

On the fourth moment of the periodic zeta-function

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Let $s = \sigma + it$ be a complex variable, and let $\mathfrak{A} = \{a_m, m \in \mathbb{Z}\}$ be a sequence of complex numbers with period $k > 0$. The periodic zeta-function $\zeta(s; \mathfrak{A})$, for $\sigma > 1$, is defined by

$$\zeta(s; \mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and by analytic continuation elsewhere. In [3], [4], [5] the asymptotics for the mean square

$$\int_0^T |\zeta(\sigma + it, \mathfrak{A})|^2 dt, \quad T \rightarrow \infty,$$

in the region $\frac{1}{2} \leq \sigma \leq 1$ was obtained. The aim of this note is to obtain the estimate for

$$I_2\left(T, \frac{1}{2}\right) = \int_0^T \left| \zeta\left(\frac{1}{2} + it, \mathfrak{A}\right) \right|^4 dt.$$

Denote by B a quantity bounded by a constant. Let $K(k) = \sum_{q=1}^k |a_q|^2$ and $\varphi(k)$, as usual, denote the Euler function.

THEOREM. *Let k be a prime number, and $T \rightarrow \infty$. Then*

$$I_2\left(T, \frac{1}{2}\right) = Bk^3 K^2(k) T \log^4 T.$$

Proof. Let χ, χ_1, χ_2 denote Dirichlet characters modulo k , $L(s, \chi)$, $L(s, \chi_1)$, $L(s, \chi_2)$ be corresponding Dirichlet L -functions, and let $\zeta(s)$ be the Riemann zeta-function. Then in [6] it was obtained that

$$\begin{aligned} \zeta^2(s, \mathfrak{A}) &= \frac{1}{\varphi^2(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1(\bmod k)} \sum_{\chi_2(\bmod k)} \bar{\chi}_1(q_1) \bar{\chi}_2(q_2) a_{q_1} a_{q_2} L(s, \chi_1) L(s, \chi_2) \\ &+ \frac{2a_k \zeta(s)}{k^s \varphi(k)} \sum_{q=1}^{k-1} \sum_{\chi(\bmod k)} \bar{\chi}(q) a_q L(s, \chi) + \frac{a_k^2}{k^{2s}} \zeta^2(s). \end{aligned}$$

Hence we find

$$\begin{aligned}
|\zeta(s, \mathfrak{A})|^4 &= \zeta^2(s, \mathfrak{A}) \overline{\zeta^2(s, \mathfrak{A})} \\
&= \left(\frac{1}{\varphi^2(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1(\bmod k)} \sum_{\chi_2(\bmod k)} \bar{\chi}_1(q_1) \bar{\chi}_2(q_2) a_{q_1} a_{q_2} L(s, \chi_1) L(s, \chi_2) \right. \\
&\quad \left. + \frac{2a_k \zeta(s)}{k^s \varphi(k)} \sum_{q=1}^{k-1} \sum_{\chi(\bmod k)} \bar{\chi}(q) a_q L(s, \chi) + \frac{a_k^2}{k^{2s}} \zeta^2(s) \right) \\
&\quad \times \left(\frac{1}{\varphi^2(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1(\bmod k)} \sum_{\chi_2(\bmod k)} \chi_1(q_1) \chi_2(q_2) \bar{a}_{q_1} \bar{a}_{q_2} \overline{L(s, \chi_1) L(s, \chi_2)} \right. \\
&\quad \left. + \frac{2\bar{a}_k \overline{\zeta(s)}}{k^{\bar{s}} \varphi(k)} \sum_{q=1}^{k-1} \sum_{\chi(\bmod k)} \chi(q) \bar{a}_q \overline{L(s, \chi)} + \frac{\overline{a_k^2 \zeta^2(s)}}{k^{2\bar{s}}} \right) \\
&= \frac{1}{\varphi^4(k)} \left| \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1(\bmod k)} \sum_{\chi_2(\bmod k)} \bar{\chi}_1(q) \bar{\chi}_2(q) a_{q_1} a_{q_2} L(s, \chi_1) L(s, \chi_2) \right|^2 \\
&\quad + \frac{4|a_k|^2 |\zeta(s)|^2}{k^{2\sigma} \varphi^2(k)} \left| \sum_{q=1}^{k-1} \sum_{\chi(\bmod k)} \bar{\chi}(q) a_q L(s, \chi) \right|^2 + \frac{|a_k|^4 |\zeta(s)|^4}{k^{4\sigma}} \\
&\quad + \frac{2\bar{a}_k \overline{\zeta(s)}}{k^{\bar{s}} \varphi^3(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1(\bmod k)} \sum_{\chi_2(\bmod k)} \bar{\chi}_1(q_1) \bar{\chi}(q_2) a_{q_1} a_{q_2} \\
&\quad \times L(s, \chi_1) L(s, \chi_2) \sum_{q=1}^{k-1} \sum_{\chi(\bmod k)} \chi(q) \bar{a}_q \overline{L(s, \chi)} \\
&\quad + \frac{\overline{a_k^2 \zeta^2(s)}}{k^{2\bar{s}} \varphi^2(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1(\bmod k)} \sum_{\chi_2(\bmod k)} \bar{\chi}_1(q_1) \bar{\chi}_2(q_2) a_{q_1} a_{q_2} L(s, \chi_1) L(s, \chi_2) \\
&\quad + \frac{2a_k \zeta(s)}{k^2 \varphi^3(k)} \sum_{q=1}^{k-1} \sum_{\chi(\bmod k)} \bar{\chi}(q) a_q L(s, \chi) \\
&\quad \times \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1(\bmod k)} \sum_{\chi_2(\bmod k)} \chi_1(q_1) \chi_2(q_2) \bar{a}_{q_1} \bar{a}_{q_2} \overline{L(s, \chi_1) L(s, \chi_2)} \\
&\quad + \frac{2a_k \zeta(s) \overline{\zeta^2(s)} a_k^2}{k^s k^{2\bar{s}} \varphi(k)} \sum_{q=1}^{k-1} \sum_{\chi(\bmod k)} \bar{\chi}(q) a_q L(s, \chi) \\
&\quad + \frac{a_k^2 \zeta^2(s)}{\varphi^2(k) k^{2s}} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1(\bmod k)} \sum_{\chi_2(\bmod k)} \chi_1(q_1) \chi_2(q_2) \bar{a}_{q_1} \bar{a}_{q_2} \overline{L(s, \chi_1) L(s, \chi_2)}
\end{aligned}$$

$$+ \frac{2\bar{a}_k a_k^2 \overline{\zeta(s)} \zeta^2(s)}{k^s k^{2s} \varphi(k)} \sum_{q=1}^{k-1} \sum_{\chi \pmod{k}} \chi(q) \bar{a}_q \overline{L(s, \chi)}.$$

Therefore

$$\begin{aligned} \int_0^T \left| \zeta\left(\frac{1}{2} + it, \mathfrak{A}\right) \right|^4 dt &= \frac{Bk^2}{\varphi^2(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{\chi_1 \pmod{k}} \sum_{\chi_2 \pmod{k}} |a_{q_1}|^2 |a_{q_2}|^2 \\ &\times \int_0^T \left| L\left(\frac{1}{2} + it, \chi_1\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi_2\right) \right|^2 dt + \frac{B|a_k|^2}{\varphi(k)} \sum_{q=1}^{k-1} \sum_{\chi \pmod{k}} |a_q|^2 \\ &\times \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt + \frac{B|a_k|^4}{k^2} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \\ &+ \frac{B|a_k|}{k^{\frac{1}{2}} \varphi^3(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} \sum_{q=1}^{k-1} |a_{q_1}| |a_{q_2}| |a_q| \sum_{\chi \pmod{k}} \sum_{\chi_1 \pmod{k}} \sum_{\chi_2 \pmod{k}} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right| \\ &\times \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| L\left(\frac{1}{2} + it, \chi_1\right) \right| \left| L\left(\frac{1}{2} + it, \chi_2\right) \right| dt \\ &+ \frac{B|a_k|^2}{k \varphi^2(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} |a_{q_1}| |a_{q_2}| \sum_{\chi_1 \pmod{k}} \sum_{\chi_2 \pmod{k}} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \\ &\times \left| L\left(\frac{1}{2} + it, \chi_1\right) \right| \left| L\left(\frac{1}{2} + it, \chi_2\right) \right| dt \\ &+ \frac{B|a_k|}{k^{\frac{1}{2}} \varphi^3(k)} \sum_{q=1}^{k-1} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} |a_q| |a_{q_1}| |a_{q_2}| \sum_{\chi \pmod{k}} \sum_{\chi_1 \pmod{k}} \sum_{\chi_2 \pmod{k}} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right| \\ &\times \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| L\left(\frac{1}{2} + it, \chi_1\right) \right| \left| L\left(\frac{1}{2} + it, \chi_2\right) \right| dt \\ &+ \frac{B|a_k|^3}{k^{\frac{3}{2}} \varphi(k)} \sum_{q=1}^{k-1} |a_q| \sum_{\chi \pmod{k}} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^3 \left| L\left(\frac{1}{2} + it, \chi\right) \right| dt \\ &+ \frac{B|a_k|^2}{k \varphi^2(k)} \sum_{q_1=1}^{k-1} \sum_{q_2=1}^{k-1} |a_{q_1}| |a_{q_2}| \sum_{\chi_1 \pmod{k}} \sum_{\chi_2 \pmod{k}} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \\ &\times \left| L\left(\frac{1}{2} + it, \chi_1\right) \right| \left| L\left(\frac{1}{2} + it, \chi_2\right) \right| dt \\ &+ \frac{B|a_q|^3}{k^{\frac{3}{2}} \varphi(k)} \sum_{q=1}^{k-1} |a_q| \sum_{\chi \pmod{k}} \left| \zeta\left(\frac{1}{2} + it\right) \right|^3 \left| L\left(\frac{1}{2} + it, \chi\right) \right| dt. \end{aligned} \tag{1}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
& \int_0^T \left| L\left(\frac{1}{2} + it, \chi_1\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi_2\right) \right|^2 dt \\
& \leq \left(\int_0^T \left| L\left(\frac{1}{2} + it, \chi_1\right) \right|^4 dt \int_0^T \left| L\left(\frac{1}{2} + it, \chi_2\right) \right|^4 dt \right)^{\frac{1}{2}}, \\
& \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt \\
& \leq \left(\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \int_0^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \right)^{\frac{1}{2}}, \\
& \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right| \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| L\left(\frac{1}{2} + it, \chi_1\right) \right| \left| L\left(\frac{1}{2} + it, \chi_2\right) \right| dt \\
& \leq \left(\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt \int_0^T \left| L\left(\frac{1}{2} + it, \chi_1\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi_2\right) \right|^2 dt \right)^{\frac{1}{2}} \\
& \leq \left(\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \int_0^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \right. \\
& \quad \left. \times \int_0^T \left| L\left(\frac{1}{2} + it, \chi_1\right) \right|^2 dt \int_0^T \left| L\left(\frac{1}{2} + it, \chi_2\right) \right|^4 dt \right)^{\frac{1}{4}}, \\
& \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi_1\right) \right| \left| L\left(\frac{1}{2} + it, \chi_2\right) \right| dt \\
& \leq \left(\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \int_0^T \left| L\left(\frac{1}{2} + it, \chi_1\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi_2\right) \right|^2 dt \right)^{\frac{1}{2}} \\
& \leq \left(\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{2}} \left(\int_0^T \left| L\left(\frac{1}{2} + it, \chi_1\right) \right|^4 dt \int_0^T \left| L\left(\frac{1}{2} + it, \chi_2\right) \right|^4 dt \right)^{\frac{1}{4}}, \\
& \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^3 \left| L\left(\frac{1}{2} + it, \chi\right) \right| dt \\
& \leq \left(\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 dt \right)^{\frac{1}{2}} \\
& \leq \left(\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \right)^{\frac{1}{2}} \left(\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \int_0^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \right)^{\frac{1}{4}}. \quad (2)
\end{aligned}$$

Let $K(k) = \sum_{q=1}^k |a_q|^2$. Moreover, it is well known that

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt = BT \log^4 T,$$

and

$$\int_0^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt = BTk \log^4 T.$$

Hence and from (1) and (2) we find

$$\begin{aligned} \int_0^T \left| \zeta \left(\frac{1}{2} + it, \mathfrak{A} \right) \right|^4 dt &= Bk^3 K^2(k) T \log^4 T + B|a_k|^2 k^{\frac{1}{2}} K(k) T \log^4 T \\ &+ \frac{B|a_k|^4}{k^2} T \log^4 T + B|a_k| k^{\frac{7}{4}} K^{\frac{3}{2}}(k) T \log^4 T \\ &+ B|a_k|^2 k^{\frac{1}{2}} K(k) T \log^4 T + B|a_k| k^{\frac{7}{4}} K^{\frac{3}{2}}(k) T \log^4 T \\ &+ \frac{B|a_k|^3}{k^{\frac{3}{4}}} K^{\frac{1}{2}}(k) T \log^4 T + B|a_k|^2 k^{\frac{1}{2}} K(k) T \log^4 T \\ &+ \frac{B|a_k|^3}{k^{\frac{3}{4}}} K^{\frac{1}{2}}(k) T \log^4 T. \end{aligned}$$

This proves the theorem.

References

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REZIUMĖ

D. Šiaučiūnas, A.A. Laurutis. Apie periodinės dzeta funkcijos ketvirtąjį momentą

Straipsnyje gautas periodinės dzeta funkcijos ketvirtojo momento įvertis kritinėje tiesėje, kai periodo ilgis k yra pirminis skaičius.