

A note on discrete limit theorems for the Matsumoto zeta-function

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In [1], [2], [3] we obtained discrete limit theorems in the sense of the weak convergence of probability measures in various spaces for the Matsumoto zeta-function $\varphi(s)$, $s = \sigma + it$. The latter function was introduced by K. Matsumoto in [4]. We recall the definition of $\varphi(s)$. Let $g(m)$ and $f(j, m)$ be positive integers, and $a_m^{(j)}$ be complex numbers. Define

$$A_m(X) = \prod_{j=1}^{g(m)} (1 - a_m^{(j)} X^{f(j,m)}),$$

and denote by p_m the m th prime number. Then the function $\varphi(s)$ is defined by the following infinite product

$$\varphi(s) = \prod_{m=1}^{\infty} A_m^{-1}(p_m^{-s}). \quad (1)$$

As K. Matsumoto in [4], we suppose that

$$g(m) \leq cp_m^\alpha, \quad |a_m^{(j)}| \leq p_m^\beta$$

with some $c > 0$ and some non-negative constants α and β . Then the product in (1) define, for $\sigma > \alpha + \beta + 1$, a holomorphic function without zeros.

Let, for positive integer N ,

$$\mu_N(\dots) = \frac{1}{N+1} \#(0 \leq m \leq N : \dots),$$

where instead of dots a condition satisfied by m is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S and let $h > 0$ be a fixed number. Then in [1], [2], [3] the weak convergence of the following probability measures

$$\begin{aligned} \mu_N(\varphi(\sigma + imh) \in A), & \quad A \in \mathcal{B}(\mathbb{C}), \\ \mu_N(\varphi(s + imh) \in A), & \quad A \in \mathcal{B}(H(D_1)), \\ \mu_N(\varphi(s + imh) \in A), & \quad A \in \mathcal{B}(M(D_2)), \end{aligned} \quad (2)$$

as $N \rightarrow \infty$, under some additional conditions on $\varphi(s)$, was investigated. Here \mathbb{C} denotes the complex plane, $H(D_1)$ is the space of analytic on $D_1 = \{s \in \mathbb{C}: \sigma > \alpha + \beta + 1\}$ functions, and $M(D_2)$ is the space of meromorphic on $D_2 = \{s \in \mathbb{C}: \sigma > \varrho\}$ functions, where $\alpha + \beta + \frac{1}{2} < \varrho < \alpha + \beta + 1$.

In this note we propose a generalization of the mentioned works. Let a positive function $w(t)$ be defined for $t \geq 0$, and let

$$U = U(N) = \sum_{m=0}^N w(m).$$

Suppose that $\lim_{N \rightarrow \infty} U(N) = +\infty$. We put

$$\mu_{N,w}(\dots) = \frac{1}{U} \sum_{\substack{m=0 \\ \dots}}^N w(m),$$

where instead of dots a condition satisfied by m is to be written. We observe that

$$\mu_{N,1}(\dots) = \mu_N(\dots).$$

Then instead of the measures (2) we can consider the weak convergence of probability measures,

$$\begin{aligned} \mu_{N,w}(\varphi(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}), \\ \mu_{N,w}(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(H(D_1)), \\ \mu_{N,w}(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(M(D_2)). \end{aligned} \quad (3)$$

It turns out that if the function $w(t)$ is non-increasing, then the weak convergence of the probability measures (2) implies that of the measures (3). This follows from the following general theorem. Let $f(t)$ be a S -valued function defined for $t \geq 0$, and

$$\begin{aligned} P_N(A) &= \mu_N(f(mh) \in A), \quad A \in \mathcal{B}(S), \\ P_{N,w}(A) &= \mu_{N,w}(f(mh) \in A), \quad A \in \mathcal{B}(S). \end{aligned}$$

Theorem. *Suppose that $w(t)$ is a continuous non-increasing function, and that P_N converges weakly to some probability measure P as $N \rightarrow \infty$. Then also $P_{N,w}$ converges weakly to P as $N \rightarrow \infty$.*

Proof. Since P_N converges weakly to P as $N \rightarrow \infty$, we have that

$$\int_S X \, dP_N \xrightarrow{N \rightarrow \infty} \int_S X \, dP \quad (4)$$

for every real bounded continuous function X on S . By the definition of P_N

$$\int_S X \, dP_N = \frac{1}{N+1} \sum_{m=0}^N X(f(mh)).$$

Therefore, putting

$$\int_S X \, dP = \kappa_X,$$

we have in view of (4) that

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N X(f(mh)) = \kappa_X. \quad (5)$$

On the other hand,

$$\int_S X \, dP_{N,w} = \frac{1}{U} \sum_{m=0}^N w(m) X(f(mh)). \quad (6)$$

Summing by parts, we find

$$\begin{aligned} \frac{1}{U} \sum_{m=0}^N w(m) X(f(mh)) &= \frac{w(N)}{U} \sum_{m=0}^N X(f(mh)) \\ &\quad - \frac{1}{U} \int_0^N \sum_{0 \leq m \leq u} X(f(mh)) \, dw(u). \end{aligned} \quad (7)$$

It follows in virtue of (5) that

$$\sum_{0 \leq m \leq u} X(f(mh)) = \sum_{0 \leq m \leq [u]} X(f(mh)) = ([u] + 1)\kappa_X + r(u)u,$$

where $r(u) \rightarrow 0$ as $u \rightarrow \infty$. Moreover, since $w(t)$ is non-increasing,

$$U = \sum_{m=0}^N w(m) \geq w(N)(N+1). \quad (8)$$

Therefore, this and (7) imply

$$\begin{aligned} \frac{1}{U} \sum_{m=0}^N w(m) X(f(mh)) &= \frac{w(N)}{U} ((N+1)\kappa_X + r(N)N) \\ &\quad - \frac{1}{U} \int_0^N (([u] + 1)\kappa_X + r(u)u) \, dw(u) \\ &= \frac{w(N)}{U} ((N+1)\kappa_X + (N+1)r(N+1)) \\ &\quad - \frac{w(u)}{U} ([u] + 1)\kappa_X \Big|_0^N + \frac{\kappa_X}{U} \int_0^N w(u) \, d([u] + 1) - \frac{1}{U} \int_0^N r(u)u \, dw(u) \\ &= o(1) + \frac{\kappa_X}{U} \sum_{m=0}^N w(m) - \frac{1}{U} \int_0^N r(u)u \, dw(u) \end{aligned}$$

$$= \kappa_X + o(1) - \frac{1}{U} \int_0^N r(u)u \, dw(u) \quad (9)$$

as $N \rightarrow \infty$. Let $N_1 = N_1(N) \rightarrow \infty$ as $N \rightarrow \infty$ be chosen so that

$$\frac{1}{U} \int_0^{N_1} r(u)u \, dw(u) = o(1) \quad \text{and} \quad N_1 w(N_1) = o(U)$$

as $N \rightarrow \infty$. Then we obtain that

$$\begin{aligned} \frac{1}{U} \int_0^N r(u)u \, dw(u) &= \frac{1}{U} \int_0^{N_1} r(u)u \, dw(u) + \frac{1}{U} \int_{N_1}^N r(u)u \, dw(u) \\ &= o(1) + \frac{B}{U} \max_{u \in [N_1, N]} |r(u)| \int_{N_1}^N u \, dw(u) \\ &= o(1) + \frac{B}{U} \max_{u \in [N_1, N]} |r(u)| u w(u) \Big|_{N_1}^N + \frac{B}{U} \max_{u \in [N_1, N]} |r(u)| \int_0^N w(u) \, du. \end{aligned} \quad (10)$$

Here B is a quantity bounded by a constant. It is easily seen that

$$\int_0^N w(u) \, du = BU.$$

This together with (8) and (10) shows that

$$\frac{1}{U} \int_0^N r(u)u \, dw(u) = o(1).$$

Therefore, (9) yields

$$\frac{1}{U} \sum_{m=0}^N w(m)X(f(mh)) = \kappa_X + o(1)$$

as $N \rightarrow \infty$, and by (6) we have that

$$\lim_{N \rightarrow \infty} \int_S X \, dP_{N,w} = \int_S X \, dP$$

for every real bounded continuous function X on S . This means that the measure $P_{N,w}$ converges weakly to P as $N \rightarrow \infty$. The theorem is proved.

We will give one corollary of theorem. Suppose that the Matsumoto zeta-function $\varphi(s)$ is meromorphically continuable to the region D_2 , all poles being included in a compact set. Moreover, we require that, for $\sigma > \varrho$, the estimates

$$\varphi(\sigma + it) = B|t|^\alpha, \quad |t| \geq t_0 > 0, \quad \alpha > 0,$$

and

$$\int_0^T |\varphi(\sigma + it)|^2 dt = BT, \quad T \rightarrow \infty,$$

should be satisfied. Let

$$\Omega = \prod_{m=1}^{\infty} \gamma_{p_m}, \quad \gamma_{p_m} = \{s \in \mathbb{C}: |s| = 1\} \quad \text{for all } m.$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where m_H denotes the Haar measure on $(\Omega, \mathcal{B}(\Omega))$, define an $H(D_2)$ -valued random element $\varphi(s, \omega)$ by the formula

$$\varphi(s, \omega) = \prod_{m=1}^{\infty} \prod_{j=1}^{g(m)} \left(1 - \frac{a_m^{(j)} \omega(p_m)}{s f^{(j,m)}(p_m)} \right)^{-1}, \quad s \in D_2,$$

where $\omega(p_m)$ is the projection of $\omega \in \Omega$ to coordinate space γ_{p_m} .

COROLLARY. *Suppose that the function $\varphi(s)$ satisfies all conditions stated above, and that $\exp\left\{\frac{2\pi k}{h}\right\}$ is irrational for all integers $k \neq 0$. If $w(t)$ is a continuous non-increasing function for $t \geq 0$ such that $\lim_{N \rightarrow \infty} U(N) = \infty$, then the probability measure*

$$\mu_{N,w}(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(M(D_2)),$$

converges weakly to the distribution of the random element $\varphi(s, \omega)$ as $N \rightarrow \infty$.

Proof follows from the Theorem and Theorem of [3].

References

- [1] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function on the complex plane, *Liet. Matem. Rink.*, **40** (4), 475–492 (2000) (in Russian) = *Lith. Math. J.*, **40** (4), 364–378 (2000).
- [2] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function in the space of analytic functions, *Liet. Matem. Rink.*, **41** (4), 441–448 (2001) (in Russian) = *Lith. Math. J.*, **41** (4), 344–350 (2001).
- [3] R. Kačinskaitė, A discrete limit theorem for the Matsumoto zeta-function in the space of meromorphic functions, *Liet. Matem. Rink.*, **42** (1), 46–67 (2002) (in Russian) = *Lith. Math. J.*, **42** (1), 37–53 (2002).
- [4] K. Matsumoto, Value-distribution of zeta-functions, *Lecture Notes in Math.*, Springer, **1434**, 178–187 (1990).

Pastaba apie diskrečias ribines teoremas Matsumoto dzeta funkcijai

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Straisnyje įrodyta diskrečioji ribinė teorema su svoriu Matsumoto dzeta funkcijai.