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The support of one random element

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In [2] a joint limit theorem for general Dirichlet series was obtained. Let $s = \sigma + it$ be a complex variable, and let, for $\sigma > \sigma_{aj}$, $f_j(s) = \sum\limits_{m=1}^{\infty} a_{mj} e^{-\lambda_{mj} s}$, $j = 1, .., n$. Here $\{a_{mj}\}$ is a sequence of complex numbers, and $\{\lambda_{mj}\}$ is an increasing sequence of real numbers, $\lim\limits_{m\to\infty} \lambda_{mj} = +\infty$, $j = 1, .., n$. We assume that the function $f_j(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_{1j}$, $\sigma_{1j} < \sigma_{aj}$, $j = 1, .., n$, and all poles in this region are included in a compact set. We also require that, for $\sigma > \sigma_{1j}$, the estimates

$$f_j(s) = O(|t|^{\delta_j}), \quad |t| \geqslant t_0, \quad \delta_j > 0, \tag{1}$$

and

$$\int_{-T}^{T} |f_j(\sigma + it)|^2 dt = O(T), \quad T \to \infty, \tag{2}$$

hold, where $j = 1, ..., n$. Moreover, we assume that

$$\lambda_{mj} \geqslant c_j (\log m)^{\theta_j} \tag{3}$$

with some positive constants c_j and θ_j, $j = 1, ..., n$. Denote by $H(D_j)$, $D_j = \{s \in \mathbb{C}: \sigma > \sigma_{1j}\}$, $j = 1, ..., n$, the space of analytic on D_j functions equipped with the topology of uniform convergence on compacta, and let $H_n = H_n(D_1, ..., D_n) = H(D_1) \times ... \times H(D_n)$. Let $\Omega = \prod\limits_{m=1}^{\infty} \gamma_m$, where $\gamma_m = \{s \in \mathbb{C}: |s| = 1\}$ for all $m \in \mathbb{N}$. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Then we have a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ where m_H is the Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m. Now define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ an H_n-valued random element $F(s_1, ..., s_n; \omega)$ by the formula

$$F(s_1, ..., s_n; \omega) = \big(f_1(s_1, \omega), ..., f_n(s_n, \omega)\big),$$

where

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj} \omega(m) e^{-\lambda_{mj} s_j}, \quad s_j \in D_j, \quad j = 1, ..., n.$$

Let $M(D_j)$ stand for the space of meromorphic on D_j functions equipped with the topology of uniform convergence on compacta, and $M_n = M_n(D_1, \ldots, D_n) = M(D_1) \times \ldots \times M(D_n)$. In [2] the following limit theorem on M_n was proved.

Theorem 1. *For $j = 1, \ldots, n$, suppose that the sets $\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$ are linearly independent over the field of rational numbers, and that for $f_j(s)$ the conditions* (1)–(3) *are satisfied. Then the probability measure*

$$\frac{1}{T} meas\{\tau \in [0,T]: \big(f_1(s_1 + i\tau), \ldots, f_n(s_n + i\tau)\big) \in A\}, \quad A \in \mathcal{B}(M_n),$$

converges weakly to the distribution P_F of the random element $F(s_1, \ldots, s_n, \omega)$.

For applications it is useful to know the support of the measure P_F, and the present note is devoted to this problem.

We need some additional conditions on the sequences $\{a_{mj}\}$ and $\{\lambda_{mj}\}$. We suppose that $\lambda_{mj} = \lambda_m, j = 1, \ldots, n$, and let $c_{mj} = a_{mj} e^{-\lambda_m \sigma_{aj}}, j = 1, \ldots, n$. Assume that there exist $r \geqslant n$ sets $\mathbb{N}_k, \mathbb{N}_{k_1} \bigcap \mathbb{N}_{k_2} = \varnothing$ for $k_1 \neq k_2, \mathbb{N} = \bigcup_{k=1}^{r} \mathbb{N}_k$, such that $c_{mj} = b_{kj}$ for $m \in \mathbb{N}_k, k = 1, \ldots, r, j = 1, \ldots, n$. We set

$$\mathcal{B} = \begin{pmatrix} b_{11}, & \ldots, & b_{1n} \\ \ldots, & \ldots, & \ldots \\ b_{r1}, & \ldots, & b_{rn} \end{pmatrix}.$$

We also suppose that the sequence of exponents $\{\lambda_m\}$ satisfies the relation

$$r(x) = \sum_{\lambda_m \leqslant x} 1 = C_1 x^{\kappa} + B, \tag{4}$$

with $\kappa \geqslant 1$ and some positive C_1, and $|B| \leqslant C_2$, and that

$$\sum_{\substack{\lambda_m \leqslant x \\ m \in \mathbb{N}_k}} 1 = \kappa_k r(x)(1 + o(1)), \quad x \to \infty, \tag{5}$$

with positive $\kappa_k, \ k = 1, \ldots, r$.

Let N be an arbitrary positive number, and let $D_{j,N} = \{s \in \mathbb{C}: \sigma_{1j} < \sigma < \sigma_{aj}, \ |t| < N\}, \ j = 1, \ldots, n$. Denote by $\widehat{F}(s_1, \ldots, s_n; \omega)$ the restriction of $F(s_1, \ldots, s_n, \omega)$ to $H_{n,N} = H(D_{1,N}) \times \ldots \times H(D_{n,N})$.

Theorem 2. *Suppose that conditions* (1), (2) *and* (4), (5) *are satisfied, and that $rank(\mathcal{B}) = n$. Then the support of the random element $\widehat{F}(s_1, \ldots, s_n; \omega)$ is the whole of $H_{n,N}$.*

We begin with one lemma on functions of exponential type. Let ν be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in $\{s \in \mathbb{C}\colon \min_{1\leqslant j\leqslant n}(\sigma_{1j}-\sigma_{aj}) < \sigma < 0,\ |t| < N\}$. Define

$$w(z) = \int_{\mathbb{C}} \mathrm{e}^{-sz}\mathrm{d}\nu(s), \quad z \in \mathbb{C}.$$

Lemma 1. *Suppose that, for some k,*

$$\sum_{\substack{m=1\\ m\in\mathbb{N}_k}}^{\infty} |w(\lambda_m)| < \infty.$$

Then $\int_{\mathbb{C}} s^l \mathrm{d}\nu(s) = 0, \quad l = 0, 1, 2, \ldots.$

Proof . Since, for $x > 0$,

$$|w(\pm x)| \leqslant \mathrm{e}^{Nx}\int_{\mathbb{C}} |\mathrm{d}\nu(s)|,$$

condition 1^0 of Theorem 6.4.12 from [1] is satisfied with $\alpha = N$. Now we fix β satisfying $0 < \beta < \frac{\pi}{N}$, and a real number ξ with

$$C_1\beta\xi > C_2, \tag{6}$$

where the constants C_1 and C_2 are given by formula (4). Define $A = \{l \in \mathbb{N}\colon \exists r \in [(l-\xi)\beta, (l+\xi)\beta]$ with $|w(r)| \leqslant r^{-\kappa}\}$. Then we have

$$\sum_{\substack{m=1\\ m\in\mathbb{N}_k}}^{\infty} |w(\lambda_m)| \geqslant \sum_{l\notin A}\sum_{m}{}' |w(\lambda_m)| \geqslant \sum_{l\notin A}\sum_{m}{}' \lambda_m^{-\kappa},$$

where $\sum_m{}'$ denotes the sum over all $m \in \mathbb{N}_k$ satisfying the inequalities $(l-\xi)\beta < \lambda \leqslant (l+\xi)\beta$. This and the hypothesis of the lemma yield

$$\sum_{l\notin A}\sum_{m\in\mathbb{N}_k} \lambda_m^{-\kappa} < \infty \tag{7}$$

with $a = (l-\xi)\beta, b = (l+\xi)\beta$. Summing by parts and applying (4) and (5), we find

$$\sum_{\substack{m\in\mathbb{N}_k\\ a<\lambda_m\leqslant b}} \lambda_m^{-\kappa} = \frac{1}{b^\kappa}\sum_{\substack{m\in\mathbb{N}_k\\ a<\lambda_m\leqslant b}} 1 + \kappa\int_a^b \Bigg(\sum_{\substack{m\in\mathbb{N}_k\\ a<\lambda_m\leqslant u}} 1\Bigg)\frac{\mathrm{d}u}{u^{\kappa+1}}$$

$$\geqslant (r(b)-r(a))\frac{\kappa_k}{b^\kappa}(1+o(1))$$
$$\geqslant \left(\frac{2C_1\kappa_k\beta^\kappa l^\kappa \kappa\xi}{b^\kappa l}+\frac{B}{l^2}-\frac{2\kappa_k C_2}{b^\kappa}\right)(1+o(1))$$

as $l\to\infty$. Therefore, (6) and (7) show that

$$\sum_{l\notin A}\frac{1}{l}<\infty. \tag{8}$$

Let $A=\{a_l\colon l\in\mathbb{N}\}$, $a_1<a_2<\dots$. Then from (8) we find

$$\lim_{l\to\infty}\frac{a_l}{l}=1. \tag{9}$$

By the definition of the set A there exists a sequence $\{\xi_l\}$ such that $(a_l-\xi)\beta<\xi_l\leqslant(a_l+\xi)\beta$ and $|w(\xi_l)|\leqslant \xi_l^{-\kappa}$. This and (9) show that $\lim\limits_{l\to\infty}\xi_l/l=\beta$, and $\limsup\limits_{l\to\infty}\frac{\log|w(\xi_l)|}{\xi_l}\leqslant 0$. Moreover, in view of (9) $|\xi_m-\xi_n|>|a_m-a_n|\delta\geqslant\delta|m-n|$ with some positive constant δ. therefore, applying Theorem 6.4.12 [1], we find that

$$\limsup_{r\to\infty}\frac{\log|w(r)|}{r}\leqslant 0. \tag{10}$$

On the other hand, by Lemma 6.4.10 [1], if $w(s)\not\equiv 0$, then $\limsup\limits_{r\to\infty}\frac{\log|w(r)|}{r}>0$, and this contradicts (10). Consequently, $w(s)\equiv 0$, and the lemma follows by differentiation.

Proof of Theorem 2. Let, for $m=1,2,\dots$,

$$\begin{aligned}\underline{f}_m(s_1,\dots,s_n;\omega(m)) &= (f_m(s_1,\omega),\dots,f_n(s_n,\omega))\\ &= (a_{m1}\omega(m)e^{-\lambda_m s_1},\dots,a_{mn}\omega(m)e^{-\lambda_m s_n}).\end{aligned}$$

From the definition of Ω it follows that $\{\omega(m)\}$ is a sequence of independent random variables with respect to the measure m_H. Hence $\{\underline{f}_m(s_1,\dots,s_n;\omega(m))\}$ is a sequence of independent $H_{n,N}$-valued random elements defined on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. The support of each random variable $\omega(m)$ is the unit circle γ. Therefore, the set $\{\underline{f}_m(s_1,\dots,s_n;a)\colon a\in\gamma\}$ is the support of the random element $\underline{f}(s_1,\dots,s_n;\omega(m))$. Hence in virtue of Lemma 5 from [3] the closure of the set of all convergent series

$$\sum_{m=1}^{\infty}\underline{f}_m(s_1,\dots,s_n;a_m),\quad a_m\in\gamma,$$

is the support of the random element $\widehat{F}(s_1,\dots,s_n;\omega)$. To prove the theorem it remains to check that the latter set is dense in $H_{n,N}$. For this we will apply Lemma 6 from [3].

Let $\mu_1, \ldots, \mu_n$ be complex measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact supports contained in $D_{1,N}, \ldots, D_{n,N}$, respectively, such that

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int_{\mathbb{C}} a_{mj} \mathrm{e}^{-\lambda_m s} \mathrm{d}\mu_j(s) \right| < \infty. \tag{11}$$

Now let $h_j(s) = s - \sigma_{aj}, j = 1, \ldots, n$. Then we have that $\mu_j h^{-1}(A) = \mu_j(h_j^{-1} A), A \in \mathcal{B}(\mathbb{C})$, is a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in $\widehat{D}_{j,N} = \{s \in \mathbb{C}: \sigma_{1j} < \sigma_{aj} < \sigma < 0, |t| \leqslant N\}$. Now (11) can be rewritten in the form

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} c_{mj} \int_{\mathbb{C}} \mathrm{e}^{-\lambda_m s} \mathrm{d}\mu_j h_j^{-1}(s) \right| < \infty.$$

This together with hypothesis on c_{mj} leads to

$$\sum_{\substack{m=1 \\ m \in \mathbb{N}_k}}^{\infty} \left| \sum_{j=1}^{n} b_{kj} \int_{\mathbb{C}} \mathrm{e}^{-\lambda_{mj} s} \mathrm{d}\mu_j h_j^{-1}(s) \right| < \infty, \quad k = 1, \ldots, r. \tag{12}$$

Taking $\widehat{\mu}_k(A) = \sum_{j=1}^{n} b_{kj} \mu_j h_j^{-1}(A),\ A \in \mathcal{B}(\mathbb{C}),\ v_k(z) = \int_{\mathbb{C}} \mathrm{e}^{-sz} \mathrm{d}\mu_k(s),\ z \in \mathbb{C}, k = 1, \ldots, r$, we write (12) in the form

$$\sum_{\substack{m=1 \\ m \in \mathbb{N}_k}}^{\infty} |w_k(A_m)| < \infty, \quad k = 1, \ldots, r.$$

Now Lemma 1 shows that $v_k(z) = 0$, and therefore $\int_{\mathbb{C}} s^l \mathrm{d}\widehat{\mu}_k(s) = 0, l = 0, 1, 2, \ldots$, $k = 1, \ldots, r$. Hence, using the definition of $\widehat{\mu}_k$ and the properties of the matrix $\mathcal{B}$, we obtain that $\int_{\mathbb{C}} s^l \mathrm{d}\mu_j h_j^{-1}(s) = 0, l = 0, 1, 2, \ldots, j = 1, 2, \ldots, n$, and this together with definition of the function h_j implies the relations

$$\int_{\mathbb{C}} s^l \mathrm{d}\mu_j(s) = 0, \quad l = 0, 1, 2, \ldots, \quad j = 1, 2, \ldots, n. \tag{13}$$

Moreover, we have [4] that there exists a sequence $\{b_m : b_m \in \gamma\}$ such that $\sum_{m=1}^{\infty} \underline{f}_m(s_1, \ldots, s_n; b_m)$ converges in $H_{n,N}$, and $\sum_{m=1}^{\infty} \sum_{j=1}^{n} \sup_{s \in K_j} |f_{mj}(s, b_m)|^2 < \infty$. Since $|b_m| = 1$, condition (13) remains also valid for $f(s_1, \ldots, s_n; b_m)$. Thus we have that all conditions of Lemma 6 [3] are satisfied, and therefore the set of all convergent series

$$\sum_{m=1}^{\infty} a_m^* \underline{f}_m(s_1, \ldots, s_n; b_m), \quad a_m^* \in \gamma,$$

is dense in $H_{n,N}$. Hence the set of all convergent series

$$\sum_{m=1}^{\infty} \underline{f}_m(s_1, \ldots, s_n; a_m), \quad a_m \in \gamma,$$

is dense in $H_{n,N}$, and the closure of this set is the whole of $H_{n,N}$. The theorem is proved.

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Vieno atsitiktinio elemento atrama

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Nagrinėjamas atsitiktinis elementas, įgyjantis reikšmes daugiamatėje analizinių funkcijų erdvėje ir įrodoma, jog šio elemento atrama yra visa minėta erdvė.