Liet. matem. rink., T. 43, spec. nr., 525-531

Convergence of the residuals based empirical characteristic functions

Vytautas MANIUŠIS, Alfredas RAČKAUSKAS (VU)
e-mail: alfredas.rackauskas@maf.vu.lt

1. Introduction

We consider the autoregressive order p ($\mathrm{AR}(p)$) process

$$X_k = \rho_1 X_{k-1} + \rho_2 X_{k-2} + \cdots + \rho_p X_{k-p} + \varepsilon_k, \tag{1}$$

where (ε_k) is a sequence of independent identically distributed (iid) random variables with zero mean. We assume that $\rho_p \neq 0$ and the roots of the polynomial $t^p - \rho_1 t^{p-1} - \cdots - \rho_p$ are less than one in absolute value. Hence the sequence (X_k) is stationary. Assume we observe data $X_{-p+1}, \ldots, X_N$. Let $\widehat{\rho}_k$ be an estimate of the coefficients ρ_k, $k = 1, \ldots, p$, based on observations $(X_k,\ -p+1 \leqslant k \leqslant N)$. The residuals are then defined by

$$\widehat{\varepsilon}_k = X_k - \widehat{\rho}_1 X_{k-1} - \widehat{\rho}_2 X_{k-2} - \cdots - \widehat{\rho}_p X_{k-p}, \quad 1 \leqslant k \leqslant N.$$

The empirical characteristic function (ECF) c_N based on ε_k is defined by $c_N(t) = N^{-1}\sum_{k=1}^{N} \exp\{it\varepsilon_k\},\ t \in \mathbb{R}$. The ECF $\widehat{c}_N$ based on residuals $\widehat{\varepsilon}_k$ is defined in the same manner.

A rich motivation to study the asymptotic behavior of the ECF's in certain functional framework is found in, e.g., [3]. In [7] ECF's of iid random variables are considered in a framework of Hölder function spaces. The present contribution extends the initial results of [7] to the setting of residuals which are not iid even for iid noise (ε_k). The paper is organized as follows. In Section 2 we study the convergence of $\widehat{c}_N$ with respect to the Hölder topology. As a corollary we obtain a limiting distribution for the large class of statistics, that are used in Section 3 to test conditional symmetry in $\mathrm{AR}(p)$ models.

2. Asymptotic results

The Hölder space $\mathcal{H}_\alpha^o[a,b],\ 0 < \alpha < 1$, consists of complex continuous functions x: $[a,b] \to \mathbb{C}$ such that $\lim_{\delta\to 0} \omega_\alpha(x,\delta) = 0$, where

$$\omega_\alpha(x,\delta) = \sup_{t,s\in[a,b],\ 0<|t-s|<\delta} \frac{|x(t)-x(s)|}{|t-s|^\alpha}.$$

The set $\mathcal{H}_\alpha^o[a,b]$ is a separable Banach space when endowed with the norm $\|x\|_{\alpha,[a,b]} = |x(a)| + \omega_\alpha(x,1)$. We shall write $\|x\|_\alpha$ for $\|x\|_{\alpha,[0,1]}$.

Theorem 1. *Assume that* $E|\varepsilon_0|^{1+\beta} < \infty$, $0 < \beta < 1$ *and*

$$\max_{1\leqslant i\leqslant p} \sqrt{N}|\widehat{\rho}_i - \rho_i| = O_P(1), \quad \textit{as } N \to \infty. \tag{2}$$

Then for all $a, b \in \mathbb{R}$ *and for all* α *such that* $0 < \alpha < \beta$,

$$\sqrt{N}\|\widehat{c}_N - c_N\|_{\alpha,[a,b]} = o_P(1), \quad \textit{as } N \to \infty.$$

Proof. Without loss of generality we take $[a,b] = [0,1]$. Set $V_k = (\rho_1 - \widehat{\rho}_1)X_{k-1} + \cdots + (\rho_p - \widehat{\rho}_p)X_{k-p}$, $k = 1,\ldots,N$. Since $\widehat{\varepsilon}_k = X_k - \widehat{\rho}_1 X_{k-1} - \cdots - \widehat{\rho}_p X_{k-p} = \varepsilon_k + V_k$, $k = 1,\ldots,N$, we have $\widehat{c}_N(t) = c_N(t) + R_N(t)$, where

$$R_N(t) = N^{-1}\sum_{k=1}^N \exp\{it\varepsilon_k\}\big[\exp\{itV_k\} - 1\big], \quad t \in \mathbb{R}.$$

Hence, the proof of the theorem reduces to showing that

$$\|\sqrt{N}R_N\|_\alpha \xrightarrow{P} 0. \tag{3}$$

Write for $t \in [0,1]$ $R_N(t) = R_{N1}(t) + itR_{N2}(t)$, where

$$R_{N1}(t) = N^{-1}\sum_{k=1}^N \exp\{it\varepsilon_k\}\big(\exp\{itV_k\} - 1 - itV_k\big)$$

and

$$R_{N2}(t) = N^{-1}\sum_{k=1}^N \exp\{it\varepsilon_k\}V_k.$$

Interpolating the inequalities $|\mathrm{e}^{ix} - 1| \leqslant |x|$ and $|\mathrm{e}^{ix} - 1 - ix| \leqslant 2^{-1}|x|^2$ which are valid for each real x, we obtain $|\mathrm{e}^{itV_k} - 1 - itV_k| \leqslant |t|^{1+\beta}|V_k|^{1+\beta} \leqslant |V_k|^{1+\beta}$ for each $0 < \beta \leqslant 1$ and $t \in [0,1]$. Applying this inequality with $0 < \beta \leqslant 1$ we obtain

$$\begin{aligned}
\|\sqrt{N}R_{N1}\|_\alpha &\leqslant N^{-1/2}\sum_{k=1}^N |V_k|^{1+\beta} \\
&= N^{-1/2}\sum_{k=1}^N \big|(\rho_1 - \widehat{\rho}_1)X_{k-1} + \cdots + (\rho_p - \widehat{\rho}_p)X_{k-p}\big|^{1+\beta} \\
&\leqslant p \max_{j=1,\ldots,p} |\sqrt{N}(\rho_j - \widehat{\rho}_j)|^{1+\beta} N^{-1-\beta/2}\sum_{k=1}^N\sum_{j=1}^p |X_{k-j}|^{1+\beta}.
\end{aligned} \tag{4}$$

It is well known (see, e.g., [5]), that there is a sequence of i.i.d. random variables $(\eta_k, k \in \mathbb{Z})$ such that $X_k = \sum_{j=0}^{\infty} a_j \eta_{k-j}$. Moreover, η_k and ε_0 have the same distributions, and there exists two constants $a > 0$ and $0 < b < 1$ such that $|a_k| \leqslant ab^k$, $0 \leqslant k < \infty$. By this it follows for each $k > 1$

$$\begin{aligned} E|X_{k-1}|^{1+\beta} &= E\Big|\sum_{j=0}^{\infty} a_j \eta_{k-j}\Big|^{1+\beta} \\ &\leqslant C \sum_{j=0}^{\infty} E|a_j \eta_{k-j}|^{1+\beta} \leqslant CE|\varepsilon_0|^{1+\beta} \sum_{j=0}^{\infty} |a_j|^{1+\beta} \end{aligned}$$

and we have by (4) that $\|\sqrt{N} R_{N1}\|_\alpha \xrightarrow{P} 0$ with any $0 < \beta \leqslant 1$. Now the proof of (3) reduces to

$$\|\sqrt{N} R_{N2}\|_\alpha \xrightarrow{P} 0. \tag{5}$$

By the definition of V_k we have

$$\sqrt{N} R_{N2} = \sqrt{N}(\rho_1 - \widehat{\rho}_1) r_{N1} + \cdots + \sqrt{N}(\rho_p - \widehat{\rho}_p) r_{Np},$$

where

$$r_{Nv}(t) = N^{-1} \sum_{k=1}^{N} \exp\{it\varepsilon_k\} X_{k-v}, \quad t \in \mathbb{R}.$$

Due to condition (2) it sufficies to prove for each $v = 1, \dots, p$

$$\|r_{Nv}\|_\alpha = o_P(1). \tag{6}$$

For this purpose we shall use an equivalent sequantial norm on $\mathcal{H}_\alpha^o[0,1]$.

For any function x: $[0,1] \to \mathbb{C}$, the second differences are defined by

$$\Delta_h^2 x(t) := x(t+h) + x(t-h) - 2x(t), \quad t, t+h \in [0,1].$$

Denote by $U_j := \{t_{j,k}, 0 \leqslant k < 2^{j-1}\}$ the set of dyadic points of level j, where $t_{j,k} := (2k+1)2^{-j}$, and define the coefficients $\lambda_{j,k}$ by $\lambda_{0,0}(x) = x(0)$, $\lambda_{0,1}(x) = x(1)$ and for $j \geqslant 1$,

$$\lambda_{j,k}(x) = -\frac{1}{2}\Delta_h^2 x(t_{j,k}), \quad 0 \leqslant k < 2^{j-1}, \quad h = 2^{-j}.$$

The sequential norm on $\mathcal{H}_\alpha^o[0,1]$ is defined by

$$\|x\|_\alpha^{\mathrm{seq}} := \sup_{j \geqslant 0} 2^{\alpha j} \max_{0 \leqslant k < 2^{j-1}} |\lambda_{j,k}(x)|. \tag{7}$$

The norm $\|x\|_\alpha$ is equivalent to the sequential norm (see [8]), i.e., there are positive constants a, b such that for every $x \in \mathcal{H}^o_\alpha[0,1]$, $a\|x\|_\alpha \leqslant \|x\|^{\mathrm{seq}}_\alpha \leqslant b\|x\|_\alpha$. Since

$$\begin{aligned}\lambda_{j,k}(r_{Nv}) &= -\frac{1}{2}\Big(r_{Nv}(t_{j,k}+h) + r_{Nv}(t_{j,k}-h) - 2r_{Nv}(t_{j,k})\Big)\\ &= -\frac{1}{2}N^{-1}\Big(\sum_{l=1}^{N} X_{l-v}(-4)\exp\{it_{j,k}\varepsilon_l\}\sin^2(h\varepsilon_l)\Big),\end{aligned}$$

using the equivalent sequential norm (7) and noting that ε_l does not depend on X_{l-v} for $v \geqslant 1$ we have, with $1 < q \leqslant 2$,

$$\begin{aligned}E\|r_{Nv}\|^q_\alpha &= E\Big(\sup_{j\geqslant 0} 2^{q\alpha j}\max_{0\leqslant k<2^{j-1}}|\lambda_{j,k}(r_{Nv})|^q\Big)\\ &\leqslant CN^{-q}\sum_{j=0}^{\infty}2^{q\alpha j}\sum_{k=0}^{2^{j-2}}E\Big|\sum_{l=1}^{N}\exp\{it_{j,k}\varepsilon_l\}\sin^2(2^{-j}\varepsilon_l)X_{l-v}\Big|^q\\ &\leqslant CN^{1-q}E|X_1|^q\sum_{j=0}^{\infty}2^{q\alpha j+j}E|\sin(2^{-j}\varepsilon_0)|^{2q}\\ &\leqslant CN^{1-q}E|X_1|^qE|\varepsilon_0|^{2\gamma q}\sum_{j=0}^{\infty}2^{-(2\gamma q-q\alpha-1)j}\end{aligned}$$

for any γ, $0 < \gamma \leqslant 1$, and (6) follows by an appropriate choice of $q \in (1, \min\{1+\beta, \beta/\alpha\}$ and $\gamma = (1+\beta)/2q$.

It is well-known that condition (2) in Theorem 1 is satisfied, if $\widehat{\rho}_k$ is the least squares estimate and $E\varepsilon_0^4 < \infty$ (see, e.g., [6], Lemma 2.1).

3. Testing for conditional symmetry

As an application of the Theorem 1, tests for conditional symmetry in AR(p) model may be considered. A rich motivation to testing conditional symmetry may be found in [1]. Distribution of X_k conditional on X_{k-1} is symmetric with respect to its conditional mean $\mu_k = \mathbf{E}\,(X_k|X_{k-1})$, if $F_k(x+\mu_k|X_{k-1}) = 1 - F_k(-x+\mu_k|X_{k-1})$ or $f_k(x+\mu_k|X_{k-1}) = f_k(-x+\mu_k|X_{k-1})$, where F_k and f_k are the conditional cumulative distribution and probability density functions of X_k respectively, with respect to X_{k-1}. In the case of AR(p) model (1), conditional symmetry is equivalent to the symmetry of ε_0 about the origin or in terms of characteristic functions to $c(t) = c(-t)$ or $\operatorname{Im} c(t) = 0$ for all $t \in \mathbb{R}$, where $c(t) = \mathbf{E}\,\exp\{it\varepsilon_0\}$. We will use the last observation to construct a class of statistics. Consider

$$\widehat{T}_N(q) = \int_{\mathbb{R}} |\operatorname{Im}\widehat{c}_N(t)|^2 q(t)\,\mathrm{d}t, \tag{8}$$

where $\operatorname{Im}\widehat{c}_N(t) = N^{-1}\sum_{k=1}^{N}\sin(t\widehat{\varepsilon}_k)$, $q(t)\colon \mathbb{R}\to\mathbb{R}$ is a nonnegative function.

Theorem 2. *Assume that ε_0 is symmetric about the origin, conditions of Theorem 1 are satisfied and*

$$\int_{\mathbb{R}} \min(1, |t|^{\alpha}) q(t)\,\mathrm{d}t < \infty \quad \textit{for all } \alpha \textit{ such that } 0 < \alpha < \beta. \tag{9}$$

Then $N\widehat{T}_N(q) \xrightarrow{\mathcal{D}} T(q) = \int_{\mathbb{R}} |a(t)|^2 q(t)\,\mathrm{d}t$, where $a(t) = \int_{\mathbb{R}} \sin(tx)\,\mathrm{d}W(F(x))$, $W(t)$ is a standard Wiener process and $F(x)$ denotes a cumulative distribution function of ε_0.

Proof. First let us observe, that

$$\sum_{j=1}^{\infty} 2^{\alpha j}\sqrt{j}\Big(\mathbf{E}\sin^4(2^{-j-1}\varepsilon_0)\Big)^{1/2} < \infty, \tag{10}$$

when $\alpha < \beta$. As shown in [7], under conditions (9) and (10) and symmetry of ε_0, $NT_N(q) \xrightarrow{\mathcal{D}} T(q)$, where $T_N(q) = \int_{\mathbb{R}} |\operatorname{Im} c_N(t)|^2 q(t)\,\mathrm{d}t$. It can be shown that for any $K \geqslant 1$

$$\begin{aligned} N|\widehat{T}_N(q) - T_N(q)| &\leqslant C\Big(\sqrt{N}\|\widehat{c}_N - c_N\|_{\alpha,[-K,K]}\Big)^2 \\ &\quad + C\sqrt{N}\|\widehat{c}_N - c_N\|_{\alpha,[-K,K]}\sqrt{N}\|c_N - c\|_{\alpha,[-K,K]} + NC_K, \end{aligned}$$

where $C_K \to 0$, as $K \to \infty$. Hence, $N|\widehat{T}_N(q) - T_N(q)| = o_P(1)$, as $N \to \infty$. The result then follows by Theorem 1 and Theorem 10 in [7].

Theorem 3. *If ε_0 is asymmetric about the origin, then*

$$\liminf_{N\to\infty} N\widehat{T}_N(q) = \infty$$

almost surely.

Proof. Proof is similar to that of Theorem 5.1 in [4].

If we take $q(t) = q_\gamma(t) = |t|^{-1-\gamma}$, $0 < \gamma < 1$, then by simple calculations $\widehat{T}_N(q_\gamma) = c_\gamma \widehat{T}_{N,\gamma}$, where

$$\widehat{T}_{N,\gamma} = N^{-2}\sum_{j,k=1}^{N}\Big(|\widehat{\varepsilon}_j + \widehat{\varepsilon}_k|^{\gamma} - |\widehat{\varepsilon}_j - \widehat{\varepsilon}_k|^{\gamma}\Big), \tag{11}$$

$$c_\gamma = \int_{\mathbb{R}} \sin^2(u/2)|u|^{-1-\gamma}\,\mathrm{d}u. \tag{12}$$

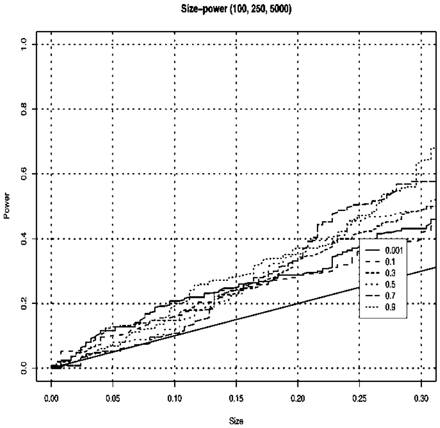

Fig. 1. $\widehat{T}_{N,\gamma}$ family.

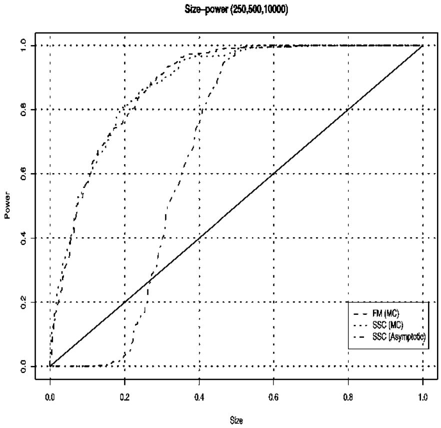

Fig. 2. $\widehat{T}_{N,0.5}$ and π tests.

Theorem 4. *If ε_0 is symmetric about the origin and conditions of Theorem 1 are satisfied, then for all γ such that $0 < \gamma < \beta$*

$$N\widehat{T}_{N,\gamma} \xrightarrow{\mathcal{D}} T_\gamma,$$

where

$$T_\gamma = \iint\limits_{\mathbb{R}^2} \left(|x+y|^\gamma - |x-y|^\gamma\right) \mathrm{d}W(F(x))\,\mathrm{d}W(F(y)). \tag{13}$$

Proof. We have $T(q_\gamma) = c_\gamma T_\gamma$. Integral (12) converges, when $0 < \gamma < 2$. The result then follows by Theorem 2.

A limited simulation study of the $\widehat{T}_{N,\gamma}$ tests using small samples ($N = 100$) was conducted for the $\mathrm{AR}(1)$ model ($\rho_1 = 0.9$). Fig. 1 shows size-power plots (see [2]) for various γ values ($\gamma = 0.001, 0.1, 0.3, 0.5, 0.7, 0.9$) based on 250 simulations of the test statistic $\widehat{T}_{N,\gamma}$ with 5000 Monte Carlo replications for each simulation of $\varepsilon_k \sim \mathcal{N}(0,1)$ under H_0 and $\varepsilon_k \sim \mathcal{N}(0, 0.25)$ under H_1. Fig. 2 compares properties of our test ($\gamma = 0.5$) with that of the π-test based on sample skewness coefficient (see [1], $N = 250$, 500 simulations and 10000 Monte Carlo replicates). The size-power curve of the asymptotic π-test (one can see its poor performance) also is plotted on Fig. 2.

References

[1] J. Bai, S. Ng, A consistent test for conditional symmetry in time series models, *J. of Econometrics*, **103**, 225–258 (2001).

[2] R. Davidson, J.G. MacKinnon, Graphical methods for investigating the size and power of test statistics, *The Manchester School*, **66**, 1–26 (1998).

[3] A. Feuerverger, R.A. Mureika, The empirical characteristic function and its applications, *Annals of Statistics*, **5**(1), 88–97 (1977).
[4] N. Henze, B. Klar, S.G. Meintanis, Invariant tests for symmetry about an unspecified point based on the empirical characteristic function, to appear in: *J. Multivariate Anal.* (2003).
[5] L. Horváth, Change in autoregressive processes, *Stochastic Process. Appl.*, **44**, 221–242 (1993).
[6] R.J. Kulperger, On the residuals of autoregressive processes and polynomial regression, *Stochastic Process. Appl.*, **21**, 107–118 (1985).
[7] A. Račkauskas, Ch. Suquet, Hölder convergences of multivariate empirical characteristic functions, *Pub. IRMA Lille*, **54-VI** (2001a).
[8] A. Račkauskas, Ch. Suquet, Hölder versions of Banach spaces valued random fields, *Georgian Math. J.*, **8**(2), 347–362 (2001b).

Liekanų empirinės charakteristinės funkcijos konvergavimas

V. Maniušis, A. Račkauskas

Ištirtas AR(p) modelio regresijos liekanų empirinio charakteristinio proceso konvergavimas Hiolderio erdvėse. Rezultatai pritaikyti autoregresijos sąlyginiam simetriškumui tikrinti.