

Zeta functions of cusp forms

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Let $F(z)$ be a holomorphic normalized eigenform of weight κ for the full modular group $SL(2, \mathbb{Z})$. In this case $F(z)$ has the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}, \quad c(1) = 1.$$

Denote by $s = \sigma + it$ a complex variable. Then the function

$$\varphi(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}$$

is called the zeta-function attached to the form $F(z)$. The latter series converges absolutely for $\sigma > \frac{\kappa+1}{2}$, and the function $\varphi(s, F)$ is holomorphic in this region. It is well known that the function $\varphi(s, F)$ is analytically continuable to an entire function. Moreover, $\varphi(s, F)$ has the Euler product expansion

$$\varphi(s, F) = \prod_{m=1}^{\infty} \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1}, \quad \sigma > \frac{\kappa+1}{2},$$

over all primes p with $\alpha(p)$ and $\beta(p)$ satisfying $c(p) = \alpha(p) + \beta(p)$. This and the Deligne estimates $|\alpha(p)| \leq p^{\frac{\kappa-1}{2}}$, $|\beta(p)| \leq p^{\frac{\kappa-1}{2}}$ show that $\varphi(s, F) \neq 0$ for $\sigma > \frac{\kappa+1}{2}$. Let $w \neq 0$ be an arbitrary complex number. Then we can define a branch of multi-valued function $\varphi^w(s, F)$, for $\sigma > \frac{\kappa+1}{2}$, by

$$\varphi^w(s, F) = \exp\{w \log \varphi(s, F)\} = \sum_{m=1}^{\infty} \frac{g_w(m)}{m^s},$$

where $g_w(m)$ is a multiplicative function,

$$g_w(p^k) = \sum_{l=0}^k d_w(p^l) \alpha^l(p) d_w(p^{k-l}) \beta^{k-l}(p),$$

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and

$$d_w(p^k) = \frac{w(w+1)\dots(w+k-1)}{k!}, \quad k = 1, 2, \dots$$

The investigation of the moments

$$\int_0^T |\varphi(\sigma + it)|^{2v} dt, \quad \sigma \geq \frac{\kappa}{2}, \quad v \geq 0,$$

require to know the behaviour of the mean-value of $g_w(m)$. Let $h_w(m) = g_w^2(m)m^{1-\kappa}$. In this note we will consider the mean value $\sum_{m \leq x} h_w(m)$ as $x \rightarrow \infty$. We set $\varepsilon(x) = \max(|w|^{-2}\varepsilon_2(x), (\log x)^{-2})$, where $\varepsilon_2(x) = \max(\varepsilon_1(x), (\log x)^{-1})$ and

$$\varepsilon_1(x) = \sup_{y \geq x} |y^{-1} \sum_{p \leq y} h_w(p) \log p - w^2| \rightarrow 0$$

as $x \rightarrow \infty$ by properties of $h_w(m)$. Denote

$$m(h_w, x) = \prod_{p \leq x} \left(1 + \sum_{\alpha=1}^{\infty} \frac{h_w(p^\alpha)}{p^\alpha}\right),$$

$$R(y) = \max(|w|^2, \frac{\log^2 y}{y}),$$

where $y \leq x$ and $\log^2 y \geq c \log x \log^2 \log x$ with some $c > 0$. We suppose that w is a complex number such that $|w| \leq \frac{1}{2}$ and $\text{Re} w^2 > 0$. Let $t = \log x / \log y$, γ_0 stand for the Euler constant, and let B be a quantity bounded by a constant.

Theorem 1. *Suppose that $\varepsilon(x) \rightarrow 0$ as $x \rightarrow 0$. Then uniformly in w, x and y*

$$\begin{aligned} \sum_{m \leq x} h_w(m) &= \frac{e^{-\gamma_0 w^2} x m(h_w, x) t^{o(|w|^2)}}{\Gamma(w^2) \log x} \left(1 + \frac{B|w|^2}{y}\right) \\ &+ \frac{B|w|^2 x m(|h_w|, x) t^{-1+o(|w|^2)}}{\log x} + o\left(\frac{x m(|h_w|, x) t^{|w|^2} R(y)}{\log x}\right) \end{aligned}$$

as $x \rightarrow \infty$.

Note that in applications we usually need that $w = w(x) \rightarrow 0$ as $x \rightarrow \infty$. The uniformity in w allows us to consider this case.

We will give a sketch of the proof only, the details will be given elsewhere.

First we observe that $h_w(m)$ is a multiplicative function, $|h_w(m)| \leq 1$, moreover, $h_w(p) = w^2 c_p^2$, where $c_p = c(p)p^{\frac{1-\kappa}{2}}$. Applying one Rankin's result [1], hence we obtain that

$$\sum_{p \leq x} h_w(p) = \frac{w^2 x}{\log x} (1 + o(1)), \quad x \rightarrow \infty. \quad (1)$$

Now we consider the mean values of $h_w(m)$ with large and small prime divisors. Let

$$S_1(x, y) = \sum_{m_1 \leq x} h_w(m_1),$$

where all prime divisors of m_1 are greater than y . Denote by $\rho(u)$, $u \geq 0$, a continuous solution of the difference – differential equation $u\rho'(u) = w^2\rho(u-1)$ with initial condition $\rho(u) = 1$ for $0 \leq u \leq 1$. Then the application of known methods and (1) shows that uniformly in w , x and y

$$S_1(x, y) = 1 + \frac{x\rho'(t)}{\log y} - w^2 \frac{y}{\log y} + \frac{B|w|^2 x \varepsilon(y) t^{|w|^2-1} m(|h_w|, x)}{\log y m(|h_w|, y)} + \frac{Bx \varepsilon(y) \log y t^{|w|^2-1}}{y}. \quad (2)$$

Denote by m_2 positive integers free of prime divisors greater than y , and let

$$S_2(x, y) = \sum_{m_2 > x} \frac{h_w(m_2)}{m_2}.$$

Then (1) implies that uniformly in w , x and y

$$S_2(x, y) = Bm(|h_w|, y) \exp\{-t \log t + B|w|^2 t\}. \quad (3)$$

Now define

$$S_3(x, y) = \sum_{m_2 \leq x} h_w(m_2).$$

Then, using the estimates

$$\sum_{\substack{p^\alpha + p^\beta \leq x \\ \alpha, \beta \geq 1}} |h_w(p^\alpha)| |h_w(p^\beta)| \log p^\alpha = B|w|^4 \sqrt{x},$$

$$\sum_{\substack{p^\alpha \leq x \\ \alpha \geq 2}} |h_w(p^\alpha)| \log p^\alpha = B|w|^2 \sqrt{x}, \quad x \rightarrow \infty,$$

we deduce that uniformly in w, x and y

$$S_3(x, y) = B|w|^2 xm(|h_w|, y) \exp\{-t \log t + B|w|^2 t\}. \quad (4)$$

Proof of the Theorem 1. First we observe that

$$\begin{aligned} \sum_{m \leq x} h_w(m) &= \sum_{m_2 \leq x} h_w(m_2) \sum_{m_1 \leq x/m_2} h_w(m_1) \\ &= \sum_{m_2 \leq x/y} h_w(m_2) \left(S_1\left(\frac{x}{m_2}, y\right) - 1 \right) + \sum_{m_2 \leq x} h_w(m_2). \end{aligned} \quad (5)$$

By (2) we obtain that the first term in the right-hand side of (5) is

$$\begin{aligned} &\frac{x}{\log y} \sum_{m_2 \leq x/y} \frac{h_w(m_2)}{m_2} \rho' \left(\frac{\log \frac{x}{m_2}}{\log y} \right) - \frac{w^2 y}{\log y} \sum_{m_2 \leq x/y} h_w(m_2) \\ &+ B \left(\frac{|w|^2 x \varepsilon(y) t^{|w|^2 - 1} m(|h_w|, x)}{\log y m(|h_w|, y)} + \frac{x \varepsilon(y) \log y t^{|w|^2 - 1}}{y} \right) \sum_{m_2 \leq x/y} \frac{|h_w(m_2)|}{m_2}. \end{aligned} \quad (6)$$

Since $\rho'(1) = w^2$, it is not difficult to see that the first term in (6) can be written in the form

$$\frac{x m(h_w, y)}{\log y} \rho'(t) - \frac{w^2 x}{\log y} \sum_{m_2 > x/y} \frac{h_w(m_2)}{m_2} - \frac{x}{\log y} \int_1^t \left(\sum_{m_2 > x/y^u} \frac{h_w(m_2)}{m_2} \right) \rho''(u) du.$$

Therefore, by (5) and (6)

$$\begin{aligned} \sum_{m \leq x} h_w(m) &= \frac{x m(|h_w|, y)}{\log y} \rho'(t) - \frac{w^2 x}{\log y} \sum_{m_2 > x/y} \frac{h_w(m_2)}{m_2} \\ &- \frac{x}{\log y} \int_1^t \left(\sum_{m_2 > x/y^u} \frac{h_w(m_2)}{m_2} \right) \rho''(u) du - \frac{w^2 y}{\log y} \sum_{m_2 \leq x/y} h_w(m_2) \\ &+ \sum_{m \leq x} h_w(m) + \frac{B|w|^2 x \varepsilon(y) t^{|w|^2 - 1} m(|h_w|, x)}{\log y} \\ &+ \frac{Bx \varepsilon(y) \log y t^{|w|^2 - 1} m(|h_w|, x)}{y}. \end{aligned} \quad (7)$$

From asymptotic properties of $\rho'(u)$ it follows that

$$\begin{aligned} \frac{x m(h_w, y)}{\log y} \rho'(t) &= \frac{e^{-\gamma_0 w^2} x m(h_w, x) t^{o(|w|^2)}}{\Gamma(w^2) \log x} \left(1 + \frac{B|w|^2}{y} \right) \\ &+ \frac{B|w|^2 x m(|h_w|, x) t^{-1+o(|w|^2)}}{\log x}, \end{aligned} \quad (8)$$

where $\Gamma(w^2)$ denotes the gamma function. All other terms in (7) must be estimated. In view of (3) and (4) we find that

$$\frac{w^2 x}{\log y} \sum_{m_2 > x/y} \frac{h_w(m_2)}{m_2} = \frac{B|w|^2 x m(|h_w|, x)}{\log x} \exp\{-c_1 t\}, \quad (9)$$

$$\sum_{m_2 \leq x} h_w(m_2) = \frac{B|w|^2 x m(|h_w|, x)}{\log x} \exp\{-c_2 t\}, \quad (10)$$

$$\frac{w^2 x}{\log y} \sum_{m_2 \leq x/y} h_w(m_2) = \frac{B|w|^2 x m(|h_w|, x)}{\log x} \exp\{-c_3 t\} \quad (11)$$

with some positive constants c_1 , c_2 and c_3 . Also, properties of $\rho(u)$ give the estimate

$$\frac{x}{\log y} \int_1^t \left(\sum_{m_2 > x/y^u} \frac{h_w(m_2)}{m_2} \right) \rho''(u) du = \frac{B|w|^2 x m(|h_w|, x) t^{-1}}{\log x}. \quad (12)$$

Moreover,

$$\frac{|w|^2 x \varepsilon(y) t^{|w|^2 - 1} m(|h_w|, x)}{\log y} = \frac{|w|^2 x \varepsilon(y) t^{|w|^2} m(|h_w|, x)}{\log y},$$

$$\frac{x \varepsilon(y) \log y t^{|w|^2 - 1} m(|h_w|, x)}{y} = \frac{x \varepsilon(y) \log^2 y t^{|w|^2} m(|h_w|, x)}{y \log x}.$$

This and (8)–(12) together with (7) prove the theorem.

References

- [1] R.A. Rankin, An Ω -result for the coefficients of cusp form, *Math. Annalen*, **203**, 239–250 (1973).

Parabolinių formų dzeta funkcijos

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Gauta parabolinių formų dzeta funkcijos laipsnių koeficientų vidurkių asimptotika.