1, \infty)\backslash\{0\}$. The marginals become independent as $\theta \to 0$. Clayton copula can be used to model negative dependence when $\theta \in [-1, 0)$. Detailed analysis of this copula can be found, for instance, in [33], [49], [52] and [55].

In this example, the marginal distribution of X is Poisson with parameter 0.3, and the marginal distribution of Y is Poisson with parameter 1.4. We take three values for the covariance parameter $\theta = \{-0.9; 0.01; 100\}$, and the corresponding correlations equal $\{-0.53; 0; 0.8\}$.

From the results of simulation it could be observed, that as in Example 4.3.2 for positively dependent claims the ruin probability is decreasing more slowly. It is also interesting to note that the value of $\psi(0)$ is largest in the case of negatively dependent claims.

Table 4.3: Values of $\psi(u)$ in Example 4.3.3

u	cor $= -0.53$ ($\Delta < 10^{-20}$)	cor $= 0$ ($\Delta < 10^{-11}$)	cor $= 0.8$ ($\Delta < 10^{-10}$)
0	0.8217	0.7977	0.7810
1	0.5064	0.6040	0.6717
2	0.3165	0.4469	0.5715
3	0.1977	0.3269	0.4669
4	0.1231	0.2383	0.3909
5	0.0766	0.1736	0.3221
6	0.0476	0.1265	0.2661
7	0.0296	0.0921	0.2195
8	0.0184	0.0671	0.1812
9	0.0115	0.0489	0.1496
10	0.0071	0.0356	0.1235
11	0.0044	0.0260	0.1019
12	0.0028	0.0189	0.0841

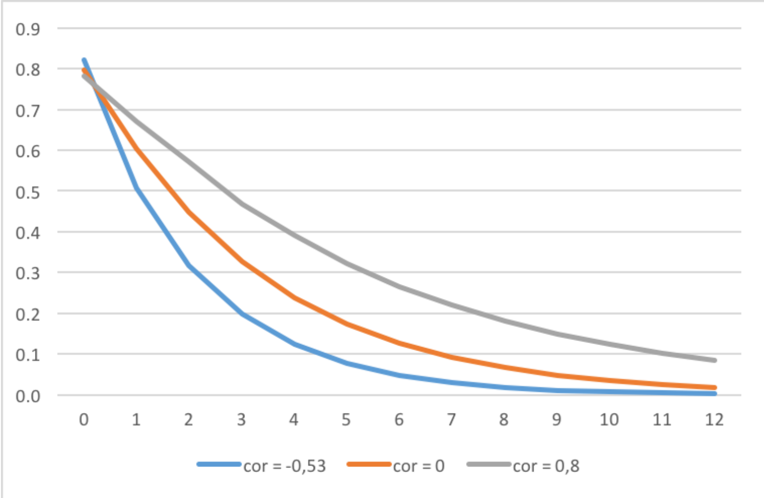

Figure 4.3: Values of $\psi(u)$ in Example 4.3.3

Example 4.3.4. *This example is the opposite case of Example 4.3.3. The marginal distribution of X is Poisson with parameter* 1.4*, and the marginal distribution of Y is Poisson with parameter* 0.3*. To model the dependence between the marginals, we use the Clayton copula with $\theta = \{-0.9; 0.01; 100\}$ again, and the corresponding correlations equal $\{-0.53; 0; 0.8\}$.*

From the simulation we can observe that the order of appearance of claims has considerable effect on the ruin probability.

Table 4.4: Values of $\psi(u)$ in Example 4.3.4

u	cor $= -0.53$ ($\Delta < 10^{-20}$)	cor $= 0$ ($\Delta < 10^{-11}$)	cor $= 0.8$ ($\Delta < 10^{-9}$)
0	0.9267	0.9023	0.8988
1	0.6940	0.7269	0.7316
2	0.4653	0.5473	0.5897
3	0.2961	0.4014	0.4859
4	0.1850	0.2926	0.4048
5	0.1151	0.2131	0.3347
6	0.0716	0.1552	0.2763
7	0.0445	0.1131	0.2280
8	0.0277	0.0824	0.1882
9	0.0172	0.0600	0.1553
10	0.0107	0.0437	0.1282
11	0.0067	0.0319	0.1059
12	0.0042	0.0232	0.0874

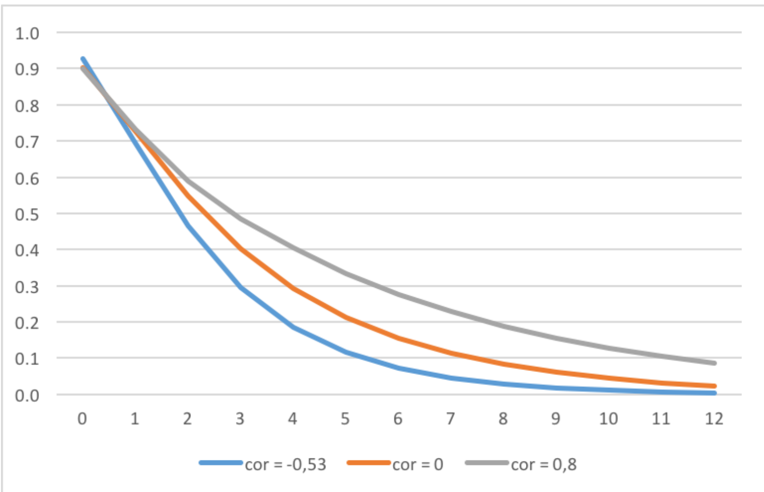

Figure 4.4: Values of $\psi(u)$ in Example 4.3.4

Example 4.3.5. *All the examples considered so far deal only with light-tailed marginals, but Theorem 4.1.1 only imposes requirement for the expectations of the marginals while higher order moments can be infinite. In this example, the distribution of the first claim X is Poisson with parameter $\lambda = 0.2$, and the second claim Y is distributed according to the Zeta* *distribution with parameter* 2.3*, that is*

$$\mathbb{P}(Y = m) = \frac{1}{\zeta(2.3)} \frac{1}{(m+1)^{2.3}}, m \in \mathbb{N}_0,$$

where ζ *denotes the Riemann zeta function. It should be noted that here Zeta distribution is not defined in the usual way, i.e. with support* $m \in \{1, 2, \ldots\}$ *and the corresponding probabilities.*

The expectation of Y is 1.74497 and the variance is infinite. Therefore, the correlation between the claims is undefined. As before, we use the Clayton copula with $\theta = \{-0.9; 0.01; 100\}$ to model the dependence between the marginals.

As can be intuitively expected, the presence of heavy-tailed marginal makes a large impact on the values of the ruin probability.

Table 4.5: Values of $\psi(u)$ in Example 4.3.5

u	$\theta = -0.9$ ($\Delta < 10^{-6}$)	$\theta = 0.01$ ($\Delta < 10^{-6}$)	$\theta = 100$ ($\Delta < 10^{-5}$)
0	0.9721	0.9715	0.9690
1	0.9611	0.9620	0.9656
2	0.9570	0.9579	0.9615
3	0.9543	0.9550	0.9584
4	0.9520	0.9527	0.9559
5	0.9500	0.9507	0.9538
6	0.9483	0.9489	0.9520
7	0.9467	0.9473	0.9503
8	0.9453	0.9458	0.9488
9	0.9439	0.9444	0.9474
10	0.9427	0.9432	0.9460
11	0.9416	0.9421	0.9448
12	0.9406	0.9410	0.9437

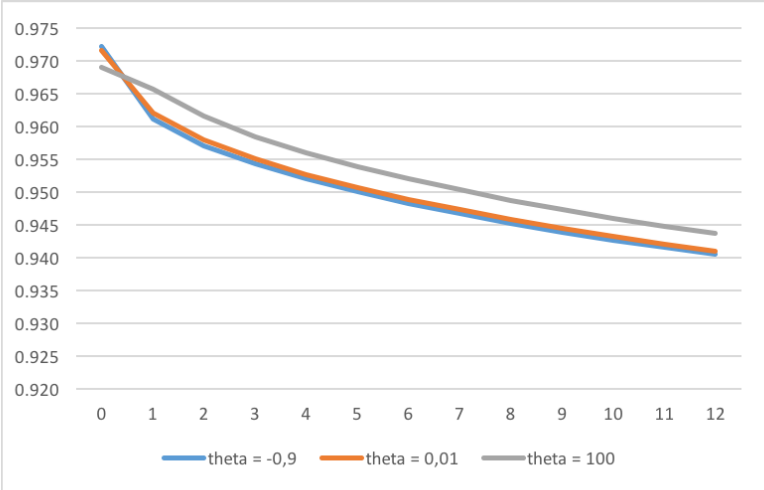

Figure 4.5: Values of $\psi(u)$ in Example 4.3.5

4.4 Conclusive remarks

In this chapter, the bi-seasonal discrete time risk model with dependent claims is introduced. We derived a recursive algorithm for calculating the values of ruin probability. Theoretical results are illustrated by some numerical examples.

The results obtained in this chapter could be improved in the following directions:

- Our results could be generalized to the models with more complex structure of claims' non-homogeneity. For instance, claims' generating random vectors of form $(X_1, X_2, \ldots, X_p)$ with $p > 2$ could be considered, thus getting p-seasonal model.
- Algorithm for the calculation of more complex risk measures, such as Gerber-Shiu expected discounted penalty function [29], could be derived for bi-seasonal discrete time risk model with dependent claims.
- Our considered model and algorithm could be fitted to real insurance data examples.

Chapter 5

Conclusions

In the last Chapter, a short summary of the results obtained is provided.

(i) Algorithm for calculating the values of the particular case of the Gerber-Shiu discounted penalty function in bi-seasonal discrete time risk model was derived.

(ii) Algorithm for computing the values of the ultimate ruin probability in bi-seasonal discrete time risk model with dependent claims was created.

(iii) The case when net profit condition is not satisfied in bi-seasonal discrete time risk model with dependent claims was investigated.

(iv) The conditions for model components under which the algorithms can be used were derived.

(v) The applicability and computational properties of the algorithms were investigated with numerical examples.

(vi) Methods were created for measuring approximation errors of the algorithms.

Appendices

Appendix A

Algorithm code of numerical examples in Chapter 3

```
library(Rmpfr)

# set the values of parameters --------------------------------------------

delta = 0.1
N = 30
K = 2
umax = 20


# initialise vectors ---------------------------------------------------

q = numeric(N)
a = mpfrArray(0, precBits = 1024, dim = c(N,1))
b = mpfrArray(0, precBits = 1024, dim = c(N,1))
d = mpfrArray(0, precBits = 1024, dim = c(N,1))
psi = mpfrArray(0, precBits = 1024, dim = c((umax+1),1))
FX = numeric(N)
FY = numeric(N)


# choose the distributions of claims (4 different distributions are
considered as described in Numerical examples section)-------------------------

x = c(0.6, 0.2, 0.2)
```

```
y = c(0.5, 0.2, 0.2, 0.1)

# x = c(0.4,0.6)
# y = c(0.1,0.6,0.3)

# y = c(0.4,0.6)
# x = c(0.1,0.6,0.3)

# lambda = 0.8
# prob = 0.7
# x = dpois(c(0:N),lambda)
# y = dgeom(c(0:N),prob)


# compute quantities related with claims' distributions ---------------------

Xmax <- length(x)-1
Ymax <- length(y)-1

X = 0:Xmax
Y = 0:Ymax

EX = sum(X * x)
EY = sum(Y * y)

x[(Xmax+2):N] = 0
y[(Ymax+2):N] = 0

for (i in 0:(Xmax+Ymax)) {
  for (k in 1:(i+1))
    q[i+1] = q[i+1] + x[k] * y[(i+2) - k]
}

FX[1] = x[1]
for (u in 1:(N-1)) {
  FX[u+1] = FX[u] + x[u+1]
}
F_X = 1 - FX

FY[1] = y[1]
for (u in 1:(N-1)) {
  FY[u+1] = FY[u] + y[u+1]
```

```
}
F_Y = 1 - FY


# calculate the coefficients of algorithm --------------------------------

a[1] = mpfr(1,1024)
a[2] = mpfr(-1,1024) / mpfr(y[1],1024)
for (n in 2:(N-1)) {
  a[n + 1] = mpfr(1,1024) / mpfr(q[1],1024) * (mpfr(exp(2 * delta),1024)
   * mpfr(a[(n + 1) - 2],1024) + mpfr(x[n],1024) * mpfr(y[1],1024) * mpfr(a[2],1024))
   for (i in 2:n)
     a[n + 1] = mpfr(a[n + 1],1024) - (mpfr(1,1024) / mpfr(q[1],1024))
      * (mpfr(q[i],1024) * mpfr(a[n - i + 2],1024))
}

b[1] = 0
b[2] = -(exp(2 * delta) - 1) / mpfr(y[1],1024)
for (n in 2:(N-1)) {
   b[n + 1] = 1 / mpfr(q[1],1024) * (exp(2 * delta)
   * mpfr(b[(n + 1) - 2],1024) + mpfr(x[n],1024) * mpfr(y[1],1024) * mpfr(b[2],1024))
   for (i in 2:n)
     b[n + 1] = mpfr(b[n + 1],1024) - (1 / mpfr(q[1],1024))
      * (mpfr(q[i],1024) * mpfr(b[n - i + 2],1024))
}

d[1] = 0
d[2] = (exp(delta) * mpfr(EX,1024) + mpfr(y[1],1024) + mpfr(EY,1024) - 1)
/ mpfr(y[1],1024)
for (n in 2:(N-1)) {
  d[n + 1] = 1 / mpfr(q[1],1024) * (exp(2 * delta) * mpfr(d[(n + 1) - 2],1024)
   + mpfr(x[n],1024) * mpfr(y[1],1024) * mpfr(d[2],1024)
   - exp(delta) * mpfr(F_X[n - 1],1024))
   for (i in 2:n)
    d[n + 1] = mpfr(d[n + 1],1024) - (1 / mpfr(q[1],1024)) * (mpfr(q[i],1024)
     * mpfr(d[n - i + 2],1024) + mpfr(x[i - 1],1024) * mpfr(F_Y[n - i + 2],1024))
}


# solve the system of linear equations ------------------------------------

eqA = array(c(mpfr(a[N-K],1024), mpfr(a[N],1024), mpfr(b[N-K],1024), mpfr(b[N],1024)
```

```
dim = c(2, 2))
eqb = array(c(mpfr(-d[N-K],1024), mpfr(-d[N],1024)))
detA = mpfr(eqA[1,1],1024) * mpfr(eqA[2,2],1024)
- mpfr(eqA[1,2],1024) * mpfr(eqA[2,1],1024)
eqA_inv = 1/detA * array(c(mpfr(eqA[2,2],1024), mpfr(-eqA[2,1],1024),
mpfr(-eqA[1,2],1024), mpfr(eqA[1,1],1024)), dim = c(2, 2))
eqx  = mpfr(eqA_inv,1024) %*% mpfr(eqb,1024)
id_mat  = mpfr(eqA_inv,1024) %*% mpfr(eqA,1024)

psi[1] = eqx[1]
S = eqx[2]


# check the accuracy of solutions -----------------------------------------

acc_psi0 = mpfr(exp(-delta),1024) * (abs(mpfr(b[N-K],1024)) + abs(mpfr(b[N],1024)))
/ abs(mpfr(detA,1024))
acc_S = exp(-delta) * (abs(a[N-K]) + abs(a[N])) / abs(detA)


# calculate the values of Gerber-Shiu function --------------------------------

psi[2] = a[2] * psi[1] + b[2] * S + d[2]
psi[3:(umax+1)] = a[3:(umax+1)] * psi[1] + b[3:(umax+1)] * S + d[3:(umax+1)]
psi2 = asNumeric(psi)
```

Appendix B

Algorithm code of numerical examples in Chapter 4

```
library(Rmpfr)
library(extraDistr)
library(VGAM)
library(pracma)

# set the values of algorithm parameters ------------------------------------------------

N = 20
umax = 12


# initialise vectors ----------------------------------------------------

q = numeric(N+1)
a = mpfrArray(0, precBits = 1024, dim = c(N,1))
b = mpfrArray(0, precBits = 1024, dim = c(N,1))
d = mpfrArray(0, precBits = 1024, dim = c(N,1))
psi = mpfrArray(0, precBits = 1024, dim = c((umax+1),1))
phi = mpfrArray(0, precBits = 1024, dim = c((umax+1),1))
h = matrix(NA, (N+1), (N+1))
C = matrix(NA, (N+1), (N+1))


# choose the distributions of claims (only Example 3 code is provided) --------------

t1 = 1.4
```

```
t2 = 0.3

x = dpois(c(0:N), t1)
y = dpois(c(0:N), t2)


# calculate values of Clayton copula ------------------------------------

u1 = ppois(c(0:N), t1)
u2 = ppois(c(0:N), t2)

t0 = 100

for (i in 1:(N+1))
  for (j in 1:(N+1))
     C[i,j] = max(u1[i]^(-t0) + u2[j]^(-t0) - 1, 0) ^ (-1/t0)


# derive matrix of (X,Y) local probabilities ------------------------------------

h[1,1] = C[1,1]

for (i in 2:(N+1))
  h[i,1] = C[i,1] - C[(i-1),1]

for (j in 2:(N+1))
  h[1,j] = C[1,j] - C[1,(j-1)]

for (i in 2:(N+1))
  for (j in 2:(N+1))
    h[i,j] = C[i,j] - C[(i-1),j] - C[i,(j-1)] + C[(i-1),(j-1)]


# compute quantities related with claims' distributions ---------------------

Xmax <- length(x)-1
Ymax <- length(y)-1
Qmax <- length(q)-1

X = 0:Xmax
Y = 0:Ymax
Q = 0:Qmax
```

```
EX = t1
EY = t2

EXY = 0
for (i in 1:(N+1))
  for (j in 1:(N+1))
    EXY = EXY + (i-1) * (j-1) * h[i,j]

DX = t1
DY = t2

sdX = sqrt(DX)
sdY = sqrt(DY)

cov = EXY - EX * EY
cor = cov / (sqrt(DX) * sqrt(DY))


x[(Xmax+2):N] = 0
y[(Ymax+2):N] = 0

for (i in 0:N) {
  for (k in 1:(i+1))
    q[i+1] = q[i+1] + h[k, ((i+2) - k)]
}

EQ = sum(Q * q)


# calculate the coefficients of algorithm ---------------------------------

a[1] = mpfr(1,1024)
a[2] = mpfr(-1,1024) / mpfr(y[1],1024)
for (n in 2:(N-1)) {
 a[n + 1] = mpfr(1,1024) / mpfr(q[1],1024) * (mpfr(a[(n + 1) - 2],1024) +
   mpfr(h[n,1],1024) * mpfr(a[2],1024))
  for (i in 2:n)
    a[n + 1] = mpfr(a[n + 1],1024) -
   (mpfr(1,1024) / mpfr(q[1],1024)) * (mpfr(q[i],1024) * mpfr(a[n - i + 2],1024))
}
aa = as.numeric(a)
```

```
b[1] = 0
b[2] = mpfr(1,1024) / mpfr(y[1],1024)
for (n in 2:(N-1)) {
 b[n + 1] = mpfr(1,1024) / mpfr(q[1],1024) * (mpfr(b[(n + 1) - 2],1024) +
   mpfr(h[n,1],1024) * mpfr(b[2],1024))
  for (i in 2:n)
    b[n + 1] = mpfr(b[n + 1],1024) -
   (mpfr(1,1024) / mpfr(q[1],1024)) * (mpfr(q[i],1024) * mpfr(b[n - i + 2],1024))
}
bb = as.numeric(b)


for (i in 1:(N-1)) {
  d[i] = 1 - (2-EQ) * (b[i+1] - b[i]) / (a[i] - a[i+1])
}
dd = as.numeric(d)


# calculate the upper estimate of \psi(0) approximation error -----------------------

print(format(abs(dd[N-1] - dd[N-2]), scientific = TRUE, digits = 16))


# calculate the values of ruin probability --------------------------------

psi[1] = d[N-1]
phi[1] = 1 - psi[1]
phi[2:(umax+1)] = a[2:(umax+1)] * phi[1] + b[2:(umax+1)] * (2-EQ)
psi[2:(umax+1)] = 1 - phi[2:(umax+1)]
psi2 = asNumeric(psi)
```

Abstract

The topic of the thesis is the calculation of risk measures in inhomogeneous discrete time risk models with independent and dependent claims. Firstly, the algorithm for calculating the values of the particular case of the Gerber-Shiu discounted penalty function in bi-seasonal discrete time risk model was derived. Also, the algorithm for computing the values of the ultimate ruin probability in bi-seasonal discrete time risk model with dependent claims was created. In this model, the case when net profit condition is not satisfied was investigated as well. In the practical part of the thesis, the applicability and computational properties of the algorithms were investigated with numerical examples. Furthermore, methods were created for measuring approximation errors of the algorithms.

The results of the thesis naturally extend the results obtained by Damarackas and Šiaulys (2014). In their paper the calculation of ruin probability in the bi-seasonal discrete time risk model was considered. In the thesis considered both more general risk measure (Gerber-Shiu function) and more general model. Bi-seasonal model with dependent claims is introduced for the first time in the thesis. Furthermore, with dependent and differently distributed claims, recursive calculation of ruin probability in any kind of discrete time risk model was not considered in the literature before. Besides that, was derived the new algorithm for calculating Gerber-Shiu function values which is both more computationally feasible and less prone to numerical errors than the existing solutions found in the literature.

The main results of the thesis were proved using the classical methods of probability theory and mathematical analysis, with an emphasis on discrete differentiation.

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