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On Joint Distribution of General Dirichlet Series *

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Abstract. In the paper a joint limit theorem in the sense of the weak convergence in the space of meromorphic functions for general Dirichlet series is proved under weaker conditions as in [1].

Keywords: distribution, general Dirichlet series, probability measure, weak convergence.

1 Introduction

Let $s = \sigma + it$ denote a complex variable, and let for $\sigma > \sigma_{aj}$,

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_{mj} s}, \quad j = 1, \ldots, n,$$

be a collection of general Dirichlet series. Here a_{mj} are complex numbers, and $\{\lambda_{mj}\}$ is an increasing sequence of positive numbers, $\lim\limits_{m\to\infty} \lambda_{mj} = +\infty$, $j = 1, \ldots, n$. In [1] a joint limit theorem for the functions $f_1(s), \ldots, f_n(s)$ has been considered. To state it we need some notation and assumptions. We assume that the functions $f_1(s), \ldots, f_n(s)$ are meromorphically continuable to the half-planes $\sigma > \sigma_{11}$, $\sigma_{11} < \sigma_{a1}, \ldots$, $\sigma > \sigma_{1n}$, $\sigma_{1n} < \sigma_{an}$, respectively,

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and all poles in the regions are included in a compact set. We also suppose that, for $\sigma > \sigma_{1j}$, the estimates

$$f_j(s) = B|t|^{\delta_j}, \quad |t| \geq t_0, \quad \delta_j > 0, \tag{1}$$

and

$$\int_{-T}^{T} \left|f_j(\sigma + it)\right|^2 dt = BT, \quad T \to \infty, \tag{2}$$

$j = 1, \dots, n$, are satisfied, where B denotes a quantity bounded by a constant. Moreover, we assume that

$$\lambda_{mj} \geq c_j (\log m)^{\theta_j} \tag{3}$$

with some positive constants c_j and θ_j, $j = 1, \dots, n$.

Denote by $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ the unit circle on the complex plane $\mathbb{C}$, and let

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \geq 1$, be the infinite-dimensional torus. With product topology and pointwise multiplication the torus Ω becomes a compact topological Abelian group. Therefore, on $\big(\Omega, \mathcal{B}(\Omega)\big)$, where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S, the probability Haar measure m_H exists, and this leads to a probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m.

Let G be a region on $\mathbb{C}$. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let $D_j = \{s \in \mathbb{C}\colon \sigma > \sigma_{1j}\}$, and put

$$H_n = H_n(D_1, \dots, D_n) = H(D_1) \times \dots \times H(D_n).$$

Now on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$ we define an H_n-valued random element $F(s_1, \dots, s_n; \omega)$ by the formula

$$F(s_1, \dots, s_n; \omega) = \big(f_1(s_1, \omega), \dots, f_n(s_n, \omega)\big),$$

where

$$f_j(s_j,\omega)=\sum_{m=1}^{\infty}a_{mj}\omega(m)e^{-\lambda_{mj}s_j},\quad s\in D_j,\quad j=1,\dots,n.$$

Now we define the space of meromorphic functions. Let $\mathbb{C}_\infty=\mathbb{C}\bigcup\{\infty\}$ be the Riemann sphere with the spherical metric given by the formulas

$$\begin{aligned}d(s_1,s_2)&=\frac{2|s_1-s_2|}{\sqrt{1+|s_1|^2}\sqrt{1+|s_2|^2}},\\ d(s,\infty)&=\frac{2}{\sqrt{1+|s|^2}},\quad d(\infty,\infty)=0,\\ &s_1,s_2,s\in\mathbb{C}.\end{aligned}$$

Let $M(G)$ stand for the space of meromorphic functions $g\colon G\to(\mathbb{C}_\infty,d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $g_n(s)\in M(G)$ converges to a function $g(s)\in M(G)$, if

$$d\big(g_n(s),g(s)\big)\to 0$$

as $n\to\infty$ uniformly on compact subsets of G. We put

$$M_n=M_n(D_1,\dots,D_n)=M(D_1)\times\dots\times M(D_n),$$

and let, for $T>0$,

$$\nu_T(\dots)=\frac{1}{T}\mathrm{meas}\big\{\tau\in[0,T]\colon\ \dots\big\},$$

where the dots denote some condition satisfied by τ. Then in [1] the following statement was given.

Theorem 1. *For $j=1,\dots,n$, suppose that the sets $\{\log 2\}\cup\bigcup_{m=1}^{\infty}\{\lambda_{mj}\}$ are linearly independent over the field of rational numbers, and that for $f_j(s)$ the conditions* (1)–(3) *are satisfied. Then the probability measure*

$$\nu_T\Big(\big(f_1(s+i\tau),\dots,f_n(s+i\tau)\big)\in A\Big),\quad A\in\mathcal{B}(M_n),$$

converges weakly to the distribution of the random element $F(s_1,\dots,s_n;\omega)$ as $T\to\infty$.

However, the proof of Theorem 1 has a gap. For its validity the hypothesis on the linear independence of the sets $\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$, $j = 1, \ldots, n$, must be replaced by that on the linear independence of the set $\{\log 2\} \cup \bigcup_{j=1}^{n} \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$. However, the main shortcoming of Theorem 1 is the presence of the number $\log 2$ in its hypotheses. This is not natural and not convenient.

The aim of this note is to consider a collection of general Dirichlet series with the same exponents and to remove the number $\log 2$ from the hypothesis on the linear independence.

Let, for $\sigma > \sigma_{aj}$,

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_m s}, \quad j = 1, \ldots, n,$$

and let in the definition of $F(s_1, \ldots, s_n; \omega)$

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj} \omega(m) e^{-\lambda_m s_j}, \quad s \in D_j, \quad j = 1, \ldots, n.$$

Theorem 2. *Suppose that the system of exponents* $\{\lambda_m\}$ *is linearly independent over the field of rational numbers, and that for* $f_j(s)$, $j = 1, \ldots, n$, *the conditions* (1)–(3) *are satisfied. Then the assertion of Theorem 1 is valid.*

The proof of the theorem is similar to that of Theorem 1 but simpler and shorter than in [1].

2 A limit theorem in H_{2n}

We begin with a limit theorem in the space

$$H_{2n} = H_{2n}(D_1, \ldots, D_n) = H^2(D_1) \times \ldots \times H^2(D_n).$$

Denote the poles of the function $f_j(s)$ in the region $\sigma > \sigma_{1j}$ by $s_{1j}, \ldots, s_{r_j j}$, $j = 1, \ldots, n$, and define

$$f_{1j}(s) = \prod_{l=1}^{r_j} \left(1 - e^{\lambda_1 (s_{lj} - s)}\right).$$

Then, clearly, $f_{1j}(s_{lj}) = 0$ for $l = 1, \dots, r_j,\ j = 1, \dots, n$. This shows that the function

$$f_{2j}(s) = f_{1j}(s) f_j(s)$$

is regular on $D_j,\ j = 1, \dots, n$. We write

$$f_{1j}(s_j, \omega) = \prod_{l=1}^{r_j} \big(1 - e^{\lambda_1(s_{lj} - s)} \omega(1)\big)$$

and

$$f_{2j}(s_j, \omega) = \sum_{l=0}^{r_j} \sum_{m=1}^{\infty} a_{m,l}^{(j)} \omega^l(1) \omega(m) e^{-(\lambda_m + l\lambda_1)s},$$
$$s_j \in D_j, \quad j = 1, \dots, n,$$

where the coefficients $a_{m,l}^{(j)}$ are defined by

$$f_{2j}(s) = \sum_{l=0}^{r_j} \sum_{m=1}^{\infty} a_{m,l}^{(j)} e^{-(\lambda_m + l\lambda_1)s}, \quad \sigma > \sigma_{aj}, \quad j = 1, \dots, n.$$

Moreover, we set

$$Q_j(s, \omega) = \big(f_{1j}(s, \omega), f_{2j}(s, \omega)\big), \quad s_j \in D_j, \quad j = 1, \dots, n,$$

and define a probability measure

$$Q_{T,j}(A) = \nu_T\Big(\big(f_{1j}(s + i\tau), f_{2j}(s + i\tau)\big) \in A\Big),$$
$$A \in \mathcal{B}\big(H^2(D_j)\big), \quad j = 1, \dots, n.$$

Lemma 1. *For $j = 1, \dots, n$, the probability measure $Q_{T,j}$ converges weakly to the distribution of the random element Q_j as $T \to \infty$.*

The lemma is Lemma 10 from [2].

Now let

$$Q = Q(s_1, \dots, s_n; \omega) = \big(Q_1(s_1, \omega), \dots, Q_n(s_n, \omega)\big),$$
$$s_j \in D_j, \quad j = 1, \dots, n,$$

and

$$Q_T(A) = \nu_T\Big(\big(f_{11}(s_1 + i\tau), f_{21}(s_1 + i\tau), \dots, f_{1n}(s_n + i\tau), f_{2n}(s_n + i\tau)\big) \in A\Big), \quad A \in \mathcal{B}(H^{2n}).$$

Lemma 2. *The probability measure Q_T converges weakly to the distribution of the random element Q as $T \to \infty$.*

Proof. First we prove that the family of probability measures $\{Q_T\}$ is relatively compact, i.e. every sequence of $\{Q_T\}$ contains a weakly convergent subsequence. By Lemma 1, for every $j = 1, \dots, n$, the probability measure

$$\nu_T\big(\big(f_{1j}(s+i\tau), f_{2j}(s+i\tau)\big) \in A\big), \quad A \in \mathcal{B}\big(H^2(D_j)\big),$$

converges weakly to the distribution of the random element $Q_j(s,\omega)$ as $T \to \infty$. Therefore, the family of probability measures $\{Q_{T,j}\}$ is relatively compact, $j = 1, \dots, n$. Since $H^2(D_j)$ is a complete separable space, by the Prokhorov theorem [3], hence we have that the family $\{Q_{T,j}\}$ is tight, i.e. for an arbitrary $\varepsilon > 0$ there exists a compact subset $K_j \subset H^2(D_j)$ such that

$$Q_{T,j}\big(H^2(D_j) \setminus K_j\big) < \frac{\varepsilon}{n}, \quad j = 1, \dots, n, \tag{4}$$

for all $T > 0$. Let a random variable η_T be defined on a probability space $(\widehat{\Omega}, \mathcal{F}, \mathbb{P})$ and have the distribution

$$\mathbb{P}(\eta_T \in A) = \frac{\text{meas}\big\{A \bigcap [0,T]\big\}}{T}, \quad A \in \mathcal{B}(R).$$

Consider the $H^2(D_j)$-valued random element $f_{T,j}(s)$ defined by

$$f_{T,j}(s) = \big((f_{1j}(s+i\eta_T), f_{2j}(s+i\eta_T)\big), \quad j = 1, \dots, n.$$

Taking into account (4), we have

$$\mathbb{P}\big(f_{T,j}(s) \in H^2(D_j) \setminus K_j\big) < \frac{\varepsilon}{n}, \quad j = 1, \dots, n. \tag{5}$$

Now let

$$f_T(s_1, \dots, s_n) = \big(f_{T,1}(s_1), \dots, f_{T,n}(s_n)\big), \quad s_j \in D_j, \quad j = 1, \dots, n,$$

and let $K = K_1 \times \dots \times K_n$. Then K is a compact subset of the space H_{2n}. Moreover, (5) yields

$$\begin{aligned} Q_T(H_{2n} \setminus K) &= \mathbb{P}\left(f_T(s_1, \dots, s_n) \in H_{2n} \setminus K\right) \\ &= \mathbb{P}\Big(\bigcup_{j=1}^{n} (f_{T,j}(s) \in H_2(D_j) \backslash K_j)\Big) \\ &\le \sum_{j=1}^{n} \mathbb{P}\left(f_{T,j}(s) \in H_2(D_j) \backslash K_j\right) < \varepsilon \end{aligned}$$

for all $T > 0$. This means that the family of probability measures $\{Q_T\}$ is tight. Hence, by the Prokhorov theorem, it is relatively compact.

Now we take arbitrary points $s_1^{(j)}, \dots, s_k^{(j)}$ in the region D_j, and put

$$\sigma_1^{(j)} = \min_{1 \le l \le k} \operatorname{Re}(s_l^{(j)}), \quad j = 1, \dots, n.$$

Clearly,

$$\sigma_2^{(j)} \stackrel{def}{=} \sigma_{1j} - \sigma_1^{(j)} < 0, \quad j = 1, \dots, n.$$

Define a region D by

$$D = \bigl\{ s \in \mathbb{C} \colon \sigma > \max_{1 \le j \le n} \sigma_2^{(j)} \bigr\}.$$

Let u_{jl}, $j = 1, \dots, n,\ l = 1, \dots, k$, be arbitrary complex numbers. Define a function $h \colon H_{2n} \to H(D)$ by the formula

$$h(g_{11}, g_{21}, \dots, g_{1n}, g_{2n}; s) = \sum_{r=1}^{2} \sum_{j=1}^{n} \sum_{l=1}^{k} u_{jl} g_{rj}(s_s^{(j)} + s),$$

where $s \in D$, and $g_{rj} \in H(D_j),\ r = 1, 2,\ j = 1, \dots, n$. Moreover, let

$$\varphi_h(s) = h\bigl(f_{11}(s_1), f_{21}(s_1), \dots, f_{1n}(s_n), f_{2n}(s_n); s\bigr).$$

We will prove that

$$\varphi_h(s + i\eta_T) \xrightarrow[T \to \infty]{\mathcal{D}} h(Q; s),$$

where $\xrightarrow[T \to \infty]{\mathcal{D}}$ means the convergence in distribution. Clearly, for all $j=1, \dots, n$,

$$f_{1j}(s) = \sum_{m=0}^{r_j} b_{mj} e^{-\lambda_1 m s}$$

is a Dirichlet polynomial. In the region of absolute convergence $\sigma > \sigma_{aj}$ we have that

$$f_{2j}(s) = \sum_{l=0}^{r_j} \sum_{m=1}^{\infty} a_{m,l}^{(j)} e^{-(\lambda_m + l\lambda_1)s}, \quad \sigma > \sigma_{aj}, \quad j = 1, \dots, n.$$

We put $r=\max_{1\le j\le n} r_j$,

$$\widehat{b}_{mj}=\begin{cases} b_{mj}, & m\le r_j,\\ 0, & m>r_j,\end{cases}$$

$$v_m=\sum_{l=1}^{k} u_{j,l}\widehat{b}_{mj}e^{-\lambda_1 m s_l^{(j)}},$$

$$\widehat{a}_{m,\theta}^{(j)}=\begin{cases} a_{m,\theta}^{(j)}, & \theta\le r_j,\\ 0, & \theta>r_j,\end{cases}$$

and

$$b_{m,\theta,l}^{(j)}=\widehat{a}_{m,\theta}^{(j)}e^{-(\lambda_m+\theta\lambda_1)s_l^{(j)}}.$$

Now suppose that

$$\sigma>\max_{1\le j\le n}\big(\sigma_2^{(j)}+(\sigma_{aj}-\sigma_{1j})\big).$$

From the definition of the function h we find

$$\begin{aligned}\varphi_h(s)&=\sum_{j=1}^{n}\sum_{l=1}^{k}u_{jl}\sum_{m=0}^{r_j}b_{mj}e^{-\lambda_1 m(s_l^{(j)}+s)}\\ &\quad+\sum_{j=1}^{n}\sum_{l=1}^{k}u_{jl}\sum_{\theta=0}^{r_j}\sum_{m=1}^{\infty}a_{m,\theta}^{(j)}e^{-(\lambda_m+\theta\lambda_1)(s_l^{(j)}+s)}\\ &=\sum_{m=0}^{r}v_m e^{-\lambda_1 ms}+\sum_{j=1}^{n}\sum_{l=1}^{k}u_{jl}\sum_{\theta=0}^{r}\sum_{m=1}^{\infty}b_{m,\theta,l}^{(j)}e^{-(\lambda_m+\theta\lambda_1)s}\\ &\overset{def}{=}D_r(s)+\widehat{D}_r(s),\end{aligned}$$

where

$$D_r(s)=\sum_{m=0}^{r}v_m e^{-\lambda_1 ms}$$

is a Dirichlet polynomial, and $\widehat{D}_r(s)$ is a linear combination of Dirichlet series satisfying conditions (1)–(3). Clearly, the function $\widehat{D}_r(s)$ is regular on D.

Since the set $\{\lambda_m\}$ is linearly independent over the field of rational numbers it can be obtained by using standard arguments, see, for example, [4], Chapter 5, that the probability measure

$$\nu_T\big(\big(D_r(s+i\tau)+\widehat{D}_r(s+i\tau)\big)\in A\big), \quad A\in\mathcal{B}\big(H(D)\big),$$

converges weakly to the distribution of the $H(D)$-valued random element

$$\varphi_h(s,\omega) \overset{def}{=} \sum_{m=0}^{r} v_m\omega^m(1)e^{-\lambda_1 ms} + \sum_{j=1}^{n}\sum_{l=1}^{k} u_{jl}\sum_{\theta=0}^{r}\sum_{m=1}^{\infty} b_{m,\theta,l}^{(j)}\omega^\theta(1)\omega(m)e^{-(\lambda_m+\theta\lambda_1)s}. \tag{6}$$

Thus, we have proved that the probability measure

$$\nu_T\left(\varphi_h(s+i\tau)\in A\right), \quad A\in\mathcal{B}\big(H(D)\big),$$

converges to the distribution of the random element (6) as $T\to\infty$. However, by the definition of h

$$\begin{aligned}\varphi_h(s,\omega) &= \sum_{j=1}^{n}\sum_{l=1}^{k} u_{jl}\sum_{m=0}^{r_j} b_{mj}\omega^m(1)e^{-\lambda_1 m(s_l^{(j)}+s)} \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{k} u_{jl}\sum_{\theta=0}^{r_j}\sum_{m=1}^{\infty} a_{m,l}^{(j)}\omega^\theta(1)\omega(m)e^{-(\lambda_m+\theta\lambda_1)(s_l^{(j)}+s)} \\ &= \sum_{r=1}^{2}\sum_{j=1}^{n}\sum_{l=1}^{k} u_{jl}f_{rf}(s_j^{(j)}+s,\omega) = h(Q;s),\end{aligned}$$

and therefore

$$\varphi_h(s+i\eta_T)\xrightarrow[T\to\infty]{\mathcal{D}} h(Q;s). \tag{7}$$

Now we are ready to prove Lemma 2. We have seen that the family of probability measures $\{Q_T\}$ is relatively compact. Hence we can find a sequence $T_1\to\infty$ such that the measure Q_{T_1} converge weakly to some probability measure Q_0 on $\big(H_{2n},\mathcal{B}(H_{2n})\big)$ as $T_1\to\infty$. This shows that there exists an H_{2n}-valued random element

$$\widehat{f}=\widehat{f}(s_1,\dots,s_n)=\big(\widehat{f}_{11}(s_1),\widehat{f}_{21}(s_1),\dots,\widehat{f}_{1n}(s_n),\widehat{f}_{2n}(s_n)\big)$$

with distribution Q_0 defined, say, on a certain probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$. Write the random element $f_T(s_1, \ldots, s_n)$ in the form

$$f_T = f_T(s_1, \ldots, s_n) = (f_{T,11}(s_1), f_{T,21}(s_1), \ldots, f_{T,1n}(s_n), f_{T,2n}(s_n)),$$

where

$$f_{T,1j}(s_j) = f_{1j}(s_j + i\eta_T),$$
$$f_{T,2j}(s_j) = f_{2j}(s_j + i\eta_T),$$

and $s_j \in D_j$, $j = 1, \ldots, n$. By the choise of T_1 we have that $f_{T_1} \xrightarrow[T_1\to\infty]{\mathcal{D}} \widehat{f}$, and therefore

$$h(f_{T_1}) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(\widehat{f}; s).$$

Hence, in view of the definition of $\varphi_h(s)$,

$$\varphi_h(s + i\eta_{T_1}) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(\widehat{f}; s). \tag{8}$$

On the other hand, (7) shows that

$$\varphi_h(s + i\eta_{T_1}) \xrightarrow[T_1\to\infty]{\mathcal{D}} h(Q; s).$$

This and (8) yield

$$h(Q; s) \stackrel{\mathcal{D}}{=} h(\widehat{f}; s). \tag{9}$$

Let the function $u\colon H(D) \to \mathbb{C}$ be given by the formula

$$u(f) = f(0), \quad f \in H(D). \tag{10}$$

The topology of the space $H(D)$ shows that the function u is measurable, and therefore by (9)

$$u\big(h(Q; s)\big) \stackrel{\mathcal{D}}{=} u\big(h(\widehat{f}; s)\big),$$

and

$$h(Q; 0) \stackrel{\mathcal{D}}{=} h(\widehat{f}; 0)$$

by (10). From this it follows that

$$\sum_{r=1}^{2}\sum_{j=1}^{n}\sum_{l=1}^{k} u_{jl} f_{rj}(s_l^{(j)},\omega) \overset{\mathcal{D}}{=} \sum_{r=1}^{2}\sum_{j=1}^{n}\sum_{l=1}^{k} u_{jl} \widehat{f}_{rj}(s_l^{(j)}) \tag{11}$$

for arbitrary complex numbers u_{jl}. It is well known [3] that all hyperplanes in $\mathbb{R}^{4nk}$ generates a determining class, and therefore they generate a determining class in $\mathbb{C}^{2nk}$. Consequently, in view of (11) the $\mathbb{C}^{2nk}$-valued random elements $f_{rj}(s_l^{(j)},\omega)$ and $\widehat{f}_{rj}(s_l^{(j)}),\ r=1,2,\ j=1,\dots,n,\ l=1,\dots,k$, have the same distributions.

Let K_j be an arbitrary compact subset of D_j, and let $v_{rj}(s)\in H(D_j)$, $r=1,2$. Now we suppose that the set $\{s_l^{(j)},\ 1\le l<\infty\}$ is dense in K_j, $j=1,\dots,n$. Consider the sets of functions

$$\begin{aligned} G = \Big\{&(g_{11},g_{21},\dots,g_{1n},g_{2n})\in H_{2n}\colon \sup_{s\in K_j}\big|g_{rj}(s)-v_{rj}(s)\big|\le\varepsilon,\\ &j=1,\dots,n,\ r=1,2\Big\} \quad \text{and}\\ G_k = \Big\{&(g_{11},g_{21},\dots,g_{1n},g_{2n})\in H_{2n}\colon \big|g_{rj}(s_l^{(j)})-v_{rj}(s_l^{(j)})\big|\le\varepsilon,\\ &j=1,\dots,n,\ l=1,\dots,k,\ r=1,2\Big\}. \end{aligned}$$

Since the distributions of the random elements $f_{rj}(s_l^{(j)},\omega)$ and $\widehat{f}_{rj}(s_l^{(j)})$, $r=1,2,\ j=1,\dots,n,\ l=1,\dots,k$, coincide we have

$$m_H\big(\omega\in\Omega\colon Q(s_1,\dots,s_n,\omega)\in G_k\big)=\mathbb{P}_0\big(\widehat{f}(s_1,\dots,s_n)\in G_k\big). \tag{12}$$

Clearly, $G_1\supset G_2\supset\dots$. This and the denseness of $\{s_l^{(j)},\ 1\le l<\infty\}$ show that $G_k\to G$ as $k\to\infty$. Hence and from (12) we deduce that

$$m_H\big(\omega\in\Omega\colon Q(s_1,\dots,s_n,\omega)\in G\big)=\mathbb{P}_0\big(\widehat{f}(s_1,\dots,s_n)\in G\big). \tag{13}$$

Since H_{2n} is a separable space, finite intersections of spheres in it form a determining class [3]. Therefore, (13) yields

$$Q\overset{\mathcal{D}}{=}\widehat{f}.$$

Since $f_{T_1}\xrightarrow[T_1\to\infty]{\mathcal{D}}\widehat{f}$, hence it follows that

$$f_{T_1}\xrightarrow[T_1\to\infty]{\mathcal{D}}Q.$$

In other words, the probability measure Q_{T_1} converges weakly to the distribution $\widehat{Q}$ of the random element Q as $T_1 \to \infty$. Since the family of probability measures $\{Q_T\}$ is relatively compact, the measure $\widehat{Q}$ is independent on the choice of the sequence Q_{T_1}. Since Q_T converges weakly to $\widehat{Q}$ as $T \to \infty$ if and only if every subsequence $\{Q_{T_1}\}$ of $\{Q_T\}$ contains another subsequence $\{Q_{T_2}\}$ weakly convergent to $\widehat{Q}$ as $T_2 \to \infty$, hence we obtain the lemma. □

3 Proof of Theorem 2

Define a metric on the spaces H_{2n} and M_n as the maximum of the metrics on the coordinate spaces. Let the function $u\colon H_{2n} \to M_n$ be given by the formula

$$u(g_{11}, g_{21}, \ldots, g_{1n}, g_{2n}) = \left(\frac{g_{21}}{g_{11}}, \ldots, \frac{g_{2n}}{g_{1n}}\right),$$

where $g_{1j}, g_{2j} \in H_j$, $j = 1, \ldots, n$. In virtue of the equality

$$d(g_1, g_2) = d\left(\frac{1}{g_1}, \frac{1}{g_2}\right)$$

for the spherical metric we have that u is a continuous function. Therefore, by Lemma 2 the probability measure

$$\begin{aligned}&\nu_T\big(\big(f_1(s_1+i\tau), \ldots, f_n(s_n+i\tau)\big) \in A\big)\\ &\quad= \nu_T\left(\left(\frac{f_{21}(s_1+i\tau)}{f_{11}(s_1+i\tau)}, \ldots, \frac{f_{2n}(s_n+i\tau)}{f_{1n}(s_n+i\tau)}\right) \in A\right), \quad A \in \mathcal{B}(M_n),\end{aligned}$$

converges weakly to the distribution of the random element

$$\left(\frac{f_{21}(s_1,\omega)}{f_{11}(s_1,\omega)}, \ldots, \frac{f_{2n}(s_n,\omega)}{f_{1n}(s_n,\omega)}\right),$$

where

$$\begin{aligned}\frac{f_{2j}(s_j,\omega)}{f_{11}(s_j,\omega)} &= \frac{\sum_{l=0}^{r_j}\sum_{m=1}^{\infty} a_{m,l}^{(j)}\omega^l(1)\omega(m)e^{-(\lambda_m+l\lambda_1)s_j}}{\prod_{l=1}^{r_j}\big(1-\omega(1)e^{\lambda_1(s_{lj}-s_j)}\big)}\\ &= \sum_{m=1}^{\infty} a_{mj}\omega(1)\omega(m)e^{-\lambda_m s_j} = f_j(s_j,\omega),\\ &\quad s_j \in D_j, \quad j = 1, \ldots, n.\end{aligned}$$

□

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