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On Value Distribution of One Epstein Zeta-Function

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Abstract

The paper is devoted to Epstein zeta-function $\zeta_3(s)$, $s = \sigma + it$, of a quadratic form of three variables. We obtain a probabilistic characterization of the function $\zeta_3(s)$ in terms of weak convergence of probability measures in the space of analytic functions defined by means of shifts $\zeta_3(s + i\psi(\tau))$, where $\psi(\tau)$ is a differentiable function increasing to infinity with a monotonic derivative satisfying certain growth conditions. The Gram function is an example of $\psi(\tau)$. We prove a limit theorem for $\zeta_3(s)$ with an explicitly given limit measure—the distribution of an analytic random element. For the proof, the mean square estimate for $\zeta_3(s)$ is applied. The obtained limit theorem characterizes the chaotic behavior of the function $\zeta_3(s)$ in the space of analytic functions and is applied for approximation of a certain class of analytic functions by shifts $\zeta_3(s + i\psi(\tau))$. This shows the impact of $\zeta_3(s)$ in approximation function theory.

Keywords: approximation of analytic functions; Epstein zeta-function; Haar measure; limit theorem; weak convergence of probability measures

MSC: 11M41

1. Introduction and Main Results

Let $s = \sigma + it$ be a complex variable, $\{a(m) : m \in \mathbb{N}\} \subset \mathbb{C}$, and let $\{\lambda_m : m \in \mathbb{N}\} \subset \mathbb{R}$ be an increasing sequence, $\lim_{m\to\infty} \lambda_m = +\infty$. The series

$$\sum_{m=1}^{\infty} \frac{a(m)}{m^s} \quad \text{and} \quad \sum_{m=1}^{\infty} a(m)\mathrm{e}^{-\lambda_m s}$$

are called the Dirichlet series; the regions of their convergence and absolute convergence are half-planes $\sigma > \sigma_0$ and $\sigma > \sigma_a$. By the Dirichlet series, analytic almost periodic functions are defined. This class of functions is not wide; however, the so-called zeta-functions (or L-functions) defined by the Dirichlet series are very important objects not only in analytic number theory but in mathematics in general. The name "zeta-functions" comes from the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1.$$

The function $\zeta(s)$ with real s was already studied by L. Euler; however, B. Riemann created analytic theory of $\zeta(s)$ [1]. The Riemann hypothesis that all zeros of $\zeta(s)$ in the region $\sigma > 0$ lie on the line $\sigma = 1/2$ remains one of the most important problems of mathematics [2].

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Let $\chi(m)$ be a Dirichlet character modulo $q \in \mathbb{N}$. Dirichlet L-functions $L(s,\chi)$ introduced by L. Dirichlet also are given by Dirichlet series:

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1.$$

Let $\alpha \in (0,1]$ be a fixed parameter. The classical Hurwitz zeta-function $\zeta(s,\alpha)$ has a more general form:

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1.$$

There are zeta-functions given by Dirichlet series of a non-classical type. For example, the Dedekind zeta-function $\zeta_{\mathbb{K}}(s)$ of a number field $\mathbb{K}$ over the field of rational numbers $\mathbb{Q}$ is given by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{(N(\mathfrak{a}))^s},$$

where the sum runs over all non-zero integral ideals of $\mathbb{K}$, and $N(\mathfrak{a})$ denotes the norm of the ideal $\mathfrak{a}$.

The above examples show that zeta-functions are given by series of the Dirichlet type. The second feature of zeta-functions is their functional equations. These equations serve for analytic continuation and investigation of value distribution in general of zeta-functions. Riemann was the first to prove [1] a very elegant symmetric functional equation for the function $\zeta(s)$. Let $\Gamma(s)$ denote the Euler gamma-function. Then Riemann obtained [1] that, for all $s \in \mathbb{C}$, the equality

$$\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\tfrac{1-s}{2}\right)\zeta(1-s)$$

is true. Now, it is known that the mentioned above zeta-functions and others also satisfy functional equations; see [3–5].

In 1903, P. Epstein raised devised a problem for himself: to construct the most general zeta-function that could satisfy the functional equation of the Riemann type. His project was successful: he defined a really general zeta-function and proved the functional equation for it. More precisely, let Q_n be a positive defined quadratic $n \times n$ matrix, and, for brevity, let $Q_n[x] = \underline{x}^T Q_n \underline{x}$ with $\underline{x} \in \mathbb{Z}^n$. In [6], Epstein defined and studied the zeta-function

$$\zeta(s;Q_n) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q_n[\underline{x}])^{-s}, \quad \sigma > \frac{n}{2}.$$

He proved for $\zeta(s;Q_n)$ the following functional equation of the Riemann type:

$$\pi^{-s}\Gamma(s)\zeta(s;Q_n) = (\det Q_n)^{-1/2}\pi^{s-n/2}\Gamma\left(\tfrac{n}{2}-s\right)\zeta\left(\tfrac{n}{2}-s;Q_n^{-1}\right), \quad s \in \mathbb{C},$$

where Q_n^{-1} is the inverse of Q_n, and continued analytically $\zeta(s;Q_n)$ to the whole complex plane, except for a simple pole at the point $s = n/2$ with residue

$$\frac{\pi^{n/2}}{\Gamma(n/2)\sqrt{\det Q_n}}.$$

We observe that $Q_n[\underline{x}]$ is a positive definite quadratic form of n variables. Thus, the function $\zeta(s;Q_n)$ is defined by a "generalized" Dirichlet series and satisfies the functional equation of the Riemann type.

The function $\zeta(s;Q_n)$ is an interesting analytic object and was studied by many authors. Among them, E. Hecke [7], M,F. Deuring [8], P.R. Taylor [9], P. Bateman and E. Groswald [10],

S. Chowla and A. Selberg [11], H. Iwaniec [12], O.M. Fomenko [13], A. Södergren [14,15], T. Nakamura and Ł. Pańkowski [16], A. Strömbergsson and A. Södergren [17], J. Andersson and A. Södergren [18], P. Ribeiro and S. Yakubovich [19], and many others.

In a partial case, when $\underline{x}^T Q_n \underline{x} \in \mathbb{Z}$ for $\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}$, the function $\zeta(s; Q_n)$ can be written as a classical Dirichlet series

$$\zeta(s; Q_n) = \sum_{m=1}^{\infty} \frac{a_{Q_n}(m)}{m^s}, \quad \sigma > \frac{n}{2}, \tag{1}$$

where $a_{Q_n}(m)$ denotes the number of $\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}$ satisfying $Q_n[\underline{x}] = m$. From this, it follows that $\zeta(s; Q_1) = 2\zeta(2s)$ with $Q_1 = (1)$. Moreover, the representation [13]

$$\zeta(s; Q_n) = \zeta(s; E_{Q_n}) + \zeta(s; F_{Q_n}),$$

where $\zeta(s; E_{Q_n})$ is the zeta-function of an Eisenstein series, and $\zeta(s; F_{Q_n})$ the zeta-function of a cusp form of weight $n/2$, is valid. If, additionally, $n \geqslant 4$ and is even, then, for $\sigma > (n-1)/2$, $\zeta(s; Q_n)$ can be written in the form [16]

$$\zeta(s; Q_n) = \sum_{l=1}^{L} \sum_{k=1}^{K} \frac{a_{lk}}{l^s k^s} L(s, \chi_l) L\left(s - \tfrac{n}{2} + 1, \widehat{\chi}_k\right) + \sum_{m=1}^{\infty} \frac{b_{Q_n}(m)}{m^s}, \tag{2}$$

where

$$L(s, \chi_l) = \sum_{m=1}^{\infty} \frac{\chi_l(m)}{m^s} \qquad \text{and} \qquad L(s, \widehat{\chi}_k) = \sum_{m=1}^{\infty} \frac{\widehat{\chi}_k(m)}{m^s}$$

are Dirichlet L-functions, $a_{lk} \in \mathbb{C}$, the series

$$\sum_{m=1}^{\infty} \frac{b_{Q_n}(m)}{m^s}$$

converges absolutely for $\sigma > (n-1)/2$, and K and L are certain positive integers.

Though the function $\zeta(s; Q_n)$ has been introduced for needs of pure mathematics, it has some practical applications in crystallography [20] and quantum field theory [21,22].

In [23], a probabilistic approach has been applied for characterization of the chaotic behavior of the function $\zeta(s; Q_n)$. Later, the same authors proved several limit theorems on weakly convergent probability measures on the complex plane for the function $\zeta(s; Q_n)$ satisfying representation (2). These theorems generalize Bohr–Jessen theorems proved for the function $\zeta(s)$ in [24,25].

A general definition of the function $\zeta(s; Q_n)$ is too complicated; it is difficult to build a similar theory to $\zeta(s)$. However, we are motivated to consider a case more general than (2) for $n = 3$. We observe that this case is not included in (2) because 3 is an odd number smaller than 4.

Thus, the object of this paper is the Epstein zeta-function

$$\zeta_3(s) = \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3 \setminus \{\underline{0}\}} \frac{1}{(m_1^2 + m_2^2 + m_3^2)^s}, \quad \sigma > \frac{3}{2}. \tag{3}$$

Denoting by $r_3(m)$ the number of representations of $m \in \mathbb{Z}$ by a sum of three squares of integers, from (3) we have

$$\zeta_3(s) = \sum_{m=1}^{\infty} \frac{r_3(m)}{m^s}, \quad \sigma > \frac{3}{2}.$$

We will obtain a limit theorem in the sense of weak convergence of probability measures in the space of analytic functions. For this, we recall some notation and definitions. Denote by $\mathcal{X}$ a topological space and by $\mathcal{B}(\mathcal{X})$ the Borel σ-field of $\mathcal{X}$. Let $\mathcal{P}_n$, $n \in \mathbb{N}$, and $\mathcal{P}$ be probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. We say that $\mathcal{P}_n$ converges weakly to $\mathcal{P}$ as $n \to \infty$, if, for every real continuous bounded function f on $\mathcal{X}$,

$$\lim_{n\to\infty} \int\limits_{\mathcal{X}} f \mathrm{d}\mathcal{P}_n = \int\limits_{\mathcal{X}} f \, \mathrm{d}\mathcal{P}.$$

For brevity, this will be denoted as $\mathcal{P}_n \xrightarrow[n\to\infty]{\mathrm{w}} \mathcal{P}$.

Now, let $\mathcal{D}_1 = \{s \in \mathbb{C} : 1 < \sigma < 3/2\}$, $\mathcal{D}_2 = \{s \in \mathbb{C} : \sigma > 3/2\}$, and $\mathcal{H}(\mathcal{D}_1)$ and $\mathcal{H}(\mathcal{D}_2)$ be the spaces of analytic functions on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, endowed with the topology of uniform convergence on compacta. For $j = 1, 2$, and a certain real function $\psi(\tau)$, define

$$P_{jT}(A) = W_T(\zeta_3(s + i\psi(\tau)) \in A), \quad A \in \mathcal{B}(\mathcal{H}(\mathcal{D}_j)),$$

where

$$W_T(\dots) = \frac{1}{T}\mathrm{meas}\{\tau \in [T, 2T] : \dots\},$$

in place of dots a condition satisfied by τ is to be written, and meas$A$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Since in the definition of P_{jT} the function $\psi(\tau)$ occurs, it is more convenient to deal with large values of τ. Therefore, we limit ourselves by the interval $[T, 2T]$. We will consider weak convergence of P_{jT} as $T \to \infty$ to certain probability measures. For their definition, we need one topological object. Set

$$\mathbb{T} = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\},$$

where $\mathbb{P}$ is the set of all prime numbers. Thus, the torus $\mathbb{T}$ is an infinite-dimensional Cartesian product of unit circles. On $\mathbb{T}$, the operation of pointwise multiplication and product topology can be defined. This makes $\mathbb{T}$ a topological group. As a product of compact sets, $\mathbb{T}$ is a compact topological group. Hence, on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, the probability Haar measure m^H exists. Let $\mathfrak{t} = (\mathfrak{t}(p) : p \in \mathbb{P})$. Notice that m^H has the invariance property, i.e., for all $A \in \mathcal{B}(\mathbb{T})$ and $\mathfrak{t} \in \mathbb{T}$,

$$m^H(A) = m^H(\mathfrak{t}A) = m^H(A\mathfrak{t}).$$

Such a construction gives a probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m^H)$. Since the Haar measure m^H is the product of Haar measures on the coordinate spaces, $\{\mathfrak{t}(p) : p \in \mathbb{P})$ is a sequence of independent random variables. On the above probability space, define $\mathcal{H}(\mathcal{D}_j)$-valued random elements

$$\zeta_{3j}(s, \mathfrak{t}) = \sum_{m=1}^{\infty} \frac{r_3(m)\mathfrak{t}(m)}{m^s},$$

where the function $\mathfrak{t}(m)$ is the extension of $\mathfrak{t}(p)$ from $\mathbb{P}$ to the set $\mathbb{N}$ by the formula

$$\mathfrak{t}(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \mathfrak{t}^l(p).$$

Denote by P_j the distribution of the random element $\zeta_{3j}(s, \mathfrak{t})$, i.e.,

$$P_j(A) = m^H\{\mathfrak{t} \in \mathbb{T} : \zeta_{3j}(s, \mathfrak{t}) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}(\mathcal{D}_j)), \ j = 1, 2.$$

A proof that $\zeta_{3j}(s, \mathfrak{t})$ is the $\mathcal{H}(\mathcal{D}_j)$-valued random element will be given in Section 2.

Now, we introduce the class of functions containing a possible example of the function $\psi(\tau)$. Denote by $\mathfrak{V}$ the class of real differentiable functions $\psi(\tau)$ defined for all large τ and satisfying the following hypotheses:

1° $\psi(\tau) \to \infty$ as $\tau \to \infty$;

2° $\psi(\tau)$ has a monotonic derivative $\psi'(\tau)$ satisfying

$$\frac{\psi(2\tau)}{\widehat{\psi}'(\tau)} \ll \tau, \quad \tau \to \infty,$$

where

$$\widehat{\psi}'(\tau) = \min(\psi'(\tau), \psi'(2\tau)).$$

We recall that the notation $a \ll_\varepsilon b$ with $a \in \mathbb{C}$, $b > 0$ indicates that there is a constant $c = c(\varepsilon) > 0$ such that $|a| \leqslant cb$.

The main result of the paper is the following theorem.

Theorem 1. *Suppose that $\psi(\tau) \in \mathfrak{V}$. Then*

$$P_{jT} \xrightarrow[T\to\infty]{\mathrm{w}} P_j, \quad j = 1, 2.$$

A proof of Theorem 1 will be given in Section 4.

Theorem 1 with $j = 1$ has applications in the theory of approximation of analytic functions by zeta-functions. The universality of zeta-functions is understood as a phenomenon when a wide class of analytic functions is approximated by shifts of one and the same zeta-function. We recall that universality of the Riemann zeta-function has been discovered by S.M. Voronin in [26] (see also [4,27–29]) and developed by numerous authors in [30–33]; for results, see a survey paper [34]. For example, every analytic non-vanishing in the strip $\mathfrak{D} = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ can be approximated uniformly on compact sets by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$ of the Riemann zeta-function. To be precise, denote by $\mathcal{K}$ the class of compact subsets of $\mathfrak{D}$ with connected complements and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic inside of K. Then the modern version of Voronin's theorem says [32,35,36]:

For every $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} W_T\left(\sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right) > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} W_T\left(\sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right)$$

exists and is positive for every $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and all but at most countably many $\varepsilon > 0$.

The case of the function $\zeta(s; Q_n)$ is more complicated, and only partial finite results are known. Voronin himself obtained [4] the universality of Epstein zeta-functions that can be written as a product of Dirichlet L-functions. In [16], the universality of the function $\zeta(s; Q_n)$ given in the form (2) has been discussed and applied for the estimation of the number of its zeros.

We recall a very interesting paper [18], where the authors investigate the general form of the Epstein zeta-functions. Let X_n be the space of all n-dimensional lattices L of $\mathbb{R}^n$ of covolume one, and, for $\sigma > n/2$

$$E_n(L,s) = \sum_{\mathfrak{m}\in L\setminus\{0\}} \frac{1}{|\mathfrak{m}|^{2s}}.$$

Then the following statement is valid [18]:

Let $n \geqslant 2$. Then, for almost all $L \in X_n$with respect to the normalized Siegel measure and any compact set $K \subset \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complement, any function $f(s)$ that is continuous on K and analytic in the interior of K, and any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \max_{s\in K}\left|E_n\left(L, \tfrac{n(s+i\tau)}{2}\right) - f(s)\right| < \varepsilon\right\} > 0.$$

Unfortunately, the latter beautiful theorem does not give any information on the universality of $E_n(L,s)$ with a concrete lattice L.

Moreover, the paper [18] contains deep universality results for $E_n(L,s)$ in the lattice aspect.

Limit theorems in the space of analytic functions is the main ingredient in Bagchi's method [31] for the proof of universality of zeta-functions. Unfortunately, Theorem 1 cannot be applied for this because the random variables $\mathfrak{t}(m)$ are not independent, and it is not clear how to find the explicit form of the support of the measure P_1. Thus, we have to limit ourselves by the following statement, which shows the good approximation properties of $\zeta_3(s)$.

Theorem 2. *Suppose that $\psi(\tau) \in \mathfrak{V}$. Then, on $\mathcal{H}(\mathcal{D}_1)$, there exists a non-empty closed set $\mathcal{F} \subset \mathcal{H}(\mathcal{D}_1)$ such that, for every compact set $K \subset \mathcal{D}_1$, every $f(s) \in \mathcal{F}$ and every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} W_T\left(\sup_{s\in K}|\zeta_3(s+i\psi(\tau)) - f(s)| < \varepsilon\right) > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} W_T\left(\sup_{s\in K}|\zeta_3(s+i\psi(\tau)) - f(s)| < \varepsilon\right)$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

Proof of Theorem 2 will be given in Section 4.

Obviously, the functions $\psi(\tau) = \tau$ and $\psi(\tau) = \tau^\alpha$, $\alpha > 0$ lie in the class $\mathfrak{V}$. A more interesting example is the following. Denote by $\theta(t)$ the increment of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s_1 = 1/2$ and $s_2 = 1/2 + it$, and consider the equation

$$\theta(t) = \pi(\tau - 1), \quad \tau \geqslant 0.$$

Since the function $\theta(t)$ is increasing and unbounded, the above equation has the unique solution t_τ, which is called the Gram function. J.-P. Gram considered [37] t_τ with $\tau \in \mathbb{N}_0$ in connection with non-trivial zeros $1/2 + i\gamma_n$ of the Riemann zeta-function. It is known [38,39] that, as $\tau \in \infty$,

$$t_\tau = \frac{2\pi\tau}{\log\tau}(1 + o(1)) \qquad \text{and} \qquad t'_\tau = \frac{2\pi}{\log\tau}(1 + o(1)).$$

This shows that $t_\tau \in \mathfrak{V}$.

Theorems 1 and 2 are new results for the function $\zeta_3(s)$. Using generalized shifts $\zeta_3(s + i\varphi(\tau))$ is a novel aspect of the paper as well.

The paper is organized as follows. Sections 2 and 3 are devoted to preliminary results. In Section 2, we prove that $\zeta_{3j}(s,\mathfrak{t})$ is a $\mathcal{H}(\mathcal{D}_j)$-valued random element and present mean value results for the approximation of $\zeta_{3j}(s)$ by absolutely convergent series, $j=1,2$. Section 3 is probabilistic; it is devoted to preparatory results for the proof of Theorem 1. In Section 4, we prove Theorems 1 and 2.

2. Preliminary Results (1)

It is well known that mean square estimates play an important role in the universality theory of zeta-functions. This is true for the function $\zeta_3(s)$ as well.

Lemma 1 (See [40], Theorem 2). *Suppose that $\sigma>1$ is fixed. Then, for every fixed $\varepsilon>0$,*

$$\int_1^T|\zeta_3(\sigma+it)|^2\,\mathrm{d}t-T\sum_{m=1}^{\infty}\frac{r_3^2(m)}{m^{2\sigma}}\ll_\sigma T^{2-\sigma+\varepsilon},\quad T\to\infty.$$

Lemma 1 is a key for the proof that $\zeta_{3j}(s,\mathfrak{t})$ is the $\mathcal{H}(\mathcal{D}_j)$-valued random element on the probability space $(\mathbb{T},\mathcal{B}(\mathbb{T}),m^H)$.

For $s\in\mathcal{D}_2$, the series for $\zeta_3(s)$ is absolutely convergent; therefore, in this case, all statements are proved similarly and even much easier than for $s\in\mathcal{D}_1$. For this reason, we will consider only the case $j=1$ and omit index 1.

Proposition 1. *$\zeta_3(s,\mathfrak{t})$ is the $\mathcal{H}(\mathcal{D})$-valued random element.*

Proof. We have to show that the series defining $\zeta_3(s,\mathfrak{t})$, for almost all $\mathfrak{t}$, converges uniformly on compact subsets of $\mathcal{D}$.

Fix $\sigma_1>1$, and consider

$$\xi_3(m,\mathfrak{t})=\frac{r_3(m)\mathfrak{t}(m)}{m^{\sigma_1}}.$$

Denote by $\mathbb{E}X$ the expectation of a random variable X. Then we have that ($\overline{z}$ is complex conjugate of $z\in\mathbb{C}$)

$$\mathbb{E}\Big(\xi_3(m,\mathfrak{t})\overline{\xi_3(m_1,\mathfrak{t})}\Big)=\frac{r_3(m)r_3(m_1)}{m^{\sigma_1}m_1^{\sigma_1}}\int_{\mathbb{T}}\mathfrak{t}(m)\overline{\mathfrak{t}(m_1)}\,\mathrm{d}m^H=\begin{cases}\frac{r_3^2(m)}{m^{2\sigma_1}} & \text{if } m=m_1,\\ 0 & \text{if } m\ne m_1.\end{cases}$$

Hence, $\{\xi_3(m,\mathfrak{t})\}$ is a sequence of complex-valued pairwise orthogonal random variables; moreover,

$$\sum_{m=1}^{\infty}\mathbb{E}|\xi_3(m,\mathfrak{t})|^2\log^2 m<\infty$$

because, by Lemma 1, the series

$$\sum_{m=1}^{\infty}\frac{r_3^2(m)\log^2 m}{m^{2\sigma_1}}$$

is convergent. Therefore, by the Rademacher theorem, see, for example, [41], the series

$$\sum_{m=1}^{\infty}\frac{r_3(m)\mathfrak{t}(m)}{m^{\sigma_1}}$$

converges for almost all $\mathfrak{t} \in \mathbb{T}$ with respect to the Haar measure m^H. This and the property of Dirichlet series imply that the series

$$\sum_{m=1}^{\infty} \frac{r_3(m)\mathfrak{t}(m)}{m^s}, \tag{4}$$

for almost all $\mathfrak{t}$ converges uniformly on compact subsets of the half plane $\{s \in \mathbb{C} : \sigma > \sigma_1\}$. Now put

$$\sigma_1 = 1 + \frac{1}{k}, \quad \text{with } k \in \mathbb{N},$$

and denote by $\mathcal{A}_k$ the set of such $\mathfrak{t} \in \mathbb{T}$ for which the series (4) converges uniformly on compact subsets of $\{s \in \mathbb{C} : \sigma > 1 + 1/k\}$. Then $m^H(\mathcal{A}_k) = 1$ for all $k \in \mathbb{N}$, Therefore, taking

$$\mathcal{A} = \bigcap_{k \in \mathbb{N}} \mathcal{A}_k,$$

we have $m^H(\mathcal{A}) = 1$, and the series (4), for $\mathfrak{t} \in \mathcal{A}$, converges uniformly on compact subsets of the region $\mathcal{D}$. Since all terms of the series (4) are analytic functions, we obtain that $\zeta_3(s, \mathfrak{t})$ is the $\mathcal{H}(\mathcal{D})$-valued random element defined on the probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m^H)$. The proposition is proved. □

Lemma 2. *Suppose that $\psi(\tau) \in \mathfrak{V}$. Then, for every fixed $\sigma > 1$, and $t \in \mathbb{R}$,*

$$\int_T^{2T} |\zeta_3(\sigma + it + i\psi(\tau))|^2 \, \mathrm{d}\tau \ll_\sigma T(1 + |t|\psi^{-1}(2T))$$

Proof. By Lemma 1,

$$\int_T^{2T} |\zeta_3(\sigma + it)|^2 \, \mathrm{d}t \ll_\sigma T.$$

Therefore, the second mean value theorem gives

$$\begin{aligned}
\int_T^{2T} |\zeta_3(\sigma + it + i\psi(\tau))|^2 \, \mathrm{d}\tau &= \int_T^{2T} \frac{1}{\psi'(t)} |\zeta_3(\sigma + it + i\psi(\tau)|^2 \, \mathrm{d}\psi(\tau) \\
&\leqslant \frac{1}{\widehat{\psi}'(T)} \int_{t+\psi(T)}^{t+\psi(2T)} |\zeta_3(\sigma + iu)|^2 \, \mathrm{d}u \\
&\ll \frac{1}{\widehat{\psi}'(T)} \left(\int_1^{|t|+\psi(2T)} - \int_1^{-|t|+\psi(T)} \right) |\zeta_3(\sigma + iu)|^2 \mathrm{d}u \\
&\ll_\sigma \frac{1}{\widehat{\psi}'(T)} (|t| + \psi(2T)) \ll_\sigma T(1 + |t|\psi^{-1}(2T))
\end{aligned}$$

by the definition of $\widehat{\psi}'(T)$. □

Now, we will deal with the mean square for

$$\zeta_3(s, \mathfrak{t}) = \sum_{m=1}^{\infty} \frac{r_3(m)\mathfrak{t}(m)}{m^s}, \quad \sigma > 1.$$

For this, we will use some elements of ergodic theory. Recall basic definitions of ergodicity. Let

$$a_\tau = \left(p^{-i\tau} : p \in \mathbb{N}\right).$$

Define $\varphi_\tau : \mathbb{T} \to \mathbb{T}$ by

$$\varphi_\tau(\mathfrak{t}) = a_\tau \mathfrak{t}, \quad \mathfrak{t} \in \mathbb{T}.$$

Then $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on $\mathbb{T}$. A set $A \in \mathcal{B}(\mathbb{T})$ is called invariant with respect to $\{\varphi_\tau : \tau \in \mathbb{R}\}$ if for every τ the sets A and $A_\tau = \varphi_\tau(A)$ differ one from another by a set of zero m^H-measure. All invariant sets constitute a sub-σ-field of the field $\mathcal{B}(\mathbb{T})$. The group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is called ergodic if the σ-field of invariant sets consists only from sets having measure m^H zero or one.

Lemma 3 (See [31], Lemma 4.6)**.** *The one-parameter group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is ergodic.*

We recall that

$$\zeta_3(s, \mathfrak{t}) = \sum_{m=1}^{\infty} \frac{r_3(m)\mathfrak{t}(m)}{m^s}, \quad s \in \mathcal{D}.$$

Lemma 4. *Suppose that $\psi(\tau) \in \mathfrak{V}$, $\sigma > 1$ is fixed and $t \in \mathbb{R}$. Then, for almost all $\mathfrak{t} \in \mathbb{T}$,*

$$\int_T^{2T} |\zeta_3(\sigma + it + i\psi(\tau), \mathfrak{t})|^2 \, \mathrm{d}\tau \ll_\sigma T(1 + |t|\psi^{-1}(2T)).$$

Proof. As in proof of Lemma 2, for $\sigma > 1$, we have

$$\int_T^{2T} |\zeta_3(\sigma + it + i\psi(\tau), \mathfrak{t})|^2 \, \mathrm{d}\tau \ll \frac{1}{\widehat{\psi}'(T)} \int_0^{|t|+\psi(2T)} |\zeta_3(\sigma + iu, \mathfrak{t})|^2 \, \mathrm{d}u. \tag{5}$$

In the proof of Proposition 1, we have seen that the random variables

$$\xi_3(m, \mathfrak{t}) = \frac{r_3(m)\mathfrak{t}(m)}{m^\sigma}, \quad \sigma > 1,$$

are pairwise orthogonal, and

$$\mathbb{E}|\xi_3(m, \mathfrak{t})|^2 = \frac{r_3^2(m)}{m^{2\sigma}}.$$

Since $\sigma > 1$, the series

$$\sum_{m=1}^{\infty} \frac{r_3^2(m)}{m^{2\sigma}}$$

is convergent. Hence, by the pairwise orthogonality,

$$\mathbb{E}|\zeta_3(\sigma, \mathfrak{t})|^2 = \sum_{m=1}^{\infty} \frac{r_3^2(m)}{m^{2\sigma}} < \infty.$$

Therefore, in view of Lemma 3 and the Birkhoff–Khintchine ergodic theorem, see, for example, [42], for almost all $\mathfrak{t} \in \mathbb{T}$,

$$\lim_{T\to\infty} \frac{1}{|t| + \psi(2T)} \int_0^{|t|+\psi(2T)} |\zeta_3(\sigma + iu, \mathfrak{t})|^2 \, \mathrm{d}u = \lim_{T\to\infty} \frac{1}{|t| + \psi(2T)} \int_0^{|t|+\psi(2T)} |\zeta_3(\sigma, \varphi_\tau(\mathfrak{t}))|^2 \, \mathrm{d}\tau$$

$$= \lim_{T\to\infty} \frac{1}{|t|+\psi(2T)} \int_0^{|t|+\psi(2T)} |\zeta_3(\sigma, a_\tau \mathfrak{t})|^2 \, \mathrm{d}\tau$$
$$= \mathbb{E}|\zeta_3(\sigma, \mathfrak{t})|^2 < \infty.$$

This together with (5) proves the lemma. □

Now we introduce an absolutely convergent Dirichlet series connected to $\zeta_3(s)$. Let $\theta > 1/2$ be a fixed number, and

$$\kappa_n(m) = \exp\Big(-\big(\tfrac{m}{n}\big)^\theta\Big), \quad m, n \in \mathbb{N}.$$

Define the series

$$\zeta_{3,n}(s) = \sum_{m=1}^{\infty} \frac{r_3(m)\kappa_n(m)}{m^s}.$$

Then this series is absolutely convergent in every half-plane $\sigma > \sigma_0$ with a finite σ_0. Define

$$v_n(s) = \frac{1}{\theta}\Gamma\big(\tfrac{s}{\theta}\big)n^s.$$

The next lemma shows a connection of $\zeta_{3,n}(s)$ with $\zeta_3(s)$.

Lemma 5. *Suppose that $s \in \mathcal{D}$. Then the representation*

$$\zeta_{3,n}(s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta_3(s+z)v_n(z) \, \mathrm{d}z \tag{6}$$

is valid.

Proof. We use the well-known Mellin formula

$$\frac{1}{2\pi i} \int_{a-\infty}^{a+i\infty} \Gamma(z)b^{-z} \, \mathrm{d}z = \mathrm{e}^{-b}, \quad a, b > 0,$$

as well as the representation for $\zeta_3(s+z)$ by an absolutely convergent Dirichlet series for $s \in \mathcal{D}$ and $\operatorname{Re} z = \theta$. □

Lemma 6. *Let $K \in \mathcal{D}$ be a compact set and $\psi(\tau) \in \mathfrak{V}$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} \sup_{s\in K} |\zeta_3(s+i\psi(\tau)) - \zeta_{3,n}(s+i\psi(\tau))| \, \mathrm{d}\tau = 0.$$

Proof. We use a method similar to that applied for the Riemann zeta-function with $\psi(\tau) = \tau$. Therefore, we will omit some clear details.

There is a number $1/2 > \delta > 0$ such that $K \subset \{s = \sigma + it : 1 + 2\delta \leqslant \sigma \leqslant 3/2 - \delta\}$. Denote $\widehat{\theta} = 1 + \delta - \sigma$, and take $\theta = 1/2 + \delta$. Then

$$\frac{\widehat{\theta}}{\theta} \leqslant -\frac{\delta}{1/2+\delta} < -\delta \qquad \text{and} \qquad \frac{\widehat{\theta}}{\theta} \geqslant \frac{-1/2+2\delta}{1/2+\delta} > -1.$$

Now, we apply the integral representation of Lemma 5 and the residue theorem. By the above choice of $\widehat{\theta}$ and θ, the integrand in (6) has simple poles at $z = 0$ and $z = 3/2 - s$ lying in the strip $\widehat{\theta} \leqslant \operatorname{Re} z \leqslant \theta$. Therefore, using the estimate

$$\Gamma(\sigma + it) \ll |t|^{\sigma - 1/2} \exp\{-\tfrac{\pi}{2}|t|\} \tag{7}$$

which is uniform in σ in every finite interval, and by the residue theorem, we find for $s \in K$ that

$$\zeta_{3,n}(s) - \zeta_3(s) = \frac{1}{2\pi i} \int\limits_{\widehat{\theta}-i\infty}^{\widehat{\theta}+i\infty} \zeta_3(s+z) v_n(z)\, \mathrm{d}z + a v_n(3/2 - s),$$

where $a = \operatorname{Res}_{s=3/2} \zeta_3(s)$. From this, we have

$$\begin{aligned} &\sup_{s\in K} |\zeta_{3,n}(s + i\psi(\tau)) - \zeta_3(s + i\psi(\tau))| \\ &\quad \ll \int\limits_{-\infty}^{\infty} |\zeta_3(1 + \delta + i\psi(\tau) + iu)| \sup_{s\in K} |v_n(1 + \delta - s + iu)|\, \mathrm{d}u \\ &\qquad + \sup_{s\in K} |v_n(3/2 - s - i\psi(\tau))|. \end{aligned} \tag{8}$$

Therefore, taking into account (7) and the bound (see Theorem 1 of [40])

$$\zeta_3(\sigma + it) \ll_\sigma |t|^{1/4}, \quad \sigma \geqslant 1,$$

we obtain by Lemma 2

$$\begin{aligned} &\left(\int\limits_{-\infty}^{-\log^2 T} + \int\limits_{\log^2 T}^{\infty} \right) |\zeta_3(1 + \delta + i\psi(\tau) + iu)| \sup_{s\in K} |v_n(1 + \delta - s + iu)|\, \mathrm{d}u \\ &\quad \ll_K \left(\int\limits_{-\infty}^{-\log^2 T} + \int\limits_{\log^2 T}^{\infty} \right) \left(\int\limits_{T}^{2T} |\zeta_3(1 + \delta + i\psi(\tau) + iu)|\, \mathrm{d}\tau \right) \exp\{-c|u|\}\, \mathrm{d}u \\ &\quad \ll_K \left(\int\limits_{-\infty}^{-\log^2 T} + \int\limits_{\log^2 T}^{\infty} \right) T(1 + |u|)^{1/2} \exp\{-c|u|\} \mathrm{d}u \ll_K \exp\{-c \log^2 T\}. \end{aligned}$$

Here and in what follows, c is a positive constant that is not necessarily the same in each recurrence. Hence, in view of (8),

$$\begin{aligned} &\frac{1}{T} \int\limits_{T}^{2T} \sup_{s\in K} |\zeta_3(s + i\psi(\tau)) - \zeta_{3,n}(s + i\psi(\tau))|\, \mathrm{d}\tau \\ &\quad \ll_K \int\limits_{-\log^2 T}^{\log^2 T} \left(\frac{1}{T} \int\limits_{T}^{2T} |\zeta_3(1 + \delta + i\psi(\tau) + iu)|\, \mathrm{d}\tau \right) \sup_{s\in K} |v_n(1 + \delta - s + iu)|\, \mathrm{d}u \\ &\qquad + \frac{1}{T} \int\limits_{T}^{2T} \sup_{s\in K} |v_n(3/2 - s - i\psi(\tau))|\, \mathrm{d}\tau + n^{-\delta} \exp\{-c \log^2 T\} \stackrel{\text{def}}{=} J_1 + J_2 + J_3. \end{aligned} \tag{9}$$

Obviously,

$$J_3 = o(1), \quad T \to \infty. \tag{10}$$

Applying (7) once more, we obtain that

$$J_2 \ll_K n^{1/2-2\delta}\frac{1}{T}\int\limits_T^{2T}\exp\{-c|t+\psi(\tau)|\}\mathrm{d}\tau \ll_K n^{1/2-2\delta}\frac{1}{T}\int\limits_T^{2T}\exp\{-c\psi(\tau)\}\,\mathrm{d}\tau$$

$$\ll_K n^{1/2-2\delta}\exp\{-\tfrac{c}{2}\psi(T)\}\frac{1}{T}\int\limits_T^{2T}\exp\{-\tfrac{c}{2}\psi(\tau)\}\,\mathrm{d}\tau \ll_K n^{1/2-2\delta}\exp\{-c\psi(T)\}. \quad (11)$$

To estimate J_1, we observe that, in virtue of Lemma 2,

$$\frac{1}{T}\int_T^{2T}|\zeta_3(1+\delta+i\psi(\tau)+iu)|\,\mathrm{d}\tau \ll \left(\frac{1}{T}\int_T^{2T}|\zeta_3(1+\delta+i\psi(\tau)+iu)|^2\,\mathrm{d}\tau\right)^{1/2}$$
$$\ll (1+|u|\psi^{-1}(2T))^{1/2}.$$

Moreover, for $s\in K$, we obtain by (7)

$$v_n(1+\delta-s+iu) \ll_K n^{-\delta}\exp\{-c|u|\}.$$

Now, the latter two estimates yield

$$J_1 \ll_K n^{-\delta},$$

and this together with (9)–(11) proves the lemma. □

Now, we consider the function

$$\zeta_3(s,\mathfrak{t}) = \sum_{m=1}^{\infty}\frac{r_3(m)\mathfrak{t}(m)}{m^s}, \quad \sigma > 3/2.$$

By Proposition 1, $\zeta_3(s,\mathfrak{t})$, for almost all $\mathfrak{t}\in\mathbb{T}$, is the analytic function for $\sigma>1$.

Define

$$\zeta_{3,n}(s,\mathfrak{t}) = \sum_{m=1}^{\infty}\frac{r_3(m)\kappa_n(m)\mathfrak{t}(m)}{m^s}$$

which is analogue of $\zeta_{3,n}(s)$. The series for $\zeta_{3,n}(s,\mathfrak{t})$ is absolutely convergent in every half-plane $\sigma>\sigma_0$.

Lemma 7. *In notation of Lemma 5, for $s\in\mathcal{D}$,*

$$\zeta_{3,n}(s,\mathfrak{t}) = \frac{1}{2\pi i}\int\limits_{\theta-i\infty}^{\theta+i\infty}\zeta_3(s+z,\mathfrak{t})v_n(z)\,\mathrm{d}z.$$

Proof. We repeat the proof of Lemma 5 with obvious changes. □

Lemma 8. *Let $K\subset\mathcal{D}$ be a compact set, and $\psi(\tau)\in\mathfrak{V}$. Then, for almost all $\mathfrak{t}$,*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int\limits_T^{2T}\sup_{s\in K}|\zeta_3(s+i\psi(\tau),\mathfrak{t})-\zeta_{3,n}(s+i\psi(\tau),\mathfrak{t})|\,\mathrm{d}\tau = 0.$$

Proof. We follow the proof of Lemma 6 by using the representation

$$\zeta_{3,n}(s,\mathfrak{t}) - \zeta_3(s,\mathfrak{t}) = \frac{1}{2\pi i}\int\limits_{\widehat{\theta}-i\infty}^{\widehat{\theta}+i\infty} \zeta_3(s+z,\mathfrak{t})v_n(z)\,\mathrm{d}z$$

resulting from Lemmas 4 and 7. □

3. Preliminary Results (2)

We return to the torus $\mathbb{T}$. On $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, define

$$Q_T(A) = W_T\Big(\Big(p^{-i\psi(\tau)} : p \in \mathbb{P}\Big) \in A\Big), \quad A \in \mathcal{B}(\mathbb{T}).$$

Lemma 9. *Suppose that* $\psi(\tau) \in \mathfrak{V}$. *Then* $Q_T \xrightarrow[T\to\infty]{\mathrm{w}} m^H$.

Proof. The characters of the group are of the form

$$\chi(\mathfrak{t}) = \prod_{p\in\mathbb{P}}^{*} \mathfrak{t}^{l_p}(p),$$

where $*$ shows that only a finite number of integers l_p are not zero. Therefore, the Fourier transform $g_T(\underline{l})$, $\underline{l} = (l_p : p \in \mathbb{P})$ of the measure Q_T is given by

$$\begin{aligned} g_T(\underline{l}) &= \int\limits_{\mathbb{T}} \prod_{p\in\mathbb{P}}^{*} \mathfrak{t}^{l_p}(p)\,\mathrm{d}Q_T = \frac{1}{T}\int\limits_{T}^{2T}\left(\prod_{p\in\mathbb{P}}^{*} p^{-il_p\psi(\tau)}\right)\mathrm{d}\tau \\ &= \frac{1}{T}\int\limits_{T}^{2T} \exp\left\{-i\psi(\tau)\sum_{p\in\mathbb{P}}^{*} l_p\log p\right\}\mathrm{d}\tau. \end{aligned} \tag{12}$$

The logarithms of prime numbers are linearly independent over the field of rational numbers; therefore,

$$a(\underline{l}) \stackrel{\mathrm{def}}{=} \sum_{p\in\mathbb{P}}^{*} l_p\log p = 0$$

if and only if $\underline{l} = \underline{0}$. Hence, by (12),

$$g_T(\underline{l}) = 1 \tag{13}$$

for $\underline{l} = \underline{0}$. Since $\psi(\tau) \in \mathfrak{V}$, we have $1/\widehat{\psi}'(\tau) \ll T\psi^{-1}(2T) = o(T)$ as $T \to \infty$. Therefore, application of the mean value theorem and (12) shows that, in the case $a(\underline{l}) \neq 0$,

$$g_T(\underline{l}) = o(1), \quad T \to \infty.$$

This together with (13) yields

$$\lim_{T\to\infty} g_T(\underline{l}) = \begin{cases} 1 & \text{if } \underline{l} = \underline{0}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m^H, the lemma is proved. □

For $A \in \mathcal{B}(\mathcal{H}(\mathcal{D}))$, define

$$P_{T,n}(A) = W_T(\zeta_{3,n}(s+i\psi(\tau)) \in A)$$

and, for $\mathfrak{t} \in \mathbb{T}$,

$$\widehat{P}_{T,n}(A) = W_T(\zeta_{3,n}(s + i\psi(\tau), \mathfrak{t}) \in A).$$

Lemma 10. *Suppose that $\psi(\tau) \in \mathfrak{V}$. Then, on $(\mathcal{H}(\mathcal{D}), \mathcal{B}(\mathcal{H}(\mathcal{D})))$, there exists a probability measure P_n such that $P_{T,n} \xrightarrow[T\to\infty]{\mathrm{w}} P_n$ and $\widehat{P}_{T,n} \xrightarrow[T\to\infty]{\mathrm{w}} P_n$.*

Proof. We apply the preservation principle of weak convergence under continuous mappings; see Section 2.3 of [43]. Define $u_n : \mathbb{T} \to \mathcal{H}(\mathcal{D})$ by

$$u_n(\mathfrak{t}) = \sum_{m=1}^{\infty} \frac{r_3(m)\kappa_n(m)\mathfrak{t}(m)}{m^s}.$$

Since the latter series is absolutely convergent, u_n is continuous. Moreover,

$$\begin{aligned} u_n\left(p^{-i\psi(\tau)} : p \in \mathbb{P}\right) &= \sum_{m=1}^{\infty} \frac{r_3(m)\kappa_n(m)\widetilde{\mathfrak{t}}(m)}{m^s} = \sum_{m=1}^{\infty} \frac{r_3(m)\kappa_n(m)}{m^{s+i\psi(\tau)}} \\ &= \zeta_{3,n}(s + i\psi(\tau)), \end{aligned}$$

where

$$\widetilde{\mathfrak{t}}(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} p^{-il\psi(\tau)}.$$

Therefore,

$$P_{T,n}(A) = W_T\left(\left(p^{-i\psi(\tau)} : p \in \mathbb{P}\right) \in u_n^{-1}A\right), \quad A \in \mathcal{B}(\mathcal{H}(\mathcal{D})).$$

Hence, by the above mentioned principle and Lemma 9, we obtain that

$$P_{T,n} \xrightarrow[T\to\infty]{\mathrm{w}} m^H u_n^{-1}, \tag{14}$$

where $m^H u_n^{-1}(A) = m^H(u_n^{-1}A)$, $A \in \mathcal{B}(\mathcal{H}(\mathcal{D}))$.

Now, define $\widehat{u}_n : \mathbb{T} \to \mathcal{H}(\mathcal{D})$ by

$$\widehat{u}_n(\mathfrak{t}) = \sum_{m=1}^{\infty} \frac{r_3(m)\kappa_n(m)\mathfrak{t}(m)\widehat{\mathfrak{t}}(m)}{m^s}$$

and $u : \mathbb{T} \to \mathbb{T}$ by

$$u(\mathfrak{t}) = \mathfrak{t}\,\widehat{\mathfrak{t}}.$$

Then $\widehat{u}_n = u_n(u)$, and

$$\widehat{P}_{T,n}(A) = W_T\left(\left(p^{-i\psi(\tau)} : p \in \mathbb{P}\right) \in \widehat{u}_n^{-1}A\right), \quad A \in \mathcal{B}(\mathcal{H}(\mathcal{D})).$$

Hence, by Lemma 9, we have

$$\widehat{P}_{T,n} \xrightarrow[T\to\infty]{\mathrm{w}} m^H \widehat{u}_n^{-1}. \tag{15}$$

The Haar measure m^H is invariant with respect to shifts by elements of $\mathbb{T}$. Therefore,

$$m^H \widehat{u}_n^{-1} = m^H(u_n u)^{-1} = (m^H u^{-1})u_n^{-1} = m^H u_n^{-1}.$$

Thus, by (15), $\widehat{P}_{T,n} \xrightarrow[T\to\infty]{\mathrm{w}} m^H u_n^{-1}$. □

Now we consider the weak convergence, as $T \to \infty$, of the measures $P_T \stackrel{\text{def}}{=} P_{1T}$ and

$$\widehat{P}_T(A) = W_T(\zeta_3(s + i\psi(\tau), \mathfrak{t}) \in A), \quad A \in \mathcal{B}(\mathcal{H}(\mathcal{D})).$$

For this, we need a certain information on the probability measure $P_n \stackrel{\text{def}}{=} m^H u_n^{-1}$. Recall that P_n is said to be tight if, for every $\varepsilon > 0$, there exists a compact set $K \subset \mathcal{H}(\mathcal{D})$ such that

$$P_n(K) > 1 - \varepsilon$$

for all $n \in \mathbb{N}$.

Lemma 11. *The measure P_n is tight.*

Proof. First we recall the metric in $\mathcal{H}(\mathcal{D})$ that induces its topology of uniform convergence on compacta. There exists a sequence of embedded compact subsets $\{K_k : k \in \mathbb{N}\} \subset \mathcal{D}$ such that [44]

$$\bigcup_{k=1}^{\infty} K_k = \mathcal{D},$$

and every compact set $K \subset \mathcal{D}$ lies in some K_k. Then

$$d(f_1, f_2) = \sum_{k=1}^{\infty} 2^{-k} \frac{\sup_{s \in K_k} |f_1(s) - f_2(s)|}{1 + \sup_{s \in K_k} |f_1(s) - f_2(s)|}, \quad f_1, f_2 \in \mathcal{H}(\mathcal{D}),$$

is the desired metric.

The series

$$\zeta_{3,n}(s) = \sum_{m=1}^{\infty} \frac{r_3(m)\kappa_n(m)}{m^s}$$

is absolutely convergent for $\sigma > 1$. Therefore,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\zeta_{3,n}(\sigma + it)|^2 \, \mathrm{d}t = \sum_{m=1}^{\infty} \frac{r_3^2(m)\kappa_n^2(m)}{m^{2\sigma}}. \tag{16}$$

Let K_k be a compact set from the definition of the metric d. Then, by the Cauchy integral formula,

$$\sup_{s \in K_k} |\zeta_{3,n}(s + i\psi(\tau))| \ll_k \int_{\gamma_k} \frac{|\zeta_{3,n}(z + i\psi(z))|}{|z - s|} \, \mathrm{d}z,$$

where γ_k is a simple closed contour lying in $\mathcal{D}$ and enclosing K_k. Hence,

$$\frac{1}{T} \int_T^{2T} \sup_{s \in K_k} |\zeta_{3,n}(s + i\psi(\tau))| \, \mathrm{d}\tau \ll_k \frac{1}{T} \int_T^{2T} \left(|\zeta_{3,n}(\sigma_k + it_k + i\psi(\tau))|^2 \, \mathrm{d}\tau \right)^{1/2}$$

with certain $\sigma_k > 1$ and bounded t_k. Therefore, applying the Cauchy–Schwartz inequality, then the mean value theorem and change of variable, in view of (16) and $\psi(\tau) \in \mathfrak{V}$, we find that

$$\sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{T} \int_T^{2T} \sup_{s \in K_k} |\zeta_{3,n}(s + i\psi(\tau))| \, \mathrm{d}\tau$$

$$\ll_k \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \left(\frac{1}{T\widehat{\psi}'(T)} \int\limits_{\psi(T)}^{\psi(2T)} |\zeta_{3,n}(\sigma_k + it)|^2 \, \mathrm{d}t \right)^{1/2}$$

$$\ll_k \sup_{n\in\mathbb{N}} \left(\sum_{m=1}^{\infty} \frac{r_3^2(m)\kappa_n^2(m)}{m^{2\sigma_k}} \right)^{1/2} \ll_k \left(\sum_{m=1}^{\infty} \frac{r_3^2(m)}{m^{2\sigma_k}} \right)^{1/2} \ll_k M_k < \infty. \quad (17)$$

On a certain probability space $(\Omega, \mathcal{B}, \mu)$, define the random variable ξ_T uniformly distributed in $[T, 2T]$ and introduce the $\mathcal{H}(\mathcal{D})$-valued random element

$$X_{T,n} = X_{T,n}(s) = \zeta_{3,n}(s + i\psi(\xi_T)).$$

Now, it is convenient to use a language of convergence in distribution. Let Y_n, $n \in \mathbb{N}$, and Y be $\mathcal{X}$-valued random elements on $(\Omega, \mathcal{B}, \mu)$, and Q_n and Q their distributions, respectively. Then we say that Y_n converges to Y in distribution ($Y_n \xrightarrow[n\to\infty]{\mathcal{D}} Y$), if $Q_n \xrightarrow[n\to\infty]{\mathrm{w}} Q$.

Lemma 10 shows that

$$X_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} X_n, \quad (18)$$

where X_n is the $\mathcal{H}(\mathcal{D})$-valued random element with the distribution P_n. Fix $\varepsilon > 0$, and set $R_k = M_k 2^k \varepsilon^{-1}$. Then, by the Chebyshev-type inequality and (17), we find

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \mu\left\{ \sup_{s\in K_k} |X_{T,n}(s)| \geqslant R_k \right\} \leqslant \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{R_k T} \int\limits_{T}^{2T} \sup_{s\in K_k} |\zeta_{3,n}(s + i\psi(\tau))| \, \mathrm{d}\tau < \frac{\varepsilon}{2^k}. \quad (19)$$

Moreover, (18) gives

$$\sup_{s\in K_k} |X_{T,n}(s)| \xrightarrow[T\to\infty]{\mathcal{D}} \sup_{s\in K_k} |X_n(s)|.$$

This and (19) lead to

$$\mu\left\{ \sup_{s\in K_k} |X_{T,n}(s)| \geqslant R_k \right\} \leqslant \frac{\varepsilon}{2^k}. \quad (20)$$

Define the set

$$K_\varepsilon = \left\{ g \in \mathcal{H}(\mathcal{D}) : \sup_{s\in K_k} |g(s)| \leqslant R_k, \ k \geqslant 1 \right\}.$$

Then the set K_ε is a compact in $\mathcal{H}(\mathcal{D})$, and, in view of (20),

$$\mu\{X_n(s) \in K_\varepsilon\} \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$. The lemma is proved. □

Now we are in position to prove limit theorems for the probability measures P_T and

$$\widehat{P}_T(A) = W_T(\zeta_3(s + i\psi(\tau), \mathfrak{t}) \in A), \quad A \in \mathcal{B}(\mathcal{H}(\mathcal{D})).$$

Lemma 12. *Suppose that $\psi(\tau) \in \mathfrak{V}$. Then, on $(\mathcal{H}(\mathcal{D}), \mathcal{B}(\mathcal{H}(\mathcal{D})))$, there exists a probability measure P such that $P_T \xrightarrow[T\to\infty]{\mathrm{w}} P$ and $\widehat{P}_T \xrightarrow[T\to\infty]{\mathrm{w}} P$ for almost all $\mathfrak{t}$.*

Proof. We preserve the notation for $X_{T,n}$ and X_n used in the previous lemma and define

$$Y_T = Y_T(s) = \zeta_3(s + i\psi(\xi_T)).$$

We check the hypotheses of Theorem 3.2 of [43] for the random elements X_n, $X_{T,n}$ and Y_T.

Since by Lemma 11, the measure P_n is tight, in view of the Prokhorov theorem (Theorem 5.1 of [43]), it is relatively compact. Therefore, there exists a sequence $n_r \to \infty$ and a measure P on $(\mathcal{H}(\mathcal{D}), \mathcal{B}(\mathcal{H}(\mathcal{D})))$ such that $P_{n_r} \xrightarrow[r\to\infty]{\mathrm{w}} P$. Hence, by the definition of X_n, we have

$$X_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} P. \tag{21}$$

Moreover, by (18),

$$X_{T,n_r} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n_r}. \tag{22}$$

It remains to check a closeness between Y_n and $X_{T,n}$. The definition of the metric d and Lemma 6 show that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} d(\zeta_3(s+i\psi(\tau)), \zeta_{3,n}(s+i\psi(\tau)))\,\mathrm{d}\tau = 0.$$

Therefore, for every $\varepsilon > 0$, the Chebyshev type inequality gives

$$\lim_{r\to\infty} \limsup_{T\to\infty} \mu\{d(Y_T, X_{T,n}) \geqslant \varepsilon\}$$
$$\leqslant \lim_{r\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon T} \int_T^{2T} d(\zeta_3(s+i\psi(\tau)), \zeta_{n_r,3}(s+i\psi(\tau)))\,\mathrm{d}\tau = 0.$$

This, (21), and (22) show that all hypotheses of Theorem 3.2 of [43] are satisfied for X_{n_r}, X_{T,n_r} and Y_T. Hence

$$Y_T \xrightarrow[T\to\infty]{\mathcal{D}} P,$$

which is equivalent to

$$P_T \xrightarrow[T\to\infty]{\mathrm{w}} P.$$

The last relation shows that the measure P is independent on the sequence n_r. Since P_n is relatively compact, from this we obtain that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{23}$$

To prove the relation

$$\widehat{P}_T \xrightarrow[T\to\infty]{\mathrm{w}} P, \tag{24}$$

define

$$\widehat{X}_{T,n} = \widehat{X}_{T,n}(s) = \zeta_{3,n}(s+i\psi(\xi_T), \mathfrak{t})$$

and

$$\widehat{Y}_T = \widehat{Y}_T(s) = \zeta_3(s+i\psi(\xi_T), \mathfrak{t}).$$

Then we have by Lemma 10 that

$$\widehat{X}_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} X_n, \tag{25}$$

and, by Lemma 8,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} d(\zeta_3(s+i\psi(\tau), \mathfrak{t}), \zeta_{3,n}(s+i\psi(\tau), \mathfrak{t}))\,\mathrm{d}\tau = 0$$

for almost all $\mathfrak{t}$. This implies that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{ d(\widehat{Y}_T, \widehat{X}_{T,n}) \geqslant \varepsilon \right\} = 0$$

for almost all $\mathfrak{t}$. Therefore, (23) and (25) together with Theorem 3.1 of [43] prove (24) for almost all $\mathfrak{t}$. The proof is complete. □

4. Proof of Main Results

Proof of Theorem 1. We observe that the measure P in Lemma 12 is independent of the function $\psi(\tau)$ because of independence of P_n. Therefore, we may suppose that $\psi(\tau) = \tau$.

Let $A \in \mathcal{B}(\mathcal{H}(\mathcal{D}))$ and ∂A denote the boundary of A. Recall that A is said to be continuity set of P if $P(\partial A) = 0$. Now, we apply the equivalent of weak convergence of probability measures in terms of continuity sets, and, in view of Lemma 12, we have

$$\lim_{T\to\infty} \widehat{P}_T(A) = P(A) \tag{26}$$

for the continuity set A of P. Fix a set A, and, on $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m^H)$, define the random variable

$$\omega(\mathfrak{t}) = \begin{cases} 1 & \text{if } \zeta_3(s,\mathfrak{t}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbb{E}(\omega) = \int_{\mathbb{T}} \omega \, \mathrm{d}m^H = m^H\{\mathfrak{t} \in \mathbb{T} : \zeta_3(s,\mathfrak{t}) \in A\} = P_{\zeta_3}(A). \tag{27}$$

Now, we apply the Birkhoff–Khintchine ergodic theorem and obtain by Lemma 3 that

$$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} \omega(\varphi_\tau(\mathfrak{t})) \, \mathrm{d}\tau = \mathbb{E}\omega \tag{28}$$

for almost all $\mathfrak{t}$. It is easily seen that

$$\frac{1}{T} \int_T^{2T} \omega(\varphi_\tau(\mathfrak{t})) \, \mathrm{d}\tau = W_T(\zeta_3(s, \varphi_\tau(\mathfrak{t})) \in A) = W_T(\zeta_3(s + i\tau, \mathfrak{t}) \in A).$$

Therefore, taking into account (27) and (28) gives

$$\lim_{T\to\infty} W_T(\zeta_3(s + i\tau, \mathfrak{t}) \in A) = P_{\zeta_3}(A)$$

for almost all $\mathfrak{t}$. This and (26) show that

$$P(A) = P_{\zeta_3}(A)$$

for all continuity sets A of the measure P. Since all continuity sets constitute the determining class, we have

$$\widehat{P}_T \xrightarrow[T\to\infty]{\mathrm{w}} P_{\zeta_3},$$

and, in view of Lemma 12, the theorem is proved. □

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Proof of Theorem 2. We take $\mathcal{F} = S_{P_{\zeta_3}}$, where $S_{P_{\zeta_3}}$ is the support of P_{ζ_3}. Then $\mathcal{F} \neq \varnothing$ is a closed subset of $\mathcal{H}(\mathcal{D})$. For $f(s) \in \mathcal{F}$, a compact set $K \subset \mathcal{D}$ and $\varepsilon > 0$, define

$$G_\varepsilon = \left\{ g \in \mathcal{H}(\mathcal{D}) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then G_ε is an open neighbourhood of element $f(s)$ of the support of the measure P_{ζ_3}. Hence, by the support property,

$$P_{\zeta_3}(G_\varepsilon) > 0. \tag{29}$$

Theorem 1 and the equivalent of weak convergence of probability measures in terms of open sets, see Theorem 2.1 of [43], imply

$$\lim_{T \to \infty} P_T(G_\varepsilon) \geqslant P_\zeta(G_\varepsilon).$$

This, (29) and the definitions of G_ε and P_T prove the first statement of the theorem.

For the proof of the second statement, we observe that the boundary ∂G_ε of the set G_ε lies in the set

$$\left\{ g \in \mathcal{H}(\mathcal{D}) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\}.$$

Therefore, the boundaries $\partial G_{\varepsilon_1}$ and $\partial G_{\varepsilon_2}$ have no common elements for different positive ε_1 and ε_2. From this, it follows that $P_{\zeta_3}(\partial G_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$, or $P_{\zeta_3}(\partial G_\varepsilon) = 0$ for all but at most countably many $\varepsilon > 0$. This shows that the set G_ε is a continuity set of the measure P_{ζ_3} for all but at most countably many $\varepsilon > 0$. Now, application of Theorem 1 and the equivalent of weak convergence in terms of continuity sets, see Theorem 2.1 of [43], together with (29), show that

$$\lim_{T \to \infty} P_T(G_\varepsilon) = P_{\zeta_3}(G_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This completes the proof of Theorem 2. □

5. Conclusions

We considered the Epstein zeta-function $\zeta_3(s)$ of the form

$$\zeta_3(s) = \sum_{(m_1, m_2, m_3) \in \mathbb{Z}^3 \setminus \{\underline{0}\}} \frac{1}{(m_1^2 + m_2^2 + m_3^2)^s}, \quad \operatorname{Re} s > \frac{3}{2},$$

and investigated the value distribution of $\zeta_3(s + i\psi(\tau))$ with a certain differentiable function $\psi(\tau)$. It turned out that $\zeta_3(s + i\psi(\tau))$ follows a strong probabilistic law in the sense of weak convergence of probability measures:

$$\frac{1}{T} \operatorname{meas}\{\tau \in [T, 2T] : \zeta_3(s + i\psi(\tau)) \in A\}$$

with A in the space of analytic functions, as $T \to \infty$, converges weakly to an explicitly given random element. From this, it follows that the shifts $\zeta_3(s + i\psi(\tau))$ approximate a certain class $\mathcal{F}$ of analytic functions defined in the strip $\{s \in \mathbb{C} : 1 < \operatorname{Re} s > 3/2\}$. These results belong to the field of pure mathematics, and we believe possible applications for the investigation of value distribution of the sequence $\{r_3(m) : m \in \mathbb{N}\}$. Unfortunately, the used method does not ensure the identification of the set $\mathcal{F}$, and this remains a big future problem. Also, we are planning to consider Epstein zeta-functions defined by quadratic forms of more than three variables (at least of five variables). The first results must be

the mean square estimate for $\zeta_5(s)$ for $\operatorname{Re} s > 5/2$, which is an important but complicated problem. In general, the results of the paper motivate us to investigate interesting Epstein zeta-functions further.

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