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Reduced game and its solutions

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Abstract. In the paper the reduced game of coalitional game is defined and its relationship with solutions is investigated.

Keywords: coalitional game, reduced game, consistency.

One of the properties of cooperative game solutions is consistency, connecting solution vectors of a cooperative game with finite set of players and its reduced game, defined by removing one or more players and assigning them the payoffs according to some specific principle (e.g., a proposed payoff vector). Consistency of a solution means that any part (defined by a coalition of the original game) of a solution payoff vector belongs to the solution set of the corresponding reduced game.

Consistency is general property that many ideas appearing in social sciences have in common. It has been shown that considerable number of concepts from allocation rules, public finance, game theory and many other fields in social sciences satisfy consistency. Particularly in game theory, many solutions have been axiomatized by consistency.

The reduced game of coalitional games were defined in [1]. In [1] also was investigated the relation between reduced game and solutions of coalitional game. In this article we are investigating another type of reduced game.

Let (N, v) – coalitional game in characteristic function form, S $(S \subset N)$ – coalition, $X(N) = \{x \mid x \in R^n \text{ and } \sum_{i \in N} x_i = v(N)\}$ – set of payoff vectors in game (N, v).

DEFINITION 1. A solution of the game (N, v) is a function σ which associates with each game (N, v) a subset $s(N, v)$ of $X(N)$.

DEFINITION 2. Let $x \in X(N)$, $S \subset N$, $S \neq \varnothing$. The reduced game with respect to S and x is the game (S, v_x):

$$v_x(T) = \begin{cases} X(T), & T = S,\ T = \varnothing, \\ \sum\limits_{M \subset N \setminus S} \frac{v(T \cup M) - X(M)}{2^{n-s} - 1}, & \text{otherwise}, \end{cases} \tag{1}$$

where $X(T) = \sum_{i \in T} x_i$.

DEFINITION 3. Let σ be a solution of the game (N, v). Then σ has the reduced game property if it satisfies the following condition: if $x \in \sigma(N, v)$ then $x^s \in \sigma(S, v_x)$, where x^s is the restriction of x to S.

THEOREM 1. *The core C of game (N, v) has the reduced game property.*

Proof. If $x \in R^n$ and $S \subset N$, we denote $X(S) = \sum_{i \in S} x_i$.

Let $x \in C(N, v)$ and $S \subset N$, $S \neq \varnothing$, $T \subset S$. Since $x \in C(N, v) \Longleftrightarrow X(T) \geqslant v(T)$, $X(N) = v(N)$, then will investigate the difference $v_x(T) - X(T)$.

If $T = S$, then

$$v_x(T) - X(T) = v_x(S) - X(S) = X(S) - X(S) = 0.$$

If $T \neq S$, then

$$\begin{aligned} v_x(T) - X(T) &= \frac{\sum_{M \subset N \setminus S} v(T \cup M) - X(M)}{2^{n-s} - 1} - X(T) \\ &= \sum_{M \subset N \setminus S} \frac{v(T \cup M) - X(T \cup M)}{2^{n-s} - 1} \leqslant 0. \end{aligned}$$

Thus we have that $X(T) \geqslant v_x(T)$, $\forall\, T \subset S$.

Hence, if $x \in C(N, v)$, then $x^s \in C(S, v_x(S))$.

DEFINITION 4. A class B of reasonable issues is set of such payoff vectors which satisfy inequality

$$x_i \leqslant b(i) = \max_K \left[v(K) - v(K \setminus i)\right], \quad \forall\, i \in N.$$

This inequality, of course, is very weak requirement and it is unlikely to be sufficient for real concept of optimum. Such payoff vectors are reasonable, not optimal. Distinction of these terms means, that not all vectors from B are optimal, but vectors which do not belong B, are not. This inequality we may interpret so: if at any stage of negotiation was formed coalition $K \setminus i$, then it may involve player i only when $x_i \leqslant v(K) - v(K \setminus i)$. So, when we define class B, we are regarding only change of coalitions from K to $K \setminus i$. Set B may be very large. On the other side, against x which don't satisfy above mentioned inequality always will be threat, for example by coalition $N \setminus i$, but counter-threat may will not be. Hence class B involve many solutions also and $N - M$ solution.

If N is a coalition then we denote $\pi = \pi(N) = \{i, j \mid i, j \in N, i \neq j\}$.

DEFINITION 5. Let σ be a solution of the game (N, v). σ has the converse reduced game property if the following condition is satisfied: if $x^s \in \sigma(S, v_x)$, $\forall S$, $S \in \pi(N)$, then $x \in \sigma(N, v)$.

THEOREM 2. *Let (N, v) be a game. Then the class B has converse reduced game (1) property.*

Proof.

$$\begin{aligned}
x_i \leqslant b_x^s(i) &= \max_{T\subset S}\left[v_x(T) - v_x(T\setminus i)\right] \\
&= \max_{T\subset S}\frac{\sum\limits_{Q\subset N\setminus S} v(T\cup Q) - X(Q) - v((T\setminus i)\cup Q) + X(Q)}{2^{n-s}-1} \\
&\leqslant \max_{T\subset S}\max_{Q\subset N\setminus S}\Big[v(T\cup Q) - v\big((T\cup Q)\setminus i\big)\Big] \\
&\leqslant \max_{K\subset N}\left[v(K) - v(K\setminus i)\right] = b(i).
\end{aligned}$$

References

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REZIUMĖ

V. Dumskis. Redukuotas lošimas ir jo sprendiniai

Straipsnyje apibrėžiamas specialaus tipo redukuotas lošimas koaliciniam lošimui bei nagrinėjamas ryšys tarp redukuoto lošimo ir koalicinio lošimo sprendinių.