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A weighted discrete limit theorem on the complex plane for the Matsumoto zeta-function

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Let $\mathbb{N}$ and $\mathbb{C}$ denote the sets of positive integers and complex numbers, respectively. For $g(m)$, $f(j,m) \in \mathbb{N}$ and $a_m^{(j)} \in \mathbb{C}$, $m \in \mathbb{N}$, $j = 1, \ldots, g(m)$, define the polynomials

$$A_m(X) = \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)} X^{f(j,m)}\right).$$

Let $s = \sigma + it$ be a complex variable, and let p_m denote the mth prime number. The Matsumoto zeta-function $\varphi(s)$ was introduced and studied in [6], and is defined by

$$\varphi(s) = \prod_{m=1}^{\infty} A_m^{-1}(p_m^{-s}).$$

Assume the conditions

$$g(m) \leqslant c p_m^{\alpha}, \quad |a_m^{(j)}| \leqslant p_m^{\beta} \tag{1}$$

with a positive constant c and non-negative constants α and β. Under the condition (1) the product in the definition of $\varphi(s)$ converges absolutely for $\sigma > \alpha + \beta + 1$ and defines an analytic function without zeros.

K. Matsumoto and A. Laurinčikas proved limit theorems for the function $\varphi(s)$. We obtained in [2]–[4] discrete limit theorems for $\varphi(s)$. Let, for $N \in \mathbb{N}$, $\mu_N(\ldots) = (N+1)^{-1}\#(0 \leqslant m \leqslant N\colon \ldots)$, where in place of dots a condition satisfied by m is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. In [2]–[4] we considered the weak convergence of probability measures

$$\begin{aligned}
&\mu_N(\varphi(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}), \quad \sigma > \alpha + \beta + \frac{1}{2},\\
&\mu_N(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(H(D_1)), \quad D_1 = \{s \in \mathbb{C}\colon \sigma > \alpha + \beta + 1\}, \qquad (2)\\
&\mu_N(\varphi(s + imh) \in A), \quad A \in \mathcal{B}(M(D_2)), \quad D_2 = \left\{s \in \mathbb{C}\colon \sigma > \alpha + \beta + \frac{1}{2}\right\}.
\end{aligned}$$

Here $H(D_1)$ and $M(D_2)$ denote the spaces of analytic and meromorphic functions, respectively. In the case of the first and third measures in (2) it was assumed that the function $\varphi(s)$ is meromorphically continuable to D_2, all poles in this region are included in a compact set, and, for $\sigma > \alpha + \beta + \frac{1}{2}$, the estimates

$$\varphi(\sigma + it) = B|t|^c, \quad |t| \geqslant t_0,\ c > 0, \tag{3}$$

and

$$\int_0^T |\varphi(\sigma + it)|^2 \, dt = BT, \quad T \to \infty,$$

are valid. Here $h > 0$ is a fixed number such that $\exp\{\frac{2\pi k}{h}\}$ is irrational for all integers $k \neq 0$, and B denotes a quantity bounded by a constant.

It is possible to generalize the last results and to obtain weighted discrete limit theorems for the function $\varphi(s)$. Let $w(x)$ be a real function of bounded variation on $[N_0, \infty)$, $N_0 \in \mathbb{N}$, such that $U = U(N, w) = \sum_{m=N_0}^{N} w(m) \to +\infty$ as $N \to \infty$, and let $v_N(\ldots) = \frac{1}{U}\sum_{m=N_0}^{N} w(m)$, where in place of dots a condition satisfied by m is to be written. The aim of this note is to obtain the weak convergence as $N \to \infty$ of the probability measure $P_N(A) = v_N(\varphi(\sigma + imh) \in A)$, $A \in \mathcal{B}(\mathbb{C})$, $\sigma > \alpha + \beta + \frac{1}{2}$. Note that if the function $w(x)$ is non-increasing, then in [5] it was proved that the weak convergence of the first measure in (2) implies that of P_N.

Let, for $\sigma > \alpha + \beta + \frac{1}{2}$, and real τ,

$$\int_\tau^{T+\tau} w(t-\tau)\big|\varphi(\sigma + it)\big|^2 \, dt = B\big(1 + |\tau|\big)^{c_2}, \quad c_2 > 0. \tag{4}$$

THEOREM 1. *Suppose that the function $\varphi(s)$ satisfies conditions* (1), (3), (4), *and $h > 0$ is a fixed number such that* $\exp\{\frac{2\pi k}{h}\}$ *is irrational for all integers $k \neq 0$. Then on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *there exists a probability measure P such that the measure P_N converges weakly to P as $N \to \infty$.*

We begin with a limit theorem for Dirichlet polynomials. For $\sigma > \alpha + \beta + 1$, the function $\varphi(s)$ can be written in the form $\varphi(s) = \sum_{m=1}^{\infty} \frac{b(m)}{m^s}$. Let, for $n \in \mathbb{N}$, $p_n(s) = \sum_{m=1}^{n} \frac{b(m)}{m^s}$.

LEMMA 2. *The probability measure $v_N(p_n(\sigma + imh) \in A)$, $A \in \mathcal{B}(\mathbb{C})$, converges weakly to some measure on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *as $N \to \infty$.*

Proof. Let $p_1, \ldots, p_r$ be prime numbers which divide $n!$. Define the probability measure $Q_N(A) = v_N((p_1^{-ihm}, \ldots, p_r^{-ihm}) \in A)$, $A \in \mathcal{B}(\Omega_r)$, where $\Omega_r = \prod_{j=1}^{r} \gamma_j$, $\gamma_j = \gamma = \{s \in \mathbb{C}\colon |s| = 1\}$, $j = 1, \ldots, r$. We will prove that the measure Q_N converges weakly to the Haar measure m_{rH} on $(\Omega_r, \mathcal{B}(\Omega_r))$. The Fourier transform of $g_N(k_1, \ldots, k_r)$, $k_1, \ldots, k_r \in \mathbb{Z}$, of the measure Q_N is

$$g_N(k_1, \ldots, k_r) = \frac{1}{U}\sum_{m=N_0}^{N} w(m) \exp\left\{imh\sum_{j=1}^{r} k_j \log p_j\right\}.$$

Clearly, $g_N(0,\ldots,0)=1$. If $(k_1,\ldots,k_r)\neq(0,\ldots,0)$, then the properties of h show that

$$\sum_{N_0\leqslant m\leqslant U}\exp\left\{imh\sum_{j=1}^{r}k_j\log p_j\right\}=\left(ih\sum_{j=1}^{r}k_j\log p_j\right)^{-1}$$
$$\times\left(\exp\left\{ih[U]\sum_{j=1}^{r}k_j\log p_j\right\}-\exp\left\{ihN_0\sum_{j=1}^{r}k_j\log p_j\right\}\right)=B.$$

Hence, summing by parts, we find that in this case

$$\begin{aligned}g_N(k_1,\ldots,k_r)&=\frac{w(N)}{U}\sum_{m=N_0}^{N}\exp\left\{imh\sum_{j=1}^{r}k_j\log p_j\right\}\\&\quad-\frac{1}{U}\int_{N_0}^{N}\sum_{N_0\leqslant m\leqslant U}\exp\left\{imh\sum_{j=1}^{r}k_j\log p_j\right\}\mathrm{d}w(u)\\&=\frac{B}{U}+\frac{B}{U}\int_{N_0}^{N}|\mathrm{d}w(u)|\to 0 \text{ as } N\to\infty.\end{aligned}$$

Consequently,

$$\lim_{N\to\infty}g_N(k_1,\ldots,k_r)=\begin{cases}1 & \text{if } (k_1,\ldots,k_r)=(0,\ldots,0),\\ 0 & \text{if } (k_1,\ldots,k_r)\neq(0,\ldots,0).\end{cases}$$

This shows that the measure Q_N converges weakly to m_{rH} as $N\to\infty$. Let $u\colon\Omega_r\to\mathbb{C}$ be given by the formula

$$u(x_1,\ldots,x_r)=\sum_{m=1}^{r}\frac{b(m)}{m^{\sigma}\prod_{p_j^l\|m}x_r^l},\qquad (x_1,\ldots,x_r)\in\Omega_r.$$

Since $p_n(\sigma+imh)=u(p_1^{-imh},\ldots,p_r^{-imh})$, and the function u is continuous, hence we obtain that the measure of the lemma converges weakly to $m_{rH}u^{-1}$ as $N\to\infty$.

Without loss of generality we can suppose that the function $\varphi(s)$ is analytic in D_2. Really, since all poles are included in a compact set, we can find $N_1\in\mathbb{N}$ such that, for $m\geqslant N_1$, $mh>\max_j\Im\varrho_j$, where ϱ_j are possible poles of the function $\varphi(s)$. Let $U_1=\sum_{m=N_1}^{N}w(m)$. Then, clearly, $U_1/U\to 1$ as $N\to\infty$. Hence we have

$$\begin{aligned}\frac{1}{U}\sum_{\substack{m=N_0\\ \varphi(\sigma+imh)\in A}}^{N}w(m)&=\frac{1}{U}\sum_{\substack{m=N_1\\ \varphi(\sigma+imh)\in A}}^{N}w(m)+\frac{1}{U}\sum_{\substack{m=N_0\\ \varphi(\sigma+imh)\in A}}^{N_1}w(m)\\&=\frac{U_1}{U_1U}\sum_{\substack{m=N_1\\ \varphi(\sigma+imh)\in A}}^{N}w(m)+\mathrm{o}(1)=\frac{1}{U_1}\sum_{\substack{m=N_1\\ \varphi(\sigma+imh)\in A}}^{N}w(m)+\mathrm{o}(1)\end{aligned}$$

uniformly in $A \in \mathcal{B}(\mathbb{C})$ as $N \to \infty$. Therefore, instead of the measure P_N we can consider the measure

$$\frac{1}{U_1} \sum_{\substack{m=N_1 \\ \varphi(\sigma+imh)\in A}}^{N} w(m).$$

Let, for $\sigma_1 > \frac{1}{2}$,

$$\varphi_n(s) = \sum_{m=1}^{\infty} \frac{b(m)}{m^s} \exp\left\{ -\left(\frac{m}{n}\right)^{\sigma_1} \right\}.$$

Then the last series converges absolutely for $\sigma > \alpha + \beta + \frac{1}{2}$.

LEMMA 3. *Let $\sigma > \alpha + \beta + \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_n such that the probability measure*

$$Q_{N,n}(A) = v_N(\varphi_n(\sigma + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_n as $N \to \infty$.

Proof. Proof of the lemma uses Lemma 2 and is similar to that of Lemma 3.1 of [2].

LEMMA 4. *Let $\sigma > \alpha + \beta + \frac{1}{2}$. Then*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{U} \sum_{m=N_0}^{N} w(m)\left|\varphi(\sigma + imh) - \varphi_n(\sigma + imh)\right| = 0.$$

Proof. We apply the condition (4) and arguments of the proof of Lemma 4.3 of [2].

Proof of Theorem 1. By Lemma 3 the probability measure $Q_{N,n}$ converges weakly to P_n as $N \to \infty$. We will prove that the family of probability measures $\{P_n\}$ is tight.

Let η_N be defined on a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ and have the distribution

$$\mathbb{P}(\eta_N = hk) = \frac{w(k)}{N - N_0}, \quad k = N_0, \dots, N.$$

Define $X_{N,n}(\sigma) = \varphi_n(\sigma + i\eta_N)$. Then, clearly,

$$\mathbb{P}(X_{N,n}(\sigma) \in A) = \frac{1}{U} \sum_{m=N_0}^{N} w(m), \quad A \in \mathcal{B}(\mathbb{C}),$$

and thus

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} X_n, \tag{5}$$

where X_n is a complex-valued random element with the distribution P_n. Since the series for $\varphi_n(s)$ is absolutely convergent for $\sigma > +\alpha + \beta + \frac{1}{2}$, we have

$$\sup_{n\geqslant 1}\limsup_{N\to\infty}\frac{1}{U}\sum_{m=N_0}^{N} w(m)\left|\varphi_n(\sigma+imh)\right| \leqslant A < \infty.$$

Now, taking $M = \frac{A}{\varepsilon}$, we find that

$$\limsup_{N\to\infty}\mathbb{P}\big(|X_{N,n}| > M\big) \leqslant \limsup_{N\to\infty}\frac{1}{MU}\sum_{m=N_0}^{N} w(m)\left|\varphi_n(\sigma+imh)\right| \leqslant \varepsilon.$$

Since $|X_{N,n}| \xrightarrow[N\to\infty]{\mathcal{D}} |X_n|$, hence we find that

$$\mathbb{P}\big(|X_n| > M\big) \leqslant \varepsilon.$$

Hence

$$\mathbb{P}\big(X_n \in H_\varepsilon\big) \geqslant 1-\varepsilon \tag{6}$$

for all $n \in \mathbb{N}$, where $H_\varepsilon = \{s \in \mathbb{C}\colon\ |s| \leqslant M\}$. Clearly, H_ε is a compact set, and by (6) $P_n(H_\varepsilon) \geqslant 1-\varepsilon$. This means that the family $\{P_n\}$ is tight, and therefore by Prokhorov's theorem it is relatively compact.

Let $Y_N(\sigma) = \varphi(\sigma + i\eta_N)$. Then by Lemma 4, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{N\to\infty}\mathbb{P}\Big(\left|X_{N,n}(\sigma) - Y_N(\sigma)\right| \geqslant \varepsilon\Big)\\
&\quad = \lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{U}\sum_{\substack{m=N_0\\ |X_{N,n}(\sigma)-Y_N(\sigma)|\geqslant\varepsilon}}^{N} w(m)\\
&\quad \leqslant \lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{\varepsilon U}\sum_{m=N_0}^{N} w(m)\left|X_{N,n}(\sigma) - Y_N(\sigma)\right| = 0.
\end{aligned} \tag{7}$$

From the relative compactness of $\{P_n\}$ it follows the existence of $n_1 \to \infty$ such that P_{n_1} converges weakly to some measure P as $n_1 \to \infty$, or

$$X_{n_1} \xrightarrow[n_1\to\infty]{\mathcal{D}} P.$$

This, (5), (7) and Theorem 4.2 of [1] shows that $Y_N \xrightarrow[N\to\infty]{\mathcal{D}} P$, and the theorem is proved.

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REZIUMĖ

R. Kačinskaitė. Diskreti ribinė teorema su svoriu kompleksinėje plokštumoje Matsumoto dzeta funkcijai

Straipsnyje įrodyta diskreti ribinė teorema su svoriu silpno tikimybinių matų konvergavimo prasme kompleksinėje plokštumoje Matsumoto dzeta funkcijai.