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Weighted discrete limit theorems for general Dirichlet polynomials

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Let

$$p_n(t) = \sum_{m=1}^{n} a_m e^{i\lambda_m t},$$

$$q_n(s) = \sum_{m=1}^{n} a_m e^{-\lambda_m s}, \quad s = \sigma + it,$$

be a general Dirichlet polynomials with complex-valued coefficients a_m and real exponents λ_m. Zeta-function usually are approximated by Dirichlet polynomials, therefore limit theorems for these polynomials is the first step to obtain limit theorems for zeta-functions. Continuous limit theorems for Dirichlet polynomials an be found in [3]. In this case the weak convergence of probability measures

$$\frac{1}{T}\text{meas}\{t \in [0,T]\colon p_n(t) \in A\},$$

and

$$\frac{1}{T}\text{meas}\{\tau \in [0,T]\colon q_n(s+i\tau) \in A\},$$

on the complex plane and on the space of analytic functions is considered. Here meas$\{A\}$ denotes the Lebesque measure of the set A. In the case of discrete limit theorems the probability measures

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N\colon p_n(kh) \in A\}$$

and

$$\frac{1}{N+1}\#\{0 \leqslant k \leqslant N\colon q_n(s+ikh) \in A\},$$

where $h > 0$ is a fixed number, are studied.

In [4] discrete limit theorems for $p_n(t)$ and $g_n(s)$ were proved, and the explicit form of limit measures was given, see also [2]. The aim of this note is to obtain weighted discrete limit theorems for general Dirichlet polynomials.

Let $w(u)$ be a positive function of bounded variation on $[0, +\infty)$ and we set

$$U = U(N, w) = \sum_{m=0}^{N} w(m).$$

Suppose that $\lim_{N\to\infty} U(N, w) = \infty$. Moreover, let, for positive integer N,

$$\mu_N(...) = \sum_{\substack{m=0 \\ \cdots}} w(m),$$

where in place of dots a condition satisfied by m is to be written. We suppose, as in [4], that the exponents λ_m are real algebraic numbers linearly independent over the field of rational numbers, and that $h > 0$ be such that $\exp\left\{\frac{2\pi}{h}\right\}$ is a rational number. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and let $\mathbb{C}$ be the complex plane. We set

$$\Omega_n = \prod_{m=1}^{n} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C}: |s| = 1\}$ for all $m = 1, ..., n$. Define a function $v: \Omega_n \to \mathbb{C}$ by the formula

$$v(x_1, ..., x_n) = \sum_{m=1}^{n} a_m x_m, \quad (x_1, ..., x_n) \in \Omega_n,$$

and denote by m_{nH} the probability Haar measure on $(\Omega_n, \mathcal{B}(\Omega_n))$.

Theorem 1. *The probability measure*

$$P_N(A) = \mu_N(p_n(mh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the measure $m_{nH}v^{-1}$ *as* $N \to \infty$.

Let G be a region on $\mathbb{C}$. Denote by $H(G)$ the space of analytic on G functions with the topology of uniform convergence on compacta. Define a function $v_1: \Omega_n \to H(G)$ by the formula

$$v_1(x_1, ..., x_n) = \sum_{m=1}^{n} a_m e^{-\lambda_m s} x_m^{-1}, \quad (x_1, ..., x_n) \in \Omega_n.$$

Theorem 2. *The probability measure*

$$Q_N(A) = \mu_N(q_n(s + imh) \in A), \quad A \in \mathcal{B}(H(G)),$$

converges weakly to the measure $m_{nH}v_1^{-1}$ *as* $N \to \infty$.

We begin the proof of Theorems 1 and 2 with the following lemma.

Lemma 1. *The probability measure*

$$\mu_N((e^{i\lambda_1 mh}, ..., e^{i\lambda_n mh}) \in A), \quad A \in \mathcal{B}(\Omega_n),$$

converges weakly to the Haar measure m_{nH} *as* $N \to \infty$.

Proof. Denote by $g_N(k_1, ..., k_n)$, $(k_1, ..., k_n) \in \mathbb{Z}^n$, and $\mathbb{Z}$ is the set of all integers, i.e., [3]

$$g_N(k_1, ..., k_n) = \int_{\Omega_n} x_1^{k_1} ... x_n^{k_n} \, \mathrm{d}Q_n, \quad (x_1, ..., x_n) \in \Omega_n.$$

Then we have that

$$g_N(k_1, ..., k_n) = \frac{1}{U} \sum_{m=0}^{N} w(m) \exp\{imh \sum_{l=1}^{n} k_l \lambda_l\}. \tag{1}$$

If $(k_1, ..., k_n) = (0, ..., 0)$, clearly, $g_n(k_1, ..., k_n) = 1$. Now suppose that $(k_1, ..., k_n) \neq (0, ..., 0)$, and let

$$S_N(k_1, ..., k_n) = \sum_{m=0}^{N} \exp\{imh \sum_{l=1}^{n} k_l \lambda_l\}$$

Since the exponents λ_n are real algebraic numbers linearly independent over the field of rational numbers, we have [4]

$$S_N(k_1, ..., k_n) = \frac{1 - \exp\{i(N+1)h \sum_{l=1}^{n} k_l \lambda_l\}}{1 - \exp\{ih \sum_{l=1}^{n} k_l \lambda_l\}}.$$

Obviously, for all $u \geqslant 0$,

$$\frac{1 - \exp\{i(N+1)h \sum_{l=1}^{n} k_l \lambda_l\}}{1 - \exp\{ih \sum_{l=1}^{n} k_l \lambda_l\}} = B.$$

Where B denotes a quantity bounded by a constant. Hence, summing by parts and taking into account that $w(u)$ is a function of bounded variation, we find that

$$\sum_{m=0}^{N} w(m) \exp\{imh \sum_{l=1}^{n} k_l \lambda_l\} = w(N)S(N) - \int_0^N S(u) \, \mathrm{d}w(u) = B.$$

This and (1) shows that

$$\lim_{N\to\infty} g_N(k_1,...,k_n) = \begin{cases} 1, & (k_1,...,k_n) = (0,...,0), \\ 0, & (k_1,...,k_n) \neq (0,...,0). \end{cases}$$

Thus we obtained that the Fourier transform of the measure Q_N converges to the Fourier transform of the Haar measure on Ω_N as $N \to \infty$. Therefore, by Theorem 1. 3. 19 from [3], the measure Q_N weakly converges to $m_n H$ as $N \to \infty$. The lemma is proved.

Proof of Theorem 1. By the definition of the function v we have that

$$p_n(mh) = v(\mathrm{e}^{i\lambda_1 mh}, ..., \mathrm{e}^{i\lambda_n mh}),$$

moreover, the function v is continuous. Therefore, by Theorem 5.1 of [1] and Lemma 1 the measure of the theorem converges weakly to the measure $m_{nH}v^{-1}$ as $N \to \infty$.

Proof of Theorem 2. The definition of the function v_1 implies

$$q_n(s + imh) = v_1(\mathrm{e}^{i\lambda_1 mh}, ..., \mathrm{e}^{i\lambda_n mh}),$$

and the function v_1 is continuous. Therefore, using Lemma 1 again, we obtain the theorem.

Now let

$$p_n(t, g) = \sum_{m=1}^{n} a_m g(m)\mathrm{e}^{i\lambda_m t},$$

$$q_n(s, g) = \sum_{m=1}^{n} a_m g(m) e^{-\lambda_m s}.$$

where $g(m)$ is an arbitrary arithmetic function, $|g(m)| = 1$. Then we have the following statements.

Theorem 3. *The probability measures* P_N *and*

$$\widehat{P}_N(A) = \mu_N(p_n(mh, g) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

both converge weakly to the same limit measure, i.e., to $m_{nH}v^{-1}$ *as* $N \to \infty$.

Theorem 4. *The probability measures* Q_N *and*

$$\widehat{Q}_N(A) = \mu_N(q_n(s + imh, g) \in A), \quad A \in \mathcal{B}(H(G)),$$

both converge weakly to the same limit measure, i.e., to $m_{nH}v_1^{-1}$ *as* $N \to \infty$.

Proof of Theorems 3 and 4 uses the proofs of Theorems 1 and 2, respectively, and completely coincides with that of Theorems 3 and 4 in [4].

Note that if $w(u) \equiv 1$, then we obtain the theorems from [4].

References

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Diskrečios ribinės teoremos su svoriu bendriesiems Dirichlet polinomams

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Įrodytos diskrečios ribinės teoremos su svoriu bendriesiems Dirichlet polinomams silpno matų konvergavimo prasme. Pateiktas išreikštinis ribinių matų pavidalas. Gauti rezultatai apibendrina autorės teoremas, kai svorio funkcija $w(u) = 1$.