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On the mean square of the periodic zeta-function

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Let $s=\sigma+it$ be a complex variable, and let $\mathfrak{a}=\{a_m:\ m\in\mathbb{Z}\}$ be a periodic with period k sequence of complex numbers. The periodic zeta-function $\zeta(s;\mathfrak{a})$ is defined, for $\sigma>1$, by

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}\,,$$

and the equality

$$\zeta(s;\mathfrak{a})=\frac{1}{k^s}\sum_{q=1}^{k}a_q\zeta\Big(s,\frac{q}{k}\Big),$$

where $\zeta(s,\alpha)$ denotes the Hurwitz zeta-function, gives an analytic continuation to the whole complex plane for $\zeta(s;\mathfrak{a})$. It is regular everywhere except, maybe, for a simple pole at $s=1$.

This note is a continuation of papers [2], [3], and [5], where the asymptotics for the mean square

$$I_T(\sigma)=\int_0^T\big|\zeta(\sigma+it;\mathfrak{a})\big|^2\,\mathrm{d}t,\qquad \frac{1}{2}\leqslant\sigma<1,$$

as $T\to\infty$ was studied. Our aim is to obtain, the asymptotics for $I_T(\sigma_T)$ for $\sigma_T\to 1/2+0$, as $T\to\infty$. Let $\sigma_T=1/2+l_T^{-1}$ where $l_T\to 0$ and $l_T\to\infty$ as $T\to\infty$. Define

$$K=K(k)=\sum_{q=1}^{k}|a_q|^2,$$

and denote by B a quantity bounded by a constant. Then we have the following statements.

Theorem 1. *Let $l_T=\mathrm{o}(\log T)$ as $T\to\infty$. Then*

$$I_T(\sigma_T)=k^{-2\sigma_T}KTl_T+T\sum_{q=1}^{k}\frac{|a_q|^2}{q^{2\sigma_T}}+Bk^{1-2\sigma_T}KTl_T\exp\Big\{-\frac{\log T}{l_T}\Big\}$$
$$+Bk^{-2\sigma_T}KT+Bk^{1-\sigma_T}KT^{1/2}\log T.$$

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Theorem 2. *Let* $\log T = \mathrm{o}(l_T)$ *as* $T \to \infty$. *Then*

$$
\begin{aligned}
I_T(\sigma_T) \;=\; & k^{-2\sigma_T} K T \log T + T \sum_{q=1}^{k} \frac{|a_q|^2}{q^{2\sigma_T}} + B k^{1-2\sigma_T} K T \\
& + B k^{-2\sigma_T} K \frac{T \log^2 T}{l_T} + B k^{1-\sigma_T} K T^{1/2} \log T.
\end{aligned}
$$

Theorem 3. *Let* $\lim_{T\to\infty} l_T / \log T = \kappa \neq 0$. *Then*

$$
\begin{aligned}
I_T(\sigma_T) \;=\; & k^{-2\sigma_T} K \frac{\kappa}{2}(1 - \mathrm{e}^{-2/\kappa}) T \log T + T \sum_{q=1}^{k} \frac{|a_q|^2}{q^{2\sigma_T}} + \mathrm{o}(k^{-2\sigma_T} K T \log T) \\
& + B k^{1-2\sigma_T} K T + B k^{1-\sigma_T} K T^{1/2} \log T
\end{aligned}
$$

as $T \to \infty$.

Proof of Theorems 1–3 is based on the approximate functional equation for the function $\zeta(s; \mathfrak{a})$ obtained in [3]. We also preserve the notation of [3]. The mentioned functional equation gives

$$
\begin{aligned}
\int_{T/2}^{T} \left|\zeta(\sigma_T + it; \mathfrak{a})\right|^2 \, \mathrm{d}t = {} & k^{-2\sigma_T} \int_{T/2}^{T} \left| \sum_{q=1}^{k} a_q \sum_{0 \leqslant m \leqslant r} \frac{1}{(m + q/k)^{\sigma_T + it}} \right|^2 \mathrm{d}t \\
& + k^{-2\sigma_T} \int_{T/2}^{T} \left(\frac{2\pi}{t}\right)^{2\sigma_T - 1} \left| \sum_{q=1}^{k} a_q \sum_{1 \leqslant m \leqslant n} \mathrm{e}^{-2\pi i m \frac{q}{k}} \frac{1}{m^{1-\sigma_T - it}} \right|^2 \mathrm{d}t \\
& + B k^{-2\sigma_T} \left| \sum_{q=1}^{k} a_q \right|^2 \int_{T/2}^{T} t^{-\sigma_T} \, \mathrm{d}t + B k^{1-2\sigma_T} K \int_{T/2}^{T} t^{\sigma_T - 2} \, \mathrm{d}t \\
& + B k^{-2\sigma_T} \left| \int_{T/2}^{T} \left(\frac{2\pi}{t}\right)^{\sigma_T - 1/2 + it} \mathrm{e}^{-it} \sum_{q_1=1}^{k} a_{q_q} \sum_{q_1=1}^{k} \bar{a}_{q_2} \right. \\
& \times \left. \sum_{0 \leqslant m_1 \leqslant r} \frac{1}{(m_1 + q_1/k)^{\sigma_T + it}} \sum_{1 \leqslant m_2 \leqslant n} \mathrm{e}^{2\pi i m_2 \frac{q_2}{k}} \frac{1}{m_2^{1-\sigma_T + it}} \, \mathrm{d}t \right| \\
& + B k^{-2\sigma_T} \left| \int_{T/2}^{T} \left(\frac{2\pi}{t}\right)^{\sigma_T/2} \sum_{q_1=1}^{k} \bar{a}_{q_1} \sum_{q_2=1}^{k} a_{q_2} \right. \\
& \times \left. \sum_{0 \leqslant m \leqslant r} \frac{1}{(m + q_1/k)^{\sigma_T - it}} \exp\left\{ i f\left(\frac{q_2}{k}, t\right) \right\} \psi\left(2y - 2n + l - \frac{q_2}{k}\right) \, \mathrm{d}t \right| \\
& + B k^{-2\sigma_T} \left| \int_{T/2}^{T} \left(\frac{2\pi}{t}\right)^{3\sigma_T/2 - 1/2 - it} \mathrm{e}^{-it} \sum_{q_1=1}^{k} \bar{a}_{q_1} \sum_{q_2=1}^{k} a_{q_2} \right.
\end{aligned}
$$

$$\times \sum_{1\leq m\leqslant n} \mathrm{e}^{2\pi i m \frac{q_1}{k}} \frac{1}{m^{1-\sigma_T+it}} \exp\Big\{if\Big(\frac{q_2}{k},t\Big)\Big\}\psi\Big(2y-2n+l-\frac{q_2}{2}\Big)\,\mathrm{d}t\Big|$$

$$+Bk^{-2\sigma_T}\Big|\int_{T/2}^{T}\sum_{q=1}^{k}a_q\sum_{0\leqslant m\leqslant r}\frac{1}{(m+q/k)^{\sigma_T+it}}\overline{R(\sigma_T+it,k)}\,\mathrm{d}t\Big|$$

$$+Bk^{-2\sigma_T}\Big|\int_{T/2}^{T}\Big(\frac{2\pi}{t}\Big)^{\sigma_T-1/2+it}\sum_{q=1}^{k}a_q\sum_{1\leqslant m\leqslant n}\mathrm{e}^{-2\pi i m\frac{q}{k}}\frac{1}{m^{1-\sigma_T-it}}\overline{R(\sigma_T+it,k)}\,\mathrm{d}t\Big|$$

$$+Bk^{-\sigma_T}\Big|\int_{T/2}^{T}\Big(\frac{2\pi}{t}\Big)^{\sigma_T/2}\sum_{q=1}^{k}a_q\mathrm{e}^{if(q/k,t)}\psi\Big(2y-2n-l-\frac{q}{k}\Big)\overline{R(\sigma_T+it,k)}\,\mathrm{d}t\Big|$$

$$\stackrel{def}{=}\sum_{j=1}^{10}I_j. \tag{1}$$

Proof of Theorem 1. Let $T_1=\max(T/2,2\pi(m_1+q_1/k),2\pi(m_2+q_1/k)^2)$ and $M_q(T)=[(T/2\pi)^{1/2}-q/k]$. Then we have that

$$\begin{aligned}
I_1 &= k^{-2\sigma_T}\sum_{q=1}^{k}|a_q|^2\sum_{0\leqslant m\leqslant M_q(T)}\int_{T_1}^{T}\frac{\mathrm{d}t}{(m+q/k)^{2\sigma_T}}\\
&\quad+k^{-2\sigma_T}\sum_{q=1}^{k}|a_q|^2\sum_{0\leqslant m_1\leqslant M_q(T)}\sum_{0\leqslant m_2\leqslant M_q(T)}\frac{1}{(m_1+q/k)^{\sigma_T}(m_2+q/k)^{\sigma_T}}\\
&\quad\times\int_{T_1}^{T}\left(\frac{m_1+q/k}{m_2+q/k}\right)^{it}\,\mathrm{d}t\\
&\quad+k^{-2\sigma_T}\sum_{q=1}^{k}a_{q_1}\sum_{\substack{q_2=1\\ q_1\neq q_2}}^{k}\bar a_{q_2}\sum_{0\leqslant m_1\leqslant M_{q_1}(T)}\sum_{0\leqslant m_2\leqslant M_{q_2}(T)}\frac{1}{(m_1+q_1/k)^{\sigma_T}(m_2+q_2/k)^{\sigma_T}}\\
&\quad\times\int_{T_1}^{T}\left(\frac{m_2+q_2/k}{m_1+q_1/k}\right)^{it}\,\mathrm{d}t\\
&\quad+k^{-2\sigma_T}\sum_{q=1}^{k}a_{q_1}\sum_{\substack{q_2=1\\ q_1\neq q_2}}^{k}\bar a_{q_2}\sum_{0<m\leqslant\max(M_{q_1}(T),M_{q_2}(T))}\frac{1}{(m+q_1/k)^{\sigma_T}(m+q_2/k)^{\sigma_T}}\\
&\quad\times\int_{T_1}^{T}\left(\frac{m+q_2/k}{m+q_1/k}\right)^{it}\,\mathrm{d}t\stackrel{def}{=}\sum_{j=1}^{4}I_{1j}.
\end{aligned}\tag{2}$$

Using the equality [4]

$$\begin{aligned}
\sum_{0\leqslant m\leqslant x}\frac{1}{(m+\alpha)^s} &= \frac{1}{\alpha^s}+\frac{[x]}{(x+\alpha)^s}-s\int_1^x\frac{u-[u]}{(u+\alpha)^{s+1}}\,\mathrm{d}u+\frac{s}{s-1}(1+\alpha)^{1-s}\\
&\quad-\frac{s}{s-1}(x+\alpha)^{1-s}+\alpha(1+\alpha)^{-s}-\alpha(x+\alpha)^{-s},
\end{aligned}\tag{3}$$

where $0 < \alpha \leqslant 1$ and $s \neq 1$, we find without difficult that

$$I_{11} = \frac{Tl}{4k^{2\sigma_T}}K + \frac{T}{2}\sum_{q=1}^{k}\frac{|a_q|^2}{q^{2\sigma_T}} + Bk^{-2\sigma_T}K + Bk^{-2\sigma_T}Tl_TK\exp\left\{-\frac{\log T}{l_T}\right\}. \quad (4)$$

From the estimate [1]

$$\sum_{0\leqslant m<n\leqslant N}\frac{1}{(m+\alpha_1)^{\beta}(n+\alpha_2)^{\theta}}\left(\log\frac{n+\alpha_1}{m+\alpha_2}\right)^{-1} = B_cN^{2-\beta-\theta}\log N, \quad (5)$$

where $0 < \alpha_2, \alpha_1 \leqslant 1$, $\beta < 1$, $\beta + \theta \leqslant c < 2$, we derive that

$$I_{12} = Bk^{-2\sigma_T}KT^{1-\sigma_T}\log T. \quad (6)$$

Similarly

$$I_{13} = Bk^{-2\sigma_T}KT^{1-\sigma_T}\log T,$$
$$I_{14} = Bk^{1-2\sigma_T}KT^{1-\sigma_T}\log T.$$

This and (2), (4) and (6) yield

$$\begin{aligned} I_1 \;=\; & \frac{Tl_T}{4}k^{-2\sigma_T}K + \frac{T}{2}\sum_{q=1}^{k}\frac{|a_q|^2}{q^{2\sigma_T}} + Bk^{-\sigma_T}Tl_TK\exp\left\{-\frac{\log T}{l_T}\right\} \\ & + Bk^{1-2\sigma_T}KT^{1-\sigma_T}\log T. \end{aligned} \quad (7)$$

Now let $T_2 = \max(T/2, 2\pi m_1^2, 2\pi m_2^2)$. Then

$$\begin{aligned} I_2 \;=\; & k^{-2\sigma_T}\sum_{q=1}^{k}|a_q|^2\sum_{1\leqslant m\leqslant M_0(T)}\frac{1}{m^{2-2\sigma_T}}\int_{T_2}^{T}\left(\frac{2\pi}{t}\right)^{2\sigma_T-1}\,\mathrm{d}t \\ & + k^{-2\sigma_T}\sum_{q=1}^{k}|a_q|^2\sum_{1\leqslant m_1,}\ \sum_{\substack{m_2\leqslant M_0(T)\\ m_1\neq m_2}}\frac{\mathrm{e}^{-2\pi i\frac{q}{k}(m_1-m_2)}}{(m_1m_2)^{1-\sigma_T}} \\ & \times\int_{T_2}^{T}\left(\frac{2\pi}{t}\right)^{2\sigma_T-1}\exp\left\{it\log\frac{m_1}{m_2}\right\}\,\mathrm{d}t \\ & + k^{-2\sigma_T}\sum_{\substack{q_1=1\\ q_1\neq q_2}}^{k}\sum_{q_2=1}^{k}a_{q_1}\bar{a}_{q_2}\sum_{1\leqslant m_1}\ \sum_{\substack{m_2\leqslant M_0(T),\\ m_1\neq m_2}}\frac{\mathrm{e}^{-\frac{2\pi i}{k}(q_1m_1-q_2m_2)}}{(m_1m_2)^{1-\sigma_T}} \end{aligned}$$

$$
\begin{aligned}
&\times \int_{T_2}^{T} \left(\frac{2\pi}{t}\right)^{2\sigma_T-1} \exp\left\{it\log\frac{m_1}{m_2}\right\} \mathrm{d}t \\
&+ k^{-2\sigma_T} \sum_{\substack{q_1=1 \\ q_1\neq q_2}}^{k} \sum_{q_2=1}^{k} a_{q_1}\bar{a}_{q_2} \sum_{1\leqslant m\leqslant M_0(T)} \frac{\mathrm{e}^{-\frac{2\pi i m}{k}(q_1-q_2)}}{m^{2-2\sigma_T}} \int_{T_2}^{T} \left(\frac{2\pi}{t}\right)^{2\sigma_T-1} \mathrm{d}t \\
&\stackrel{def}{=} \sum_{j=1}^{4} I_{2j}. \qquad (8)
\end{aligned}
$$

The application of (3) shows that

$$
I_{21} = k^{-2\sigma_T} K\frac{T}{4}\, l_T\left(1 + B\exp\left\{-\frac{\log T}{l_T}\right\} + Bl_T^{-1}\right), \qquad (9)
$$

$$
I_{24} = Bk^{1-2\sigma_T} KTl_T \exp\left\{-\frac{\log T}{l_T}\right\}. \qquad (10)
$$

Since

$$
\int_{T_2}^{T} \left(\frac{t}{2\pi}\right)^{1-2\sigma_T} \exp\left\{it\log\frac{m_1}{m_2}\right\} \mathrm{d}t = BT^{1-2\sigma_T}\left|\log\frac{m_1}{m_2}\right|^{-1},
$$

in view of (5) we obtain

$$
\begin{aligned}
I_{22} &= Bk^{-2\sigma_T} KT^{1-\sigma_T}\log T, \\
I_{23} &= Bk^{1-2\sigma_T} KT^{1-\sigma_T}\log T.
\end{aligned}
$$

Hence and from (8)–(10) we have

$$
I_2 = k^{-2\sigma_T} K\frac{Tl_T}{4} + Bk^{1-2\sigma_T} KTl_T\exp\left\{-\frac{\log T}{l_T}\right\} + Bk^{-2\sigma_T} KT. \qquad (11)
$$

It is easily seen that

$$
I_3 = Bk^{1-2\sigma_T} KT^{1-\sigma_T}, \qquad (12)
$$

$$
I_4 = Bk^{1-2\sigma_T} K. \qquad (13)
$$

Repeating the proof of Theorem 2 from [3], we find that

$$
\begin{aligned}
I_5 &= Bk^{1-2\sigma_T} KT^{1/2}\log T, \\
I_6 &= Bk^{1-2\sigma_T} KT^{1-\sigma_T}\log T + Bk^{1-\sigma_T} KT^{(1-\sigma_T)/2}, \\
I_7 &= Bk^{1-2\sigma_T} KT^{1-\sigma_T}\log T, \\
I_8 + I_9 + I_{10} &= Bk^{1-\sigma_T} KT^{1/2}.
\end{aligned}
$$

This and (10, (7), (11)–(13) show that

$$\int_{T/2}^{T} \left|\zeta(\sigma_T + it; \mathfrak{a})\right|^2 \, \mathrm{d}t = k^{-2\sigma_T} K \frac{T l_T}{2} + \frac{T}{2} \sum_{q=1}^{k} \frac{|a_q|^2}{q^{2\sigma_T}} + B k^{1-2\sigma_T} K T l_T \exp\left\{ -\frac{\log T}{l_T} \right\} + B k^{-2\sigma_T} K T + B k^{1-\sigma_T} K T^{1/2} \log T.$$

Taking $2^{-j}T$ instead of T and summing over all $j \geqslant 0$, hence we deduce Theorem 1.

Proof of Theorem 2. Clearly, it suffices to calculate the integrals I_1 and I_2 in (1). In this case we find that

$$I_1 = k^{-2\sigma_T} K \frac{T \log T}{4} + \frac{T}{2} \sum_{q=1}^{k} \frac{|a_q|^2}{q^{2\sigma_T}} + B k^{-2\sigma_T} K T + B k^{2-\sigma_T} K \frac{T \log T}{l_T} + B k^{1-2\sigma_T} K T^{1-\sigma_T} \log T,$$

$$I_2 = k^{-2\sigma_T} K \frac{T \log T}{4} + B k^{1-2\sigma_T} K T + B k^{-2\sigma_T} + K \frac{T \log T}{l_T},$$

and the proof is completed in the same way as Theorem 1.

Proof of Theorem 3. In this case we have

$$I_1 = k^{-2\sigma_T} K \frac{\kappa}{4} (1 - \mathrm{e}^{-1/\kappa} T \log T + \frac{T}{2} \sum_{q=1}^{k} \frac{|a_q|^2}{q^{2\sigma_T}} + \mathrm{o}(k^{-2\sigma_T} K T \log T) + B k^{1-2\sigma_T} K T^{1-\sigma_T} \log T,$$

$$I_2 = k^{-2\sigma_T} K \frac{\kappa}{4} (\mathrm{e}^{-1/\kappa} - \mathrm{e}^{-2/\kappa}) T \log T + \mathrm{o}(k^{-2\sigma_T} K T \log T) + B k^{1-2\sigma_T} K T.$$

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Apie periodinės dzeta funkcijos kvadrato vidurkį

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Gauta periodinės dzeta funkcijos antrojo momento asimptotika arti kritinės tiesės.