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A limit theorem in the space of meromorphic functions for general Dirichlet series

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Let $s = \sigma + it$ be a complex variable, and let $\mathbb{R}$ and $\mathbb{C}$ denote the set of all real numbers and of the set of all complex numbers, respectively. The series

$$\sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s}, \tag{1}$$

where $a_m \in \mathbb{C}$ and $\lambda_m \in \mathbb{R}$, $0 < \lambda_1 < \lambda_2 < \ldots$, $\lim_{m\to\infty} \lambda_m = +\infty$, is called a general Dirichlet series with coefficients a_m and exponents λ_m.

In [1], [2], [4] probabilistic limit theorems for the function $f(s)$ given in some half-plane by the series (1) were obtained. Denote by σ_a the abscissa of the absolute convergence of (1). Then $f(s)$ is an analytic function for $\sigma > \sigma_a$. Suppose that $f(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_1$, $\sigma_1 < \sigma_a$, and that all poles in this region are included in a compact set. Moreover, we require that, for $\sigma > \sigma_1$, the estimates

$$f(s) = \mathrm{O}(|t|^a), \quad |t| \geqslant t_0, \quad a > 0, \tag{2}$$

and

$$\int_{-T}^{T} |f(\sigma + it)|^2 \,\mathrm{d}t = \mathrm{O}(T), \quad T \to \infty, \tag{3}$$

should be satisfied. We also assume that exponents λ_m satisfy the inequality

$$\lambda_m \geqslant c(\log m)^{\delta} \tag{4}$$

with some positive constants c and δ.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let $d(s_1, s_2)$ denote the spherical metric defined by

$$d(s_1, s_2) = \frac{2|s_2 - s_1|}{\sqrt{1+|s_1|^2}\sqrt{1+|s_2|^2}}, \quad d(s, \infty) = \frac{2}{\sqrt{1+|s|^2}}, \quad d(\infty, \infty) = 0,$$

$s_1, s_2, s \in \mathbb{C}$. Denote $D = \{s \in \mathbb{C}\colon \sigma > \sigma_1\}$, and let $M(D)$ stand for the space of meromorphic functions $g\colon D \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. Similarly, by $H(D)$ we denote the space of analytic on D functions with the same topology as in the case of $M(D)$.

For the identification of a limit measure we need the following topological structure. Denote by $\gamma = \{s \in \mathbb{C}: |s| = 1\}$ the unit circle on $\mathbb{C}$, and let

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \geqslant 1$. With the product topology and pointwise multiplication the infinitedimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(S)$ is the class of Borel sets of the space S, the probability Haar measure m_H exists, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(m)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_m, and define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ the $H(D)$-valued random element $f(s, \omega)$ by

$$f(s, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m s}, \quad s \in D, \quad \omega \in \Omega.$$

Denote by P_ξ the distribution of the random element ξ. Moreover, let, as usual, $\nu_T(\ldots) = T^{-1}$ meas $\{\tau \in [0, T] : \ldots\}$, where meas $\{A\}$ is the Lebesgue measure of the set $A \subset \mathbb{R}$, and the dots denote some condition satisfied by τ. Then in [4] the following limit theorem was obtained: if the set $\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_m\}$ is linearly independent over the field of rational numbers, and if $f(s)$ satisfies conditions (2), (3) and (4), then the probability measure

$$P_T(A) = \nu_T(f(s + i\tau) \in A), \quad A \in \mathcal{B}(M(D)),$$

weakly converges to P_f as $T \to \infty$.

The aim of this note is to remove $\log 2$ from the set $\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_m\}$.

Theorem. *Suppose that the system of exponents λ_m is linearly independent over the field of rational numbers, and that the function $f(s)$ satisfies conditions* (2), (3) *and* (4). *Then the probability measure P_T weakly converges to P_f as $T \to \infty$.*

A full proof of the theorem is sufficiently long, it will be given elsewhere. There we present only a sketch of the proof.

Denote the poles of the function $f(s)$ in the half-plane D by $s_1, \ldots, s_r$. Without loss of generality we can assume that every of these poles has order 1. Let

$$f_1(s) = \prod_{j=1}^{r} \left(1 - \mathrm{e}^{\lambda_1 (s_j - s)}\right),$$

and $f_2(s) = f_1(s) f(s)$. Then $f_2(s)$ is a regular function on D. Moreover, we have, for $\sigma > \sigma_a$,

$$f_2(s) = \sum_{A\subseteq\{1,\dots,r\}} \sum_{m=1}^{\infty} a_m \mathrm{e}^{\lambda_1} \sum_{j\in A} s_j (-1)^{|A|} \mathrm{e}^{-(\lambda_m+|A|\lambda_1)s}$$
$$= \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{m,j} \mathrm{e}^{-(\lambda_m+j\lambda_1)s}$$

with some coefficients $a_{m,j}$ satisfying $a_{m,j} = \mathrm{O}(|a_m|)$, $m = 1, 2, \dots, j = 1, \dots, r$. Here the first sum runs over all subsets A of $\{1, \dots, r\}$, and $|A|$ denotes the number of elements of A. Clearly, the function $f_2(s)$ satisfies the estimates of type (2) and (3).

First we consider the Dirichlet polynomials

$$p_n(s) = \sum_{j=0}^{r}\sum_{m=1}^{n} a_{m,j} \mathrm{e}^{-(\lambda_m+j\lambda_1)s}.$$

Let $\Omega_n = \prod_{m=1}^{n} \gamma_m$ with $\gamma_m = \gamma$ for $m = 1, \dots, n$. Denote the Haar measure on $(\Omega_n, \mathcal{B}(\Omega_n))$ by m_{nH}, and let the function $h\colon \Omega_n \to H(D)$ be defined by the formula

$$h(x_1, \dots, x_n) = \sum_{j=0}^{r}\sum_{m=1}^{n} \mathrm{e}^{-(\lambda_m+j\lambda_1)s} x_1^{-j} x_m^{-1}, \quad (x_1, \dots, x_n) \in \Omega_n.$$

Moreover, let $u(m)$ be an arithmetic function, $|u(m)| \leqslant 1$, and let

$$p_n(s, n) = \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{m,j} u^j(1) u(m) \mathrm{e}^{-(\lambda_m+j\lambda_1)s}.$$

Lemma 1. *The probability measures*

$$\nu_T \left(p_n(s + i\tau) \in A\right), \quad A \in \mathcal{B}(H(D)),$$

and

$$\nu_T \left(p_n(s + i\tau, u) \in A\right), \quad A \in \mathcal{B}(H(D)),$$

weakly converge to the measure $m_{nH}h^{-1}$ *as* $T \to \infty$.

Proof uses the continuity of the function h and is based on the weak convergence of the probability measure $\nu_T\left((\mathrm{e}^{i\lambda_1\tau}, \dots, \mathrm{e}^{i\lambda_n\tau}) \in A\right)$, $A \in \mathcal{B}(\Omega_n)$.

The next step of the proof of the theorem consists of the approximation in the mean of the functions $f_2(s)$ and $f_2(s, \omega)$ by absolutely convergent Dirichlet series. Let, for $\sigma_2 > \sigma_a - \sigma_1$,

$$g_n(s) = \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{m,j} \exp\left\{-\mathrm{e}^{(\lambda_m-\lambda_n)\sigma}\right\} \mathrm{e}^{-(\lambda_m+j\lambda_1)s},$$

$$g_n(s,\omega)=\sum_{j=0}^{r}\sum_{m=1}^{\infty}a_{m,j}\omega^j(1)\omega(m)\exp\left\{-\mathrm{e}^{(\lambda_m-\lambda_n)\sigma_2}\right\}\mathrm{e}^{-(\lambda_m+j\lambda_1)s},$$

and

$$f_2(s,\omega)=\sum_{j=0}^{r}\sum_{m=1}^{\infty}a_{m,j}\omega^j(1)\omega(m)\mathrm{e}^{-(\lambda_m+j\lambda_1)s},\quad \omega\in\Omega,\quad s\in D.$$

Note that both the latter series converge absolutely for $\sigma>\sigma_1$.

Lemma 2. *Let K be a compact subset of D. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}|f_2(s+i\tau)-g_n(s+i\tau)|\,\mathrm{d}\tau=0,$$

and

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}|f_2(s+i\tau,\omega)-g_n(s+i\tau,\omega)|\,\mathrm{d}\tau=0.$$

Proof of the first relation of the lemma uses estimate (3), and for the proof of the second relation we apply some elements of the ergodic theory.

The next lemma is devoted to limit theorems for the functions $g_n(s)$ and $g_n(s,\omega)$.

Lemma 3. *There exists a probability measure P_n on $(H(D),\mathcal{B}(H(D)))$ such that both measures $\nu_T\left(g_n(s+i\tau)\in A\right)$ and $\nu_T\left(g_n(s+i\tau,\omega)\in A\right)$, $A\in\mathcal{B}(H(D))$, weakly converge to P_n as $T\to\infty$.*

Proof of the lemma is based on Lemma 1 and on absolute convergence of the series defining $g_n(s)$ and $g_n(s,\omega)$.

Now we are able to limit theorems for $f_2(s)$ and $f_2(s,\omega)$.

Lemma 4. *There exists a probability measure P on $(H(D),\mathcal{B}(H(D)))$ such that both measures $\nu_T\left(f_2(s+i\tau),A\right)$ and $\nu_T\left(f_2(s+i\tau,\omega)\in A\right)$, $A\in\mathcal{B}(H(D))$, weakly converge to P as $T\to\infty$.*

Proof of the lemma is a consequence of Lemmas 2 and 3.

The application of some ideas related to the ergodic theory shows that the limit measure P in Lemma 4 coincides with P_{f_2}.

Proof of the theorem. It remains to pass from the function $f_2(s)$ to $f(s)$. First we observe that the probability measure $\nu_T\left(f_1(s+i\tau)\in A\right)$, $A\in\mathcal{B}(H(D))$, weakly converges to the distribution of the $H(D)$-valued random element

$$f_1(s,\omega)=\prod_{j=1}^{r}\left(1-\omega(1)\mathrm{e}^{\lambda_1(s_j-s)}\right),$$

as $T \to \infty$. Next, we define the $H^2(D)$-valued random element $F(s,\omega)$ by $F(s,\omega) = (f_1(s,\omega), f_2(s,\omega))$. Then by standard method, see, for example, [3], we show that the probability measure

$$P_{T,f_1,f_2}(A) = \nu_T\left(f_1(s+i\tau), f_2(s+i\tau) \in A\right), \quad A \in \mathcal{B}(H^2(D)),$$

weakly converges to P_F as $T \to \infty$.

Let the function h: $H^2(D) \to M(D)$ be given by the formula

$$h(g_1, g_2) = \frac{g_2}{g_1}, \quad g_1, g_2 \in H(D).$$

It is not difficult to verify that the metric d satisfies

$$d(g_1, g_2) = d\left(\frac{1}{g_1}, \frac{1}{g_2}\right).$$

Therefore, the function h is continuous. Hence from above remark we deduce that the measure $P_T = P_{T,f_1,f_2}h^{-1}$ weakly converges to the measure

$$m_H\left(\omega \in \Omega\colon \frac{f_2(s,\omega)}{f_1(s,\omega)} \in A\right), \quad A \in \mathcal{B}(M(D_1)).$$

However,

$$\begin{aligned} f_2(s,\omega) &= \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{m,j}\omega^j(1)\omega(m)\mathrm{e}^{-(\lambda_m + j\lambda_1)s} \\ &= \prod_{j=1}^{r}\left(1 - \omega(1)\mathrm{e}^{\lambda_1(s_j - s)}\right)\sum_{m=1}^{\infty} a_m\omega(m)\mathrm{e}^{-\lambda_m s}, \end{aligned}$$

and hence the theorem follows.

References

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Ribinė teorema meromorfinių funkcijų erdvėje bendrosioms Dirichlet eilutėms

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Patikslinta viena ribinė teorema meromorfinių funkcijų erdvėje bendrosioms Dirichlet eilutėms.