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Tail asymptotics of a sum of heavy-tailed distributions with application to risk theory

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Contents

Notation

$f(x) \underset{x\to\infty}{\sim} g(x)$	means $\lim_{x\to\infty} f(x)/g(x) = 1$.
$f(x) \underset{x\to\infty}{\lesssim} g(x)$	means $\limsup_{x\to\infty} f(x)/g(x) \leqslant 1$.
$f(x) = o\left(g(x)\right)$	means $\lim_{x\to\infty} f(x)/g(x) = 0$.
$f(x) = O(g(x))$	means $\limsup_{x\to\infty} f(x)/g(x) < \infty$.
$f(x) \asymp g(x)$	means $0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leqslant \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.
r.v.	random variable.
d.f.	distribution function.
$F^{-1}(y)$	$:= \inf\{x \in \mathbb{R} : F(x) \geqslant y\}$. The quantile function of a d.f. F.
$F * G(x)$	the convolution of d.f.s F and G.
$\mathbb{1}_A$	the indicator function of an event A.
X^+	$:= X\mathbb{1}_{\{X\geqslant 0\}}$. The positive part of a r.v. X.
X^-	$:= -X\mathbb{1}_{\{X<0\}}$. The negative part of a r.v. X.
$\overline{F}(x)$	$:= 1 - F(x)$. The survival function of a d.f. F.
$\mathbb{N}$	the set of positive integers $\{1, 2, \dots\}$.
$\mathbb{R}$	the set of real numbers $(-\infty, \infty)$.
$\mathbb{P}(A)$	the probability of an event A.
$\mathbb{E}(X)$	the mean of a r.v. X.

Chapter 1

Introduction

1.1 Research topic and actuality

Let $\{X_1, X_2, \ldots, X_n\}$ be a collection of real-valued and heavy-tailed random variables (r.v.s), called primary random variables, and let $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be n nonnegative and nondegenerated at zero random variables, called random weights. Throughout the thesis, it is supposed that the r.v.s $\{X_1, X_2, \ldots, X_n\}$ and $\{\theta_1, \theta_2, \ldots, \theta_n\}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The randomly weighted sum

$$S_n^{\theta X} = \sum_{i=1}^{n} \theta_i X_i \tag{1.1.1}$$

is the main object of our consideration.

Such randomly weighted sums are often encountered in actuarial and financial context. For instance, in the discrete-time risk model, the real-valued r.v. X_k, $k \in \{1, 2, \ldots, n\}$, can be interpreted as a net loss of an insurance company during the k-th time period, and the random weight θ_k, $k \in \{1, 2, \ldots, n\}$, can be regarded as a stochastic discount factor for the first k time periods. In this situation, the sum $S_n^{\theta X}$ is the present value of the total net loss of the insurance company during the first n time periods.

Another interpretation of the sum (1.1.1) relates to the portfolio construction. Suppose that an investment portfolio consists of n dependent sources of risk (financial assets, risk factors, business lines, etc.) with losses X_k and weights θ_k, $k \in \{1, 2, \ldots, n\}$, over some time period. If the portfolio is actively managed, then the weights and their dependence structure are unknown for future time periods.

In this case, the randomly weighted sum (1.1.1) can be used to model the total amount of future losses potentially incurred by the investment portfolio.

The sum (1.1.1) has been an attractive research topic in the recent works of applied probability. The majority of such works examine the asymptotic behaviour of the tail probability $\mathbb{P}(S_n^{\theta X} > x)$. Tang and Tsitsiashvili [56] proved that, if the primary r.v.s $\{X_1, X_2, \ldots, X_n\}$ are independent and identically distributed according to a subexponential distribution (see Definition 1.2 in Section 1.2) and the random weights $\{\theta_1, \theta_2, \ldots, \theta_n\}$ (independent of $\{X_1, X_2, \ldots, X_n\}$) take values in the interval $[a, b]$, $0 < a < b < \infty$, then

$$\mathbb{P}\Big(\max_{1\leqslant k\leqslant n} S_k^{\theta X} > x\Big) \underset{x\to\infty}{\sim} \mathbb{P}(S_n^{\theta X} > x) \underset{x\to\infty}{\sim} \mathbb{P}\Big(\max_{1\leqslant k\leqslant n} \theta_k X_k > x\Big) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \mathbb{P}(\theta_k X_k > x). \tag{1.1.2}$$

After studying the asymptotic behaviour (1.1.2) it is natural to consider the asymptotic behaviour of the conditional tail expectation $\mathbb{E}(S_n^{\theta X} | S_n^{\theta X} > x)$ and the related quantities $\mathbb{E}S_n^{\theta X}\mathbb{1}_{\{S_n^{\theta X}>x\}}$, $\mathbb{E}\theta_l X_l \mathbb{1}_{\{S_n^{\theta X}>x\}}$, $l \in \{1, 2, \ldots, n\}$, as $x \to \infty$. To our knowledge, the paper [58] of Tang and Yuan is the first work in this direction. Below we present one result of this work (see Theorem 4).

Theorem 1.1. *Let $\{X_1, X_2, \ldots, X_n\}$ be real-valued independent r.v.s with d.f.s $\{F_1, F_2, \ldots, F_n\}$ respectively and $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be nonnegative nondegenerated at zero r.v.s independent of $\{X_1, X_2, \ldots, X_n\}$ and mutually arbitrarily dependent. If, in addition, $F_k \in \mathcal{L} \cap \mathcal{D}$, $\mathbb{E}\,\theta_k^{\beta_k} < \infty$, $\beta_k > \mathcal{M}_{F_k}$ and $\mathbb{P}(\theta_k X_k > x) = O\big(\mathbb{P}(\theta_1 X_1 > x)\big)$ for all $k \in \{1, 2, \ldots, n\}$, then*

$$\mathbb{E}\,\theta_1 X_1 \mathbb{1}_{\{S_n^{\theta X}>x\}} \underset{x\to\infty}{\sim} \mathbb{E}\,\theta_1 X_1 \mathbb{1}_{\{\theta_1 X_1 > x\}}\,, \tag{1.1.3}$$

where $\mathcal{L}$ is the class of long-tailed distributions (see Definition 1.3), $\mathcal{D}$ is the class of dominatedly varying distributions (see Definition 1.4), and $\mathcal{M}_F$ is the upper Matuszewska index (see Section 1.2 for definition).

Under the additional condition $\mathbb{P}(\theta_k X_k > x) \asymp \mathbb{P}(\theta_1 X_1 > x)$, $k \in \{1, 2, \ldots, n\}$, Theorem 1.1 implies that

$$\mathbb{E}S_n^{\theta X}\mathbb{1}_{\{S_n^{\theta X}>x\}} \underset{x\to\infty}{\sim} \sum_{i=1}^{n} \mathbb{E}\theta_i X_i \mathbb{1}_{\{\theta_i X_i > x\}}. \tag{1.1.4}$$

In the case where d.f.s $\{F_1, F_2, \ldots, F_n\}$ have regularly varying tails (see Definition 1.5 in Section 1.2), Tang and Yuan [58] obtained an asymptotic formula for the *Conditional Tail Expectation (CTE) of confidence level* q (also called Conditional Value at Risk (CVaR) or Expected Shortfall (ES))

$$\mathrm{CTE}_q(S_n^{\theta X}) = \mathbb{E}\big(S_n^{\theta X} \mid S_n^{\theta X} > \mathrm{VaR}_q(S_n^{\theta X})\big),$$

where $\mathrm{VaR}_q(Z) = F_Z^{-1}(q)$. Below we formulate the obtained result (see Corollary 1 in [58]).

Theorem 1.2. *Let r.v.s* $\{X_1, X_2, \ldots, X_n\}$ *and* $\{\theta_1, \theta_2, \ldots, \theta_n\}$ *satisfy the requirements of Theorem 1.1. Assume that, for each* $k \in \{1, 2, \ldots, n\}$, $\overline{F}_k(x) \underset{x\to\infty}{\sim} c_k\overline{F}(x)$ *for some constant* $c_k > 0$. *If* $F \in \mathcal{R}_{-\alpha}$ *with* $\alpha > 1$ *and* $\mathbb{E}\theta_k^\beta < \infty$ *for some* $\beta > \alpha$ *and all* $k \in \{1, 2, \ldots, n\}$, *then*

$$\mathbb{E}\big(\theta_l X_l | S_n^{\theta X} > \mathrm{VaR}_q(S_n^{\theta X})\big) \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha - 1} \frac{c_l \mathbb{E}\,\theta_l^\alpha}{\Big(\sum\limits_{k=1}^{n} c_k \mathbb{E}\,\theta_k^\alpha\Big)^{1-1/\alpha}} \mathrm{VaR}_q(Z) \tag{1.1.5}$$

for each fixed $l \in \{1, 2, \ldots, n\}$, *and*

$$\mathrm{CTE}_q\big(S_n^{\theta X}\big) \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha - 1} \Big(\sum_{k=1}^{n} c_k \mathbb{E}\,\theta_k^\alpha\Big)^{1/\alpha} \mathrm{VaR}_q(Z). \tag{1.1.6}$$

Here, in both asymptotic relations, Z *is a nonnegative r.v. distributed according to the d.f.* F.

The asymptotic results for both tail probability and tail expectation to some extent were generalized in the case of nonnegative primary r.v.s or for real-valued r.v.s with various types of tail independence, such as those described in Definitions 1.6-1.8 in Section 1.2 (see Albrecher et al. [2], Alink et al. [3], Andrulytė et al. [4], Asimit et al. [5], Cai and Li [12], Chen et al. [13], Chen et al. [14], Chen and Yuen [15], Cheng [16], Danilenko et al. [21], Dindienė and Leipus [23, 24], Gao and Wang [27], Geluk and Tang [28], Hashorva and Li [30], Hazra and Maulik [31], Huang et al. [33], Joe and Li [35], Jordanova and Stehlik [36], Li [41], Liu and Wang [43], Liu et al. [44], Nyrhinen [48, 49], Olvera-Cravioto [50], Tang and Tsitsiashvili [55], Tang and Yuan [57, 58], Wang et al. [60], Yang et al. [61], Yang and Konstantinides [62], Yang et al. [63, 66], Yang and Wang [67], Yang and Yuen [69], Yang et al. [70], Yi et al. [71], Zhang et al. [72] among others). For real-valued r.v.s with tail dependence (with positive limits in Definitions 1.6-1.8) only the asymptotic tail probability, to our knowledge, was investigated by Kortschak and Albrecher [39].

From asymptotic formulas like (1.1.5) and (1.1.6) we can obtain asymptotic decomposition of CTE. Recall that if $S_n^Z := \sum_{i=1}^n Z_i$, then the capital allocated to line $l \in \{1, \ldots, n\}$ according to the Euler principle (see, for instance, [22] or Section 6.3.2 of [46]) is

$$\mathrm{AC}_{ql}\left(S_n^Z\right) := \mathbb{E}\left(Z_l | S_n^Z > \mathrm{VaR}_q\left(S_n^Z\right)\right),$$

and the contribution of individual risk $l \in \{1, \ldots, n\}$ to the total CTE is

$$\mathrm{CIR}_{ql}\left(S_n^Z\right) := \frac{\mathrm{AC}_{ql}\left(S_n^Z\right)}{\mathrm{CTE}_q(S_n^Z)}.$$

The limiting behaviour of these measures when $q \uparrow 1$ can give valuable insight into asymptotic properties of aggregate risk. If the convergence is fast enough, the asymptotic measures could also be used as approximations for high confidence levels and serve as an alternative to a simulations based approach.

The various properties, relations and generalizations of the presented quantities can be found in [1], [6], [10], [22], [38], [45], [46], and references therein. For the asymptotic relationships between VaR and CTE in the case of heavy-tailed distributions, see Fougère and Mercadier [26], Joe and Lei [34], and Li and Zhu [42], among others.

1.2 Preliminaries

Heavy-tailed distribution classes

Definition 1.1. *A d.f. F on $\mathbb{R}$ is said to be heavy-tailed, written as $F \in \mathcal{K}$, if for every $\varepsilon > 0$, $\int_0^\infty e^{\varepsilon x} \mathrm{d}F(x) = \infty$.*

Definition 1.2. *We recall that a d.f. F on $\mathbb{R}$ is said to be subexponential and written $F \in \mathcal{S}$ if its positive part $F^+(x) = F(x)\mathbb{1}_{[0,\infty)}(x)$ satisfies the following relation: $\lim_{x\to\infty} (F^+ * F^+(x))/F^+(x) = 2$.*

Definition 1.3. *A d.f. F on $\mathbb{R}$ is said to be long-tailed, written as $F \in \mathcal{L}$, if $\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = 1$ for each fixed $y \in \mathbb{R}$.*

Definition 1.4. *A d.f. F on $\mathbb{R}$ is said to be dominatedly varying tailed, written as $F \in \mathcal{D}$, if for arbitrary fixed* $0 < y < 1$, $\limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x) < \infty$.

Each d.f. from $\mathcal{D}$ can be characterized by the upper Matuszewska index

$$\mathcal{M}_F = \inf_{y>1}\left\{ -\frac{1}{\log y}\log \liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}\right\} = -\lim_{y\to\infty}\frac{1}{\log y}\log\left(\liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}\right),$$

which is well defined for an arbitrary d.f. F with an ultimate right tail $\big(\overline{F}(x) > 0,\ x \in \mathbb{R}\big)$. It is well known that $F \in \mathcal{D}$ if and only if $0 \leqslant \mathcal{M}_F < \infty$ (see, for instance, Bingham et al. [9] or Cline and Samorodnitsky [18]). Another useful index, which is called *L-index*, was introduced by Yang and Wang [67]. This index L_F describes the behaviour of a d.f. F with an infinite tail and is defined by the equality

$$L_F = \lim_{y\downarrow 1}\liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)}. \qquad (1.2.1)$$

By the concepts presented above it follows that $F \in \mathcal{D}$ if and only if $L_F > 0$, and we have

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = \frac{1}{L_F}.$$

Example 1.1. Let a r.v. X with a distribution function F be distributed according to the generalized Peter-and-Paul distribution with parameter a, i.e.

$$\overline{F}(x) \;=\; (5^a-1)\sum_{2^{ak}>x}\frac{1}{5^{ak}} = (5^a)^{-\lfloor \log x/\log 2^a\rfloor}, \quad x \geqslant 1.$$

It can be checked that F belongs to the class $\mathcal{D}$ with L-index $L_F = \frac{1}{5^a}$.

Definition 1.5. *A d.f. F on $\mathbb{R}$ is said to be regularly varying tailed with index $-\alpha$ for some $\alpha \geqslant 0$, written as $F \in \mathcal{R}_{-\alpha}$, if for each $y > 0$,* $\lim_{x\to\infty} \overline{F}(xy)/\overline{F}(x) = y^{-\alpha}$.

Let $\mathcal{R}$ be the union of $\mathcal{R}_{-\alpha}$ over the range $0 \leqslant \alpha < \infty$. Then it is well known that $\mathcal{R} \subset \mathcal{L}\cap\mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{K}$. See, for instance, Embrechts et al. [25]. Moreover, if $F \in \mathcal{R}_{-\alpha}$, then we have

$$\mathcal{M}_F = \inf_{y>1}\left\{-\frac{\log y^{-\alpha}}{\log y}\right\} = \alpha$$

and

$$L_F = \lim_{y\downarrow 1}\frac{1}{y^\alpha} = 1.$$

Example 1.2. Let a r.v. X with a distribution function F be distributed according to the Lomax distribution with shape parameter $\alpha > 1$ and scale parameter $\lambda > 0$, i.e.

$$F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \quad x \geq 0.$$

It is also called the Pareto Type II distribution and is widely used to model insurance claims. Since

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \frac{1}{y^{\alpha}}$$

for any fixed $0 < y < 1$, we conclude that $F \in \mathcal{R}_{-\alpha}$.

Example 1.3. One of the most common distributions used for modelling financial returns is t-location-scale distribution $t(\mu, \sigma, \nu)$ with a density function

$$f(x) = \frac{1}{\sigma} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu + \left(\frac{x-\mu}{\sigma}\right)^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter, and $\nu > 0$ is the tail parameter (lower values imply heavier tails). If F is the d.f. of a t-location-scale distribution $t(\mu, \sigma, \nu)$, then $F \in \mathcal{R}_{-\nu}$. Indeed, using L'Hôpital's rule, for any fixed $y > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{yf(xy)}{f(x)} = \lim_{x\to\infty} y \left(\frac{\nu + \left(\frac{xy-\mu}{\sigma}\right)^2}{\nu + \left(\frac{x-\mu}{\sigma}\right)^2}\right)^{-\frac{\nu+1}{2}} = y^{-\nu}.$$

The following lemma presents a useful property of regularly varying distributions (see [52, Proposition 0.8 (vi)] or [5, Lemma 2.1]).

Lemma 1.1. *Let ξ and η be two r.v.s such that the d.f.s F_ξ and F_η belong to the class $\mathcal{R}_{-\alpha}$ for some positive α. Then, for some $c > 0$, $\overline{F}_\eta(x) \underset{x\to\infty}{\sim} c\overline{F}_\xi(x)$ if and only if* $\mathrm{VaR}_q(\eta) \underset{q\uparrow 1}{\sim} c^{1/\alpha}\mathrm{VaR}_q(\xi)$.

Tail dependence structures

Definition 1.6. *Two random variables ξ_1 and ξ_2 with d.f.s F_{ξ_1} and F_{ξ_2} are said to be asymptotically independent (AI) if*

$$\begin{aligned}
\lim_{q\uparrow 1} \mathbb{P}\big(F_{\xi_1}(\xi_1) > q \,|\, F_{\xi_2}(\xi_2) > q\big) &= \lim_{q\uparrow 1} \mathbb{P}\big(F_{\xi_1}(\xi_1) < 1 - q \,|\, F_{\xi_2}(\xi_2) > q\big) \\
&= \lim_{q\uparrow 1} \mathbb{P}\big(F_{\xi_2}(\xi_2) > q \,|\, F_{\xi_1}(\xi_1) > q\big) \\
&= \lim_{q\uparrow 1} \mathbb{P}\big(F_{\xi_2}(\xi_2) < 1 - q \,|\, F_{\xi_1}(\xi_1) > q\big) = 0.
\end{aligned}$$

The following dependence structure was introduced by Chen and Yuen [15].

Definition 1.7. *Two random variables* ξ_1 *and* ξ_2 *with d.f.s* F_{ξ_1} *and* F_{ξ_2} *are said to be quasi-asymptotically independent (QAI) if*

$$\lim_{x\to\infty}\frac{\mathbb{P}\big(\xi_1^+>x,\xi_2^+>x\big)}{\overline{F}_{\xi_1}(x)+\overline{F}_{\xi_2}(x)}=\lim_{x\to\infty}\frac{\mathbb{P}\big(\xi_1^+>x,\xi_2^->x\big)}{\overline{F}_{\xi_1}(x)+\overline{F}_{\xi_2}(x)}=\lim_{x\to\infty}\frac{\mathbb{P}\big(\xi_1^->x,\xi_2^+>x\big)}{\overline{F}_{\xi_1}(x)+\overline{F}_{\xi_2}(x)}=0.$$

Geluk and Tang [28] introduced the following slightly stronger dependence structure.

Definition 1.8. *Two random variables* ξ_1 *and* ξ_2 *with d.f.s* F_{ξ_1} *and* F_{ξ_2} *are said to be strongly quasi-asymptotically independent (SQAI) if*

$$\begin{aligned}\lim_{\min\{x,y\}\to\infty}\mathbb{P}\big(\xi_1^+>x\,|\,\xi_2>y\big) &= \lim_{\min\{x,y\}\to\infty}\mathbb{P}\big(\xi_1^->x\,|\,\xi_2>y\big)\\ &= \lim_{\min\{x,y\}\to\infty}\mathbb{P}\big(\xi_2^+>x\,|\,\xi_1>y\big)\\ &= \lim_{\min\{x,y\}\to\infty}\mathbb{P}\big(\xi_2^->x\,|\,\xi_1>y\big)=0.\end{aligned}$$

Various properties of QAI and SQAI r.v.s as well as related dependence structures are discussed, for instance, in [28, 41, 44, 60] and references therein.

Copula concept

Due to Sklar's theorem (see for instance Theorem 2.3.3. in [47]) every multivariate distribution can be expressed in terms of its marginal distribution functions and a copula function C, characterizing the dependence structure, i.e. for a random vector $\{X_1,\ldots,X_n\}$ with distribution functions $\{F_1,\ldots,F_n\}$

$$\mathbb{P}(X_1\leqslant x_1,\ldots,X_n\leqslant x_n)=C(F_1(x_1),\ldots,F_n(x_n)),$$

where copula C is unique if marginal distributions are continuous. The survival copula of copula C is denoted by $\hat{C}(u_1,\ldots,u_n)$ and satisfies the following equality

$$\mathbb{P}(X_1>x_1,\ldots,X_n>x_n)=\hat{C}(\bar{F}_1(x_1),\ldots,\bar{F}_n(x_n)).$$

More details regarding the copula concept can be found in Nelsen [47].

Bivariate Sarmanov distribution

Definition 1.9. *Let ξ be a real-valued r.v. with d.f. F_ξ, and let η be a nonnegative r.v. with d.f. F_η. We say that a random vector (ξ, η) follows a bivariate Sarmanov distribution if*

$$\mathbb{P}\left((\xi,\eta)\in B\right) = \iint\limits_{B} \left(1 + r\varphi(x)\psi(y)\right) \mathrm{d}F_\xi(x)\,\mathrm{d}F_\eta(y) \tag{1.2.2}$$

for each Borel set $B \subset \mathbb{R}\times[0,\infty)$. Here r is a real constant, and $\varphi : \mathbb{R}\to\mathbb{R}$, $\psi : [0,\infty)\to\mathbb{R}$ are two measurable functions satisfying the following conditions:

- $\mathbb{E}\,\varphi(\xi) = \mathbb{E}\,\psi(\eta) = 0$*;*
- $1 + r\varphi(x)\psi(y) \geqslant 0$ *for all*

$$\begin{aligned} x \in D_\xi &= \left\{x\in\mathbb{R} : \mathbb{P}(\xi\in(x-\delta,x+\delta))>0 \textit{ for all } \delta>0\right\},\\ y \in D_\eta &= \left\{y\in[0,\infty) : \mathbb{P}(\eta\in(y-\delta,y+\delta))>0 \textit{ for all } \delta>0\right\}. \end{aligned}$$

We remark that in cases: $r = 0$; $\varphi(x) = 0,\ x\in D_\xi$; $\psi(y) = 0,\ y \in D_\eta$, r.v.s. ξ and η are independent. Thus, we say that a random vector (ξ,η) follows a proper Sarmanov distribution, if $r \neq 0$ and the kernel functions are not identical to zeroes on D_ξ and D_η respectively. If (ξ,η) follows a bivariate Sarmanov distribution with the coefficient r and the kernel functions φ,ψ, then we write $(\xi,\eta)\in\mathcal{S}(r,\varphi,\psi)$. In addition, we write $(\xi,\eta)\in\mathcal{S}^*(r,\varphi,\psi)$ if (ξ,η) follows a bivariate Sarmanov distribution with the supplementary condition

$$\liminf_{\substack{x\to\infty\\ x\in D_\xi}} \inf_{y\in D_\eta} \left(1 + r\varphi(x)\psi(y)\right) > 0. \tag{1.2.3}$$

In the above form, the Sarmanov bivariate distribution was introduced by Sarmanov [53]. We can obtain various copulas from formula (1.2.2) by choosing suitable kernel functions φ and ψ. For example, if we choose $\varphi(x) = 1-2F_\xi(x)$ and $\psi(y) = 1-2F_\eta(y)$, then from (1.2.2) we get vector (ξ,η) distributed according to the well known Farlie–Gumbel–Morgenstern copula. The properties and possible generalisations of the original bivariate Sarmanov distribution can be found in Lee [40], Bairamov et al. [7] and Vernic [59]. The use of bivariate and multivariate Sarmanov distribution in various applied studies are described in [8], [19], [20], [32], [51], [54], for instance.

Vague convergence criterion

Definition 1.10. *If* $\{\mu_n, n \geqslant 1\}$ *is a sequence of measures on a locally compact Hausdorff space* $\mathbb{B}$ *with countable base, then* μ_n *converges vaguely to some measure* μ*, written as* $\mu_n \xrightarrow{v} \mu$*, if for all continuous functions* f *with compact support we have*

$$\lim_{n\to\infty} \int_{\mathbb{B}} f \mathrm{d}\mu_n = \int_{\mathbb{B}} f \mathrm{d}\mu.$$

The following statement is called the vague convergence criterion (*VCC*). See, for instance, Proposition 3.12 of Resnick [52] or Proposition A2.12 of Embrechts et al. [25].

Lemma 1.2. $\mu_n \xrightarrow{v} \mu$ *if and only if* $\lim\limits_{n\to\infty} \mu_n(A) = \mu(A)$ *for any relatively compact set* $A \in \mathcal{F}$ *such that* $\mu(\partial A) = 0$*, where* ∂A *is the boundary of set* A*.*

A thorough background on vague convergence is given by Kallenberg [37] and Resnick [52].

1.3 Aims and problems

In order to obtain asymptotic formulas for capital allocation, the asymptotic properties of both tail probability and tail expectation need to be investigated. There are plenty of results for both in the case of nonnegative risks or for real-valued risks with some type of tail independence between primary r.v.s. Therefore the aim of the thesis is to further generalize the results for real-valued r.v.s assuming different distribution classes and dependence structures.

1.4 Methods

Methods of general probability theory, integral calculus and measure convergence are used in the thesis. Numerical computations and simulation studies were performed using software environment MATLAB.

1.5 Novelty

All results presented in section 1.6 are new.

1.6 Main results

In the following two theorems we analyse the case where primary r.v.s $\{X_1, X_2, \ldots, X_n\}$ and random weights $\{\theta_1, \theta_2, \ldots, \theta_n\}$ follow some dependence structure. We leave the random vectors $\{(X_1, \theta_1), (X_2, \theta_2) \ldots, (X_n, \theta_n)\}$ independent, but we introduce the following dependence structure between X_k and θ_k $(k = 1, 2, \ldots, n)$.

Assumption 1.1. *For each $k = 1, 2, \ldots, n$, there exists a measurable function $h_k\colon [0, \infty) \mapsto (0, \infty)$ such that $\mathbb{P}(X_k > x \,|\, \theta_k = t) \underset{x\to\infty}{\sim} \overline{F}_k(x) h_k(t)$ uniformly for all $t \geqslant 0$ as $x \to \infty$, i.e.,*

$$\lim_{x\to\infty} \sup_{t\geq 0} \left| \frac{\mathbb{P}(X_k > x \mid \theta_k = t)}{\overline{F}_k(x)\, h_k(t)} - 1 \right| = 0.$$

When t is not a possible value of some θ_k, the conditional probability $\mathbb{P}(X_k > x \,|\, \theta_k = t)$ is understood as unconditional, and therefore $h_k(t) = 1$ for such t.

In addition, we note that the relation above implies $\mathbb{E}\,(h_k(\theta_k)) = 1$ for each $k = 1, 2, \ldots, n$ if d.f. F_k has an ultimate right tail, i.e., $\overline{F}_k(x) > 0$ for all $x \in \mathbb{R}$.

Theorem 1.3. *Let $\{X_1, X_2, \ldots, X_n\}$ be real-valued r.v.s having d.f.s $\{F_1, F_2, \ldots, F_n\}$, respectively, with ultimate right tails, and let the weights $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be nonnegative nondegenerated at zero r.v.s. Also assume that the vectors $\{(X_1, \theta_1), (X_2, \theta_2), \ldots, (X_n, \theta_n)\}$ are mutually independent and satisfy Assumption 1.1. In addition, for any $k \in \{1, 2, \ldots, n\}$, we suppose that $F_k \in \mathcal{L} \cap \mathcal{D}$ and $\max\left\{\mathbb{E}\theta_k^{p_k} h_k(\theta_k),\, \mathbb{E}\theta_k^{p_k}\right\} < \infty$ for some $p_k > \mathcal{M}_{F_k}$. Then, we have*

$$\mathbb{E}\,\theta_l X_l \mathbb{1}_{\{S_n^{\theta X} > x\}} \underset{x\to\infty}{\sim} \mathbb{E}\,\theta_l X_l \mathbb{1}_{\{\theta_l X_l > x\}}$$

for each $l \in \{1, 2, \ldots, n\}$ under the assumption that $\overline{F}_k(x) = O\big(\overline{F}_l(x)\big)$ for $k \in \{1, 2, \ldots, n\}$. Consequently, relation (1.1.4) *holds if the requirement $\overline{F}_k(x) \asymp \overline{F}_1(x)$ is satisfied for each $k \in \{1, 2, \ldots, n\}$.*

For the next result, we restrict the dependence structure between r.v.s X_k and θ_k, $k \in \{1, 2, ..., n\}$, to a bivariate Sarmanov distribution (see Definition 1.9). For such dependence structure, we obtain an asymptotic formula for $\mathrm{CTE}_q\big(S_n^{\theta X}\big)$ similar to (1.1.6).

Theorem 1.4. *Let $\{X_1, X_2, \ldots, X_n\}$ be real-valued r.v.s having d.f.s $\{F_1, F_2, \ldots, F_n\}$, respectively, with ultimate right tails, and let the weights $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be nonnegative nondegenerated at zero r.v.s. Also assume that the vectors $\{(X_1, \theta_1), (X_2, \theta_2), \ldots, (X_n, \theta_n)\}$ are mutually independent and for each $k \in \{1, 2, ..., n\}$, the random vector (X_k, θ_k) follows a bivariate Sarmanov distribution $S^*(r_k, \varphi_k, \psi_k)$. In addition, we suppose that:*

- *there exists a d.f. $F \in \mathcal{R}_{-\alpha}$, $\alpha > 1$, and positive constants $c_1, ..., c_n$ such that $\overline{F}_k(x) \underset{x\to\infty}{\sim} c_k \overline{F}(x)$ for $k = 1, 2, ..., n$;*
- *$\max\limits_{1\leqslant k\leqslant n} \mathbb{E}\, \theta_k^p < \infty$ for some $p > \alpha$;*
- *for every $k \in \{1, 2, ..., n\}$, the function ψ_k is uniformly continuous, and there exists $d_k < \infty$ such that $\lim\limits_{x\to\infty} \varphi_k(x) = d_k$.*

Then, for each fixed $l \in \{1, 2, ..., n\}$, we have

$$\mathbb{E}\big(\theta_l X_l | S_n^{\theta X} > \mathrm{VaR}_q(S_n^{\theta X})\big) \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha - 1} \frac{c_l \tau_l}{\big(\sum_{k=1}^{n} c_k \tau_k\big)^{1-\frac{1}{\alpha}}} \mathrm{VaR}_q(Z)\,,$$

where $\mathrm{VaR}_q(Z) = F^{-1}(q)$ for $q \in (0, 1)$, and $\tau_k = \mathbb{E}\theta_k^\alpha + r_k d_k\, \mathbb{E}\, \psi_k(\theta_k)\, \theta_k^\alpha$ for all $k \in \{1, 2, \ldots, n\}$. Consequently,

$$\mathrm{CTE}_q\big(S_n^{\theta X}\big) \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha - 1} \left(\sum_{k=1}^{n} c_k \tau_k\right)^{1/\alpha} \mathrm{VaR}_q(Z).$$

For the following results, we allow primary r.v.s to be dependent with the assumption of QAI or SQAI (see Definitions 1.7 and 1.8) and obtain asymptotic bounds for the tail probability and tail expectation in the case of dominatedly varying distributions (see Definition 1.4).

Theorem 1.5. *Let $\{X_1, X_2, \ldots, X_n\}$ be a collection of n pairwise QAI real-valued r.v.s with corresponding d.f.s $\{F_1, F_2, \ldots, F_n\}$ such that $F_i \in \mathcal{D}$ for all $i \in \{1, 2, \ldots, n\}$. Let $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be a collection of arbitrarily dependent, nonnegative and nondegenerated at zero r.v.s. If the collections $\{X_1, X_2, \ldots, X_n\}$*

and $\{\theta_1, \theta_2, \ldots, \theta_n\}$ *are independent and* $\max_{1\leqslant k\leqslant n}\{\mathbb{E}\theta_k^p\}$ *is finite for some power* $p > \max_{1\leqslant k\leqslant n}\{\mathcal{M}_{F_k}\}$*, then*

$$L_n^X \sum_{k=1}^n \mathbb{P}\big(\theta_k X_k > x\big) \underset{x\to\infty}{\lesssim} \mathbb{P}\big(S_n^{\theta X} > x\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^X} \sum_{k=1}^n \mathbb{P}\big(\theta_k X_k > x\big),$$

where $L_n^X = \min\{L_{F_1}, L_{F_2}, \ldots, L_{F_n}\}$.

A similar result has been established in Theorem 1 by Yi et al. [71]. Nevertheless, we present the proof of Theorem 1.5 in Section 2.4 for the following three reasons. Firstly, Theorem 1 in [71] is formulated without a proof. Secondly, it has an extra condition. Thirdly, the proof of our Theorem 1.5 is based on the lemmas, which will be useful to obtain the following theorem.

Theorem 1.6. *Let all the conditions of Theorem 1.5 be satisfied with the pairwise SQAI r.v.s* $\{X_1, X_2, \ldots, X_n\}$. *If* $\mathbb{P}(\theta_k X_k > x) = O\big(\mathbb{P}(\theta_l X_l > x)\big)$ *for all* $k \in \{1, 2, \ldots, n\}$ *and some* $l \in \{1, 2, \ldots, n\}$ *under the condition* $\mathbb{E}\theta_l |X_l| < \infty$*, then*

$$L_{F_l}\mathbb{E}\big(\theta_l X_l \mathbb{1}_{\{\theta_l X_l > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\theta_l X_l \mathbb{1}_{\{S_n^{\theta X} > x\}}\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_{F_l}}\mathbb{E}\big(\theta_l X_l \mathbb{1}_{\{\theta_l X_l > x\}}\big),$$

and, consequently,

$$L_n^X \sum_{k=1}^n \mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{\theta_k X_k > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\big(S_n^{\theta X} \mathbb{1}_{\{S_n^{\theta X} > x\}}\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^X} \sum_{k=1}^n \mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{\theta_k X_k > x\}}\big),$$

if $\mathbb{E}\theta_1 |X_1| < \infty$ *and* $\mathbb{P}(\theta_k X_k > x) \underset{x\to\infty}{\asymp} \mathbb{P}(\theta_1 X_1 > x)$ *for all* $k \in \{1, 2, \ldots, n\}$.

Both theorems above together with inequality (2.4.1) imply immediately the following assertion on the asymptotic behaviour of the expectation of the truncated randomly weighted sum $S_n^{\theta X}$. We formulate the assertion below for the case where the collection $\{X_1, X_2, \ldots, X_n\}$ consists of pairwise SQAI r.v.s.

Corollary 1.1. *Let* $\{X_1, X_2, \ldots, X_n\}$ *be a collection of* n *pairwise SQAI real-valued r.v.s with corresponding d.f.s* $\{F_1, F_2, \ldots, F_n\}$ *such that* $F_i \in \mathcal{D}$ *for all* $i \in \{1, 2, \ldots, n\}$. *Let* $\{\theta_1, \theta_2, \ldots, \theta_n\}$ *be a collection of arbitrarily dependent, nonnegative and nondegenerate at zero r.v.s such that* $\max_{1\leqslant k\leqslant n}\mathbb{E}\theta_k^p < \infty$ *for some* $p > \max_{1\leqslant k\leqslant n}\{\mathcal{M}_{F_k}\}$.

(i) *If collections* $\{X_1, X_2, \ldots, X_n\}$ *and* $\{\theta_1, \theta_2, \ldots, \theta_n\}$ *are independent and*

$\mathbb{P}(\theta_k X_k > x) = O\big(\mathbb{P}(\theta_l X_l > x)\big)$ *for all* $k \in \{1, 2, \ldots, n\}$ *and some* $l \in \{1, 2, \ldots, n\}$ *under condition* $\mathbb{E}\theta_l |X_l| < \infty$, *then*

$$L_n^X \frac{L_{F_l} \mathbb{E}\big(\theta_l X_l \mathbb{1}_{\{\theta_l X_l > x\}}\big)}{\sum\limits_{k=1}^{n} \mathbb{P}(\theta_k X_k > x)} \underset{x\to\infty}{\lesssim} \mathbb{E}\big(\theta_l X_l | S_n^{\theta X} > x\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^X} \frac{\mathbb{E}\big(\theta_l X_l \mathbb{1}_{\{\theta_l X_l > x\}}\big)}{L_{F_l} \sum\limits_{k=1}^{n} \mathbb{P}(\theta_k X_k > x)}.$$

(ii) *If collections* $\{X_1, X_2, \ldots, X_n\}$ *and* $\{\theta_1, \theta_2, \ldots, \theta_n\}$ *are independent,* $\mathbb{E}\theta_1 |X_1| < \infty$ *and* $\mathbb{P}(\theta_k X_k > x) \underset{x\to\infty}{\asymp} \mathbb{P}(\theta_1 X_1 > x)$ *for all* $k \in \{1, 2, \ldots, n\}$, *then*

$$\mathbb{E}\big(S_n^{\theta X} | S_n^{\theta X} > x\big) \underset{x\to\infty}{\lesssim} \left(\frac{1}{L_n^X}\right)^2 \frac{\sum\limits_{k=1}^{n} \mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{\theta_k X_k > x\}}\big)}{\sum\limits_{k=1}^{n} \mathbb{P}(\theta_k X_k > x)} \leqslant \left(\frac{1}{L_n^X}\right)^2 \max_{1\leqslant k\leqslant n} \mathbb{E}\big(\theta_k X_k | \theta_k X_k > x\big),$$

$$\mathbb{E}\big(S_n^{\theta X} | S_n^{\theta X} > x\big) \underset{x\to\infty}{\gtrsim} \big(L_n^X\big)^2 \frac{\sum\limits_{k=1}^{n} \mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{\theta_k X_k > x\}}\big)}{\sum\limits_{k=1}^{n} \mathbb{P}(\theta_k X_k > x)} \geqslant \big(L_n^X\big)^2 \min_{1\leqslant k\leqslant n} \mathbb{E}\big(\theta_k X_k | \theta_k X_k > x\big).$$

In the next theorem we assume $\theta_1 = \cdots = \theta_n = 1$ and analyse the case where primary r.v.s have tail dependence and regularly varying distributions (see Definition 1.5). We note that although Asimit et al. [5] and Joe and Li [35] allow tail dependence for nonnegative r.v.s, their assumptions do not apply for dependence structures obtained through mixtures of both tail dependence and tail independence. Therefore we generalize the results for real-valued r.v.s with wider conditions for dependence.

We denote before the formulation of the statement r.v.s $X_k^{(i)} = i_k X_k \mathbb{1}_{\{i_k X_k > 0\}}$ for each $k \in \{1, 2, \ldots, n\}$ and each collection $i = \{i_1, \ldots, i_n\} \in I = \{-1, 1\}^n \diagdown \{-1\}^n$.

Theorem 1.7. *Let* $\{X_1, X_2, \ldots, X_n\}$ *be a collection of* n *real-valued and continuous at zero r.v.s with d.f.s* $\{F_1, F_2, \ldots, F_n\}$ *such that* $F_k \in \mathcal{R}_{-\alpha}$, $\alpha > 1$, *and* $0 < \lim\limits_{t\to\infty} \frac{\mathbb{P}(-X_k > t)}{\overline{F_k}(t)} < \infty$ *for all* $k \in \{1, 2, \ldots, n\}$. *Additionally, for all* $k \in \{1, 2, \ldots, n\}$ *and all collections* $i = \{i_1, \ldots, i_n\} \in I = \{-1, 1\}^n \diagdown \{-1\}^n$, *assume that*

$$\lim_{t\to\infty} \frac{\mathbb{P}(X_1^{(i)} > tx_1, \ldots, X_n^{(i)} > tx_n)}{\overline{F}_1(t)} := H^{(i)}(\underline{x})$$

exists for all $\underline{x} = (x_1, \ldots, x_n) \in [0, \infty]^n \diagdown \{0\}^n$ *such that*

$$H^{(i)}(\underline{x}) = a^{(i)} \mu_D^{(i)}((x_1, \infty] \times \cdots \times (x_n, \infty]) + (1 - a^{(i)}) \mu_I^{(i)}((x_1, \infty] \times \cdots \times (x_n, \infty])$$

for some constant $a^{(i)} \in [0,1]$, where $\mu_D^{(i)}$ is a Radon measure (i.e. finite on compact sets), such that $\mu_D^{(i)}((x_1,\infty]\times\cdots\times(x_n,\infty])$ is continuous on $[0,\infty]^n\setminus\{0\}^n$ (i.e. the measure $\mu_D^{(i)}$ does not put any mass on the boundary of the domain) and $\mu_I^{(i)}$ is a Radon measure that puts mass only on the coordinate axes. Further assume that for all $z>0$ and each $k\in\{1,2,\ldots,n\}$ measure $\mu_D^{(i)}$ satisfies the following equality

$$\mu_D^{(i)}(\underline{x} : x_k > z) = c_k z^{-\alpha} \tag{1.6.1}$$

for each $i \in I$ such that $i_k = 1$, where $c_1,\ldots,c_n$ are positive constants such that $\overline{F}_k(t) \underset{t\to\infty}{\sim} c_k\overline{F_1}(t)$, $k\in\{1,2,\ldots,n\}$.

Then, for all $k\in\{1,2,\ldots,n\}$ we have

$$\mathbb{E}[X_k|S_n^X > t] \underset{t\to\infty}{\sim} \frac{\sum\limits_{i\in I} i_k a^{(i)} \int\limits_0^\infty \mu_D^{(i)}\left(A_k^{(i)}(z)\right)\mathrm{d}z + \frac{\alpha}{\alpha-1}\left(1-\sum\limits_{\substack{i\in I\\ i_k=1}} a^{(i)}\right)c_k}{\sum\limits_{i\in I} a^{(i)}\mu_D^{(i)}\left(A^{(i)}\right) + \sum\limits_{k=1}^n\left(1-\sum\limits_{\substack{i\in I\\ i_k=1}} a^{(i)}\right)c_k}\, t, \tag{1.6.2}$$

where

$$S_n^X := \sum_{k=1}^n X_k, \quad A_k^{(i)}(z) := \left\{\underline{x} : x_k > z, \sum_{j=1}^n i_j x_j > 1\right\}, \quad A^{(i)} := \left\{\underline{x} : \sum_{j=1}^n i_j x_j > 1\right\}.$$

Remark 1.1. Theorem 1.7 imediately gives the asymptotic relations for $\mathrm{AC}_{qk}\left(S_n^X\right)$ and $\mathrm{CIR}_{qk}\left(S_n^X\right)$ as $q\uparrow 1$, if we replace t by $\mathrm{VaR}_q(S_n^X)$. Also, since $\mathbb{P}\left(S_n^X > t\right) \underset{t\to\infty}{\sim} D\overline{F_1}(t)$, where D equals the denominator in (1.6.2) (see the proof of Theorem 1.7), the distribution of S_n^X belongs to the class $\mathcal{R}_{-\alpha}$ and, due to Lemma 1.1, $\mathrm{VaR}_q(S_n^X) \underset{q\uparrow 1}{\sim} D^{\frac{1}{\alpha}}\mathrm{VaR}_q(X_1)$. Moreover, $\mathrm{CTE}_q(S_n^X) \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha-1}\mathrm{VaR}_q(S_n^X)$ (see, for instance, Alink et al. [3] or Joe and Li [35]), which implies that the constants on the right hand side of (1.6.2) sum to $\frac{\alpha}{\alpha-1}$.

1.7 Publications

- Yang Y., Ignatavičiūtė E. and Šiaulys J. (2015). Conditional tail expectation of randomly weighted sums with heavy-tailed distributions. *Statistics and Probability Letters*, 105:20–28.

- Jaunė E., Ragulina O. and Šiaulys J. (2018). Expectation of the truncated randomly weighted sums with dominatedly varying summands. *Lithuanian Mathematical Journal* (accepted).

The result stated in Theorem 1.7 is submitted as:

- Jaunė E. and Šiaulys J. Asymptotic risk decomposition for regularly varying distributions with tail dependence.

1.8 Conferences

The results of the thesis were presented in the following conferences:

- International Conference "Modern Stochastics: Theory and Applications. IV", Kyiv, Ukraine, 2018 05 24-26.
- 59th Conference of Lithuanian Mathematical Society, Kaunas, Lithuania, 2018 06 18-19.

1.9 Structure of the thesis

The rest of the thesis is organized as follows. In Chapter 2 we prove the main results. A few relative examples demonstrating the applicability of the main results are discussed in Chapter 3. In Chapter 4 we carry out certain simulation studies verifying the accuracy of the results of Theorem 1.7 and revealing the spead of convergence. Finally, Chapter 5 concludes the thesis.

1.10 Acknowledgements

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Chapter **2**

Proofs of the main results

2.1 Auxiliary lemmas

This section consists of several lemmas which we use to prove Theorems 1.3 and 1.4.

Lemma 2.1. *(see Proposition 2.2.1 in [9] and Lemma 3.5 in [55]) Let V be a d.f. belonging to class $\mathcal{D}$. For any $p > \mathcal{M}_V$ there exist positive constants $c = c(V)$ and $d = d(V)$ such that*

$$\frac{\overline{V}(y)}{\overline{V}(x)} \leqslant c\left(\frac{y}{x}\right)^{-p} \text{for all } x \geqslant y \geqslant d.$$

In addition, for any $p > \mathcal{M}_V$ it holds that $x^{-p} = o\left(\overline{V}(x)\right)$.

Lemma 2.2. *(see Lemma 2.1 in [64]) Let ξ be a real-valued r.v. with d.f. F_ξ, and let η be a nonnegative nondegenerated at zero r.v. with d.f. F_η. Assume that there exists a measurable function $h\colon [0,\infty) \mapsto (0,\infty)$ such that*

$$\lim_{x\to\infty} \sup_{t\geq 0} \left| \frac{\mathbb{P}(\xi > x \,|\, \eta = t)}{\overline{F}_\xi(x)h(t)} - 1 \right| = 0.$$

If $F_\xi \in \mathcal{L}$ and $\overline{F}_\eta(x) = o\left(\overline{F}_\xi(cx)\right)$ for some $c > 0$, then d.f. $\mathbb{P}(\xi\eta \leqslant x)$ belongs to the class $\mathcal{L}$.

Lemma 2.3. *(see Lemma 2 in [65]) Let ξ and η be random variables satisfying the basic conditions of* Lemma 2.2. *If $F_\xi \in \mathcal{D}$ and $\overline{F}_\eta(x) = o\left(\overline{F}_\xi(x)\right)$, then d.f. $\mathbb{P}(\xi\eta \leqslant x)$ belongs to the class $\mathcal{D}$ as well.*

Lemma 2.4. *(see Theorem 1 in [65]) Let* $\{X_1, X_2, \dots, X_n\}$ *be real-valued r.v.s having d.f.s* $\{F_1, F_2, \dots, F_n\}$ *respectively with ultimate right tails and let weights* $\{\theta_1, \theta_2, \dots, \theta_n\}$ *be nonnegative nondegenerated at zero r.v.s. Also assume that vectors* $\{(X_1, \theta_1), (X_2, \theta_2), \dots, (X_n, \theta_n)\}$ *are mutually independent and satisfy Assumption 1.1. If, in addition,* $F_k \in \mathcal{L} \cap \mathcal{D}$ *and* $\mathbb{P}(\theta_k > x) = o\left(\overline{F}_k(x)\right)$ *for all* $k \in \{1, 2, \dots, n\}$*, then*

$$\mathbb{P}\left(\max_{1\leqslant k\leqslant n} S_k^{\theta X} > x\right) \underset{x\to\infty}{\sim} \mathbb{P}\left(S_n^{\theta X} > x\right) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \mathbb{P}\left(\theta_k X_k > x\right).$$

Remark 2.1. In the statement of Theorem 1 of [65] it is assumed that r.v.s $\{\theta_1, \theta_2, \dots, \theta_n\}$ are positive. But the statement of this theorem still holds in the case of nonnegative and nondegenerated at zero r.v.s $\{\theta_1, \theta_2, \dots, \theta_n\}$. To obtain this we only remark that $(\theta_k X_k)^+ = \theta_k X_k^+$ if r.v. θ_k is nonnegative (see page 524 in [65]).

Lemma 2.5. *Suppose that* ξ *is a real-valued r.v. with a d.f.* F_ξ *having an ultimate right tail, and* η *is a nonnegative r.v. with a d.f.* F_η *such that* $(\xi, \eta) \in S^*(r, \varphi, \psi)$ *(see Definition 1.9). If* ψ *is uniformly continuous on* D_η *and* $\lim_{x\to\infty} \varphi(x) = d < \infty$*, then there exists a measurable function* $h : [0, \infty) \mapsto (0, \infty)$ *such that* $\sup_{y\in[0,\infty)} h(y) < \infty$*, and*

$$\mathbb{P}(\xi > x \mid \eta = y) \underset{x\to\infty}{\sim} \overline{F}_\xi(x) h(y) \tag{2.1.1}$$

uniformly for all $y \geqslant 0$.

Proof. If ξ and η are independent, then $h \equiv 1$, and the statement of the lemma holds. If (ξ, η) is a proper Sarmanov distribution, then

$$\sup_{x\in D_\xi} |\varphi(x)| < \infty \text{ and } \sup_{y\in D_\eta} |\psi(y)| < \infty \tag{2.1.2}$$

via Proposition 1.1. of [68].

Let in such a case

$$h(y) = \begin{cases} 1 + rd\psi(y) & if \quad y \in D_\eta, \\ 1 & if \quad y \notin D_\eta \end{cases}.$$

Applying (1.2.3) and (2.1.2), we get that

$$0 < \inf_{y\in[0,\infty)} h(y) \leqslant \sup_{y\in[0,\infty)} h(y) < \infty.$$

It remains to prove the asymptotic relation (2.1.1). If $y \notin D_\eta$, then

$$\mathbb{P}(\xi > x \mid \eta = y) = \overline{F}_\xi(x)\, h(y). \tag{2.1.3}$$

If $y \in D_\eta$, then

$$\mathbb{P}(\xi > x \mid \eta = y) = \lim_{\delta \downarrow 0} \frac{\int\limits_x^\infty \int\limits_{y-\delta}^{y+\delta} (1 + r\varphi(u)\psi(v))\, \mathrm{d}F_\eta(v)\, \mathrm{d}F_\xi(u)}{\int\limits_{y-\delta}^{y+\delta} \mathrm{d}F_\eta(v)}$$

due to the basic equality (1.2.2).

Applying (2.1.2) and additional conditions for the kernel functions φ, ψ, we obtain

$$\begin{aligned}
&\limsup_{x\to\infty} \sup_{y \in D_\eta} \frac{\mathbb{P}(\xi > x \mid \eta = y)}{\overline{F}_\xi(x)\, h(y)} \\
&= \limsup_{x\to\infty} \sup_{y \in D_\eta} \lim_{\delta\downarrow 0} \frac{\int\limits_x^\infty \int\limits_{y-\delta}^{y+\delta} \Big(1 + rd\psi(v) + r\big(\varphi(u) - d\big)\psi(v)\Big)\, \mathrm{d}F_\eta(v)\, \mathrm{d}F_\xi(u)}{\overline{F}_\xi(x)\, h(y) \int\limits_{y-\delta}^{y+\delta} \mathrm{d}F_\eta(v)} \\
&\leqslant \limsup_{x\to\infty} \sup_{y \in D_\eta} \lim_{\delta\downarrow 0} \left(\frac{\int\limits_{y-\delta}^{y+\delta} (1 + rd\psi(v))\, \mathrm{d}F_\eta(v)}{h(y) \int\limits_{y-\delta}^{y+\delta} \mathrm{d}F_\eta(v)} + |r| \sup_{\substack{u \geqslant x \\ u \in D_\xi}} |\varphi(u) - d| \frac{\sup\limits_{v \in D_\eta} |\psi(v)|}{\inf\limits_{y \in D_\eta} h(y)} \right) \\
&= \sup_{y \in D_\eta} \lim_{\delta\downarrow 0} \frac{\int\limits_{y-\delta}^{y+\delta} (h(y) + rd(\psi(v) - \psi(y)))\mathrm{d}F_\eta(v)}{h(y) \int\limits_{y-\delta}^{y+\delta} \mathrm{d}F_\eta(v)} \\
&\leqslant 1 + |rd| \lim_{\delta\downarrow 0} \sup_{\substack{y, v \in D_\eta \\ |v-y| \leqslant \delta}} |\psi(v) - \psi(y)| \frac{1}{\inf\limits_{y \in D_\eta} h(y)} \\
&= 1.
\end{aligned}$$

The last estimate and equality (2.1.3) imply that

$$\limsup_{x\to\infty} \sup_{y \in [0,\infty)} \frac{\mathbb{P}(\xi > x \mid \eta = y)}{\overline{F}_\xi(x)\, h(y)} \leqslant 1.$$

In a similar way it can be shown that

$$\liminf_{x\to\infty} \inf_{y \in [0,\infty)} \frac{\mathbb{P}(\xi > x \mid \eta = y)}{\overline{F}_\xi(x)\, h(y)} \geqslant 1.$$

Hence, the desired relation (2.1.1) follows, and Lemma 2.5 is proved. □

The next lemma is a version of the well-known Breiman's lemma (see [11]) for dependent random variables. The proof of Lemma 2.6 can be found in [68].

Lemma 2.6. *Let $(\xi,\eta)\in S(r,\varphi,\psi)$, $F_\xi\in\mathcal{R}_{-\alpha}$ with $\alpha\geqslant 0$, $\mathbb{E}\eta^p<\infty$ for some $p>\alpha$ and $\lim\limits_{x\to\infty}\varphi(x)=d<\infty$. Then*

$$\mathbb{P}(\xi\eta>x)\underset{x\to\infty}{\sim}\big(\,\mathbb{E}\,\eta^\alpha+rd\,\mathbb{E}\,\psi(\eta)\eta^\alpha\,\big)\,\overline{F}_\xi(x).$$

The following assertion follows from Theorem 1.1 immediately with $\theta_1=\theta_2=\ldots=\theta_n\equiv 1$.

Lemma 2.7. *Let $\{X_1,X_2,\ldots,X_n\}$ be real-valued independent r.v.s with d.f.s $\{F_1,F_2,\ldots,F_n\}$ respectively and $S_n=\sum_{i=1}^n X_i$. Further, let $F_k\in\mathcal{L}\cap\mathcal{D}$ and $\overline{F}_k(x)=O\big(\overline{F}_1(x)\big)$ for each $k\in\{1,2,\ldots,n\}$. Then*

$$\mathbb{E}\,X_1\mathbb{1}_{\{S_n>x\}}\underset{x\to\infty}{\sim}\mathbb{E}\,X_1\mathbb{1}_{\{X_1>x\}}.$$

If, in addition, $\overline{F}_k(x)\asymp\overline{F}_1(x)$ for $k\in\{1,2,\ldots,n\}$, then

$$\mathbb{E}S_n\mathbb{1}_{\{S_n>x\}}\underset{x\to\infty}{\sim}\sum_{i=1}^n\mathbb{E}X_i\mathbb{1}_{\{X_i>x\}}.$$

2.2 Proof of Theorem 1.3

It is sufficient to consider the case where $l=1$ and $n\geqslant 2$. We suppose that $\overline{F}_k(x)=O\big(\overline{F}_1(x)\big)$, $k\in\{1,2,\ldots,n\}$, and we prove that

$$\mathbb{E}\,\theta_1X_1\mathbb{1}_{\{S_n^{\theta X}>x\}}\underset{x\to\infty}{\sim}\mathbb{E}\,\theta_1X_1\mathbb{1}_{\{\theta_1X_1>x\}}.\tag{2.2.1}$$

Lemma 2.7 is the crucial assertion in our proof of (2.2.1). For any $k\in\{1,2,\ldots,n\}$, an arbitrary positive c, and $p_k>\mathcal{M}_{F_k}$, we have that

$$\mathbb{P}(\theta_k>x)\leqslant\frac{\mathbb{E}\,\theta_k^{p_k}}{x^{p_k}}=\frac{\mathbb{E}\theta_k^{p_k}}{(cx)^{p_k}}c^{p_k}=o\left(\overline{F}_k(cx)\right)\tag{2.2.2}$$

via Markov's inequality and Lemma 2.1. Hence, by Lemmas 2.2 and 2.3 we obtain that d.f. $\mathbb{P}(\,\theta_kX_k\leqslant x)$ belongs to the class $\mathcal{L}\cap\mathcal{D}$ for each fixed $k\in\{1,2,\ldots,n\}$.

Random vectors $(X_1,\theta_1),(X_2,\theta_2),\ldots,(X_n,\theta_n)$ are independent. Therefore $\{\theta_1X_1,\theta_2X_2,\ldots,\theta_nX_n\}$ are independent r.v.s. The desired relation (2.2.1) follows from Lemma 2.7 if we establish that

$$\mathbb{P}(\theta_kX_k>x)=O\big(\mathbb{P}(\theta_1X_1>x)\big),\ k\in\{1,2,\ldots,n\}.\tag{2.2.3}$$

Let $k \in \{2, 3, \ldots, n\}$ be temporarily fixed. If $x \geqslant 2y \geqslant 2$, then we have

$$\mathbb{P}(\theta_k X_k > x) = \Bigg(\int\limits_{[0,1]} + \int\limits_{\left(1, \frac{x}{y}\right]} + \int\limits_{\left(\frac{x}{y}, \infty\right)} \Bigg) \mathbb{P}\left(X_k > \frac{x}{u} \,\Big|\, \theta_k = u \right) \mathrm{d}\,\mathbb{P}(\theta_k \leqslant u).$$

If y is sufficiently large, then Assumption 1.1 implies that

$$\mathbb{P}\left(X_k > \frac{x}{u} \,\Big|\, \theta_k = u \right) \leqslant 2\overline{F}_k\left(\frac{x}{u}\right) h_k(u)$$

for all $u \leqslant \frac{x}{y}$. By Lemma 2.1 and Markov's inequality we have

$$\begin{aligned} \mathbb{P}(\theta_k X_k > x) &\leqslant 2\Bigg(\int\limits_{[0,1]} + \int\limits_{\left(1, \frac{x}{y}\right]} \Bigg) \overline{F}_k\left(\frac{x}{u}\right) h_k(u)\,\mathrm{d}\,\mathbb{P}(\theta_k \leqslant u) + \mathbb{P}\left(\theta_k > \frac{x}{y}\right) \\ &\leqslant 2\overline{F}_k(x) \int\limits_{[0,1]} h_k(u)\,\mathrm{d}\,\mathbb{P}(\theta_k \leqslant u) + 2c_{1k}\overline{F}_k(x) \int\limits_{\left(1, \frac{x}{y}\right]} u^{p_k} h_k(u)\,\mathrm{d}\,\mathbb{P}(\theta_k \leqslant u) \\ &\quad + \left(\frac{y}{x}\right)^{p_k} \mathbb{E}\,\theta_k^{p_k}, \end{aligned}$$

where $x \geqslant 2y$, y is sufficiently large, exponent $p_k > \mathcal{M}_{F_k}$ is from conditions of Theorem 1.3, and c_{1k} is a positive constant originating from Lemma 2.1.

Since $\mathbb{E}\,h_k(\theta_k) = 1$ and $x^{-p_k} = o\left(\overline{F}_k(x)\right)$, the last estimate implies that

$$\begin{aligned} \limsup_{x\to\infty} \frac{\mathbb{P}(\theta_k X_k > x)}{\overline{F}_k(x)} &\leqslant 2\left(1 + c_{1k}\mathbb{E}\,\theta_k^{p_k} h_k(\theta_k)\right) + y^{p_k}\mathbb{E}\,\theta_k^{p_k} \limsup_{x\to\infty} \frac{1}{x^{p_k}\overline{F}_k(x)} \\ &= 2\left(1 + c_{1k}\mathbb{E}\,\theta_k^{p_k} h_k(\theta_k)\right) < \infty \end{aligned}$$

due to the conditions of Theorem 1.3.

Therefore

$$\limsup_{x\to\infty} \frac{\mathbb{P}(\theta_k X_k > x)}{\overline{F}_1(x)} < \infty \tag{2.2.4}$$

because $F_k \in \mathcal{L} \cap \mathcal{D}$ and $\overline{F}_k(x) = O\left(\overline{F}_1(x)\right)$.

On the other hand, by $\mathbb{E}\,h_1(\theta_1) = 1$ we can choose a constant $c_1 \in (0,1)$ such that $\mathbb{E}\,h_1(\theta_1)\mathbb{I}_{\{c_1 < \theta_1 < 1/c_1\}} > 0$. Since

$$\mathbb{P}(\theta_1 X_1 > x) \geqslant \int\limits_{c_1}^{1/c_1} \frac{\mathbb{P}\left(X_1 > \frac{x}{u} \,|\, \theta_1 = u\right)}{\overline{F}_1\left(\frac{x}{u}\right) h_1(u)}\, \overline{F}_1\left(\frac{x}{u}\right) h_1(u)\,\mathrm{d}\,\mathbb{P}(\theta_1 \leqslant u),$$

the Fatou lemma implies that

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{\mathbb{P}(\theta_1 X_1 > x)}{\overline{F}_1(x)} \\
&\quad \geqslant \liminf_{x\to\infty} \frac{\overline{F}_1\left(\frac{x}{c_1}\right)}{\overline{F}_1(x)} \int\limits_{c_1}^{1/c_1} \liminf_{x\to\infty} \frac{\mathbb{P}\left(X_1 > \frac{x}{u} \,|\, \theta_1 = u\right)}{\overline{F}_1\left(\frac{x}{u}\right)\, h_1(u)}\, h_1(u)\, \mathrm{d}\,\mathbb{P}(\theta_k \leqslant u) \\
&= \mathbb{E}\, h_1(\theta_1) 1\!\!1_{\{c_1 < \theta_1 < 1/c_1\}} \liminf_{x\to\infty} \frac{\overline{F}_1\left(x/c_1\right)}{\overline{F}_1(x)} > 0
\end{aligned}$$

due to the choice of c_1, Assumption 1.1 and the definition of class $\mathcal{D}$. The last estimate proves that

$$\limsup_{x\to\infty} \frac{\overline{F}_1(x)}{\mathbb{P}(\theta_1 X_1 > x)} < \infty. \tag{2.2.5}$$

For fixed k, the asymptotic bound (2.2.3) follows now from (2.2.4) and (2.2.5) immediately. Theorem 1.3 is proved.

2.3 Proof of Theorem 1.4

The conditions of Theorem 1.4 and Lemma 2.5 imply that random variables $X_1, ..., X_n$ and $\theta_1, ..., \theta_n$ satisfy Assumption 1.1 with $h_k(t) = 1 + r_k d_k \psi_k(t),\ t \in [0, \infty),\ k \in \{1, 2, ..., n\}$. According to Lemma 2.1,

$$\mathbb{P}(\theta_k > x) \leqslant \frac{\mathbb{E}\,\theta_k^p}{x^p} = o\left(\overline{F}_k(x)\right)$$

for each $k = 1, 2, ..., n$. Therefore, due to Lemma 2.4,

$$\mathbb{P}\left(S_n^{\theta X} > x\right) \underset{x\to\infty}{\sim} \sum_{k=1}^{n} \mathbb{P}\left(\theta_k X_k > x\right).$$

On the other hand, Lemma 2.6 implies that

$$\mathbb{P}\left(\theta_k X_k > x\right) \underset{x\to\infty}{\sim} \tau_k \overline{F}_k(x) \underset{x\to\infty}{\sim} \tau_k c_k \overline{F}(x)$$

for each $k \in \{1, ..., n\}$. Hence, we have

$$\mathbb{P}\left(S_n^{\theta X} > x\right) \underset{x\to\infty}{\sim} \overline{F}(x) \sum_{k=1}^{n} \tau_k c_k. \tag{2.3.1}$$

Let now $l \in \{1, 2, ..., n\}$ be fixed. Applying Theorem 1.3 we get:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\mathbb{E}\,\theta_l X_l \mathbb{1}_{\{S_n^{\theta X}>x\}}}{x\overline{F}(x)} &= \lim_{x\to\infty} \frac{\mathbb{E}\,\theta_l X_l \mathbb{1}_{\{\theta_l X_l>x\}}}{x\overline{F}(x)} = \lim_{x\to\infty} \frac{-\int\limits_x^\infty u \mathrm{d}\mathbb{P}\left(\theta_l X_l > u\right)}{x\overline{F}(x)} \\
&= \lim_{x\to\infty} \frac{x\mathbb{P}\left(\theta_l X_l > x\right) + x\int\limits_1^\infty \mathbb{P}\left(\theta_l X_l > xy\right)\mathrm{d}y}{x\overline{F}(x)} \\
&= \lim_{x\to\infty} \left(\frac{\mathbb{P}\left(\theta_l X_l > x\right)}{\overline{F}(x)} + \int\limits_1^\infty \frac{\mathbb{P}\left(\theta_l X_l > xy\right)}{\overline{F}(x)}\mathrm{d}y \right). \qquad (2.3.2)
\end{aligned}$$

Lemma 2.6 implies that $\mathbb{P}\left(\theta_l X_l > xy\right) \underset{x\to\infty}{\sim} \tau_l c_l \overline{F}(xy) \underset{x\to\infty}{\sim} \tau_l c_l y^{-\alpha}\overline{F}(x)$ for each $y \geqslant 1$. Therefore, from (2.3.2) and the Lebesgue dominated convergence theorem we obtain

$$\lim_{x\to\infty} \frac{\mathbb{E}\,\theta_l X_l \mathbb{1}_{\{S_n^{\theta X}>x\}}}{x\overline{F}(x)} = \tau_l c_l \left(1 + \int\limits_1^\infty y^{-\alpha}\mathrm{d}y\right) = \frac{\alpha}{\alpha-1}\,\tau_l c_l. \qquad (2.3.3)$$

Relations (2.3.1) and (2.3.3) imply that

$$\frac{\mathbb{E}\,\theta_l X_l \mathbb{1}_{\{S_n^{\theta X}>x\}}}{\mathbb{P}\left(S_n^{\theta X} > x\right)} \underset{x\to\infty}{\sim} \frac{\alpha}{\alpha-1}\,\frac{\tau_l c_l}{\sum\limits_{k=1}^n \tau_k c_k}\,x.$$

If we choose $x = \mathrm{VaR}_q\left(S_n^{\theta X}\right) = F^{-1}_{S_n^{\theta X}}(q)$, then the last asymptotic relation implies that

$$\frac{\mathbb{E}\,\theta_l X_l \mathbb{1}_{\{S_n^{\theta X}>x\}}}{\mathbb{P}\left(S_n^{\theta X} > x\right)} \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha-1}\,\frac{\tau_l c_l}{\sum\limits_{k=1}^n \tau_k c_k}\,F^{-1}_{S_n^{\theta X}}(q) \qquad (2.3.4)$$

with the quantile function $F^{-1}_{S_n^{\theta X}}$ of d.f. $F_{S_n^{\theta X}}(x) = \mathbb{P}(S_n^{\theta X} \leqslant x)$. According to the conditions of our theorem F is regularly varying with index $\alpha > 1$. Due to relation (2.3.1) d.f. $F_{S_n^{\theta X}}$ is also regularly varying with the same index. The obtained asymptotic relation (2.3.1) and Lemma 1.1 imply that

$$F^{-1}_{S_n^{\theta X}}(q) \underset{q\uparrow 1}{\sim} \left(\sum_{k=1}^n \tau_k c_k\right)^{\frac{1}{\alpha}} F^{-1}(q).$$

The last relation and asymptotic relation (2.3.4) imply the assertion of Theorem 1.4.

2.4 Proof of Theorem 1.5

In this section, we prove Theorem 1.5. It is clear that assertion of this theorem follows immediately from the following four lemmas.

Lemma 2.8. *Let ξ_1, ξ_2, ..., ξ_n be n pairwise QAI real-valued r.v.s. If the d.f.s $F_{\xi_1} \in \mathcal{D}$, $F_{\xi_2} \in \mathcal{D}$, ..., $F_{\xi_n} \in \mathcal{D}$, then*

$$L_n^\xi \sum_{k=1}^n \overline{F}_{\xi_k}(x) \underset{x\to\infty}{\lesssim} \mathbb{P}\big(S_n^\xi > x\big) \underset{x\to\infty}{\lesssim} \frac{1}{L_n^\xi} \sum_{k=1}^n \overline{F}_{\xi_k}(x),$$

where $L_n^\xi = \min\{L_{F_{\xi_1}}, L_{F_{\xi_2}}, \ldots, L_{F_{\xi_n}}\}$ and $S_n^\xi = \xi_1 + \xi_2 + \ldots + \xi_n$.

Proof. The assertion of the lemma follows from the bounds

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathbb{P}\big(S_n^\xi > x\big)}{\sum\limits_{k=1}^n \overline{F}_{\xi_k}(x)} &\leqslant \limsup_{x\to\infty} \frac{\sum\limits_{k=1}^n \overline{F}_{\xi_k}((1-\delta)x)}{\sum\limits_{k=1}^n \overline{F}_{\xi_k}(x)} \\
&\leqslant \max_{1\leqslant k\leqslant n} \left\{ \limsup_{x\to\infty} \frac{\overline{F}_{\xi_k}((1-\delta)x)}{\overline{F}_{\xi_k}(x)} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\mathbb{P}\big(S_n^\xi > x\big)}{\sum\limits_{k=1}^n \overline{F}_{\xi_k}(x)} &\geqslant \liminf_{x\to\infty} \frac{\sum\limits_{k=1}^n \overline{F}_{\xi_k}((1+\delta)x)}{\sum\limits_{k=1}^n \overline{F}_{\xi_k}(x)} \\
&\geqslant \min_{1\leqslant k\leqslant n} \left\{ \liminf_{x\to\infty} \frac{\overline{F}_{\xi_k}((1+\delta)x)}{\overline{F}_{\xi_k}(x)} \right\},
\end{aligned}$$

which hold for any $\delta \in (0,1)$. The first steps of the above inequalities can be obtained along the lines of the proof of Theorem 3.1 by Chen and Yuen [15], while the second steps follow from the inequality

$$\min_{1\leqslant i\leqslant m} \left\{ \frac{a_i}{b_i} \right\} \leqslant \frac{\sum\limits_{i=1}^m a_i}{\sum\limits_{i=1}^m b_i} \leqslant \max_{1\leqslant i\leqslant m} \left\{ \frac{a_i}{b_i} \right\} \tag{2.4.1}$$

provided that $m \in \mathbb{N}$ and $a_i \geqslant 0$, $b_i > 0$ for all $i \in \{1, 2, \ldots, m\}$. □

Lemma 2.9. *Let ξ and η be independent r.v.s such that $F_\xi \in \mathcal{D}$, $\eta \geqslant 0$ a.s., $\mathbb{P}(\eta = 0) < 1$ and $\mathbb{E}\eta^p < \infty$ for some $p > \mathcal{M}_{F_\xi}$. Then the d.f. $F_{\xi\eta}$ of the product $\xi\eta$ belongs to the class $\mathcal{D}$ as well.*

Proof. The assertion of the lemma follows from Lemma 3.9 by Tang and Tsitsiashvili [55]. We remark only that in that lemma, it is supposed that the r.v. η is strictly positive. If η is nonnegative and nondegenerate at zero, then we can correctly define a new r.v. $\widehat{\eta}$ with d.f.

$$\mathbb{P}(\widehat{\eta} \leqslant x) = \mathbb{P}(\eta \leqslant x | \eta > 0).$$

By Lemma 3.9 in [55], the d.f. $F_{\widehat{\eta}\xi}$ belongs to the class $\mathcal{D}$. But

$$\frac{\mathbb{P}(\eta\xi > xy)}{\mathbb{P}(\eta\xi > x)} = \frac{\mathbb{P}(\widehat{\eta}\xi > xy)}{\mathbb{P}(\widehat{\eta}\xi > x)}$$

for all positive x and y. Consequently, $F_{\xi\eta} \in \mathcal{D}$ as well. □

Lemma 2.10. *Let ξ and η be independent r.v.s satisfying conditions of Lemma 2.9. Then $L_{F_{\eta\xi}} \geqslant L_{F_\xi}$.*

Proof. A similar assertion for the class of consistently varying d.f.s $\mathcal{C} \subset \mathcal{D}$ is proved in [15, Lemma 3.1]. Here we present a more direct argument in comparison with that in [15]. Let $a \in (0,1)$ and $b > 0$ be such that $\mathbb{P}(\eta > b) > 0$.

For such numbers a and b, we have

$$\begin{aligned}
\frac{1}{L_{F_{\eta\xi}}} &= \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\mathbb{P}(\eta\xi > xy)}{\mathbb{P}(\eta\xi > x)} \\
&\leqslant \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\mathbb{P}(\eta\xi > xy, \eta \leqslant x^a)}{\mathbb{P}(\eta\xi > x)} + \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\mathbb{P}(\eta\xi > xy, \eta > x^a)}{\mathbb{P}(\eta\xi > x)} \\
&\leqslant \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\mathbb{P}(\eta\xi > xy, \eta \leqslant x^a)}{\mathbb{P}(\eta\xi > x, \eta \leqslant x^a)} + \limsup_{x\to\infty}\frac{\mathbb{P}(\eta > x^a)}{\mathbb{P}(\eta\xi > x, \eta > b)} \\
&\leqslant \lim_{y\uparrow 1}\limsup_{x\to\infty}\sup_{0<w\leqslant x^a}\frac{\overline{F}_\xi(xy/w)}{\overline{F}_\xi(x/w)} + \frac{\mathbb{E}\eta^p}{\mathbb{P}(\eta > b)}\limsup_{x\to\infty}\frac{1}{x^{ap}\mathbb{P}(\xi > x/b)} \\
&\leqslant \lim_{y\uparrow 1}\limsup_{x\to\infty}\sup_{z\geqslant x^{1-a}}\frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)} \\
&+ \frac{\mathbb{E}\eta^p}{\mathbb{P}(\eta > b)}\limsup_{x\to\infty}\frac{1}{x^{ap}\overline{F}_\xi(x)}\limsup_{x\to\infty}\frac{\overline{F}_\xi(x)}{\overline{F}_\xi(x/b)}.
\end{aligned} \tag{2.4.2}$$

Since $F_\xi \in \mathcal{D}$, it is obvious that $\overline{F}_\xi(x) = O\big(\overline{F}_\xi(x/b)\big)$. Moreover,

$$\lim_{x\to\infty} x^q\overline{F}_\xi(x) = \infty \tag{2.4.3}$$

for any $q > \mathcal{M}_{F_\xi}$ (see, for instance, [55, Lemma 3.5]).

If $p > \mathcal{M}_{F_\xi}$, then there exists $a \in (0,1)$ such that $ap > \mathcal{M}_{F_\xi}$. For this particular a, the second term on the right-hand side of (2.4.2) is equal to zero. Consequently,

$$\frac{1}{L_{F_{\eta\xi}}} \leqslant \lim_{y\uparrow 1}\limsup_{x\to\infty}\sup_{z\geqslant x^{1-a}}\frac{\overline{F}_\xi(zy)}{\overline{F}_\xi(z)} = \frac{1}{L_{F_\xi}},$$

and the assertion of the lemma follows. □

Lemma 2.11. *Let ξ_1 and ξ_2 be two QAI (SQAI) r.v.s with d.f.s $F_{\xi_1} \in \mathcal{D}$ and $F_{\xi_2} \in \mathcal{D}$. Let η_1 and η_2 be two nonnegative, nondegenerated at zero r.v.s such that the vectors (ξ_1, ξ_2) and (η_1, η_2) are independent. If $\max\left\{\mathbb{E}\eta_1^p, \mathbb{E}\eta_2^p\right\} < \infty$ for some $p > \max\left\{\mathcal{M}_{\xi_1}, \mathcal{M}_{\xi_2}\right\}$, then the r.v.s $\eta_1\xi_1$ and $\eta_2\xi_2$ are QAI (SQAI).*

Proof. We consider only the case where the r.v.s ξ_1 and ξ_2 are QAI because the proof in the case of SQAI r.v.s is identical. In this case, the assertion of the lemma can be proved along the lines of Lemma 3.1 in [15]. But here we present a more direct proof. For $x > 0$ and $a \in (0,1)$, we have

$$
\begin{aligned}
&\mathbb{P}\big((\eta_1\xi_1)^+ > x, (\eta_2\xi_2)^+ > x, \eta_1 \leqslant x^a, \eta_2 \leqslant x^a\big) \\
&\quad = \iint\limits_{(0,\,x^a]^2} \mathbb{P}\Big(\xi_1 > \frac{x}{u}, \xi_2 > \frac{x}{v}\Big) \mathrm{d}\mathbb{P}(\eta_1 \leqslant u, \eta_2 \leqslant v) \\
&\quad \leqslant \sup_{\substack{0<u\leqslant x^a \\ 0<v\leqslant x^a}} \frac{\mathbb{P}\Big(\xi_1 > \frac{x}{u}, \xi_2 > \frac{x}{v}\Big)}{\mathbb{P}\Big(\xi_1 > \frac{x}{u}\Big) + \mathbb{P}\Big(\xi_2 > \frac{x}{v}\Big)} \big(\mathbb{P}(\eta_1\xi_1 > x) + \mathbb{P}(\eta_2\xi_2 > x)\big).
\end{aligned}
$$

Hence, applying considerations similar to those in (2.4.2), we get

$$
\begin{aligned}
\frac{\mathbb{P}\big((\eta_1\xi_1)^+ > x, (\eta_2\xi_2)^+ > x\big)}{\mathbb{P}(\eta_1\xi_1 > x) + \mathbb{P}(\eta_2\xi_2 > x)} \quad &\leqslant \quad \sup_{z \geqslant x^{1-a}} \frac{\mathbb{P}(\xi_1 > z, \xi_2 > z)}{\mathbb{P}(\xi_1 > z) + \mathbb{P}(\xi_2 > z)} \\
&\quad + \max\left\{\frac{\mathbb{P}(\eta_1 > x^a)}{\mathbb{P}(\eta_1\xi_1 > x)}, \frac{\mathbb{P}(\eta_2 > x^a)}{\mathbb{P}(\eta_2\xi_2 > x)}\right\} \\
&\leqslant \quad \sup_{z \geqslant x^{1-a}} \frac{\mathbb{P}(\xi_1 > z, \xi_2 > z)}{\mathbb{P}(\xi_1 > z) + \mathbb{P}(\xi_2 > z)} \\
&\quad + \max_{i\in\{1,2\}} \left\{\frac{\mathbb{E}\eta_i^p}{\mathbb{P}(\eta_i > b)} \frac{1}{x^{ap}\overline{F}_{\xi_i}(x)} \frac{\overline{F}_{\xi_i}(x)}{\overline{F}_{\xi_i}(x/b)}\right\},
\end{aligned}
$$

where $x > 0$, $a \in (0,1)$, $p > \max\{\mathcal{M}_{F_{\xi_1}}, \mathcal{M}_{F_{\xi_2}}\}$ and $b > 0$ is such that

$$
\min\big\{\mathbb{P}(\eta_1 > b), \mathbb{P}(\eta_2 > b)\big\} > 0.
$$

The last inequality, conditions of the lemma and relation (2.4.3) imply that

$$
\mathbb{P}\big((\eta_1\xi_1)^+ > x, (\eta_2\xi_2)^+ > x\big) = o\big(\mathbb{P}(\eta_1\xi_1 > x) + \mathbb{P}(\eta_2\xi_2 > x)\big) \quad \text{as} \quad x \to \infty.
$$

The similar considerations give

$$
\mathbb{P}\big((\eta_1\xi_1)^+ > x, (\eta_2\xi_2)^- > x\big) = o\big(\mathbb{P}(\eta_1\xi_1 > x) + \mathbb{P}(\eta_2\xi_2 > x)\big)
$$

and

$$
\mathbb{P}\big((\eta_1\xi_1)^- > x, (\eta_2\xi_2)^+ > x\big) = o\big(\mathbb{P}(\eta_1\xi_1 > x) + \mathbb{P}(\eta_2\xi_2 > x)\big)
$$

as $x \to \infty$, and the assertion of the lemma follows. $\square$

2.5 Proof of Theorem 1.6

In this section, we prove Theorem 1.6. Conditions of this theorem, as well as Lemma 2.9, imply that the d.f.s $F_{\theta_k X_k}$ belong to the class $\mathcal{D}$ for all $k \in \{1, 2, \ldots, n\}$. Moreover, Lemma 2.11 implies that $\{\theta_1 X_1, \theta_2 X_2, \ldots, \theta_n X_n\}$ is a collection of pairwise SQAI real-valued r.v.s if $\{X_1, X_2, \ldots, X_n\}$ are pairwise SQAI. In addition, the conditions of Theorem 1.6 state that $\overline{F}_{\theta_k X_k}(x) = O\big(\overline{F}_{\theta_l X_l}(x)\big)$ for all $k \in \{1, 2, \ldots, n\}$ and for some particular index $l \in \{1, 2, \ldots, n\}$. Hence, the assertion of Theorem 1.6 follows immediately from Lemmas 2.13 and 2.14 below. In Lemma 2.13, we get the upper asymptotic bound, while in Lemma 2.14, we obtain the lower asymptotic bound. We start this section with a simple technical assertion, which will be used in the proofs of the main lemmas.

Lemma 2.12. *Let ξ and η be two nonnegative random variables such that $\mathbb{E}\xi$ is finite. Then*

$$\mathbb{E}\big(\xi \mathbb{1}_{\{\xi > x,\, \eta > y\}}\big) = x\, \mathbb{P}(\xi > x, \eta > y) + \int_x^\infty \mathbb{P}(\xi > u, \eta > y)\, \mathrm{d}u$$

for any pair of positive numbers x and y.

Proof. Let F_ξ and F_η be the d.f.s of ξ and η, respectively, and let $F_{\xi,\eta}$ be the joint d.f. of ξ and η. Then, for all $x > 0$ and $y > 0$, we have

$$\begin{aligned}
\mathbb{E}\big(\xi \mathbb{1}_{\{\xi > x,\, \eta > y\}}\big) &= \int_x^\infty u \int_y^\infty \mathrm{d}F_{\xi,\eta}(u, v) \\
&= \int_x^\infty u\, \mathrm{d}F_\xi(u) - \int_x^\infty u\, \mathrm{d}F_{\xi,\eta}(u, y) \\
&= \int_\infty^x u\, \mathrm{d}\overline{F}_\xi(u) + \int_x^\infty u\, \mathrm{d}\big(F_\eta(y) - F_{\xi,\eta}(u, y)\big) \\
&= \Big[u\, \overline{F}_\xi(u)\Big]_\infty^x - \int_\infty^x \overline{F}_\xi(u)\, \mathrm{d}u \\
&\quad + \Big[u\big(F_\eta(y) - F_{\xi,\eta}(u, y)\big)\Big]_x^\infty - \int_x^\infty \big(F_\eta(y) - F_{\xi,\eta}(u, y)\big)\, \mathrm{d}u \\
&= x\, \overline{F}_\xi(x) - \lim_{u \to \infty} u \overline{F}_\xi(u) + \int_x^\infty \overline{F}_\xi(u)\, \mathrm{d}u \\
&\quad - x\big(\mathbb{P}(\eta \leqslant y) - \mathbb{P}(\xi \leqslant x, \eta \leqslant y)\big) + \lim_{u \to \infty} u\big(F_\eta(y) - F_{\xi,\eta}(u, y)\big) \\
&\quad - \int_x^\infty \big(\mathbb{P}(\eta \leqslant y) - \mathbb{P}(\xi \leqslant u, \eta \leqslant y)\big)\, \mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&= x\big(\mathbb{P}(\xi > x) - \mathbb{P}(\xi > x, \eta \leqslant y)\big) \\
&+ \int_x^{\infty} \big(\mathbb{P}(\xi > u) - \mathbb{P}(\xi > u, \eta \leqslant y)\big)\,\mathrm{d}u \\
&= x\,\mathbb{P}(\xi > x, \eta > y) + \int_x^{\infty} \mathbb{P}(\xi > u, \eta > y)\,\mathrm{d}u.
\end{aligned}
$$

Here we used the fact that $\mathbb{E}\xi < \infty$, which implies that $\lim\limits_{u\to\infty} u\overline{F}_\xi(u) = 0$ and

$$
0 \leqslant \lim_{u\to\infty} u\big(F_\eta(y) - F_{\xi,\eta}(u,y)\big) = \lim_{u\to\infty} u\,\mathbb{P}(\xi > u, \eta \leqslant y) \leqslant \lim_{u\to\infty} u\,\overline{F}_\xi(u) = 0.
$$

Lemma is proved. □

Lemma 2.13. *Let ξ_1, ξ_2, …, ξ_n be n pairwise SQAI real-valued r.v.s. If $F_{\xi_1} \in \mathcal{D}$, $F_{\xi_2} \in \mathcal{D}$, …, $F_{\xi_n} \in \mathcal{D}$ and $\overline{F}_{\xi_k}(x) = O\big(\overline{F}_{\xi_l}(x)\big)$ for all $k \in \{1,2,\dots,n\}$ and some $l \in \{1,2,\dots,n\}$ such that $\mathbb{E}\xi_l^+ < \infty$, then*

$$
\mathbb{E}\Big(\xi_l \mathbb{1}_{\left\{S_n^\xi > x\right\}}\Big) \underset{x\to\infty}{\lesssim} \frac{1}{L_{F_l}} \mathbb{E}\big(\xi_l \mathbb{1}_{\{\xi_l > x\}}\big),
$$

where, as usual, $S_n^\xi = \xi_1 + \xi_2 + \ldots + \xi_n$.

Proof. Obviously, we can suppose that $n \geqslant 2$ and choose the special index $l = 1$. Hence, we conclude that all basic requirements of Lemma 2.13 are satisfied with the condition

$$
\overline{F}_{\xi_k}(x) = O\big(\overline{F}_{\xi_1}(x)\big), \tag{2.5.1}
$$

which holds for all $k \in \{1,2,\dots,n\}$, and we must prove that

$$
\mathbb{E}\Big(\xi_1 \mathbb{1}_{\left\{S_n^\xi > x\right\}}\Big) \underset{x\to\infty}{\lesssim} \frac{1}{L_{F_1}} \mathbb{E}\big(\xi_1 \mathbb{1}_{\{\xi_1 > x\}}\big). \tag{2.5.2}
$$

It is evident that

$$
\begin{aligned}
\mathbb{E}\Big(\xi_1 \mathbb{1}_{\left\{S_n^\xi > x\right\}}\Big) &\leqslant \mathbb{E}\Big(\xi_1^+ \mathbb{1}_{\left\{S_n^{\xi^+} > x\right\}}\Big) \\
&\leqslant \mathbb{E}\big(\xi_1 \mathbb{1}_{\{\xi_1 > (1-\varepsilon)x\}}\big) + \mathcal{I}
\end{aligned} \tag{2.5.3}
$$

for any $x > 0$, where

$$
S_n^{\xi^+} = \sum_{k=1}^{n} \xi_k^+
$$

and

$$
\begin{aligned}
\mathcal{I} &= \mathbb{E}\Big(\xi_1^+ \mathbb{1}_{\left\{\xi_1^+ \leqslant (1-\varepsilon)x,\, S_n^{\xi^+} > x\right\}}\Big) \\
&= \mathbb{E}\Big(\xi_1^+ \mathbb{1}_{\left\{\xi_1^+ \leqslant \varepsilon x,\, S_n^{\xi^+} > x\right\}}\Big) + \mathbb{E}\Big(\xi_1^+ \mathbb{1}_{\left\{\varepsilon x < \xi_1^+ \leqslant (1-\varepsilon)x,\, S_n^{\xi^+} > x\right\}}\Big) \\
&\leqslant \varepsilon x\,\mathbb{P}\big(S_n^{\xi^+} > x\big) + (1-\varepsilon)x\,\mathbb{P}\big(\varepsilon x < \xi_1^+ \leqslant x,\, S_n^{\xi^+} > x\big)
\end{aligned} \tag{2.5.4}
$$

with any $\varepsilon \in (0,1/2)$.

Using Lemma 2.8 and the additional requirement (2.5.1) we obtain

$$\mathbb{P}\big(S_n^{\xi+} > x\big) \leqslant \frac{2}{L_n^{\xi}} \sum_{k=1}^{n} \mathbb{P}(\xi_k > x) \leqslant c_1 \, \mathbb{P}(\xi_1 > x) \leqslant \frac{c_1}{x} \, \mathbb{E}\big(\xi_1 \mathbb{1}_{\{\xi_1 > x\}}\big) \qquad (2.5.5)$$

provided that x is large enough, where the positive constant $c_1 = c_1(n)$ may depend on n but not on x.

The collection $\{\xi_1, \xi_2, \ldots, \xi_n\}$ consists of pairwise SQAI r.v.s. Hence, for any $\varepsilon \in (0,1/2)$, we get

$$\begin{aligned}
\mathbb{P}\big(\varepsilon x < \xi_1^+ \leqslant (1-\varepsilon)x,\, S_n^{\xi^+} > x\big) &\leqslant \mathbb{P}\Bigg(\xi_1^+ > \varepsilon x, \bigcup_{k=2}^{n} \Big\{\xi_k^+ > \frac{\varepsilon x}{n-1}\Big\}\Bigg) \\
&\leqslant \sum_{k=2}^{n} \mathbb{P}\big(\xi_1^+ > \varepsilon x,\, \xi_k^+ > \frac{\varepsilon x}{n-1}\big) \\
&\leqslant \Delta_1(\varepsilon x) \sum_{k=2}^{n} \overline{F}_{\xi_1}(\varepsilon x),
\end{aligned}$$

where

$$\Delta_1(x) = \Delta_1(n,x) = \max_{2 \leqslant k \leqslant n} \frac{\mathbb{P}(\xi_1^+ > x, \xi_k^+ > \frac{x}{n-1})}{\overline{F}_{\xi_1}(x)} \underset{x\to\infty}{\to} 0.$$

Since the d.f. F_{ξ_1} belongs to the class $\mathcal{D}$, the additional condition (2.5.1) implies that

$$\begin{aligned}
\mathbb{P}\big(\varepsilon x < \xi_1^+ \leqslant (1-\varepsilon)x,\, S_n^{\xi^+} > x\big) &\leqslant c_2 \Delta_1(\varepsilon x)\, \overline{F}_{\xi_1}(x) \\
&\leqslant c_2\, \Delta_1(\varepsilon x)\, \frac{1}{x}\, \mathbb{E}\big(\xi_1 \mathbb{1}_{\{\xi_1 > x\}}\big), \quad (2.5.6)
\end{aligned}$$

where x is large enough and $c_2 = c_2(\varepsilon, n) > 0$.

Substituting the derived bounds (2.5.4)–(2.5.6) into (2.5.3) yields

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\mathbb{E}\Big(\xi_1 \mathbb{1}_{\big\{S_n^{\xi} > x\big\}}\Big)}{\mathbb{E}\big(\xi_1 \mathbb{1}_{\{\xi_1 > x\}}\big)} &\leqslant (1-\varepsilon) \limsup_{x\to\infty} \max\left\{\frac{\overline{F}_{\xi_1}((1-\varepsilon)x)}{\overline{F}_{\xi_1}(x)}, \sup_{y \geqslant x} \frac{\overline{F}_{\xi_1}((1-\varepsilon)y)}{\overline{F}_{\xi_1}(y)}\right\} \\
&+ \varepsilon c_1 + (1-\varepsilon) c_2 \limsup_{x\to\infty} \Delta_1(\varepsilon x)
\end{aligned}$$

for any $\varepsilon \in (0,1/2)$ due to Lemma 2.12. The last inequality implies immediately the desired asymptotic relation (2.5.2). □

Lemma 2.14. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be n pairwise SQAI real-valued r.v.s. If $F_{\xi_1} \in \mathcal{D}$, $F_{\xi_2} \in \mathcal{D}, \ldots, F_{\xi_n} \in \mathcal{D}$ and $\overline{F}_{\xi_k}(x) = O\big(\overline{F}_{\xi_l}(x)\big)$ for all $k \in \{1,2,\ldots,n\}$ and some particular index $l \in \{1,2,\ldots,n\}$ such that $\mathbb{E}|\xi_l| < \infty$, then*

$$L_{F_{\xi_l}} \mathbb{E}\big(\xi_l \mathbb{1}_{\{\xi_l > x\}}\big) \underset{x\to\infty}{\lesssim} \mathbb{E}\Big(\xi_l \mathbb{1}_{\{S_n^{\xi} > x\}}\Big),$$

where $L_{F_{\xi_l}}$ is the L-index of the d.f. F_{ξ_l} defined by equality (1.2.1).

Proof. If $n = 1$, then the assertion of the lemma is evident. Therefore, in what follows, we deal with the case $n \geqslant 2$. In addition, as in the proof of Lemma 2.13, we can choose the particular index $l = 1$. Hence, we suppose that $\overline{F}_{\xi_k}(x) = O\big(\overline{F}_{\xi_1}(x)\big)$ for all indices $k \in \{1, 2, \dots, n\}$, and we must prove that

$$\mathbb{E}\Big(\xi_1 1\!\!\mathrm{I}_{\{S_n^\xi > x\}}\Big) \underset{x\to\infty}{\gtrsim} L_{F_{\xi_1}} \mathbb{E}\big(\xi_1 1\!\!\mathrm{I}_{\{\xi_1 > x\}}\big). \tag{2.5.7}$$

For any $x > 0$ and an arbitrary $\delta \in (0, 1)$, we have

$$\begin{aligned}
\mathbb{E}\Big(\xi_1 1\!\!\mathrm{I}_{\{S_n^\xi > x\}}\Big) &= \mathbb{E}\Big(\xi_1^+ 1\!\!\mathrm{I}_{\{S_n^\xi > x\}}\Big) - \mathbb{E}\Big(\xi_1^- 1\!\!\mathrm{I}_{\{S_n^\xi > x\}}\Big) \\
&= \mathbb{E}\Big(\xi_1^+ 1\!\!\mathrm{I}_{\{\xi_1^+ > (1+\delta)x,\, S_n^\xi > x\}}\Big) + \mathbb{E}\Big(\xi_1^+ 1\!\!\mathrm{I}_{\{\xi_1^+ \leqslant (1+\delta)x,\, S_n^\xi > x\}}\Big) \\
&\quad - \mathbb{E}\Big(\xi_1^- 1\!\!\mathrm{I}_{\{\xi_1^- \leqslant x,\, S_n^\xi > x\}}\Big) - \mathbb{E}\Big(\xi_1^- 1\!\!\mathrm{I}_{\{\xi_1^- > x,\, S_n^\xi > x\}}\Big) \\
&\geqslant \mathbb{E}\Big(\xi_1^+ 1\!\!\mathrm{I}_{\{\xi_1^+ > (1+\delta)x\}}\Big) - \mathbb{E}\Big(\xi_1^+ 1\!\!\mathrm{I}_{\{\xi_1^+ > (1+\delta)x,\, S_n^\xi \leqslant x\}}\Big) \\
&\quad - \mathbb{E}\Big(\xi_1^- 1\!\!\mathrm{I}_{\{\xi_1^- \leqslant x,\, S_n^\xi > x\}}\Big) - \mathbb{E}\Big(\xi_1^- 1\!\!\mathrm{I}_{\{\xi_1^- > x,\, S_n^\xi > x\}}\Big) \\
&=: \mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3 - \mathcal{J}_4.
\end{aligned} \tag{2.5.8}$$

Since $\mathbb{E}\xi_1^+ < \infty$, using Lemma 2.12 and inequality (2.4.1) we obtain

$$\begin{aligned}
\frac{\mathcal{J}_1}{\mathbb{E}\big(\xi_1 1\!\!\mathrm{I}_{\{\xi_1 > x\}}\big)} &= \frac{(1+\delta)x\overline{F}_{\xi_1}((1+\delta)x) + \int\limits_{(1+\delta)x}^{\infty} \overline{F}_{\xi_1}(y)\mathrm{d}y}{x\overline{F}_{\xi_1}(x) + \int\limits_x^\infty \overline{F}_{\xi_1}(y)\mathrm{d}y} \\
&\geqslant (1+\delta)\min\left\{\frac{\overline{F}_{\xi_1}((1+\delta)x)}{\overline{F}_{\xi_1}(x)},\ \frac{\int\limits_x^\infty \Big(\overline{F}_{\xi_1}((1+\delta)y)/\overline{F}_{\xi_1}(y)\Big)\overline{F}_{\xi_1}(y)\mathrm{d}y}{\int\limits_x^\infty \overline{F}_{\xi_1}(y)\mathrm{d}y}\right\} \\
&\geqslant (1+\delta)\min\left\{\frac{\overline{F}_{\xi_1}((1+\delta)x)}{\overline{F}_{\xi_1}(x)},\ \inf_{y\geqslant x}\frac{\overline{F}_{\xi_1}((1+\delta)y)}{\overline{F}_{\xi_1}(y)}\right\}.
\end{aligned}$$

Consequently,

$$\liminf_{x\to\infty} \frac{\mathcal{J}_1}{\mathbb{E}\big(\xi_1 1\!\!\mathrm{I}_{\{\xi_1 > x\}}\big)} \geqslant (1+\delta)\liminf_{x\to\infty}\left(\inf_{y\geqslant x}\frac{\overline{F}_{\xi_1}((1+\delta)y)}{\overline{F}_{\xi_1}(y)}\right). \tag{2.5.9}$$

For the third term in (2.5.8), we have

$$\begin{aligned}\mathcal{J}_3 &= \mathbb{E}\Big(\xi_1^-\mathbb{I}_{\{\xi_1^-\leqslant\delta x,\, S_n^\xi>x\}}\Big)+\mathbb{E}\Big(\xi_1^-\mathbb{I}_{\{\delta x<\xi_1^-\leqslant x,\, S_n^\xi>x\}}\Big)\\ &\leqslant \delta x\mathbb{P}\big(S_n^\xi>x\big)+\mathbb{E}\Big(\xi_1^-\mathbb{I}_{\{\delta x<\xi_1^-\leqslant x,\, S_n^\xi>x,\,\xi_1>0\}}\Big)\\ &\quad+\mathbb{E}\Big(\xi_1^-\mathbb{I}_{\{\delta x<\xi_1^-\leqslant x,\, S_n^\xi>x,\,\xi_1\leqslant 0\}}\Big)\\ &\leqslant \delta x\mathbb{P}\Bigg(\sum_{k=1}^n\xi_k^+>x\Bigg)+x\mathbb{P}\Bigg(\xi_1^->\delta x,\,\sum_{k=2}^n\xi_k^+>x\Bigg).\end{aligned}\tag{2.5.10}$$

Applying arguments similar to those in the derivation of formula (2.5.5), using Lemma 2.8 and the condition $\overline{F}_{\xi_k}(x)=O\big(\overline{F}_{\xi_1}(x)\big)$, $k\in\{1,2,\ldots,n\}$, we conclude that

$$\mathbb{P}\Bigg(\sum_{k=1}^n\xi_k^+>x\Bigg)\leqslant\frac{c_3}{x}\mathbb{E}\big(\xi_1\mathbb{I}_{\{\xi_1>x\}}\big)\tag{2.5.11}$$

for large enough x and some positive constant $c_3=c_3(n)$ not depending on x.

Since $\{\xi_1,\xi_2,\ldots,\xi_n\}$ are SQAI (consequently, QAI) r.v.s, we deduce that

$$\begin{aligned}\mathbb{P}\Bigg(\xi_1^->\delta x,\,\sum_{k=2}^n\xi_k^+>x\Bigg) &\leqslant \mathbb{P}\Bigg(\xi_1^->\delta x,\bigcup_{k=2}^n\Big\{\xi_k^+>\frac{x}{n-1}\Big\}\Bigg)\\ &\leqslant \sum_{k=2}^n\mathbb{P}\big(\xi_1^->\delta x,\,\xi_k^+>\delta x\big)\\ &\leqslant \Delta_2(\delta x)\sum_{k=2}^n\mathbb{P}\big(\overline{F}_{\xi_1}(\delta x)+\overline{F}_{\xi_k}(\delta x)\big)\end{aligned}$$

for large enough x and an arbitrary $\delta\in(0,1/(n-1))$, where

$$\Delta_2(x)=\Delta_2(n,x)=\max_{2\leqslant k\leqslant n}\frac{\mathbb{P}(\xi_1^->x,\xi_k^+>x)}{\overline{F}_{\xi_1}(x)+\overline{F}_{\xi_k}(x)}\underset{x\to\infty}{\to}0.$$

Applying arguments similar to those in the derivation of formula (2.5.6), for large enough x, we get

$$\mathbb{P}\Bigg(\xi_1^->\delta x,\,\sum_{k=2}^n\xi_k^+>x\Bigg)\leqslant c_4\Delta_2(\delta x)\frac{1}{x}\mathbb{E}\big(\xi_1\mathbb{I}_{\{\xi_1>x\}}\big)\tag{2.5.12}$$

with some positive constant $c_4=c_4(n)$.

Substituting (2.5.11) and (2.5.12) into (2.5.10) yields

$$\limsup_{x\to\infty}\frac{\mathcal{J}_3}{\mathbb{E}\big(\xi_1\mathbb{I}_{\{\xi_1>x\}}\big)}\leqslant\delta c_3\tag{2.5.13}$$

for any $\delta\in(0,1/(n-1))$.

For any $x>0$ and $\widehat{\delta}\in(0,\delta/(n-1))$, the second term in (2.5.8) can be bounded above in the following way:

$$\begin{aligned}
\mathcal{J}_2 &= \mathbb{E}\left(\xi_1^+ 1\!\!1_{\{\xi_1^+>(1+\delta)x,\, \sum_{k=1}^n \xi_k^+ - \sum_{k=2}^n \xi_k^- \leqslant x\}}\right)\\
&\leqslant \mathbb{E}\left(\xi_1^+ 1\!\!1_{\{\xi_1^+>(1+\delta)x,\, \sum_{k=2}^n \xi_k^- > \delta x\}}\right)\\
&\leqslant \sum_{k=2}^n \mathbb{E}\left(\xi_1^+ 1\!\!1_{\{\xi_1^+>(1+\delta)x,\, \xi_k^- > \delta x/(n-1)\}}\right)\\
&\leqslant \sum_{k=2}^n \mathbb{E}\left(\xi_1^+ 1\!\!1_{\{\xi_1^+>\widehat{\delta}x,\, \xi_k^- > \widehat{\delta}x\}}\right). \qquad (2.5.14)
\end{aligned}$$

For any $x>0$ and $\widetilde{\delta}\in(0,1/(n-1))$, the last term in (2.5.8) can be bounded above likewise:

$$\begin{aligned}
\mathcal{J}_4 &= \mathbb{E}\left(\xi_1^- 1\!\!1_{\{\xi_1^->x,\, \sum_{k=2}^n \xi_k^+ - \sum_{k=1}^n \xi_k^- > x\}}\right)\\
&\leqslant \sum_{k=2}^n \mathbb{E}\left(\xi_1^- 1\!\!1_{\{\xi_1^->\widetilde{\delta}x,\, \xi_k^+ > \widetilde{\delta}x\}}\right).
\end{aligned}$$

Since $\mathbb{E}|\xi_1|<\infty$, using Lemma 2.12 and inequality (2.4.1) we obtain

$$\frac{\mathbb{E}\left(\xi_1^+ 1\!\!1_{\{\xi_1^+>\widehat{\delta}x,\, \xi_k^->\widehat{\delta}x\}}\right)}{\mathbb{E}\big(\xi_1 1\!\!1_{\{\xi_1>x\}}\big)} = \frac{\widehat{\delta}x\mathbb{P}(\xi_1^+>\widehat{\delta}x,\, \xi_k^->\widehat{\delta}x)+\int_{\widehat{\delta}x}^{\infty}\mathbb{P}(\xi_1^+>u,\, \xi_k^->\widehat{\delta}x)\mathrm{d}u}{x\mathbb{P}(\xi_1>x)+\int_x^{\infty}\mathbb{P}(\xi_1>u)\mathrm{d}u}$$

$$= \frac{\widehat{\delta}x\mathbb{P}(\xi_1>\widehat{\delta}x,\, \xi_k^->\widehat{\delta}x)+\int_{\widehat{\delta}x}^{\infty}\mathbb{P}(\xi_1>u,\, \xi_k^->\widehat{\delta}x)\mathrm{d}u}{x\mathbb{P}(\xi_1>\widehat{\delta}x)\frac{\overline{F}_{\xi_1}(x)}{\overline{F}_{\xi_1}(\widehat{\delta}x)}+\frac{1}{\widehat{\delta}}\int_{\widehat{\delta}x}^{\infty}\mathbb{P}(\xi_1>u)\frac{\overline{F}_{\xi_1}(u/\widehat{\delta})}{\overline{F}_{\xi_1}(u)}\mathrm{d}u}$$

$$\leqslant \widehat{\delta}\max\left\{\frac{\overline{F}_{\xi_1}(\widehat{\delta}x)}{\overline{F}_{\xi_1}(x)}\frac{\mathbb{P}(\xi_1>\widehat{\delta}x,\, \xi_k^->\widehat{\delta}x)}{\mathbb{P}(\xi_1>\widehat{\delta}x)},\ \sup_{u\geqslant\widehat{\delta}x}\frac{\overline{F}_{\xi_1}(u)}{\overline{F}_{\xi_1}(u/\widehat{\delta})}\frac{\mathbb{P}(\xi_1>u,\, \xi_k^->\widehat{\delta}x)}{\mathbb{P}(\xi_1>u)}\right\}$$

for any $x>0$, $\widehat{\delta}\in(0,\delta/(n-1))$ and $k\in\{2,\ldots,n\}$.

The d.f. F_{ξ_1} belongs to the class $\mathcal{D}$, and the r.v.s $\{\xi_1,\xi_k\}$ are SQAI. Therefore, for large enough x and any $\widehat{\delta}\in(0,\delta/(n-1))$, we have

$$\frac{\mathbb{E}\left(\xi_1^+ 1\!\!1_{\{\xi_1^+>\widehat{\delta}x,\xi_k^->\widehat{\delta}x\}}\right)}{\mathbb{E}\big(\xi_1 1\!\!1_{\{\xi_1>x\}}\big)} \leqslant \Delta_{3,k}(\widehat{\delta}x), \qquad (2.5.15)$$

where $\Delta_{3,k}(\widehat{\delta}x)\underset{x\to\infty}{\to}0$ for any $k\in\{2,\ldots,n\}$.

Substituting (2.5.15) into (2.5.14) gives

$$\limsup_{x\to\infty}\frac{\mathcal{J}_2}{\mathbb{E}\big(\xi_1 1\!\!1_{\{\xi_1>x\}}\big)}=0. \qquad (2.5.16)$$

Similarly, we can show that

$$\limsup_{x\to\infty} \frac{\mathcal{J}_4}{\mathbb{E}\big(\xi_1 1\!\!1_{\{\xi_1>x\}}\big)} = 0. \tag{2.5.17}$$

Equality (2.5.8) as well as asymptotic relations (2.5.9), (2.5.13), (2.5.16) and (2.5.17) yield

$$\liminf_{x\to\infty} \frac{\mathbb{E}\Big(\xi_1 1\!\!1_{\{S_n^{\xi}>x\}}\Big)}{\mathbb{E}\big(\xi_1 1\!\!1_{\{\xi_1>x\}}\big)} \geqslant (1+\delta)\liminf_{x\to\infty}\left(\inf_{y\geqslant x}\frac{\overline{F}_{\xi_1}((1+\delta)y)}{\overline{F}_{\xi_1}(y)}\right) - \delta c_3 \tag{2.5.18}$$

for any $\delta \in (0,1)$. Letting $\delta \downarrow 0$ in (2.5.18) and taking into account the definition of $L_{F_{\xi_1}}$ we obtain (2.5.7), which completes the proof. □

2.6 Proof of Theorem 1.7

This section deals with the proof of theorem 1.7. We first note that the conditions of the theorem imply that

$$\frac{\mathbb{P}\big((X_1^{(i)}/t,\dots,X_n^{(i)}/t)\in\cdot\big)}{\overline{F}_1(t)} \xrightarrow{v} a^{(i)}\mu_D^{(i)} + (1-a^{(i)})\mu_I^{(i)}$$

holds on $[0,\infty]^n \diagdown \{0\}^n$ (see Section 3.4 in Resnick [52]). Moreover, since $\mu_D^{(i)}$ is continuous on its domain and $\mu_I^{(i)}$ puts mass only on the coordinate axes, the vague convergence criterion applies for any set $A \in [0,\infty]^n$ whose closure either does not intersect with any of the coordinate axes or the intersection consists only of distinct points, since they have zero measure.

Moreover, since $i_k^2 = 1$ for all $k \in \{1,2,\dots,n\}$ and all $i \in I$, we have

$$\begin{aligned} \mathbb{E}[X_k|S_n^X>t] &= \frac{1}{\mathbb{P}(S_n^X>t)}\sum_{i\in I}\mathbb{E}X_k 1\!\!1_{\left\{S_n^X>t,\,\bigcap\limits_{j=1}^{n}\{i_jX_j>0\}\right\}} \\ &= \frac{\sum\limits_{i\in I} i_k\mathbb{E}X_k^{(i)} 1\!\!1_{\left\{\sum\limits_{j=1}^{n} i_jX_j^{(i)}>t,\,\bigcap\limits_{j=1}^{n}\{X_j^{(i)}>0\}\right\}}}{\sum\limits_{i\in I}\mathbb{P}\left(\sum\limits_{j=1}^{n} i_jX_j^{(i)}>t,\,\bigcap\limits_{j=1}^{n}\{X_j^{(i)}>0\}\right)}. \end{aligned} \tag{2.6.1}$$

We will first deal with the denominator of the last fraction. Observe that for

$t > 0$

$$\sum_{i\in I}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$
$$= \sum_{i\in I}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j:\, i_j=1}\{X_j^{(i)} \leqslant t\}, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$
$$+ \sum_{i\in I}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcup_{j:\, i_j=1}\{X_j^{(i)} > t\}, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$
$$= \sum_{i\in I}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j:\, i_j=1}\{X_j^{(i)} \leqslant t\}, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$
$$+ \sum_{i\in I}\sum_{j:\, i_j=1}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, X_j^{(i)} > t, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$

$$+ \sum_{i\in I}\sum_{l=2}^{|J|}(-1)^{l+1}\sum_{\{j_1,\ldots,j_l\}\in J}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{r=1}^{l}\{X_{j_r}^{(i)} > t\}, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$
$$= \sum_{i\in I}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j:\, i_j=1}\{X_j^{(i)} \leqslant t\}, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$
$$+ \sum_{i\in I}\sum_{j:\, i_j=1}\mathbb{P}\left(X_j^{(i)} > t, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$
$$- \sum_{i\in I}\sum_{j:\, i_j=1}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} \leqslant t, X_j^{(i)} > t, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)$$
$$+ \sum_{i\in I}\sum_{l=2}^{|J|}(-1)^{l+1}\sum_{\{j_1,\ldots,j_l\}\in J}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{r=1}^{l}\{X_{j_r}^{(i)} > t\}, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right), \tag{2.6.2}$$

where $J = \{j : i_j = 1\}$. Let A_k be the k-th axis, i.e.

$$A_k = \{\underline{x} : x_1 = \cdots = x_{k-1} = x_{k+1} = \cdots = x_n = 0\},$$

and let $A_k(x)$ denote a point on the k-th axis where $x_k = x$. Then it is obvious that for all $k \in \{1, 2, \ldots, n\}$ and $j \in J$ we have

$$\left\{\underline{x}:\sum_{l=1}^{n} i_l x_l \geqslant 1, \bigcap_{l:i_l=1}\{x_l \leqslant 1\}\right\}\bigcap A_k = \begin{cases} A_k(1), & \text{if } i_k = 1 \\ \emptyset & \text{if } i_k = -1 \end{cases},$$

$$\left\{\underline{x}:\sum_{l=1}^{n} i_l x_l \leqslant 1, x_j \geqslant 1\right\}\bigcap A_k = \begin{cases} \emptyset, & \text{if } k \neq j \\ A_k(1) & \text{if } k = j \end{cases}$$

and

$$\left\{\underline{x}:\sum_{l=1}^{n} i_l x_l > 1, \bigcap_{\substack{r=1 \\ j_r\in J}}^{l}\{x_{j_r} > 1\}\right\}\bigcap A_k = \emptyset,$$

if $l > 2$. In addition, note that

$$\sum_{i\in I}\sum_{j:\,i_j=1}\mathbb{P}\left(X_j^{(i)} > t, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right) = \sum_{k=1}^{n}\mathbb{P}\left(X_k > t\right). \tag{2.6.3}$$

Therefore, by applying Lemma 1.2 and the fact that for all $i \in I$ measure $\mu_D^{(i)}$ does not put any mass on the boundary of its domain we get

$$\begin{aligned}
\lim_{t\to\infty} & \frac{\sum_{i\in I}\mathbb{P}\left(\sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)}{\overline{F}_1(t)} \\
&= \sum_{i\in I} a^{(i)}\mu_D^{(i)}\left(\underline{x}:\sum_{j=1}^{n} i_j x_j > 1, \bigcap_{j:i_j=1}\{x_j \leqslant 1\}\right) \\
&+ \sum_{k=1}^{n} c_k - \sum_{i\in I}\sum_{j:i_j=1} a^{(i)}\mu_D^{(i)}\left(\underline{x}:\sum_{j=1}^{n} i_j x_j \leqslant 1, x_j > 1\right) \\
&+ \sum_{i\in I}\sum_{l=2}^{|J|}(-1)^{l+1}\sum_{\{j_1,\dots,j_l\}\in J} a^{(i)}\mu_D^{(i)}\left(\underline{x}:\sum_{j=1}^{n} i_j x_j > 1, \bigcap_{r=1}^{l}\{x_{j_r} > 1\}\right) \\
&= \sum_{i\in I} a^{(i)}\mu_D^{(i)}\left(\underline{x}:\sum_{j=1}^{n} i_j x_j > 1\right) + \sum_{k=1}^{n}\left(1 - \sum_{\substack{i\in I \\ i_k=1}} a^{(i)}\right)c_k,
\end{aligned} \tag{2.6.4}$$

where in the last step we reversed the logic of expression (2.6.2) for measures $\mu_D^{(i)}$, $i \in I$, used equality (1.6.1) with $z = 1$ and the fact that

$$\sum_{i\in I}\sum_{j:\,i_j=1} a^{(i)}c_j = \sum_{k=1}^{n}\sum_{\substack{i\in I \\ i_k=1}} a^{(i)}c_k.$$

Now we will deal with the numerator of (2.6.1). Obviously, for $t > 0$ we have

$$\sum_{i\in I} i_k \mathbb{E} X_k^{(i)} \mathbb{I}_{\left\{\sum\limits_{j=1}^n i_j X_j^{(i)} > t, \bigcap\limits_{j=1}^n \{X_j^{(i)} > 0\}\right\}}$$

$$= \int_0^\infty \sum_{i\in I} i_k \mathbb{P}\left(X_k^{(i)} > z, \sum_{j=1}^n i_j X_j^{(i)} > t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right) \mathrm{d}z$$

$$= t\int_0^\infty \sum_{i\in I} i_k \mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^n i_j X_j^{(i)} > t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right) \mathrm{d}z. \tag{2.6.5}$$

First, observe that for $z > 1$ we have

$$\sum_{i\in I} i_k \mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^n i_j X_j^{(i)} > t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right)$$

$$= \sum_{\substack{i\in I\\ i_k=1}} \mathbb{P}\left(X_k^{(i)} > tz, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right)$$

$$- \sum_{\substack{i\in I\\ i_k=1}} \mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^n i_j X_j^{(i)} \leqslant t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right)$$

$$- \sum_{\substack{i\in I\\ i_k=-1}} \mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^n i_j X_j^{(i)} > t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right) \tag{2.6.6}$$

and for $0 < z \leqslant 1$ we have

$$\sum_{i\in I} i_k \mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^n i_j X_j^{(i)} > t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right)$$

$$= \sum_{\substack{i\in I\\ i_k=1}} \mathbb{P}\left(tz < X_k^{(i)} \leqslant t, \sum_{j=1}^n i_j X_j^{(i)} > t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right)$$

$$+ \sum_{\substack{i\in I\\ i_k=1}} \mathbb{P}\left(X_k^{(i)} > t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right)$$

$$- \sum_{\substack{i\in I\\ i_k=1}} \mathbb{P}\left(X_k^{(i)} > t, \sum_{j=1}^n i_j X_j^{(i)} \leqslant t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right)$$

$$- \sum_{\substack{i\in I\\ i_k=-1}} \mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^n i_j X_j^{(i)} > t, \bigcap_{j=1}^n \{X_j^{(i)} > 0\}\right). \tag{2.6.7}$$

Similarly as in (2.6.3) for all $z > 0$ we have

$$\sum_{\substack{i\in I\\ i_k=1}} \mathbb{P}\left(X_k^{(i)} > tz, \bigcap_{j=1}^{n} \{X_j^{(i)} > 0\} \right) = \mathbb{P}\left(X_k > tz\right). \tag{2.6.8}$$

For other sets in (2.6.6) and (2.6.7) observe that for all $l \in \{1, 2, \ldots, n\}$

$$\left\{ \underline{x} : x_k \geqslant z, \sum_{j=1}^{n} i_j x_j \geqslant 1 \right\} \bigcap A_l = \emptyset,$$

if $i_k = -1$,

$$\left\{ \underline{x} : x_k \geqslant z, \sum_{j=1}^{n} i_j x_j \leqslant 1 \right\} \bigcap A_l = \emptyset,$$

if $i_k = 1$ and $z > 1$, and

$$\left\{ \underline{x} : z \leqslant x_k \leqslant 1, \sum_{j=1}^{n} i_j x_j \geqslant 1 \right\} \bigcap A_l = \begin{cases} \emptyset, & \text{if } l \neq k \\ A_l(1) & \text{if } l = k \end{cases},$$

$$\left\{ \underline{x} : x_k \geqslant 1, \sum_{j=1}^{n} i_j x_j \leqslant 1 \right\} \bigcap A_l = \begin{cases} \emptyset, & \text{if } l \neq k \\ A_l(1) & \text{if } l = k \end{cases},$$

if $i_k = 1$ and $0 < z \leqslant 1$.

Therefore we can apply Lemma 1.2 for (2.6.6) and (2.6.7) and by using (1.6.1) and (2.6.8) we get

$$\lim_{t\to\infty} \frac{\sum_{i\in I} i_k \mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j=1}^{n} \{X_j^{(i)} > 0\} \right)}{\overline{F}_1(t)} = \sum_{i\in I} i_k a^{(i)} \mu_D^{(i)} \left(\underline{x} : x_k > z, \sum_{j=1}^{n} i_j x_j > 1 \right) + \left(1 - \sum_{\substack{i\in I\\ i_k=1}} a^{(i)} \right) c_k z^{-\alpha} \tag{2.6.9}$$

for $z > 1$ and

$$\lim_{t\to\infty} \frac{\sum_{i\in I} i_k \mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j=1}^{n} \{X_j^{(i)} > 0\} \right)}{\overline{F}_1(t)} = \sum_{i\in I} i_k a^{(i)} \mu_D^{(i)} \left(\underline{x} : x_k > z, \sum_{j=1}^{n} i_j x_j > 1 \right) + \left(1 - \sum_{\substack{i\in I\\ i_k=1}} a^{(i)} \right) c_k \tag{2.6.10}$$

for $0 < z \leqslant 1$.

To get the assertion of the theorem, we need to integrate the quantity

$$\frac{\mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right)}{\overline{F}_1(t)}$$

for each $i \in I$ on the interval $[0, \infty)$. We note that this quantity is obviously bounded on the interval $[0, 1]$, while on interval $[1, \infty)$ it is bounded by an integrable function due to

$$\mathbb{P}\left(X_k^{(i)} > tz, \sum_{j=1}^{n} i_j X_j^{(i)} > t, \bigcap_{j=1}^{n}\{X_j^{(i)} > 0\}\right) \leqslant \mathbb{P}\left(X_k^{(i)} > tz\right)$$

and Proposition 0.8 of Resnick [52]. Therefore, we can apply the Lebesgues dominated convergence theorem and pass the limit through the sign of integral in expression (2.6.5). After integrating, equalities (2.6.9) and (2.6.10), together with equality (2.6.4) and basic relations (2.6.1) and (2.6.5), imply the desired relation (1.6.2). Theorem 1.7 is proved. $\square$

Chapter **3**

Examples

In this chapter we present several theoretical and practical examples to demonstrate the applicability of the main results for particular collections of the primary r.v.s $\{X_1, X_2, \ldots, X_n\}$ and the random weights $\{\theta_1, \theta_2, \ldots, \theta_n\}$. In the first example the d.f.s of the primary r.v.s belong to the class $\mathcal{D}$. In other examples we restrict to regularly varying tails.

Example 3.1. Let us consider three pairwise SQAI r.v.s X_1, X_2, X_3 with d.f.s F_1, F_2, F_3, respectively. Suppose that these primary r.v.s are distributed according to the generalized Peter-and-Paul distributions with parameters $1, \frac{1}{2}$ and 2 respectively (see Example 1.1), i.e.

$$\begin{aligned}
\overline{F}_1(x) &= 4\sum_{2^k>x}\frac{1}{5^k} = 5^{-\lfloor \log x/\log 2\rfloor}, \quad x \geqslant 1,\\
\overline{F}_2(x) &= (\sqrt{5}-1)\sum_{\sqrt{2}^k>x}\frac{1}{\sqrt{5}^k} = \sqrt{5}^{\,-\lfloor \log x/\log\sqrt{2}\rfloor}, \quad x \geqslant 1,\\
\overline{F}_3(x) &= 24\sum_{4^k>x}\frac{1}{25^k} = 25^{-\lfloor \log x/\log 4\rfloor}, \quad x \geqslant 1,
\end{aligned}$$

with the L-indexes

$$L_{F_1} = \frac{1}{5},\ L_{F_2} = \frac{1}{\sqrt{5}},\ L_{F_3} = \frac{1}{25}.$$

Consequently, $L_3^X = 1/25$ in this case.

Let the random weights $\{\theta_1, \theta_2, \theta_3\}$ be independent of $\{X_1, X_2, X_3\}$, mutually arbitrarily dependent and identically distributed according to the discrete law $\mathbb{P}(\theta_1 = 1/2) = \mathbb{P}(\theta_1 = 1) = 1/2$.

After some calculations we conclude that all the conditions of Theorems 1.5, 1.6 and Corollary 1.1 are satisfied. Theorem 1.5 implies the following asymptotic formulas:

$$\begin{aligned}
\mathbb{P}\big(S_3^{\theta X} > x\big) &\underset{x\to\infty}{\lesssim} \frac{25}{2}\Bigg(5^{-\lfloor \log 2x/\log 2\rfloor} + 5^{-\lfloor \log x/\log 2\rfloor} + \sqrt{5}^{-\lfloor \log 2x/\log\sqrt{2}\rfloor} \\
&\qquad + \sqrt{5}^{-\lfloor \log x/\log\sqrt{2}\rfloor} + 25^{-\lfloor \log 2x/\log 4\rfloor} + 25^{-\lfloor \log x/\log 4\rfloor}\Bigg) \\
&\leqslant 15\,(30+\sqrt{5})5^{-\log x/\log 2} < 484\,x^{-\log 5/\log 2}, \\
\mathbb{P}\big(S_3^{\theta X} > x\big) &\underset{x\to\infty}{\gtrsim} \frac{1}{50}\Bigg(5^{-\lfloor \log 2x/\log 2\rfloor} + 5^{-\lfloor \log x/\log 2\rfloor} + \sqrt{5}^{-\lfloor \log 2x/\log\sqrt{2}\rfloor} \\
&\qquad + \sqrt{5}^{-\lfloor \log x/\log\sqrt{2}\rfloor} + 25^{-\lfloor \log 2x/\log 4\rfloor} + 25^{-\lfloor \log x/\log 4\rfloor}\Bigg) \\
&\geqslant \frac{9}{125}\,x^{-\log 5/\log 2}.
\end{aligned}$$

Next, Theorem 1.6 and Lemma 2.12 yield

$$\begin{aligned}
\mathbb{E}\big(S_3^{\theta X}\mathbb{I}_{\{S_3^{\theta X}>x\}}\big) &\underset{x\to\infty}{\lesssim} \frac{25}{4}\sum_{k=1}^{3}\mathbb{E}\big(X_k\mathbb{I}_{\{X_k>2x\}}\big) + \frac{1}{2}\sum_{k=1}^{3}\mathbb{E}\big(X_k\mathbb{I}_{\{X_k>x\}}\big) \\
&= \frac{25}{4}\Bigg(2x\left(5^{-\lfloor\frac{\log 2x}{\log 2}\rfloor} + \sqrt{5}^{-\lfloor\frac{\log 2x}{\log\sqrt{2}}\rfloor} + 25^{-\lfloor\frac{\log 2x}{\log 4}\rfloor}\right) \\
&\qquad + \int_{2x}^{\infty}\left(5^{-\lfloor\frac{\log u}{\log 2}\rfloor} + \sqrt{5}^{-\lfloor\frac{\log u}{\log\sqrt{2}}\rfloor} + 25^{-\lfloor\frac{\log u}{\log 4}\rfloor}\right)\mathrm{d}u\Bigg) \\
&+ \frac{25}{4}\Bigg(x\left(5^{-\lfloor\frac{\log x}{\log 2}\rfloor} + \sqrt{5}^{-\lfloor\frac{\log x}{\log\sqrt{2}}\rfloor} + 25^{-\lfloor\frac{\log x}{\log 4}\rfloor}\right) \\
&\qquad + \int_{x}^{\infty}\left(5^{-\lfloor\frac{\log u}{\log 2}\rfloor} + \sqrt{5}^{-\lfloor\frac{\log u}{\log\sqrt{2}}\rfloor} + 25^{-\lfloor\frac{\log u}{\log 4}\rfloor}\right)\mathrm{d}u\Bigg) \\
&\leqslant 15(30+\sqrt{5})\,\frac{\log 5}{\log 5-\log 2}\,x\cdot 5^{-\frac{\log x}{\log 2}} < 850\,x^{1-\log 5/\log 2}, \\
\mathbb{E}\big(S_3^{\theta X}\mathbb{I}_{\{S_3^{\theta X}>x\}}\big) &\underset{x\to\infty}{\gtrsim} \frac{1}{100}\sum_{k=1}^{3}\mathbb{E}\big(X_k\mathbb{I}_{\{X_k>2x\}}\big) + \frac{1}{50}\sum_{k=1}^{3}\mathbb{E}\big(X_k\mathbb{I}_{\{X_k>x\}}\big) \\
&\geqslant \frac{9}{125}\frac{\log 5}{\log 5-\log 2}\,x\cdot 5^{-\frac{\log x}{\log 2}} > \frac{1}{8}\,x^{1-\log 5/\log 2}.
\end{aligned}$$

At the end of this example we only remark that we need an assertion similar to Lemma 1.1 for further consideration of the values related to the risk measures in this model.

Example 3.2. Let us consider two independent sequences of random variables $\{X_1, X_2, \ldots, X_n\}$ and $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Suppose that the r.v.s $X_1, X_2, \ldots, X_n$ are independent and the r.v.s $\theta_1, \theta_2, \ldots, \theta_n$ follow an arbitrary dependence structure. For each $k \in \{1, 2, \ldots, n\}$, the r.v. X_k is assumed to have the Lomax distribution (see Example 1.2) with shape parameter $\alpha_k > 1$ and scale parameter $\lambda_k > 0$, i.e.

$$F_k(x) = 1 - \left(1 + \frac{x}{\lambda_k}\right)^{-\alpha_k}, \quad x \geq 0, \quad k \in \{1, 2, \ldots, n\}.$$

Since $L_{F_k} = 1$ for all $k \in \{1, 2, \ldots, n\}$, $L_n^X = \min_{1 \le k \le n} \{L_{F_k}\} = 1$, which yields $\mathbb{P}\big(S_n^{\theta X} > x\big) \underset{x \to \infty}{\sim} \sum_{k=1}^{n} \mathbb{P}(\theta_k X_k > x)$ for any collection $\{\theta_1, \theta_2, \ldots, \theta_n\}$ satisfying the conditions of Theorem 1.5.

Next, let $\mathbb{P}(\theta_k = z_{ki}) = p_{ki}$ for all $k \in \{1, 2, \ldots, n\}$ and $i \in \{1, 2, \ldots, i_k\}$, where $i_k \in \mathbb{N}$. Here z_{ki} are real numbers such that $0 < z_{k1} < z_{k2} < \ldots < z_{ki_k} < \infty$ and $\sum_{i=1}^{i_k} p_{ki} = 1$ for all $k \in \{1, 2, \ldots, n\}$. It is obvious that $\max_{1 \le k \le n} \{\mathbb{E}\theta_k^p\}$ is finite for all $p > \max_{1 \le k \le n} \{\alpha_k\}$. Consequently, the collection $\{\theta_1, \theta_2, \ldots, \theta_n\}$ meets the conditions of Theorem 1.5.

By the law of total probability, we get

$$\begin{aligned} \mathbb{P}(\theta_k X_k > x) &= \sum_{i=1}^{i_k} \mathbb{P}(\theta_k X_k > x \,|\, \theta_k = z_{ki})\, \mathbb{P}(\theta_k = z_{ki}) = \sum_{i=1}^{i_k} p_{ki}\, \mathbb{P}(X_k > x / z_{ki}) \\ &= \sum_{i=1}^{i_k} p_{ki} \left(\frac{\lambda_k z_{ki}}{\lambda_k z_{ki} + x} \right)^{\alpha_k} \underset{x \to \infty}{\sim} \frac{\lambda_k^{\alpha_k}}{x^{\alpha_k}} \sum_{i=1}^{i_k} p_{ki} z_{ki}^{\alpha_k} = \frac{\lambda_k^{\alpha_k}\, \mathbb{E}\theta_k^{\alpha_k}}{x^{\alpha_k}}. \end{aligned} \tag{3.1}$$

Therefore, by Theorem 1.5,

$$\mathbb{P}\big(S_n^{\theta X} > x\big) \underset{x \to \infty}{\sim} \sum_{k=1}^{n} \frac{\lambda_k^{\alpha_k}\, \mathbb{E}\theta_k^{\alpha_k}}{x^{\alpha_k}} \underset{x \to \infty}{\sim} \frac{\sum_{k=1}^{n} \lambda_k^{\alpha_k}\, \mathbb{E}\theta_k^{\alpha_k}\, \mathbb{1}_{\{\alpha_k = \alpha_{min}\}}}{x^{\alpha_{min}}},$$

where $\alpha_{min} = \min_{1 \le k \le n} \{\alpha_k\}$.

To apply Theorem 1.6, the condition $\mathbb{P}\left(\theta_k X_k > x\right) \underset{x \to \infty}{\asymp} \mathbb{P}\left(\theta_1 X_1 > x\right)$ must hold for all $k \in \{1, 2, \ldots, n\}$. It follows easily from (3.1) that this condition is true provided that $\alpha_1 = \alpha_2 = \ldots = \alpha_n$. Hence, from now on, we suppose that $\alpha := \alpha_1 = \alpha_2 = \ldots = \alpha_n$.

Since $L_n^X = 1$, by Theorem 1.6, we get $\mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{S_n^{\theta X} > x\}}\big) \underset{x \to \infty}{\sim} \mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{\theta_k X_k > x\}}\big)$ for all $k \in \{1, 2, \ldots, n\}$ and $\mathbb{E}\big(S_n^{\theta X} \mathbb{1}_{\{S_n^{\theta X} > x\}}\big) \underset{x \to \infty}{\sim} \sum_{k=1}^{n} \mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{\theta_k X_k > x\}}\big)$.

By the law of total expectation, we obtain

$$\begin{aligned}
\mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{\theta_k X_k > x\}}\big) &= \sum_{i=1}^{i_k} \mathbb{E}\big(\theta_k X_k \mathbb{1}_{\{\theta_k X_k > x\}} \,|\, \theta_k = z_{ki}\big)\, \mathbb{P}(\theta_k = z_{ki}) \\
&= \sum_{i=1}^{i_k} z_{ki} p_{ki}\, \mathbb{E}\big(X_k \mathbb{1}_{\{X_k > x/z_{ki}\}}\big) \\
&= \sum_{i=1}^{i_k} z_{ki} p_{ki} \int_{x/z_{ki}}^{\infty} \frac{\alpha y}{\lambda_k} \left(1 + \frac{y}{\lambda_k}\right)^{-(\alpha+1)} \mathrm{d}y \\
&= \sum_{i=1}^{i_k} p_{ki} \left(x \left(1 + \frac{x}{\lambda_k z_{ki}}\right)^{-\alpha} + \frac{\lambda_k z_{ki}}{\alpha - 1} \left(1 + \frac{x}{\lambda_k z_{ki}}\right)^{1-\alpha} \right) \\
&\underset{x\to\infty}{\sim} \sum_{i=1}^{i_k} p_{ki} \left(x \left(\frac{\lambda_k z_{ki}}{x}\right)^{\alpha} + \frac{\lambda_k z_{ki}}{\alpha - 1} \left(\frac{\lambda_k z_{ki}}{x}\right)^{\alpha-1} \right) \\
&= \frac{\alpha \lambda_k^\alpha}{(\alpha-1)x^{\alpha-1}} \sum_{i=1}^{i_k} p_{ki} z_{ki}^\alpha = \frac{\alpha \lambda_k^\alpha\, \mathbb{E}\theta_k^\alpha}{(\alpha-1)x^{\alpha-1}}.
\end{aligned}$$

Thus, we have

$$\mathbb{E}\big(S_n^{\theta X} \mathbb{1}_{\{S_n^{\theta X} > x\}}\big) \underset{x\to\infty}{\sim} \frac{\alpha}{(\alpha-1)x^{\alpha-1}} \sum_{k=1}^{n} \lambda_k^\alpha\, \mathbb{E}\theta_k^\alpha.$$

In addition, we get

$$\mathbb{P}\big(S_n^{\theta X} > x\big) \underset{x\to\infty}{\sim} \frac{1}{x^\alpha} \sum_{k=1}^{n} \lambda_k^\alpha\, \mathbb{E}\theta_k^\alpha. \tag{3.2}$$

The last two asymptotic relations imply that

$$\mathbb{E}\big(S_n^{\theta X} \,|\, S_n^{\theta X} > x\big) \underset{x\to\infty}{\sim} \frac{\alpha}{\alpha-1}\, x. \tag{3.3}$$

Moreover, for any $l \in \{1, 2, \ldots, n\}$, we have

$$\mathbb{E}\big(\theta_l X_l \,|\, S_n^{\theta X} > x\big) \underset{x\to\infty}{\sim} \frac{\alpha}{\alpha-1}\, x\, \frac{\lambda_l^\alpha\, \mathbb{E}\theta_l^\alpha}{\sum_{k=1}^{n} \lambda_k^\alpha\, \mathbb{E}\theta_k^\alpha}. \tag{3.4}$$

If we choose $x = \mathrm{VaR}_q\big(S_n^{\theta X}\big)$ in (3.2), then, using Lemma 1.1, we conclude that

$$\mathrm{VaR}_q\big(S_n^{\theta X}\big) \underset{q\uparrow 1}{\sim} (1-q)^{-1/\alpha} \left(\sum_{k=1}^{n} \lambda_k^\alpha\, \mathbb{E}\theta_k^\alpha \right)^{1/\alpha}.$$

Therefore, (3.3) implies that

$$\mathrm{CTE}_q\big(S_n^{\theta X}\big) \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha-1} (1-q)^{-1/\alpha} \left(\sum_{k=1}^{n} \lambda_k^\alpha\, \mathbb{E}\theta_k^\alpha \right)^{1/\alpha}.$$

Furthermore, by (3.4), for any fixed $l \in \{1, 2, \ldots, n\}$, we obtain

$$\mathrm{AC}_{ql}\big(S_n^{\theta X}\big) \underset{q\uparrow 1}{\sim} \frac{\alpha}{\alpha-1} (1-q)^{-1/\alpha} \lambda_l^\alpha \mathbb{E}\theta_l^\alpha \left(\sum_{k=1}^{n} \lambda_k^\alpha\, \mathbb{E}\theta_k^\alpha \right)^{1/\alpha-1}.$$

Example 3.3. Let us consider an investment portfolio of n financial assets. Let X_k, $k \in \{1, 2, \dots, n\}$, be their negative returns (so that positive values represent losses) over some time period in the future with distribution functions F_k, $k \in \{1, 2, \dots, n\}$, and let θ_k, $k \in \{1, 2, \dots, n\}$, be their weights. We are interested in asymptotic CTE_q of the portfolio and its composition.

We assume that X_k, $k \in \{1, 2, \dots, n\}$, are pairwise SQAI and have t-location-scale distributions with parameters (μ_k, σ_k, ν_k) (see Example 1.3). It can be easily shown that for all $k \in \{1, \dots, n\}$ $\lim\limits_{x\to\infty} \frac{\overline{F_k}(x)}{\overline{F_1}(x)} = \left(\frac{\sigma_k}{\sigma_1}\right)^{\nu}$ provided that $\nu_k = \nu_1 = \nu$. If $\nu_k > \nu_1$, the limit is 0. If $\nu_k < \nu_1$, the limit is infinite. It means that the tail risk of the portfolio is dominated by the risks with the heaviest tails, and the risks with lighter tails are asymptotically irrelevant. In practice, in order to account for the estimation errors, the tail parameters, estimated to be similar, could be assumed to be equal. In this case, the tail risk of the whole portfolio will be asymptotically equal to the tail risk of the portfolio consisting only of the risks with the heaviest tails. Alternatively, all distributions could be conservatively assumed to have the tails as heavy as their heaviest estimate, i.e. $\nu = \min\limits_k(\hat{\nu}_k)$, where $\hat{\nu}_k$, $k \in \{1, 2, \dots, n\}$, are the estimated tail parameters. Therefore, in the rest of this example, we assume $\nu_k = \nu$ for all $k \in \{1, 2, \dots, n\}$. To apply Theorem 1.6, we also assume that $\nu > 1$.

Further, we assume that the weights θ_k are uniformly distributed on $[(1 - \lambda_k)a_k, (1 + \lambda_k)a_k]$, $k \in \{1, 2, \dots, n\}$. Constants $(a_1, \dots, a_n)$ can be interpreted as a target portfolio allocation (benchmark), and $(\lambda_1, \dots, \lambda_n)$ can be interpreted as limits for deviation. Then

$$\mathbb{E}\theta_k^{\nu} = \frac{1}{2\lambda_k a_k} \int\limits_{(1-\lambda_k)a_k}^{(1+\lambda_k)a_k} x^{\nu} \mathrm{d}x = \frac{a_k^{\nu}\left((1+\lambda_k)^{\nu+1} - (1-\lambda_k)^{\nu+1}\right)}{2\lambda_k(\nu+1)}.$$

Using the same way of considerations as in the previous example, for $l \in \{1, 2, \dots, n\}$, we get

$$\mathrm{AC}_{ql}\big(S_n^{\theta X}\big) \underset{q\uparrow 1}{\sim} \frac{\nu}{\nu-1} \frac{1}{\sigma_l} \left(\sum_{k=1}^{n} (\sigma_k a_k)^{\nu} \frac{(1+\lambda_k)^{\nu+1} - (1-\lambda_k)^{\nu+1}}{2\lambda_k(\nu+1)} \right)^{\frac{1}{\nu}} \mathrm{VaR}_q(X_l),$$

and the asymptotic contributions of individual risks $l \in \{1, 2, \dots, n\}$ are

$$\mathrm{CIR}_{ql}\big(S_n^{\theta X}\big) \underset{q\uparrow 1}{\sim} \frac{(\sigma_l a_l)^{\nu} \frac{1}{\lambda_l}\left((1+\lambda_l)^{\nu+1} - (1-\lambda_l)^{\nu+1}\right)}{\sum\limits_{k=1}^{n} (\sigma_k a_k)^{\nu} \frac{1}{\lambda_k}\left((1+\lambda_k)^{\nu+1} - (1-\lambda_k)^{\nu+1}\right)}.$$

Example 3.4. Let us consider an investment portfolio described in Example 3.3 with $\theta_k \equiv a_k$, $k \in \{1, 2, \dots, n\}$, where $\{a_1, \dots, a_n\}$ are positive constants. We are interested in asymptotic capital allocation of the aggregate risk (negative return) of the portfolio $S_n^{aX} := \sum_{i=1}^n a_i X_i$.

Assume that, as in Example 3.3, r.v.s X_k, $k \in \{1, 2, \dots, n\}$, have t-location-scale distributions with parameters (μ_k, σ_k, ν). To apply Theorem 1.7, we also assume that $\nu > 1$. r.v.s $a_k X_k$ then obviously have t-location-scale distributions with parameters $(a_k\mu_k, a_k\sigma_k, \nu)$ and $c_k := \lim\limits_{t\to\infty} \frac{\mathbb{P}(a_k X_k > t)}{\mathbb{P}(a_1 X_1 > t)} = \left(\frac{a_k\sigma_k}{a_1\sigma_1}\right)^\nu$, $k \in \{1, \dots, n\}$. Moreover, since the density function of a t-location-scale distribution is symmetric around its location parameter, both right and left tails are equally heavy with $\lim\limits_{t\to\infty} \frac{\mathbb{P}(-a_k X_k > t)}{\mathbb{P}(a_k X_k > t)} = 1$ for all $k \in \{1, \dots, n\}$.

The dependence structure assumed in Theorem 1.7 is very wide. It allows complex structures with different tail dependence in each orthant. It also allows dependence structures obtained through mixtures of both tail dependence and tail independence which allows the tail dependence to be modelled separately from non-tail dependence. One way to achieve such a flexible dependence structure is to choose a complex copula function with enough parameters to model all desirable properties separately. Another approach is choosing simple copula functions with few parameters capturing one or several dependence properties and joining them together in a weighted sum. Since there is a wide variety of simple copula functions, the latter approach is more intuitive and easier to apply.

Therefore for each $i = \{i_1, \dots, i_n\} \in I := \{-1, 1\}^n \diagdown \{-1\}^n$, we denote $X_k^{(i)} = i_k X_k \mathbb{1}_{\{i_k X_k > 0\}}$ for all $k \in \{1, 2, \dots, n\}$ and denote the copula function of a random vector $\{i_1 X_1, \dots, i_n X_n\}$ by $C^{(i)}$, where C is the copula function of a random vector $\{X_1, \dots, X_n\}$ and it can be expressed as follows

$$C = w_0 C_0 + \sum_{i\in I} w_i C_i, \tag{3.5}$$

where $w_0 \geqslant 0$, $w_i \geqslant 0$, $i \in I$, $w_0 + \sum\limits_{i\in I} w_i = 1$, C_0 is a copula function without tail dependence on any of the relevant orthants (i.e. only for $i = \{-1\}^n$) and C_i, $i \in I$, is a copula function with tail dependence only in the orthant i. More specifically, for all positive $x_1, \dots, x_n$ and all $i \in I$, we assume that the limit

$$\lim_{u\downarrow 0} \frac{\hat{C}_j^{(i)}(ux_1, \dots, ux_n)}{u}$$

is positive, if $j = i$, and equals zero otherwise (including $j = 0$).

In the bivariate case the most popular copula functions with tail dependence are Clayton and Gumbel. Since, unlike Gumbel copula, the Clayton copula has tail dependence in only one orthant, we choose its rotated multivariate versions, i.e.

$$\hat{C}_i^{(i)}(u_1, \ldots, u_n) := \left(\sum_{k=1}^{n} u_k^{-\theta_i} - (n-1) \right)^{-\frac{1}{\theta_i}}, \quad \theta_i > 0, \quad i \in I,$$

with the limit

$$\lim_{u \downarrow 0} \frac{\hat{C}_i^{(i)}(ux_1, \ldots, ux_n)}{u} = \left(\sum_{k=1}^{n} x_k^{-\theta_i} \right)^{-\frac{1}{\theta_i}}.$$

It is easy to show that the limit above equals zero for all other orthants. For example, if we take the orthant where only the j-th variable differs from $i \in I$, i.e. $j = \{i_1, \ldots, i_{j-1}, -i_j, i_{j+1}, \ldots, i_n\}$, then, since the marginal distributions are continuous, we have

$$\begin{aligned}
&\hat{C}_i^{(j)}(u_1, \ldots, u_n) \\
&\quad = \mathbb{P}(\overline{F}_{i_1X_1}(i_1X_1) > u_1, \ldots, \overline{F}_{-i_jX_j}(-i_jX_j) > u_j, \ldots, \overline{F}_{i_nX_n}(i_nX_n) > u_n) \\
&\quad = \mathbb{P}(\overline{F}_{i_1X_1}(i_1X_1) > u_1, \ldots, \overline{F}_{i_jX_j}(i_jX_j) < 1 - u_j, \ldots, \overline{F}_{i_nX_n}(i_nX_n) > u_n) \\
&\quad = \hat{C}_i^{(i)}(u_1, \ldots, u_{j-1}, 1, u_{j+1}, \ldots, u_n) - \hat{C}_i^{(i)}(u_1, \ldots, u_{j-1}, 1 - u_j, u_{j+1}, \ldots, u_n)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{u \downarrow 0} \frac{\hat{C}_i^{(j)}(ux_1, \ldots, ux_n)}{u} &= \lim_{u \downarrow 0} \left(\sum_{k \neq j} x_k^{-\theta_i} - (n-2)u^{\theta_i} \right)^{-\frac{1}{\theta_i}} \\
&- \lim_{u \downarrow 0} \left(\sum_{k \neq j} x_k^{-\theta_i} + (1 - ux_j)^{-\theta_i} u^{\theta_i} - (n-1)u^{\theta_i} \right)^{-\frac{1}{\theta_i}} \\
&= 0.
\end{aligned}$$

Similarly, the same result can be obtained for other orthants.

Further it can be shown that for all $\underline{x} = (x_1, \ldots, x_n) \in (0, \infty]^n$

$$\begin{aligned}
H^{(i)}(\underline{x}) &:= \lim_{t \to \infty} \frac{\mathbb{P}(a_1X_1^{(i)} > tx_1, \ldots, a_nX_n^{(i)} > tx_n)}{\mathbb{P}(a_1X_1 > t)} \\
&= \lim_{t \to \infty} \frac{\hat{C}^{(i)}(\mathbb{P}(i_1a_1X_1 > tx_1), \ldots, \mathbb{P}(i_na_nX_2 > tx_n))}{\mathbb{P}(a_1X_1 > t)} \\
&= \lim_{t \to \infty} \frac{\hat{C}^{(i)}(x_1^{-\nu}c_1\mathbb{P}(a_1X_1 > t), \ldots, x_n^{-\nu}c_n\mathbb{P}(a_1X_1 > t))}{\mathbb{P}(a_1X_1 > t)} \\
&= w_i \left(\sum_{k=1}^{n} x_k^{\nu\theta_i} c_k^{-\theta_i} \right)^{-\frac{1}{\theta_i}} \qquad (3.6)
\end{aligned}$$

and $H^{(i)}(0,\dots,0,x_k,0,\dots,0) = x_k^{-\nu}c_k$ for all $i \in I$ and positive x_k, $k = 1,\dots,n$. Hence, if for all $(\underline{x}) \in [0,\infty]^n \setminus \{0\}^n$ and all $i \in I$ we denote

$$\begin{aligned}
\mu_D^{(i)}((x_1,\infty]\times\cdots\times(x_n,\infty]) &:= \left(\sum_{k=1}^{n} x_k^{\nu\theta_i} c_k^{-\theta_i}\right)^{-\frac{1}{\theta_i}}, \\
\mu_I^{(i)}((x_1,\infty]\times\cdots\times(x_n,\infty]) &:= \begin{cases} 0, & \text{if } x_k > 0, \quad \forall k \in \{1,\dots,n\} \\ x_k^{-\nu}c_k, & \text{if } x_k > 0, \quad x_l = 0, \quad \forall l \in \{1,\dots,n\}\setminus\{k\} \end{cases},
\end{aligned}$$

then $H^{(i)}(\underline{x}) = w_i\mu_D^{(i)}((x_1,\infty]\times\cdots\times(x_n,\infty]) + (1-w_i)\mu_I^{(i)}((x_1,\infty]\times\cdots\times(x_n,\infty])$ for all $\underline{x} \in [0,\infty]^n \setminus \{0\}^n$ and all $i \in I$ and all assumptions of Theorem 1.7 are satisfied. Therefore, by applying (1.6.2) and Remark 1.1 for $k = 1,\dots,n$ we get

$$\mathrm{AC}_{qk}\left(S_n^{aX}\right) \underset{q\uparrow 1}{\sim} C_k \mathrm{VaR}_q\left(S_n^{aX}\right) \underset{q\uparrow 1}{\sim} C_k D^{\frac{1}{\nu}} \mathrm{VaR}_q(a_1X_1)$$

and

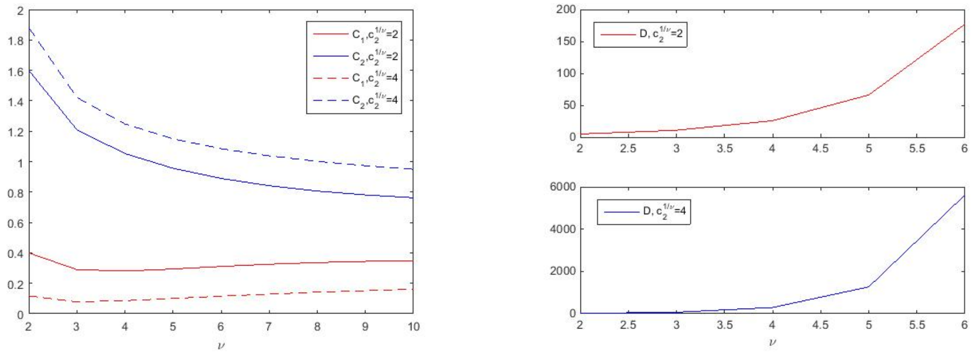
$$\mathrm{CIR}_{qk}\left(S_n^{aX}\right) \underset{q\uparrow 1}{\sim} C_k \Big/ \frac{\nu}{\nu-1},$$

where

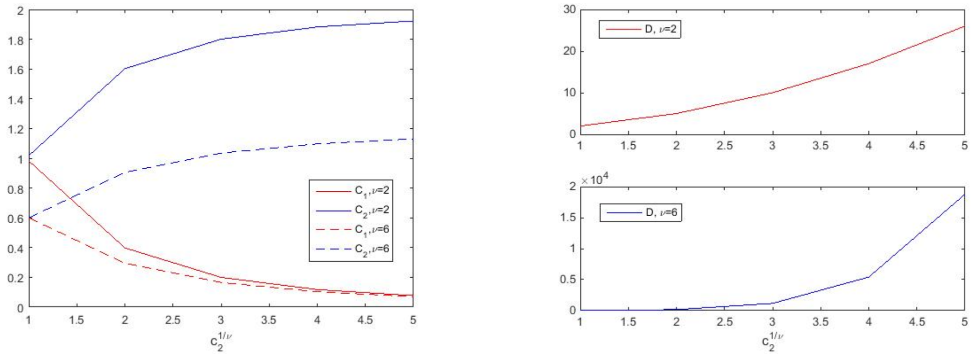
$$\begin{aligned}
C_k &= D^{-1}\left(\sum_{i\in I} i_k w_i \int_0^\infty \mu_D^{(i)}\left(A_k^{(i)}(z)\right)\mathrm{d}z + \frac{\nu}{\nu-1}\left(1 - \sum_{\substack{i\in I\\ i_k=1}} w_i\right)c_k\right), \\
D &= \sum_{i\in I} w_i \mu_D^{(i)}\left(A^{(i)}\right) + \sum_{k=1}^{n}\left(1 - \sum_{\substack{i\in I\\ i_k=1}} w_i\right)c_k, \\
A_k^{(i)}(z) &= \left\{\underline{x} : x_k > z, \sum_{j=1}^{n} i_j x_j > 1\right\}, \quad A^{(i)} = \left\{\underline{x} : \sum_{j=1}^{n} i_j x_j > 1\right\}, \\
c_k &= \left(\frac{a_k\sigma_k}{a_1\sigma_1}\right)^{\nu}.
\end{aligned}$$

Two practical questions immediately arise: how sensitive are the asymptotic constants to model parameters and how fast is the convergence? To give some insight into the first question Figure 3.1 presents some numerically computed constants for particular sets of parameters in the case $n = 2$. The second question is partially answered in the next section.

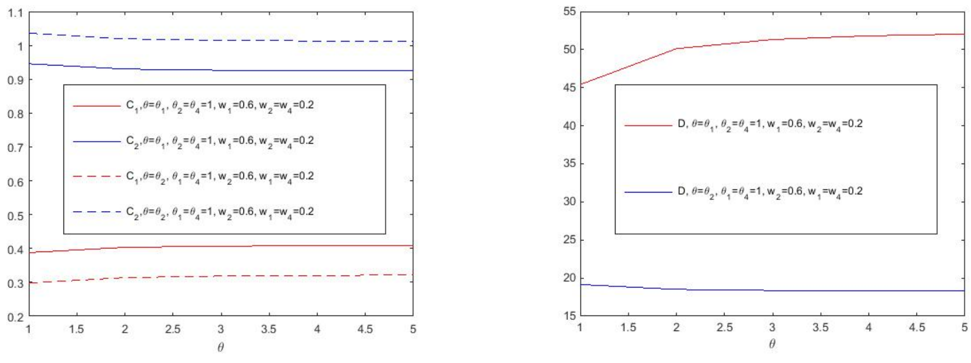
Remark 3.1. From relations in (3.6) we can see that if any of the constants c_k equal zero (i.e. some risks have lighter tails), the limit in (3.6) also equals zero, since any copula function has zero value if at least one of its parameters equals zero. This means that introducing lighter tails not only makes them irrelevant in the asymptotic analysis, but it also eliminates any tail dependence between other variables in all respective orthants (half of them, if only one tail is lighter, and all of them, if both tails are lighter).



a: $\theta_k = 2$, $w_k = 0.2$, $k = 1, 2, 4$.



b: $\theta_k = 1$, $w_k = 0.2$, $k = 1, 2, 4$.



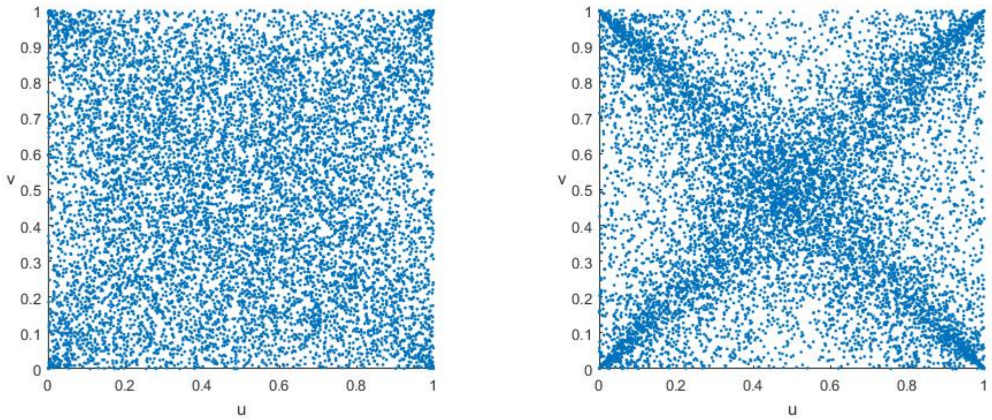
c: $\nu = 4$, $c_2^{1/\nu} = 2$.

Figure 3.1: Asymptotic constants in the case $n = 2$, where k represents the k-th quartile.

Chapter **4**

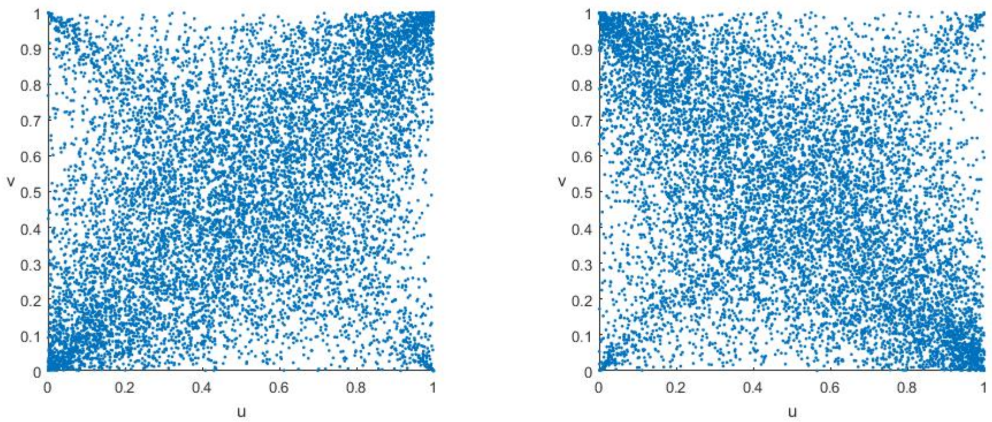
Simulation study

In this chapter we perform a simulation study for the bivariate case of the model presented in Example 3.4 with several sets of parameters. The copula function of (X_1, X_2) is a sum of five copulas with weights $w_1, \dots, w_5$. First four copulas are Clayton copulas, rotated appropriately so that there is tail dependence in all four quartiles, with parameters θ_k in k-th quartile. The fifth copula is Gaussian with correlation ρ, which is known to have no tail dependence (see McNeil et al. [46]). Figures 4.1 and 4.2 show scatter plots of u and v generated by $C(u, v)$ with different copula parameters, where C is a copula function of (X_1, X_2). For each set of model parameters we simulate 100 samples of (X_1, X_2) of size 5,000,000, compute empirical estimates of constants $C_{qk} := AC_{qk}(S_2^{aX})/\mathrm{VaR}_q(S_2^{aX})$, $k = 1, 2$, and $D_q := \left(\mathrm{VaR}_q(S_2^{aX})/\mathrm{VaR}_q(a_1 X_1)\right)^{\nu}$ for different confidence levels q (from 0.95 to 0.9999) and compare them with numerically computed asymptotic constants. In each case $a_1 = a_2 = 1$, $\sigma_1 = 1$ and $\sigma_2 = 2$. The results are presented in Figures 4.3-4.7, where solid lines represent sample averages and dashed lines represent 5-th and 95-th percentiles.



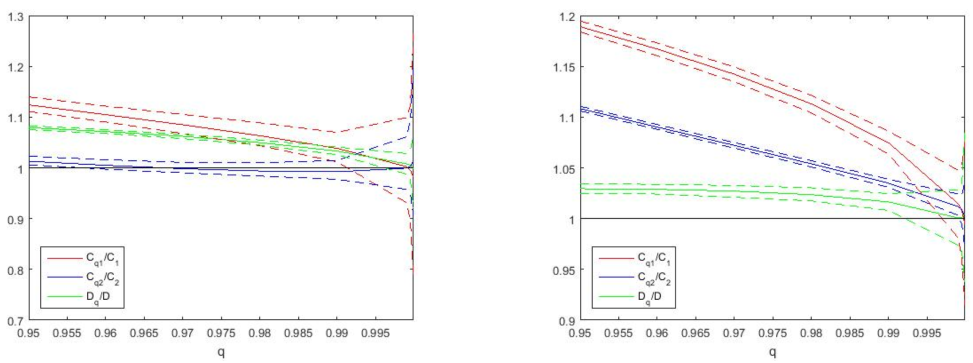
a: $\theta_k = 1,\ k = 1, \ldots, 4.$ b: $\theta_k = 5,\ k = 1, \ldots, 4.$

Figure 4.1: $w_k = 0.2,\ k = 1, \ldots, 5,\ \rho = 0.$



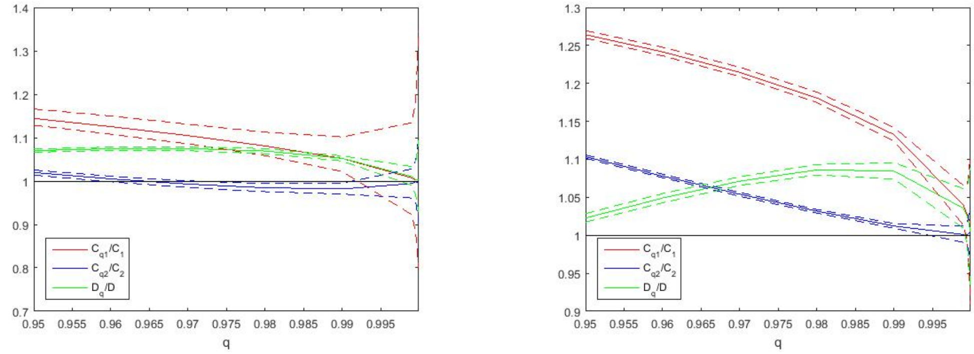
a: $\rho = 0.7.$ b: $\rho = -0.7.$

Figure 4.2: $w_1 = w_2 = w_3 = w_4 = 0.1,\ w_5 = 0.6,\ \theta_k = 3,\ k = 1, \ldots, 4.$



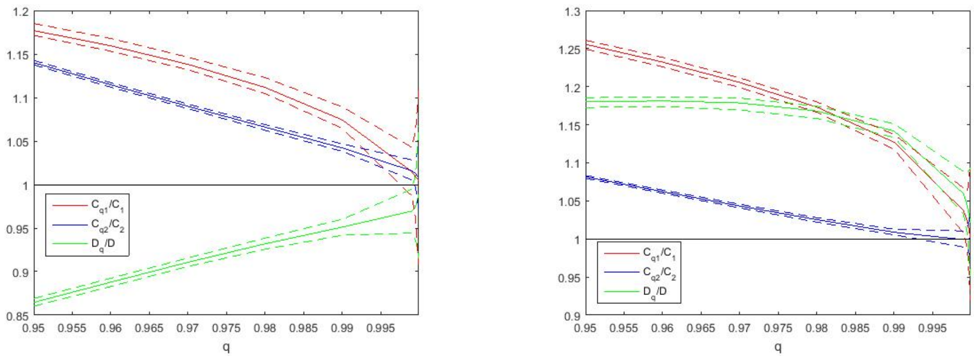
a: $\nu = 2.$ b: $\nu = 4.$

Figure 4.3: $w_k = 0.2,\ k = 1, \ldots, 5,\ \theta_k = 1,\ k = 1, \ldots, 4,\ \rho = 0,\ \mu_1 = \mu_2 = 0.$



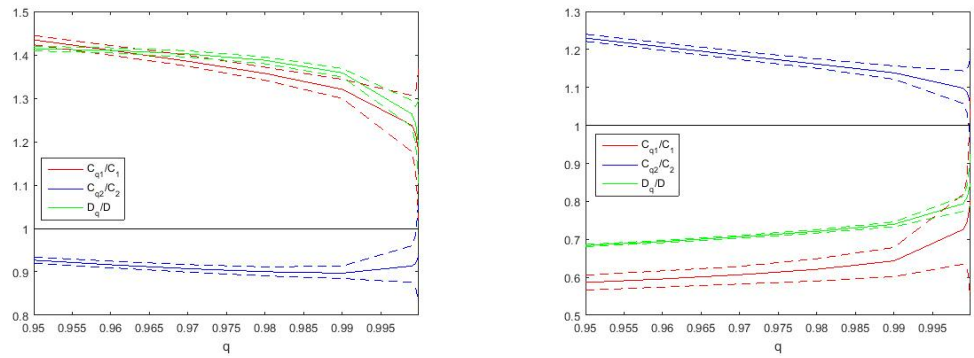
a: $\nu = 2$. b: $\nu = 4$.

Figure 4.4: $w_k = 0.2$, $k = 1, \ldots, 5$, $\theta_k = 5$, $k = 1, \ldots, 4$, $\rho = 0$, $\mu_1 = \mu_2 = 0$.



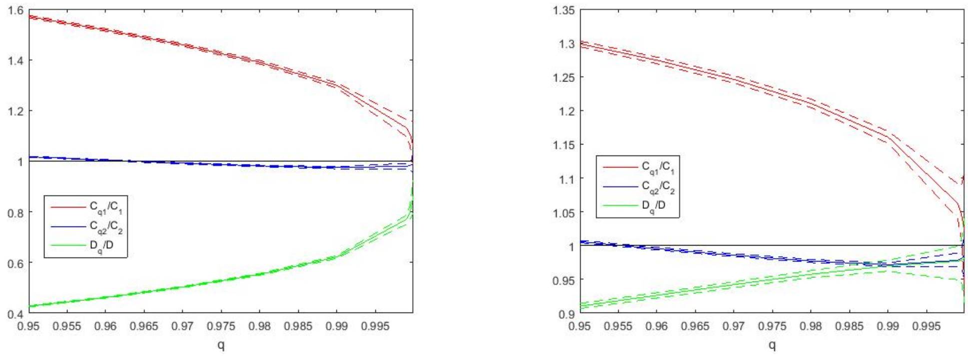
a: $\theta_1 = \theta_3 = 1$, $\theta_2 = \theta_4 = 5$. b: $\theta_1 = \theta_3 = 5$, $\theta_2 = \theta_4 = 1$.

Figure 4.5: $w_k = 0.2$, $k = 1, \ldots, 5$, $\rho = 0$, $\nu = 4$, $\mu_1 = \mu_2 = 0$.



a: $\rho = 0.7$. b: $\rho = -0.7$.

Figure 4.6: $w_1 = w_2 = w_3 = w_4 = 0.1$, $w_5 = 0.6$, $\theta_k = 3$, $k = 1, \ldots, 4$, $\nu = 2$, $\mu_1 = \mu_2 = 0$.



a: $\mu_1 = 0.5$, $\mu_2 = -0.5$. b: $\mu_1 = \mu_2 = 0.5$.

Figure 4.7: $w_k = 0.2$, $k = 1, \ldots, 5$, $\theta_k = 2$, $k = 1, \ldots, 4$, $\rho = 0$, $\nu = 4$.

We can see in Figures 4.3 and 4.4 that the rate of convergence of the constants C_{qk} decreases when ν increases, i.e. when the tails get lighter, and is quite similar for various combinations of tail dependence (see Figure 4.5). For the constant D_q, on the other hand, the rate of convergence depends on the combination of all tail parameters. Strong non-tail dependence significantly reduces the rate of convergence for all constants (see Figure 4.6), while shifting the means slows down the convergence only if the shifts are to the opposite directions (see Figure 4.7). It is also worth mentioning that in most cases the approximation of the constant C_{qk} with its asymptotic equivalent C_k is more accurate for the position with higher scale parameter ($a_k\sigma_k$), i.e. the riskier part of the portfolio is approximated better.

Chapter **5**

Conclusions

In the thesis we investigate the asymptotic properties of both tail probability and tail expectation of a randomly weighted sum

$$S_n^{\theta X} = \sum_{i=1}^{n} \theta_i X_i,$$

where $\{X_1, X_2, \ldots, X_n\}$ are real-valued and heavy-tailed r.v.s, called primary r.v.s, and $\{\theta_1, \theta_2, \ldots, \theta_n\}$ are nonnegative and nondegenerated at zero r.v.s, called random weights.

This sum has been an attractive research topic in the recent works of applied probability with the focus on nonnegative primary r.v.s or real-valued primary r.v.s with various types of tail independence. We further generalize the results for real-valued r.v.s assuming different distribution classes and dependence structures.

In Theorems 1.3 and 1.4 we introduce some dependence structure between r.v.s $\{X_1, X_2, \ldots, X_n\}$ and random weights $\{\theta_1, \theta_2, \ldots, \theta_n\}$ but leave the random vectors $\{(X_1, \theta_1), (X_2, \theta_2) \ldots, (X_n, \theta_n)\}$ being independent. In Theorem 1.4 we restrict the dependence to a bivariate Sarmanov distribution and obtain asymptotic capital allocation formulas for regularly varying distributions.

Further, in Theorems 1.5 and 1.6 we allow primary r.v.s to be dependent with the assumption of QAI or SQAI but independent from random weights and obtain asymptotic bounds for the tail probability and tail expectation in the case of dominatedly varying distributions.

Finally, in Theorem 1.7 we assume $\theta_1 = \cdots = \theta_n = 1$ and obtain asymptotic capital allocation formulas in the case where primary r.v.s have tail dependence and regularly varying distributions. Results of this theorem are verified by a simulation study for a multivariate Clayton copula with t-location-scale marginal distributions.

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