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Herbrand expansions of some formulas of modal logic S4

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We shall consider formulas of quantified modal logic. G.Mints described in [1] a reduction of an arbitrary formula F of quantified modal logic to a finite set of such formulas $G_1, ..., G_m$ that $\vdash F$ is derivable in *S4* if and only if $G_1, ..., G_m \vdash$ is derivable in S4. Moreover, the formula G_i $i = 1, 2, ..., m$ has one of the following forms:
$\Box\forall^*(L_1 \vee L_2)$, $\Box\forall^*(L_1 \vee L_2 \vee L_3)$, $\Box\forall^*\exists(L_1 \vee L_2)$, $\Box\forall^*(L_1 \vee \Box L_2)$,
$\Box\forall^*(L_1 \vee \Diamond L_2)$, L,
where $L_1, L_2, ...$ are the literals of classical logic and $\forall^*$ is a complex of universal qantifiers. In addition, the formulas can contain the constants. We assume for simplicity that the variables bounded by universal quantifiers are denoted by $x, x_1, x_2, ...$ and the variables bounded by existential quantifiers are denoted by $y, z, y_1, z_1, ...$

We will consider below the formulas of the more general form. We examine the sequents $F_1, ..., F_n \vdash$, in which F_i $i = 1, ..., n$ are the formulas having the following form

$$\forall^*\Box^i\forall^*\exists^*(A_1 \vee ... \vee A_s \vee \Box B_1 \vee ... \vee \Box B_u \vee \Diamond C_1 \vee ... \vee \Diamond C_v), \tag{1}$$

where $A_1, B_1, C_1, A_2, B_2, C_2, ...$ are literals of classical logic. $i = 0, 1$. $\Box^1 F = \Box F$. $\Box^0 = F$. In addition, the formulas (1) can contain the constants.

DEFINITION 1. The formulas of the form $L, \Box L, \Diamond L$, where L is a literal of classical logic, will be called modal literals.

We can define the sequents under consideration and in the other way. There are the sequents whose antecedents contain only one closed formula of the following form:

$$\exists^*(\forall^*\Box\forall^*\exists y_1...\exists y_p(\wedge_{i=1}^s D_i) \wedge \forall^*\exists z_1...\exists z_q(\wedge_{i=1}^j K_i)), \tag{2}$$

where D_i, K_i are the clauses of modal literals. Each individual variable y_i, z_i occurs only in one clause.

M. Fitting in [2] described the skolemization for the modal logic *K*. M.Cialdea in [3] presented another method of skolemization for the modal systems *D, T, S4*. The Herbrand properties are proved in [2], [3] for correponding modal logics. We will use below only the skolemization described in [3].

We shall present the Herbrand expansions for the formulas of the form (2). In this case we can obtain a simple method of presentation of Herbrand expansion. The quantifier-free formulas D_i, K_i in (2) have the following form

$$A_1 \vee ... \vee A_s \vee \Box B_1 \vee ... \vee \Box B_u \vee \Diamond C_1 \vee ... \vee \Diamond C_v \quad (3)$$

We shall use the notations as in [3]. We denote a set containing only one formula (2) by *S*.

DEFINITION 2. The expression (term, formula) will be called ground if it does not contain individual variables.

Theorem 1. *The set of Herbrand expansion of a formula* (2) *of the quantified modal logic S4 consists of ground formulas of the form* $F\sigma$, *where* F *is a formula of the form* (3) *and* σ *is a substitution of the ground terms.*

Proof. Since the modal degree $\mu(S)$ of a formula (2) is equal to number 2, a formula $S^{\mu(S)}$ has the form

$$\exists^k(\forall^{l_1}\Box^2\forall^r\exists^p(\wedge_{i=1}^s D_i') \wedge \forall^{l_2}\exists^b(\wedge_{i=1}^j K_i'))),$$

where D', K' are obtained from D_i, K_i replacing in D_i, K_i all occurrences of $\Box$ by $\Box^2$. The formulas S^+ has the form

$$\exists^k(\forall^{l_1}(\Box^2\forall^r\exists^p(\wedge_{i=1}^s D_i'') \wedge \Box\forall^r\exists^p(\wedge_{i=1}^s D_i'') \wedge \forall^r\exists^p(\wedge_{i=1}^s D_i'')) \wedge \forall^{l_2}\exists^q(\wedge_{i=1}^j K_i'')).$$

The clauses D_i'', K_i'' are of the form

$$A_1 \vee ... \vee A_s \vee (\Box^2 B_1 \wedge \Box B_1 \wedge B_1) \vee ... \vee (\Box^2 B_u \wedge \Box B_u \wedge B_u) \vee \Diamond C_1 \vee ... \vee \Diamond C_v . \quad (4)$$

We bring all quantifiers in front of formula.

$$\exists^k\forall x_1 ... \forall x_{l+r}\exists^h((\Box^2\forall x'_{l+r+1} ... \forall x'_{l+2r}\exists^p(\wedge_{i=1}^s D_i^1)$$
$$\wedge(\Box\forall x''_{l+r+1} ... \forall x''_{l+2r}\exists^p(\wedge_{i=1}^s D_i^2) \wedge (\wedge_{i=1}^s D_i^3) \wedge (\wedge_{i=1}^j K_i^1)).$$

$l = \max\{l_1, l_2\}$, $h = p + q$. The clauses D_i^1, D_i^2, D_i^3 can differ from D_i'' only on that the some individual variables are renamed. We obtain K_i^1 in a similar manner.

Therefore we may successively eliminate all occurrences of the existential quantifiers. For this reason we delete $\exists^k$ and we replace the occurrences of corresponding individual variables by new constants $a_1^0, a_2^0, ..., a_k^0$. The modal degrees of the constants are equal to 0. We delete $\exists^k$ and we change the occurrences of corresponding individual variables by the new $l + r$-place functional symbols $f_1^0(x_1, ..., x_{l+r}), ..., f_h^0(x_1, ..., x_{l+r})$. the modal degree of functional symbols is 0. The degree will be denoted by a superscript of the functional symbol. A term f_i^0 occurs only in one subformula of type (4). Let us the variables

$x_{j_1}, ..., x_{j_n}$ form a complete list of individual variables occurring in this subformula. We introduce a new functional symbol $g_i^0(x_{j_1}, ..., x_{j_n})(i = 1, ..., h)$.

We delete similarly two occurrences of $\exists^p$ and we change the occurrences of corresponding individual variables by the new $l + 2r$-place functional symbols of modal degree 2 and of modal degree 1:

$$f_1^2(x_1, ..., x_{l+r}, x'_{l+r+1}, ..., x'_{l+2r}), ..., \; f_p^2(x_1, ..., x_{l+r}, x'_{l+r+1}, ..., x'_{l+2r}),$$
$$f_1^1(x_1, ..., x_{l+r}, x''_{l+r+1}, ..., x''_{l+2r}), ..., \; f_p^1(x_1, ..., x_{l+r}, x''_{l+r+1}, ..., x''_{l+2r}).$$

We introduce similarly and the new functional symbols $g_i^1(x_{j_1}, ..., x_{j_n}), g_i^2(x_{j_1}, ..., x_{j_n})$. We replace all occurrences of variables x_i, respectively, x'_i and x''_i by the variables x_i^0, respectively, x_i^2 and x_i^1. We delete the complexes of quantifier beginning with $\forall$. The obtained formula is denoted by

$$F(x_1^0, ..., x_{l+r}^0, x_{l+r+1}^1, ..., x_{l+2r}^1, x_{l+r+1}^2, ..., x_{l+2r}^2).$$

Using the obtained terms we introduce a set of ground terms. We denote by H_j^i $(i = 0, 1, 2;\ j \geqslant 0)$ a set of ground terms in which the modal degree of terms does not exceed i and the height of terms does not exceed j. In addition, we assume that a set H_0^i contains a new constant a of modal degree 0 not occurring in a initial formula, that is, the constant a differs from $a_1^0, ..., a_k^0$. A set $\cup_{j=0}^{\infty} H_j^i$ will be denoted by H^i.

Let us fix a natural number j. The propositional Herbrand j-th expansion of a formula $F(x_1^0, ..., x_{l+2r}^2)$ is a conjunction of formulas of propositional modal logic $M = \wedge_{t_1, ..., t''_{l+2r}} F(t_1, ..., t''_{l+2r})$, where $t_1, ..., t_{l+r} \in H_j^0$, $t'_{l+r+1}, ..., t'_{l+2r} \in H_j^1$, $t''_{l+r+1}, ..., t''_{l+2r} \in H_j^2$.

According to Theorem 3 of [3], a formula (2) of quantifier modal logic is *S4*-contradictory if and only if for some j a propositional formula M is *S4*-contradictory. This means that a sequent $M \vdash$ is derivable in *S4*. We replace a formula M by a sequence of propositional formulas. For this reason we change:

1) the formulas of type (4) by $\wedge_{(0 \leqslant j_1, ..., j_u \leqslant 2)}(A_1 \vee ... \vee A_s \vee \Box^{j_1} B_1 \vee ... \vee \Box^{j_u} B_u \vee \Diamond C_1 \vee ... \vee \Diamond C_v)$,
2) the formulas of type $\Box\Box F$ by $\Box F$,
3) the formulas of type $\Box(A \wedge B)$ by $\Box A$ and $\Box B$,
4) all conjunctions by the comma,
5) the terms f_j^i by g_j^i.

The obtained formula is derivable in *S4* if and only if $M \vdash$ is derivable in *S4*.

Herbrand expansion can be defined using only an initial formula. First of all, to every constant occurring in (1) assign a modal degree 0. We assume that there not two bound variables with the same name in a set of formulas under consideration. We eliminate in (1) all occurrences of existential quantifier. After the elimination we obtain three formulas (if $i = 1$) or one formula (if $i = 0$).

Let $\exists^*$ be $\exists y_1 ... \exists y_k$. We delete $\exists^*$ and we replace in the matrix all occurrences of y_j $(j = 1, ..., k)$ by $f_{y_j}^r(x_1, ..., x_n)$, where $x_1, .., x_n$ is a complete list of universal individual variables occurring in a considered formula. If $i = 1$ in (1), then we construct three formulas with $r = 0, r = 1$ and $r = 2$. In addition, in the case $r = 0$ we delete the occurences of modal operator $\Box^i$. If $i = 0$, then $r = 0$ also. The obtained set of formulas will be called *a set of initial formulas.*

We can obtain all propositional formulas from *j*-th expansion using only two substitution rules. We apply the rules of substitution only to formulas of a set of initial formulas.

Substitution rules:

$$(S1) \qquad \frac{\forall x_1 ... \forall x_m \Box \forall x_{m+1} ... \forall x_n F(x_1, ..., x_m, x_{m+1}, .., x_n)}{\Box F(t_1, ..., t_m, t_{m+1}, .t_n)}$$

a) if F contains a functional symbol of modal degree $i = 1, 2$, then $t_1, .., t_m \in H_j^0; t_{m+1}, ..., t_n \in H_j^i$,

b) if F does not contain the functional symbols, then $t_1, ..., t_m \in H_j^0; t_{m+1}, ..., t_n \in H_j^2$.

$$(S2) \qquad \frac{\forall x_1 ... \forall x_n F(x_1, ..., x_n)}{F(t_1, ..., t_n)}$$

$t_1, ..., t_n \in H_j^0$.

Remark. A formula F in the rules (S1), (S2) is a quantifier-free formula.

Theorem is proved.

References

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Herbrand skleidinys vienai modalumo logikos S4 formulių klasei

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Darbe aprašyta viena kvantorinės modalumo logikos *S4* formulių klasė, kurios Herbrand skleidinį galima gauti iš atitinkamų skulemizuotų formulių naudojant tik keitinius.