The Joint Universality for L-Functions of Elliptic Curves*

V. Garbaliauskienė[1], R.Kačinskaitė[1], A. Laurinčikas[2]

[1]Šiauliai University, P. Višinskio st. 19, LT-77156 Šiauliai, Lithuania

[2]Vilnius University, Naugarduko st. 24, LT-03225 Vilnius, Lithuania
antanas.laurincikas@maf.vu.lt

Received: 15.09.2004
Accepted: 13.10.2004

Abstract. A joint universality theorem in the Voronin sense for L-functions of elliptic curves over the field of rational numbers is proved.

Keywords: elliptic curve, function of exponential type, limit theorem, probability measure, random element, universality.

1 Introduction

Let E be an elliptic curve over the field of rational numbers given by the Weierstrass equation

$$y^2 = x^3 + ax + b,$$

where a and b are rational integers. Suppose that the discriminant of E $\Delta = -16(4a^3 + 27b^2) \neq 0$. It is known that then E is non-singular.

For each prime p, denote by $\nu(p)$ the number of solutions of the congruence

$$y^2 \equiv x^3 + ax + b (\mathrm{mod}\ p),$$

and denote $\lambda(p) = p - \nu(p)$. By the classical result of H. Hasse

$$\left|\lambda(p)\right| \leq 2\sqrt{p}. \tag{1}$$

*Partially supported by grant from Lithuanian Foundation of Studies and Science.

To study the numbers $\lambda(p)$, H. Hasse and H. Weil introduced and studied the L-function attached to E. Let $s = \sigma + it$ be a complex variable. Then the later L-function is defined by

$$L_E(s) = \prod_{p \nmid \Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \prod_{p \mid \Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1},$$

in view of (1) the product being absolutely convergent for $\sigma > \frac{3}{2}$. By the Shimura-Taniyama theorem proved in [1] the function $L_E(s)$ is analytically continuable to an entire function and satisfies the functional equation

$$\left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L_E(s) = \eta \left(\frac{\sqrt{q}}{2\pi}\right)^{2-s} \Gamma(2-s) L_E(2-s),$$

where q is a positive integer composed of prime factors of the discriminant Δ, $\eta = \pm 1$ is the root number, and $\Gamma(s)$ denotes the Euler gamma-function.

In [2] the universality in the Voronin sense of the function $L_E(s)$ has been obtained. Denote by meas$\{A\}$ the Lebesque measure of the set A, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T} \operatorname{meas}\{\tau \in [0, T]\colon \ \ldots\},$$

where in place of dots a conditinion satisfied by τ is to be written. Let $\mathbb{C}$ be the complex plane, and $D = \{s \in \mathbb{C}\colon 1 < \sigma < \frac{3}{2}\}$.

Theorem A. *Suppose that E is a non-singular elliptic curve over the field of rational numbers. Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \nu_T \Big(\sup_{s \in K} \big| L_E(s + i\tau) - f(s) \big| < \varepsilon \Big) > 0.$$

In [2] also the universality of $L_E^k(s)$, $k = 2, 3, \ldots$, and, under the analogue of the Riemann hypothesis for $L_E(s)$, of $L_E^{-k}(s)$, $k = 1, 2, \ldots$, was considered.

The aim of this paper is to obtain the joint universality for L-functions of elliptic curves.

Let $n > 1$ be an positive integer. Consider n elliptic curves $E_1, \dots, E_n$ given by the Weierstrass equations

$$y^2 = x^3 + a_j x + b_j,$$

with $\Delta_j = -16(4a_j^3 + 27b_j^2) \neq 0,\ j = 1, \dots, n$. Let, as above,

$$\lambda_j(p) = p - \nu_j(p),$$

where $\nu_j(p)$ is the number of solutions of the congruence

$$y^2 \equiv x^3 + a_j x + b_j (\operatorname{mod} p), \quad j = 1, \dots, n.$$

Define

$$L_{E_j}(s) = \prod_{p \nmid \Delta_j} \left(1 - \frac{\lambda_j(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \prod_{p \mid \Delta_j} \left(1 - \frac{\lambda_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, n.$$

To state a joint universality theorem for the functions $L_{E_j}(s)$ we need some additional conditions. Let P be the set of all prime numbers and let P_l, $l = 1, \dots, r$, $r \geq n$, be sets of prime numbers such that $P_{l_1} \bigcap P_{l_2} = \varnothing$ for $l_1 \neq l_2$, and

$$P = \bigcup_{l=1}^{r} P_l.$$

Moreover, we suppose that, for $x \to \infty$,

$$\sum_{\substack{p \leq x \\ p \in P_l}} \frac{1}{p} = \varkappa_l \log \log x + b_l + \rho_l(x), \tag{2}$$

where $\varkappa_1 + \ldots + \varkappa_r = 1$, $\varkappa_l > 0$, $\rho_l(x) = O(\log^{-\theta_l} x)$ with $\theta_l > 1$, and b_l is some real number, $l = 1, \dots, r$. Denote

$$B_j(p) = \frac{\lambda_j(p)}{\sqrt{p}},$$

and suppose that $B_j(p)$ is constant for $p \in P_l$, i. e., for $p \in P_l$

$$\begin{aligned} &B_1(p) = B_{l1}, \\ &\ldots\ldots\ldots\ldots \\ &B_n(p) = B_{ln}. \end{aligned}$$

Let

$$B_{rn} = \begin{pmatrix} B_{11} & \dots & B_{1n} \\ \dots & \dots & \dots \\ B_{r1} & \dots & B_{rn} \end{pmatrix}.$$

Theorem 1. *Suppose that* $\operatorname{rank}(B_{rn}) = n$. *Let* K_j *be a compact subset of the strip* D *with connected complement, and let* $f_j(s)$ *be a continuous non-vanishing on* K_j *function which is analytic in the interior of* K_j, $j = 1, \dots, n$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{l\le j\le n} \sup_{s\in K_j} \big|L_{E_j}(s+i\tau) - f_j(s)\big| < \varepsilon \Big) > 0.$$

Joint universality theorems for Dirichlet L-functions independently were proved by S.M. Voronin [3], S.M. Gonek [4] and B.Bagchi [5], [6]. For Dirichlet series with multiplicative coefficients they were obtained in [7]. The joint universality for Lerch zeta-functions, for Matsumoto zeta-functions, and for zeta-functions attached to certain cusp forms were proved in [8], [9] and [10], respectively. Joint universality theorems for twists of Dirichlet series with Dirichlet characters were investigated in [11] and [12]. Finally, theorems of a such type for some classes of general Dirichlet series were obtained in [13] and [14]. A survey on universality is given in [15] and [16]. A large part of the work [17] is also devoted to universality of Dirichlet series.

2 A limit theorem

Let $V > 0$, and

$$D_V = \Big\{ s \in \mathbb{C}\colon 1 < \sigma < \frac{3}{2}, \quad |t| < V \Big\}.$$

In this section we state a joint limit theorem for functions $L_{E_1}, \dots, L_{E_n}$ on the space of analytic on D_V functions. Denote by $H(G)$ the space of analytic on the region G functions equipped with the topology of uniform convergence on compacta, and let

$$H^m(G) = \underbrace{H(G) \times \dots \times H(G)}_{m}, \quad m \ge 2.$$

Moreover, by $\mathcal{B}(S)$ we denote the class of Borel sets of the space S. We will consider the weak convergence of the probability measure

$$P_T(A) = \nu_T\left(\left(L_{E_1}(s+i\tau), \ldots, L_{E_n}(s+i\tau)\right) \in A\right), \quad A \in \mathcal{B}\big(H^n(D_V)\big),$$

as $T \to \infty$.

Let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ be the unit circle on the complex plane, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for any prime p. With the product topology and operation of pointwise multiplication the set Ω is a compact topological Abelian group, therefore the probability Haar measure m_H on $\big(\Omega, \mathcal{B}(\Omega)\big)$ exists. This gives a probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and define on the probability space $\big(\Omega, \mathcal{B}(\Omega), m_H\big)$ the $H^n(D_V)$-valued random element $L(s,\omega)$ by

$$L(s,\omega) = \big(L_{E_1}(s,\omega), \ldots, L_{E_n}(s,\omega)\big), \tag{3}$$

where

$$L_{E_j}(s,\omega) = \prod_{p \nmid \Delta_j} \big(1 - \frac{\lambda_j(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\big)^{-1} \prod_{p \mid \Delta_j} \big(1 - \frac{\lambda_j(p)\omega(p)}{p^s}\big)^{-1},$$

$j = 1, \ldots, n$. Let P_L be the distribution of the random element $L(s,\omega)$, i. e.,

$$P_L(A) = m_H\big(\omega \in \Omega\colon L(s,\omega) \in A\big), \quad A \in \mathcal{B}\big(H^n(D_V)\big).$$

Lemma 1. *The probability measure P_T converges weakly to P_L as $T \to \infty$.*

Proof. The function $L_{E_j}(s)$, for $\sigma > \frac{3}{2}$, can be written in the form

$$L_{E_j}(s) = \prod_{p \mid \Delta_j} \left(1 - \frac{\lambda_j(p)}{p^s}\right)^{-1} \prod_{p \nmid \Delta_j} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1},$$

where

$$\alpha_j(p) + \beta_j(p) = \lambda_j(p),$$

and by (2)

$$|\alpha_j(p)| \le 2\sqrt{p}, \quad |\beta_j(p)| \le 2\sqrt{p} \quad j = 1, \dots, n.$$

Therefore, $L_{E_j}(s)$ is the Matsumoto zeta-function with $\alpha = 0$ and $\beta = \frac{1}{2}$, for definitions, see [18] and [19]. Since by the Shimura-Taniyama theorem $L_{E_j}(s)$ coincides with L-function attached to a newform of level 2, we have that, for $\sigma > 1$, the estimates

$$L_{E_j}(\sigma + it) = O\big(|t|^{\alpha_j}\big), \quad |t| \ge t_0, \quad \alpha_j > 0,$$

and

$$\int_0^T \big|L_{E_j}(\sigma + it)\big|^2 \, \mathrm{d}\, t = O(T), \quad T \to \infty,$$

are satisfied. Therefore, by Theorem 2 of [9] we have that the probability measure

$$\nu_T\left(\big(L_{E_1}(s + i\tau), \dots, L_{E_n}(s + i\tau)\big) \in A\right), \quad A \in \mathcal{B}\big(H^n(D)\big),$$

weakly converges to the distribution of the $H^n(D)$-valued random element defined by (3) as $T \to \infty$. The function $h\colon H^n(D) \to H^n(D_V)$ defined by the coordinatewise restriction is continuous, therefore by Theorem 5.1 of [20] hence we obtain the lemma. □

3 A denseness lemma

To prove Theorem 1 we need the support of the measure P_L in Lemma 1. For this we will consider the random element $L(s, \omega)$ and its support.

Let $a_p \in \gamma$. For $j = 1, \dots, n$, we define

$$f_{jp}(s, a_p) = \begin{cases} -\log\Big(1 - \dfrac{\lambda_j(p)a_p}{p^s} + \dfrac{a_p^2}{p^{2s-1}}\Big), & \text{if} \quad p \nmid \Delta_j, \\ -\log\Big(1 - \dfrac{\lambda_j(p)a_p}{p^s}\Big), & \text{if} \quad p|\Delta_j, \end{cases}$$

and

$$\underline{f}_p(s, a_p) = \big(f_{1p}(s, a_p), \dots, f_{np}(s, a_p)\big).$$

Lemma 2. *Suppose that* $\operatorname{rank}(B_{rn}) = n$. *Then the set of all convergent series* $\sum\limits_{p} \underline{f}(s, a_p)$ *is dense in* $H^n(D_V)$.

For the proof of the lemma we will use the following statements.

Lemma 3. *Let* μ *be a complex Borel measure on* $\big(\mathbb{C}, \mathcal{B}(\mathbb{C})\big)$ *with compact support contained in* $\{s \in \mathbb{C}\colon \sigma > \sigma_0\}$, *and let*

$$f(z) = \int\limits_{\mathbb{C}} \mathrm{e}^{\,sz}\, \mathrm{d}\,\mu(s), \quad z \in \mathbb{C}.$$

If $f(z) \not\equiv 0$, *then* $\limsup\limits_{x\to\infty} \dfrac{\log\big|f(x)\big|}{x} > \sigma_0$.

The lemma is Lemma 5.2.2 of [5]. Its proof is also given [21], Lemma 6.4.10.

Lemma 4. *Let* $f(s)$ *be a function of exponential type such that*

$$\limsup_{x\to\infty} \frac{\log|f(x)|}{x} > -1.$$

Then, for $l = 1, \dots, r$,

$$\sum_{p\in P_l} \big|f(\log p)\big| = \infty.$$

The proof is based on the property (2) of the sets P_l as well as on the following lemma.

Lemma 5. *Let* $f(s)$ *be an entire function of exponential type, and let* $\{\lambda_m\}$ *be a sequence of complex numbers. Let* α, β *and* δ *be positive real numbers such that*

(i) $\limsup\limits_{x\to\infty} \dfrac{\log|f(\pm ix)|}{x} \le \alpha$;

(ii) $|\lambda_m - \lambda_n| \ge \delta|m-n|$;

(iii) $\lim\limits_{m\to\infty} \dfrac{\lambda_m}{m} = \beta$;

(iv) $\alpha\beta < \pi$.

Then $\limsup\limits_{m\to\infty} \dfrac{\log|f(\lambda_m)|}{|\lambda_m|} = \limsup\limits_{r\to\infty} \dfrac{\log|f(r)|}{r}$.

The lemma is a special version of the Bernstein theorem. The proof is given in [21].

Proof of Lemma 4. Since $f(s)$ is a function of exponential type, there exists an $\alpha > 0$ such that

$$\limsup_{x\to\infty} \frac{\log\left|f(\pm ix)\right|}{x} \le \alpha.$$

We fix a postitive number β such that $\alpha\beta < \pi$. Suppose, on the contrary, that for some l, $1 \le l \le r$, the series

$$\sum_{p\in P_l} \left|f(\log p)\right| \tag{4}$$

converges.

Define the subset A of the set $\mathbb{N}$ of positive integers by

$$A = \Big\{m \in \mathbb{N}\colon \exists r \in \big((m-\frac{1}{4})\beta, (m+\frac{1}{4})\beta\big] \quad \text{and} \quad \left|f(r)\right| \le \mathrm{e}^{-r}\Big\}.$$

Then we have that

$$\sum_{p\in P_l} \left|f(\log p)\right| \ge \sum_{m\notin A} {\sum_m}' \left|f(\log p)\right| \ge \sum_{m\notin A} {\sum_m}' \frac{1}{p}, \tag{5}$$

where $\sum_m'$ denotes a sum over prime numbers $p \in P_l$ such that

$$(m-\frac{1}{4})\beta < \log p \le (m+\frac{1}{4})\beta.$$

In view of (2) we find

$$\begin{aligned} {\sum_m}' \frac{1}{p} &= \sum_{\substack{p\in P_l \\ p\le \exp\left\{(m+\frac{1}{4})\beta\right\}}} \frac{1}{p} - \sum_{\substack{p\in P_l \\ p\le \exp\left\{(m-\frac{1}{4})\beta\right\}}} \frac{1}{p} \\ &= \varkappa_l \log\frac{m+\frac{1}{4}}{m-\frac{1}{4}} + O\big((m-\frac{1}{4})^{\theta_l}\big) = \frac{\varkappa_l}{2m} + O\left(\frac{1}{m^{\theta_l}}\right). \end{aligned}$$

This, the convergence of the series (4) and (5) yield

$$\sum_{m\notin A} \left(\frac{\varkappa_l}{2m} + O\Big(\frac{1}{m^{\theta_l}}\Big)\right) = \sum_{m\notin A} {\sum_m}' \frac{1}{p} \le \sum_{p\in P_l} \left|f(\log p)\right| < \infty.$$

Hence, clearly, since $\varkappa_l > 0$,

$$\sum_{m \notin A} \frac{1}{m} < \infty. \tag{6}$$

Suppose that $A = \{a_m \in \mathbb{N} \colon a_1 < a_2 < \dots\}$. Then (6) shows that

$$\lim_{m \to \infty} \frac{a_m}{m} = 1. \tag{7}$$

Moreover, by the definition of the set A, there exists a sequence $\{\lambda_m\}$ such that

$$(a_m - \frac{1}{4})\beta < \lambda_m \leq (a_m + \frac{1}{4})\beta, \tag{8}$$

and

$$\left|f(\lambda_m)\right| \leq \mathrm{e}^{-\lambda_m}. \tag{9}$$

Therefore, by (7) and (8)

$$\lim_{m \to \infty} \frac{\lambda_m}{m} = \beta,$$

and

$$|\lambda_m - \lambda_n| \geq \beta |a_m - a_n| - \frac{1}{2}\beta \geq \delta |m - n|$$

with some $\delta > 0$, and in view of (9)

$$\limsup_{m \to \infty} \frac{\log \left|f(\lambda_m)\right|}{|\lambda_m|} \leq -1. \tag{10}$$

So, all hypotheses of Lemma 5 are satisfied, and we have by (10) that

$$\limsup_{r \to \infty} \frac{\log \left|f(r)\right|}{r} \leq -1.$$

However, this contradicts the hypothesis of the lemma. Hence, the series (4) must be divergent, and the lemma is proved. □

Lemma 6. *Let* $\{\underline{f}_m\} = \left\{(f_{1m}, \dots, f_{nm})\right\}$ *be a sequence in* $H^n(D_V)$ *which satisfies:*

(i) *If* $\mu_1, \dots, \mu_n$ *are complex Borel measures on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *with compact supports contained in* D_V *such that*

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int_{\mathbb{C}} f_{jm} \, \mathrm{d}\, \mu_j \right| < \infty,$$

then

$$\int_{\mathbb{C}} s^r \, \mathrm{d}\, \mu_j(s) = 0 \quad \textit{for} \quad j = 1, \dots, n \quad \textit{and} \quad r = 0, 1, 2, \dots;$$

(ii) *The series* $\sum_{m=1}^{\infty} \underline{f}_m$ *converges in* $H^n(D_V)$*;*

(iii) *For any compacts* $K_1, \dots, K_n \subset D_V$*,*

$$\sum_{m=1}^{\infty} \sum_{j=1}^{n} \sup_{s \in K_j} \left| f_{jm}(s) \right|^2 < \infty.$$

Then the set of all convergent series $\sum_{m=1}^{\infty} a_m \underline{f}_m$ *with* $a_m \in \gamma$ *is dense in* $H^n(D_V)$*.*

The lemma is a special case of Lemma 5 from [10], where its proof is given. Now we are ready to prove Lemma 2.

Proof of Lemma 2. Let p_0 be a fixed positive number. We define

$$\underline{f}_p(s) = \begin{cases} \underline{f}_p(s, 1), & \text{if} \quad p > p_0, \\ 0, & \text{if} \quad p \leq p_0. \end{cases}$$

First we observe that there exists a sequence $\{\hat{a}_p \colon \hat{a}_p \in \gamma\}$ such that the series

$$\sum_p \hat{a}_p \underline{f}_p \tag{11}$$

converges in $H^n(D_V)$. Really, in view of (1)

$$f_{jp}(s, 1) = \frac{\lambda_j(p)}{p^s} + r_{jp}(s),$$

where $r_{jp}(s) = O\left(p^{1-2\sigma}\right)$, $j = 1, \ldots, n$. Hence we have that for compact subsets $K_1, \ldots, K_n$ of D_V,

$$\sum_{j=1}^{n} \sum_{p} \sup_{s \in K_j} \left| r_{jp}(s) \right| < \infty.$$

In the proof that $L_{E_j}(s, \omega)$, $j = 1, \ldots, n$, is an $H(D_V)$-valued random element it is proved that the series

$$\sum_{p} \frac{\lambda_j(p)\omega(p)}{p^s}, \quad j = 1, \ldots, n,$$

converges uniformly on compact subsets of D_V for almost all $\omega \in \Omega$, see, for example, [19], where the Matsumoto zeta-functions were considered. Hence the series

$$\sum_{p} \left(\frac{\lambda_1(p)\omega(p)}{p^s}, \ldots, \frac{\lambda_n(p)\omega(p)}{p^s} \right)$$

converges in $H^n(D_V)$ for almost all $\omega \in \Omega$. Consequently, there exists a sequence $\{\hat{a}_p \colon \hat{a}_p \in \gamma\}$ such that the series (11) converges in $H^n(D_V)$.

Now we will prove that the set all convergent series

$$\sum_{p} a_p \underline{f}_p, \quad a_p \in \gamma, \tag{12}$$

is dense in $H^n(D_V)$. To prove this, it suffices to show that the set of all convergent series

$$\sum_{p} b_p \underline{g}_p, \quad b_p \in \gamma, \tag{13}$$

where $g_p = \hat{a}_p \underline{f}_p$, is dense in $H^n(D_V)$. For this we will apply Lemma 6 for the sequence $\{\underline{g}_p\}$.

By the definition of $\underline{g}_p$ we have that the series $\sum\limits_{p} \underline{g}_p$ converges in $H^n(D_V)$. Moreover, in virtue of (1), for any compacts $K_1, \ldots, K_n \subset D_V$,

$$\sum_{p} \sum_{j=1}^{n} \sup_{s \in K_j} |g_{jp}(s)|^2 < \infty.$$

Therefore, the hypotheses ii) and iii) of Lemma 6 are satisfied, and it remains to verify the hypothesis i).

Let $\mu_1, \dots, \mu_n$ be complex Borel measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact supports contained in D_V such that

$$\sum_p \left| \sum_{j=1}^n \int_{\mathbb{C}} b_p g_{jp} \, \mathrm{d}\, \mu_j \right| < \infty. \tag{14}$$

Let $D_{0V} = \left\{ s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1, \quad |t| < V \right\}$, and let $h(s) = s - \frac{1}{2}$. Define $\mu_j h^{-1}(A) = \mu_j(h^{-1}A)$, $A \in \mathcal{B}(\mathbb{C})$, $j = 1, \dots, n$. Then, clearly, $\mu_j h^{-1}$ is a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D_{0V}, $j = 1, \dots, n$. This and (14) show that, for every $l = 1, \dots, r$,

$$\sum_{p \in P_l} \left| \sum_{j=1}^n b_{lj} \int_{\mathbb{C}} p^{-s} \, \mathrm{d}\, \mu_j h^{-1}(s) \right| < \infty. \tag{15}$$

We put

$$\nu_l(s) = \sum_{j=1}^n b_{lj} \mu_j h^{-1}(s).$$

Then (15) yields, for every $l = 1, \dots, r$,

$$\sum_{p \in P_l} \left| \rho_l(\log p) \right| < \infty,$$

where

$$\rho_l(z) = \int_{\mathbb{C}} \mathrm{e}^{-sz} \, \mathrm{d}\, \nu_l(s), \quad z \in \mathbb{C}.$$

Clearly, we have that, for $r > 0$,

$$\left| \rho_l(r \, \mathrm{e}^{i\varphi}) \right| \le \mathrm{e}^{Vr} \int_{\mathbb{C}} \left| \mathrm{d}\, \nu_l(s) \right|.$$

Hence

$$\limsup_{r \to \infty} \frac{\log \left| \rho_l(r \, \mathrm{e}^{i\varphi}) \right|}{r} \le V$$

uniformly in φ, $0 < \varphi \leq \pi$. This shows that $\rho_l(z)$, $l = 1, \ldots, r$, is a function of exponential type.

In view of Lemmas 3 and 4 we find that $\rho(z) \equiv 0$ for every $l = 1, \ldots, r$. Hence it follows by differentiation that

$$\int_{\mathbb{C}} s^k \,\mathrm{d}\, \nu_l(s) = 0 \tag{16}$$

for all $l = 1, \ldots, r$ and $k = 0, 1, 2, \ldots$. Now let

$$x_j = x_j(k) = \int_{\mathbb{C}} s^k \,\mathrm{d}\, \mu_j h^{-1}(s).$$

Then the definition of $\nu_j(s)$ and (16) give the following system of equations

$$\sum_{j=1}^{n} b_{lj} x_j = 0, \quad l = 1, \ldots, r.$$

Since rank$(B_{rn}) = n$, the later system has only a solution $x_j = 0$, $j = 1, \ldots, n$. Thus we have that

$$\int_{\mathbb{C}} s^k \,\mathrm{d}\, \mu_j h^{-1}(s) = 0$$

for all $j = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$. From this it follows that

$$\int_{\mathbb{C}} s^k \,\mathrm{d}\, \mu_j(s) = 0$$

for all $j = 1, \ldots, n$ and $k = 0, 1, 2, \ldots$. This shows that all hypotheses of Lemma 6 hold, therefore the set of all convergent series (13) is dense in $H^n(D_V)$, hence the same is true for the set of all convergent series (12).

Now let $\underline{x}(s) = \big(x_1(s), \ldots, x_n(s)\big)$ be an arbitrary element of $H^n(D_V)$, $K_1, \ldots, K_n$ be compact subsets of D_V, and let ε be an arbitrary positive number. We fix p_0 such that

$$\sum_{j=1}^{n} \sup_{s \in K_j} \sum_{p > p_0} \sum_{k=2}^{\infty} \frac{\big|\lambda_j(p)\big|^k}{k p^{k\sigma}} < \frac{\varepsilon}{4}. \tag{17}$$

The denseness of all convergent series implies the existence of the sequence $\{\widetilde{a}_m : \widetilde{a}_m \in \gamma\}$ such that

$$\sup_{1\le j\le n}\sup_{s\in K_j}\Big|\underline{x}(s)-\sum_{p\le p_0}\underline{f}_p(s)-\sum_{p\ge p_0}\widetilde{a}_p\underline{f}_p(s)\Big|<\frac{\varepsilon}{2}. \tag{18}$$

We take

$$a_p=\begin{cases}1, & \text{if} \quad p\le p_0,\\ \widetilde{a}_p, & \text{if} \quad p>p_0.\end{cases}$$

Then (17) and (18) yield

$$\begin{aligned}\sup_{1\le j\le n}\sup_{s\in K_j}&\Big|\underline{x}(s)-\sum_{p}\underline{f}_p(s,a_p)\Big|\\ &=\sup_{1\le j\le n}\sup_{s\in K_j}\Big|\underline{x}(s)-\sum_{p\le p_0}\underline{f}_p(s,a_p)-\sum_{p>p_0}\underline{f}_p(s,a_p)\Big|\\ &\le\sup_{1\le j\le n}\sup_{s\in K_j}\Big|\underline{x}(s)-\sum_{p\le p_0}\underline{f}_p(s)-\sum_{p\ge p_0}\widetilde{a}_p\underline{f}_p(s)\Big|\\ &\quad+\sup_{1\le j\le n}\sup_{s\in K_j}\Big|\sum_{p>p_0}\widetilde{a}_p\underline{f}_p(s)-\sum_{p>p_0}\underline{f}_p(s,\widetilde{a}_p)\Big|<\varepsilon.\end{aligned}$$

Since $\underline{x}(s)$, $K_1,\dots,K_n$ and ε are arbitrary, the lemma is proved. ☐

4 The support of P_L

Let

$$S=\big\{f\in H(D_V)\colon f(s)\ne 0 \quad \text{or} \quad f(s)\equiv 0\big\}.$$

Lemma 7. *The support of the measure P_L is the set S^n.*

The proof of Lemma 7 relies on Lemma 2, the Hurwitz theorem and the following statement. We denote by S_X the support of the random element X.

Lemma 8. *Let $\{X_n\}$ be a sequence of independent $H^n(D_V)$-valued random elements such that the series $\sum\limits_{m=1}^{\infty} X_m$ converges almost surely. Then the support* *of the sum of the later series is the closure of the set of all* $\underline{f} \in H^n(D_V)$ *which may be written as a convergent series*

$$\underline{f} = \sum_{m=1}^{\infty} \underline{f}_m, \quad \underline{f}_m \in S_{X_m}.$$

The lemma is a special case of Lemma 4 from [10], where its proof can be find.

Proof of Lemma 7. Since $\{\omega(p)\}$ is a sequence of independent random variables, $\{\underline{f}_p(s, \omega(p))\}$ is a sequence of independent $H^n(D_V)$-valued random elements defined on the probability space $(\mathbb{C}, \mathcal{B}(\mathbb{C}), m_H)$. The support of each $\omega(p)$ is the unit circle γ. Therefore, the support of $\underline{f}_p(s, \omega(p))$ is the set

$$\{\underline{f} \in H(D_V) \colon \underline{f}(s) = \underline{f}(s, a), a \in \gamma\}.$$

Hence by Lemma 8 the support of the $H^n(D_V)$-valued random element

$$\big(\log L_{E_1}(s, \omega), \dots, \log L_{E_n}(s, \omega)\big) \tag{19}$$

is the closure of the set of all convergent series $\sum\limits_p \underline{f}_p(s, a_p)$. However, by Lemma 2, the latter set is dense in $H^n(D_V)$. Hence the support of the random element (19) is $H^n(D_V)$. The map $h \colon H^n(D_V) \to H^n(D_V)$ given by the formula

$$h\big(f_1(s), \dots, f_n(s)\big) = \big(\mathrm{e}^{f_1(s)}, \dots, \mathrm{e}^{f_n(s)}\big), \quad f_1, \dots, f_n \in H^n(D_V),$$

is a continuous function which sends the element (19) to $L(s, \omega)$, and $H^n(D_V)$ to $\big(S \setminus \{0\}\big)^n$. Therefore, the support S_L of the random element $L(s, \omega)$ contains the set $\big(S \setminus \{0\}\big)^n$. However, the support of a random element is a closed set. In view of the Hurwitz theorem, see, for example, [22], Section 3.4.5, the closure of $S \setminus \{0\}$ is S. Thus, $S_L \supseteq S^n$. On the other the factors of the product defining $L_{E_j}(s, \omega)$, $j = 1, \dots, n$, do not vanish for $s \in D_V$. Hence $L_{E_j}(s, \omega)$, $j = 1, \dots, n$, is an almost surely convergent product of non-vanishing factors, and therefore, the Hurwitz theorem shows that $L_{E_j}(s, \omega) \in S$, $j = 1, \dots, s$, almost surely. Hence the relation $S_L \subset S^n$ holds, and we have that $S_L = S^n$. ◻

5 Proof of Theorem 1

Proof of Theorem 1 is based on Lemmas 1 and 7 as well as on the Mergelyan theorem which is the following lemma.

Lemma 9. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $f(s)$ be a continuous on K function which is analytic in the interior of K. Then $f(s)$ can be approximated uniformly on K by polynomials in s.*

Proof of the lemma can be found in [23].

Proof of Theorem 1. Clearly, there exists $V > 0$ such that the sets $K_1, \dots, K_n$ are contained in D_V. First we suppose that the functions $f_1(s), \dots, f_n(s)$ are non-zero analytically continuable to D_V. Let $G = \big\{(g_1, \dots, g_n)\colon (g_1, \dots, g_n) \in H^n(D_V)\big\}$, and

$$\sup_{1\le j\le n}\sup_{s\in K_j}\big|g_j(s) - f_j(s)\big| < \varepsilon.$$

The set G is open. Therefore, the properties of the weak convergence of probability measures [20] and Lemma 1 show that

$$\liminf_{T\to\infty}\nu_T\,\Big(\big(L_{E_1}(s+i\tau), \dots, L_{E_n}(s+i\tau)\big) \in 0\Big) \ge P_L(G).$$

However, the properties of the support and Lemma 7 show that $P_L(G) > 0$. Therefore, in this case

$$\liminf_{T\to\infty}\nu_T\Big(\sup_{1\le j\le n}\sup_{s\in K_j}\big|L_{E_j}(s+i\tau) - f_j(s)\big| < \varepsilon\Big) > 0. \tag{20}$$

Now we suppose that the functions $f_1(s), \dots, f_n(s)$ satisfy the hypotheses of Theorem 1. By Lemma 9 there exist polynomials $p_1(s), \dots, p_n(s)$ which are non-vanishing on $K_1, \dots, K_n$, respectively, such that

$$\sup_{1\le j\le n}\sup_{s\in K_j}\big|p_j(s) - f_j(s)\big| < \frac{\varepsilon}{4}. \tag{21}$$

Each polynomial $p_j(s)$, $j = 1, \dots, n$, has finitely many zeros. Therefore, there exitsts a region G_j with connected complement such $K_j \subset G_j$ and $p_j(s) \ne 0$ for $s \in G_j$, $j = 1, \dots, n$. Thus we can consider a continuouns branch of $\log p_j(s)$

on G_j, and $\log p_j(s)$ is analytic function in the interior of G_j, $j = 1, \ldots, n$. By Lemma 9 again there exist polynomials $g_1(s), \ldots, g_n(s)$ such that

$$\sup_{1\leq j\leq n}\sup_{s\in K_j}\left|p_j(s) - \mathrm{e}^{\,q_j(s)}\right| < \frac{\varepsilon}{4}.$$

This and (21) show that

$$\sup_{1\leq j\leq n}\sup_{s\in K_j}\left|f_j(s) - \mathrm{e}^{\,q_j(s)}\right| < \frac{\varepsilon}{2}. \tag{22}$$

However, $\mathrm{e}^{\,q_j(s)}$, $j = 1, \ldots, s$. Therefore, in view of (20),

$$\liminf_{T\to\infty}\nu_T\Big(\sup_{1\leq j\leq n}\sup_{s\in K_j}\left|L_{E_j}(s+i\tau) - \mathrm{e}^{\,q_j(s)}\right| < \frac{\varepsilon}{2}\Big) > 0.$$

This together with (22) proves the theorem. □

References

1. Breuil C., Conrad B., Diamond F., Taylor R. “On the modularity of ellipitic curves over $\mathbb{Q}$: wild 3-adic exercises”, *J. Amer. Math. Soc.*, **14**, p. 843–939, 2001

2. Garbaliauskienė V., Laurinčikas A. *Some analytic properties for L-functions of elliptic curves*, Preprint **16**, Vilnius University, Department of Math. and Inform., 2003

3. Voronin S.M. “On functional independence of Dirichlet L-functions”, *Acta Arith.*, **27**, p. 493–503, 1975 (in Russian)

4. Gonek S.M. *Analytic properties of zeta and L-functions*, Ph.D. Thesis, University of Michigan, 1979

5. Bagchi B. *The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series*, Ph.D. Thesis, Calcutta, Indian Statistical Institute, 1981

6. Bagchi B. “A joint universality theorem for Dirichlet L-functions,” *Math. Z.*, **181**, p. 319–334, 1982

7. Laurinčikas A. “On zeros of linear combinations of Dirichlet series”, *Liet. Matem. Rink.*, **26**, p. 468–477, 1986 (in Russian); *Lith. Math. J.*, **26**, p. 244–251, 1986

8. Laurinčikas A., Matsumoto K. “The joint universality and the functional independence for Lerch zeta-functions”, *Nagoya Math. J.*, **157**, p. 211–227, 2000

9. Laurinčikas A. “On the zeros of linear combinations of Matsumoto zeta-functions”, *Liet. Matem. Rink.*, **38**, p.185–204, 1998 (in Russian); *Lith. Math. J.*, **38**, p. 144–159, 1998

10. Laurinčikas A., Matsumoto K. “The joint universality of zeta-functions attached to certain cusp forms”, *Fiz. matem. fak. moksl. sem. darbai*, **5**, p. 58–75, Šiauliai University, 2002

11. Šleževičienė R. “The joint universality for twists of Dirichlet series with multiplicative coefficients”, In: *Analytic and Probab. Methods in Number Theory, Proc. of third Intern. Conf. in konow of J. Kubilius, Palanga 2001*, Dubickas A. et al. (Eds.), TEV, Vilnius, p. 303–319, 2002

12. Laurinčikas A., Matsumoto K. “The joint universality of twisted automorphic L-functions”, *J. Math. Soc. Japan*, **56**(3), p. 923–939, 2004 (to appear)

13. Laurinčikas A. “The joint universality for general Dirichlet series”, *Annales Univ. Sci. Budapest., Sect. Comp.*, **22**, p. 235–251, 2003

14. Laurinčikas A. “The joint universality of general Dirichlet series”, *Izv. ANR, ser. matem.* (in Russian) (to appear)

15. Laurinčikas A. “The universality of zeta-functions”, *Acta Appl. Math.*, **78**, p. 251–271, 2003

16. Grosse-Erdmann K.-G. “Universal families and hypercyclic operators”, *Bull Amer. Math. Soc.*, **36**, p. 345–381, 1999

17. Steuding J. *Value-distribution of L-functions and allied zeta-functions – with an emphasis on aspects of universality*, Habilitationsschift, J.W.Goethe-Universitat Frankfurt, 2003

18. Matsumoto K. “Value-distribution of zeta-functions”, *Lecture Notes in Math.*, **1434**, p. 178–187, Springer, 1990

19. Laurinčikas A. “On the Matsumoto zeta-function”, *Acta Arith.*, **84**, p. 1–16, 1998

20. Billingsley P. *Convergence of probability measures*, New York, John Wiley, 1968

21. Laurincikas A. *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht, Boston, London, 1996

22. Titchmarsh E.C. *The Theory of Functions*, Oxford University Press, Oxford, 1939

23. Walsh J.L. “Interpolation and Approximation by Rational Functions in the Complex Domain”, *Amer. Math. Soc. Collog. Publ.*, **20**, 1960