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Inferences on Generalized Qi Estimators

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Abstract

Assuming that all observations are available for analysis, we consider block-type estimators for a positive extreme value index. These estimators are presented as some function of generalized Qi statistics. Weak consistency and asymptotic normality of the estimators under consideration are proved. Also, we compare theoretically the considered estimators with corresponding peaks-over-threshold-type estimators, which are the same function of generalized Hill statistics. We prove that these estimators do not dominate each other. The practical validation of asymptotic comparison results for finite samples is done by means of Monte Carlo simulation for several heavy-tailed models.

Keywords: asymptotic normality; extreme value index; Hill estimator; generalized Hill statistics

MSC: 62F12; 62G32; 60F05

1. An Introduction and Main Results

Let $(X_1, X_2, \ldots, X_n)$ be a sample of independent and identically distributed (i.i.d.) random variables from an unknown distribution function (d.f.) F. In the present paper, we will formulate our assumptions in terms of a quantile-type function U (associated with F), which is the generalized inverse function of the function $1/(1-F)$, i.e., $U(t) = F^{\leftarrow}(1-1/t)$, $t > 1$. We suppose that the function U belongs to the class of regularly varying functions at infinity with an index $\gamma > 0$, denoted by $U \in RV_\gamma$, i.e., for all $x > 0$,

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\gamma. \tag{1}$$

The parameter γ is a positive extreme value index (EVI). The estimation of the EVI is one of the main problems in extreme value theory. In this paper, we assume that all observations are available for analysis and consider the family of semiparametric block-type estimators (of a positive γ) defined as

$$\hat{\gamma}_n(m_n, \ell_n, (r)_s) = g(Q_n(m_n, \ell_n, r_1), \ldots, Q_n(m_n, \ell_n, r_s)), \tag{2}$$

where $(r)_s = (r_1, \ldots, r_s)$, $s \in \mathbb{N}$, is a real vector of tuning parameters such that $r_i \neq r_j$, $i \neq j$, while

$$g : \mathcal{I}(r_1) \times \cdots \times \mathcal{I}(r_s) \to (0, \infty), \quad \mathcal{I}(r_i) = \begin{cases} (0, +\infty), & r_i \geq 0, \\ (0, -1/r), & r_i < 0 \end{cases}$$

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is some function to be specified below. In order to define the block-type statistics $Q_n(m_n, \ell_n, r)$, $r \in \mathbb{R}$, let us assume that ℓ_n and m_n are sequences of natural numbers such that

$$1 \le \ell_n < m_n \le [n/2], \tag{3}$$

where $[\cdot]$ denotes the integer part. We divide the observations $X_1, \ldots, X_n$ into non-overlapping blocks of size m_n:

$$B_i = \{X_{(i-1)m_n+1}, \ldots, X_{im_n}\}, \quad 1 \le i \le [n/m_n].$$

Let $X^{(i)}_{1,m_n} \le X^{(i)}_{2,m_n} \le \cdots \le X^{(i)}_{m_n,m_n}$ denote the order statistics of the observations in the block B_i. Then, $Q_n(m_n, \ell_n, r)$, $r \in \mathbb{R}$, is a family of block-type statistics defined as

$$Q_n(m_n, \ell_n, r) = \frac{1}{[n/m_n]} \sum_{i=1}^{[n/m_n]} \frac{1}{\ell_n} \sum_{j=1}^{\ell_n} h_r\left(\frac{X^{(i)}_{m_n-j+1,m_n}}{X^{(i)}_{m_n-\ell_n,m_n}}\right), \tag{4}$$

where

$$h_r(x) = \begin{cases} (x^r - 1)/r, & r \ne 0, \\ \ln(x), & r = 0. \end{cases}$$

Statistics $Q_n(m_n, \ell_n, r)$, $r \in \mathbb{R}$, have been suggested in [1]. The asymptotic properties (including weak consistency and asymptotic normality) of $Q_n(m_n, \ell_n, r)$, $r \in \mathbb{R}$, were investigated in [1] assuming that

$$\ell_n \equiv \tilde{n}, \tag{5}$$

where $\tilde{n}$ is some natural number and m_n is an intermediate sequence, i.e.,

$$m_n \to \infty, \quad \frac{n}{m_n} \to \infty, \quad n \to \infty. \tag{6}$$

The idea of extending the block maxima approach and using the two largest observations $X^{(i)}_{m_n,m_n}$ and $X^{(i)}_{m_n-1,m_n}$ in each block B_i was proposed in [2]. In particular, the paper [2] contains the results on statistics $Q_n(m_n, 1, -1)$. The statistics $Q_n(m_n, 1, r)$, $r \in \mathbb{R}$, were considered in [3]. The asymptotic properties of statistics, which were obtained by replacing h_r with an arbitrary function h satisfying several regularity conditions in statistics $Q_n(m_n, 1, r)$, were investigated in [4]. We believe that $Q_n(m_n, \ell_n, r)$, $r \in \mathbb{R}$ with m_n and ℓ_n satisfying (3), can be called generalized Qi statistics because these statistics generalize the estimator $\hat{\gamma}_n(m_n, \tilde{n}) = Q_n(m_n, \tilde{n}, 0)$, proposed in [5]. Accordingly, we will say that (2)–(3) are generalized Qi estimators for a positive EVI.

Replacing (3) with the conditions $[n/2] < m_n \le n$, $1 \le \ell_n < m_n$ eliminates blocks from the statistics $Q_n(m_n, \ell_n, r)$, $r \in \mathbb{R}$, and changes the type: $Q_n(m_n, \ell_n, r)$, $r \in \mathbb{R}$, are the statistics of peaks over the threshold. One can note that in the case where $[n/2] < m_n < n$, $1 \le \ell_n < m_n$, the statistics $Q_n(m_n, \ell_n, r)$, $r \in \mathbb{R}$, miss some observations. Therefore, it is convenient to consider only the case $m_n = n$. The so-called generalized Hill statistics $H_n(k_n, r) = Q_n(n, k_n, r)$, $r \in \mathbb{R}$, where k_n is an intermediate sequence, were proposed in [6]. The Hill estimator (introduced in [7]), which is one of the first estimators (together with the estimator proposed in [8]) and the most popular estimator of a positive EVI, coincides with $H_n(k_n, 0)$. We refer to the paper [9], in which the asymptotic normality of the Hill estimator is proved. We also refer to the papers [10,11], which are related to the investigation of the asymptotic normality of statistics $H_n(k_n, r)$, $r \in \mathbb{R}$. The statistics considered in [12] are linear combinations of statistics $H_n(k_n, r)$, $r \in \mathbb{R}$. Under several restrictions, the asymptotic behavior of so-called residual estimators (statistics $H_n(k_n, r)$, where h_r is replaced by an arbitrary function h) is discussed in [13]. The multivariate central

limit theorem for generalized Hill statistics can be found in [14]. An application of statistics $H_n(k_n, -1)$ and $H_n(k_n, 0)$ in the estimation of $\gamma \in \mathbb{R}$ is provided in [15].

The aim of this paper is to investigate the asymptotic behavior of the estimators $\hat{\gamma}_n(m_n, \ell_n, (r)_s)$ under assumptions (6). The assumptions (6) on a sequence m_n imply that each block width m_n and the number of blocks $[n/m_n]$ in (4) grow together with the number of observations n. As for assumptions on a sequence ℓ_m, we note that random thresholds $X^{(i)}_{m_n-\ell_n,m_n}$, $1 \le i \le [n/m_n]$, can be (i) extreme order statistics, (ii) intermediate order statistics, and (iii) central order statistics. See, e.g., [16] for the classification of order statistics. Under either assumption (i) or assumption (ii), together with (1), it can be proved that $X^{(i)}_{m_n-\ell_n,m_n}$ tends to infinity in probability, while under assumption (iii) and (1), $X^{(i)}_{m_n-\ell_n,m_n}$ converges in probability to $F^{\leftarrow}(1-p)$ as $n \to \infty$. This means that the sequence of thresholds $X^{(i)}_{m_n-\ell_n,m_n}$ allows for the use of relatively small observations in (2). For this reason, we exclude the case (iii) from our further investigation. We note that the assumptions (i) and (ii) are equivalent to (5) and

$$\ell_n \to \infty, \quad \frac{m_n}{\ell_n} \to \infty, \quad n \to \infty, \tag{7}$$

respectively.

The asymptotic properties (including weak consistency and asymptotic normality) of the estimators (2) were considered in [17] in the case when (5) and (6) hold. But for the sake of completeness, we include the results from [17] in Theorems 1–2 and Corollary 1 below. Our first result states that $Q_n(m_n, \ell_n, r)$ is a weakly consistent estimator of $\Lambda_r(\gamma) = \gamma/(1-\gamma r)$, $\gamma r < 1$.

Theorem 1. *Suppose that $X_1, \dots, X_n$ are i.i.d. random variables with d.f. F, whose quantile-type function U satisfies condition $U \in RV_\gamma$ for some $\gamma > 0$. Let*

$$\gamma r < 1. \tag{8}$$

Let $m = m_n$ be an intermediate sequence and let one of the assumptions (5) and (7) be satisfied. Then,

$$Q_n(m_n, \ell_n, r) \xrightarrow{\mathrm{P}} \Lambda_r(\gamma), \ n \to \infty. \tag{9}$$

Here and below, $\xrightarrow{\mathrm{P}}$ denotes the convergence in probability.

The function g in (2) is uniquely determined when $s = 1$. The inverse of Λ_r has the form $\Lambda_r^{\leftarrow}(t) = t/(1+rt)$. By substituting

$$g(t) = \Lambda_r^{\leftarrow}(t) \tag{10}$$

into (2), we obtain

$$\hat{\gamma}_n(m_n, \ell_n, r) = \frac{Q_n(m_n, \ell_n, r)}{1 + rQ_n(m_n, \ell_n, r)}. \tag{11}$$

Put

$$r = \begin{cases} r_1, & s = 1, \\ \max\{r_1, \dots, r_s\}, & s \ge 2. \end{cases}$$

Corollary 1. *Suppose that the assumptions of Theorem 1 hold. Suppose also that one of two conditions holds: (a) if $s = 1$, then a function g is defined by (10); or (b) if $s \ge 2$, then a function g is such that*

$$g(\Lambda_{r_1}(\gamma), \dots, \Lambda_{r_s}(\gamma)) = \gamma \tag{12}$$

and g is continuous at $(\Lambda_{r_1}(\gamma), \dots, \Lambda_{r_s}(\gamma))$.

Then,

$$\hat{\gamma}_n(m_n, \ell_n, (r)_s) \xrightarrow{\mathrm{p}} \gamma,\ n \to \infty.$$

To prove the asymptotic normality for (2), we need the so-called second-order regular variation condition. We recall that the function U satisfies the second-order regular variation condition if for some positive or negative function A with $A(t) \to 0$, $t \to \infty$,

$$\lim_{t\to\infty} \frac{1}{A(t)} \left(\frac{U(tx)}{U(t)} - x^\gamma \right) = x^\gamma h_\rho(x), \quad x > 0, \tag{13}$$

where $\rho < 0$ is the second-order parameter. It is known that $|A(t)| \in RV_\rho$. See [18]. For example, any quantile-type function U belonging to the Hall–Welsh class ([19,20]), i.e., such that, with $C > 0$, $\rho < 0$ and $\beta \in \mathbb{R} \setminus \{0\}$,

$$U(t) = Ct^\gamma \left(1 + \frac{\gamma\beta}{\rho} t^\rho + o(t^\rho) \right), \quad t \to \infty, \tag{14}$$

satisfies (13).

Before formulating our next result, we need to introduce the notation. Let us denote

$$\begin{aligned}
\nu(\rho, t) &= \frac{1}{(1-t-\rho)(1-t)}, \quad t < 1, \\
\Xi(\rho, t_1, \dots, t_s) &= (\nu(\rho, t_1), \dots, \nu(\rho, t_s)), \quad t_i < 1,\ 1 \le i \le s, \\
\sigma(t_1, t_2) &= \frac{\gamma^2}{(1-t_1-t_2)(1-t_1)(1-t_2)}, \quad t_1 < 1/2,\ t_2 < 1/2, \\
S(t_1, \dots, t_s) &= \begin{pmatrix} \sigma(t_1,t_1) & \dots & \sigma(t_1,t_s) \\ \dots & \dots & \dots \\ \sigma(t_s,t_1) & \dots & \sigma(t_s,t_s) \end{pmatrix}, \quad t_i < 1/2,\ 1 \le i \le s.
\end{aligned}$$

In the case $s = 1$, a function g has a derivative at the point $\Lambda_r(\gamma)$ and $g'(\Lambda_r(\gamma)) = (1-\gamma r)^2$. As for the case $s \ge 2$, we denote

$$g'(\Lambda_{r_1}(\gamma), \dots, \Lambda_{r_s}(\gamma)) = \left(\frac{\partial g(t_1, \dots, t_s)}{\partial t_1} \ \dots \ \frac{\partial g(t_1, \dots, t_s)}{\partial t_s} \right) \Bigg|_{(t_1,\dots,t_s)=(\Lambda_{r_1}(\gamma),\dots,\Lambda_{r_s}(\gamma))}.$$

Theorem 2. *Suppose that $X_1, \dots, X_n$ are i.i.d. random variables with d.f. F, whose quantile-type function U satisfies condition (13) for some $\gamma > 0$ and $\rho < 0$. Suppose that the tuning parameters $(r)_s$, $s \ge 1$, satisfy the condition*

$$\gamma r < 1/2. \tag{15}$$

Suppose that m_n is an intermediate sequence, i.e., (6) holds. Suppose that one of two conditions holds: (a) if $s = 1$, then a function g is defined by (10); or (b) if $s \ge 2$, then a function g satisfies condition (12), and all partial derivatives $\partial g(t_1, \dots, t_s)/\partial t_i$ exist for $(t_1, \dots, t_s)$ in a neighborhood of $(\Lambda_{r_1}(\gamma), \dots, \Lambda_{r_s}(\gamma))$ and are continuous at

$$(\Lambda_{r_1}(\gamma), \dots, \Lambda_{r_s}(\gamma)).$$

(i) Let ℓ_n satisfy condition (5) and

$$\lim_{n\to\infty} \sqrt{\frac{n}{m_n}} A(m_n) = \mu_1 \in (-\infty, +\infty).$$

Then,

$$\sqrt{\frac{n}{m_n}}(\hat{\gamma}_n(m_n,\ell_n,(r)_s)-\gamma)\xrightarrow{\mathrm{d}}\mathcal{N}\left(\frac{\mu_1\kappa(\gamma,\rho,(r)_s)\Gamma(1+\tilde{n}-\rho)}{\Gamma(1+\tilde{n})},\frac{\varsigma(\gamma,(r)_s)}{\tilde{n}}\right) \tag{16}$$

as $n\to\infty$, *where*

$$\begin{aligned}\kappa(\gamma,\rho,(r)_s) &= g'(\Lambda_{r_1}(\gamma),\ldots,\Lambda_{r_s}(\gamma))(\Xi(\rho,\gamma r_1,\ldots,\gamma r_s))^T, &(17)\\ \varsigma(\gamma,(r)_s) &= g'(\Lambda_{r_1}(\gamma),\ldots,\Lambda_{r_s}(\gamma))S(\gamma r_1,\ldots,\gamma r_s)(g'(\Lambda_{r_1}(\gamma),\ldots,\Lambda_{r_s}(\gamma)))^T &(18)\end{aligned}$$

and the transpose of g' *is denoted by* $(g')^T$. *Here and below,* $\xrightarrow{\mathrm{d}}$ *denotes convergence in distribution.*

(ii) Let ℓ_n *satisfy condition (7) and*

$$\lim_{n\to\infty}\sqrt{\frac{n}{m_n/\ell_n}}A\left(\frac{m_n}{\ell_n}\right)=\mu_2\in(-\infty,+\infty). \tag{19}$$

Then,

$$\sqrt{\frac{n}{m_n/\ell_n}}(\hat{\gamma}_n(m_n,\ell_n,(r)_s)-\gamma)\xrightarrow{\mathrm{d}}\mathcal{N}(\mu_2\kappa(\gamma,\rho,(r)_s),\varsigma(\gamma,(r)_s)),\ n\to\infty. \tag{20}$$

Here, $\kappa(\gamma,\rho,(r)_s)$ *and* $\varsigma(\gamma,(r)_s)$ *are the same as in (16).*

Remark 1. *By applying Theorem 2(i), we immediately obtain the theorem in [2], Theorem 1 in [5], Theorem 2 in [3] and Theorem 1 in [21]. Also, applying Theorem 2(ii), we obtain Theorem 3 in [21].*

The paper is organized as follows. In the next section, we compare estimators (2) with the corresponding peaks-over-threshold-type estimators theoretically. Monte Carlo simulation results are presented in Section 3. A discussion on the application of the main results can be found in Section 4. Appendix A is devoted to the proofs of the results.

2. A Theoretical Comparison of the Estimators

In this section, we provide a comparison of the estimators (2) with the estimators

$$\tilde{\gamma}_n(k_n,(r)_s)=g(H_n(k_n,r_1),\ldots,H_n(k_n,r_s)) \tag{21}$$

following closely the methodology provided in [9].

In the case $s=1$, the estimators (21) have the form

$$\tilde{\gamma}_n(k_n,r)=\frac{H_n(k_n,r)}{1+rH_n(k_n,r)},\quad r\in\mathbb{R}. \tag{22}$$

It should be noted that the family of estimators $\tilde{\gamma}_n(k_n,r)$, $r\in\mathbb{R}$, was introduced in [6,10,11] at approximately the same time.

Assume that the conditions of Theorem 2(ii) are satisfied. It is known that the asymptotic relation (20) remains valid if $n/(m_n/\ell_n)$ is replaced by the intermediate sequence k_n and $\hat{\gamma}_n(m_n,\ell_n,r)$ is replaced by $\tilde{\gamma}_n(k_n,r)$. Cf. [6,10,11] for the case $s=1$ and [14] when $s\geq 2$.

Fix a collection of different tuning parameters $(r)_s$ such that (15) holds. Similarly to [9], we define the asymptotic mean squared errors (AMSEs)

$$\begin{aligned}\mathrm{AMSE}(\hat{\gamma}_n(m_n,\ell_n,(r)_s)) &= A^2(m_n/\ell_n)\kappa^2(\gamma,\rho,(r)_s)+\varsigma(\gamma,(r)_s)\frac{m_n/\ell_n}{n},\\ \mathrm{AMSE}(\tilde{\gamma}_n(k_n,(r)_s)) &= A^2(n/k_n)\kappa^2(\gamma,\rho,(r)_s)+\varsigma(\gamma,(r)_s)\frac{1}{k_n},\end{aligned}$$

with the restriction $\kappa(\gamma,\rho,(r)_s)\neq 0$. This requirement is essential in the comparison scheme considered in [9]. It enables us to balance the decay rates of squared asymptotic bias and asymptotic variance.

We know, say from [17], that an intermediate sequence $k_n^*((r)_s)$, which minimizes AMSE for the estimator $\tilde{\gamma}_n(k_n,(r)_s)$ (in other words, the optimal choice of the sequence k_n in the AMSE sense), satisfies the asymptotic relation

$$k_n^*((r)_s)\sim\left(\frac{\varsigma(\gamma,(r)_s)}{\kappa^2(\gamma,\rho,(r)_s)}\right)^{1/(1-2\rho)}\frac{n}{a^{\leftarrow}(1/n)},\quad n\to\infty, \tag{23}$$

where $a^{\leftarrow}$ is the inverse of $a\in \mathrm{RV}_{2\rho-1}$ given by

$$A^2(t)\sim\int_t^{\infty}a(x)\mathrm{d}x,\quad t\to\infty. \tag{24}$$

Here and below, we write $a_n\sim b_n$ if $a_n/b_n\to 1$ as $n\to\infty$. Also,

$$\mathrm{AMSE}(\tilde{\gamma}_n(k_n^*((r)_s),(r)_s))\sim\frac{1-2\rho}{-2\rho}\left(\kappa^2(\gamma,\rho,(r)_s)\varsigma^{-2\rho}(\gamma,(r)_s)\right)^{1/(1-2\rho)}\frac{a^{\leftarrow}(1/n)}{n} \tag{25}$$

as $n\to\infty$. We replace $k_n^*((r)_s)$ with $n\ell_n^*((r)_s))/m_n^*((r)_s)$ in (23) to conclude that any two sequences of integers $m_n^*((r)_s)$ and $\ell_n^*((r)_s)$ are the optimal choices of the sequences m_n and ℓ_n for the estimator (2) if $1\le\ell_n^*((r)_s)<m_n^*((r)_s)\le[n/2]$, $m_n^*((r)_s)$ is an intermediate sequence, $\ell_n^*((r)_s)$ is an intermediate sequence with respect to $m_n^*((r)_s)$ and

$$\frac{m_n^*((r)_s)}{\ell_n^*((r)_s)}\sim\left(\frac{\varsigma(\gamma,(r)_s)}{\kappa^2(\gamma,\rho,(r)_s)}\right)^{1/(2\rho-1)}a^{\leftarrow}(1/n),\quad n\to\infty. \tag{26}$$

Thus, the asymptotic relation (25) remains valid with $\tilde{\gamma}_n(k_n^*((r)_s),(r)_s)$ replaced by

$$\hat{\gamma}_n(m_n^*((r)_s),\ell_n^*((r)_s),(r)_s).$$

This implies that for any $\gamma>0$, $\rho<0$ and collection of different tuning parameters $(r)_s$ satisfying (15), the following relation holds

$$\lim_{n\to\infty}\frac{\mathrm{AMSE}(\hat{\gamma}_n(m_n^*((r)_s),\ell_n^*((r)_s),(r)_s))}{\mathrm{AMSE}(\tilde{\gamma}_n(k_n^*((r)_s),(r)_s))}=1, \tag{27}$$

i.e., the estimators $\tilde{\gamma}_n(k_n^*((r)_s),(r)_s)$ and $\hat{\gamma}_n(m_n^*((r)_s),\ell_n^*((r)_s),(r)_s)$ do not dominate each other in the whole area $\{(\gamma,\rho):\ \gamma>0,\rho<0\}$. This is the main conclusion of this paper. This conclusion contradicts the assertion (see [5,22]) that block-type estimators may not be as efficient as corresponding POT-type estimators.

Let us consider several particular cases of the result (27). Taking $s=1$ in (27), we see that the generalized Hill estimator $\tilde{\gamma}_n(k_n^*(r),r)$ and the estimator $\hat{\gamma}_n(m_n^*(r),\ell_n^*(r),r)$ do not outperform each other. We recall that the estimators $\hat{\gamma}_n(m_n,1,r)$, $r\in\mathbb{R}$, were introduced by [3]. The optimal choice of the tuning parameter r is

$$r^*=\frac{1}{\gamma\left(2-\rho+\sqrt{(2-\rho)^2-2}\right)},$$

see [3,6,11].

The families of estimators (11) and (22) generalize several estimators. Firstly, substituting $r=0$ in (11) and in (22), we derive the Qi estimator $\hat{\gamma}_n(m_n,\ell_n,0)$ and the Hill estimator $\tilde{\gamma}_n(k_n,0)$. Secondly, substituting $r=-1$ gives the estimators $\hat{\gamma}_n(m_n,\ell_n,-1)$ and

$\tilde{\gamma}_n(k_n,-1)$. The last estimator was proposed in [12], while the estimator $\hat{\gamma}_n(m_n,1,-1)$ can be found in [2].

Let $s=2$. Substituting $(r_1,r_2)=(r,0)$, $r\neq 0$ and $g(t_1,t_2)=(t_1-t_2)/(rt_1)$ into (11) and (21), we turn back to the estimators

$$\hat{\gamma}_n(m_n(r,0),\ell_n(r,0),(r,0))=\frac{Q_n(m_n,\ell_n,r)-Q_n(m_n,\ell_n,0)}{rQ_n(m_n,\ell_n,r)}$$

and

$$\tilde{\gamma}_n(k_n(r,0),(r,0))=\frac{H_n(k_n,r)-H_n(k_n,0)}{rH_n(k_n,r)}.$$

The family of estimators $\hat{\gamma}_n(m_n(r,0),\ell_n(r,0),(r,0))$, $r\neq 0$, was studied in [1]. The estimators $\tilde{\gamma}_n(k_n(r,0),(r,0))$, $r\neq 0$, were introduced by [23]. From (27), it follows that the estimators $\hat{\gamma}_n(m_n^*(r^*,0),\ell_n^*(r^*,0),(r^*,0))$ and $\tilde{\gamma}_n(k_n^*(r^*,0),(r^*,0))$ do not dominate each other in the area $\{(\gamma,\rho):\ \gamma>0,\rho<0\}$. Here, $r^*=\rho$ is the optimal choice of parameter r. Cf. [23].

A pair of estimators that do not prevail over each other can be constructed using the mixed moment estimator, introduced in [15]. In particular, take $(r_1,r_2)=(-1,0)$ and $g(t_1,t_2)=(t_2-t_1-t_1^2)/t_1^2$ in (11) and (21) to derive the estimator

$$\hat{\gamma}_n(m_n(-1,0),\ell_n(-1,0),(-1,0))=\frac{Q_n(m_n,\ell_n,0)-Q_n(m_n,\ell_n,-1)}{(Q_n(m_n,\ell_n,-1))^2}-1$$

and the mixed moment estimator

$$\tilde{\gamma}_n(k_n(-1,0),(-1,0))=\frac{H_n(k_n,0)-H_n(k_n,-1)}{(H_n(k_n,-1))^2}-1.$$

Since $\kappa(\gamma,-1,(-1,0))=0$ for the mixed moment estimator $\tilde{\gamma}_n(k_n(-1,0),(-1,0))$, the relation (27) holds for $(\gamma,\rho)\in(0,+\infty)\times(-\infty,0)\setminus(0,+\infty)\times(-1-\epsilon,-1+\epsilon)$, where $\epsilon>0$.

Assume now that the conditions of Theorem 2(i) hold. In [17], it was proved that

$$\lim_{n\to\infty}\frac{\mathrm{AMSE}(\hat{\gamma}_n(m_n^\star((r)_s),\tilde{n},(r)_s))}{\mathrm{AMSE}(\tilde{\gamma}_n(k_n^*((r)_s),(r)_s))}=\left(\frac{\Gamma^2(\tilde{n}+1-\rho)}{\Gamma^2(\tilde{n}+1)(\tilde{n})^{-2\rho}}\right)^{1/(1-2\rho)},\tag{28}$$

where the optimal choice of the sequence m_n satisfies the relation

$$m_n^\star((r)_s)\sim\left(\frac{\Gamma^2(\tilde{n}+1-\rho)}{\Gamma(\tilde{n})\Gamma(\tilde{n}+1)}\right)^{1/(1-2\rho)}\left(\frac{\kappa^2(\gamma,\rho,(r)_s)}{\varsigma(\gamma,(r)_s)}\right)^{1/(1-2\rho)}a^{\leftarrow}(1/n),\quad n\to\infty.$$

From Lemma 1 in [17], we know that the right-hand side of (28) is greater than one. Thus, from (27) and (28) follows a natural conclusion: the efficiency of the block-type estimators (2) is better when the number of upper statistics ℓ_n (in each block) increases together with the number of observations n.

3. A Monte Carlo Simulation

For Monte Carlo simulations, we need more information regarding the function $a^{\leftarrow}$. See (24). For this reason, we assume that the quantile-type function U satisfies (14), or, equivalently, that the d.f. F satisfies

$$1-F(x)=\left(\frac{x}{C}\right)^{-1/\gamma}\left(1+\frac{\beta}{\rho}\left(\frac{x}{C}\right)^{\rho/\gamma}+o\left(\left(\frac{x}{C}\right)^{\rho/\gamma}\right)\right),\quad x\to\infty.$$

Assuming (14), one can check that (13) holds with $A(t) = \beta\gamma t^{\rho}$. Now, using (24), we find

$$a^{\leftarrow}(t) = \left(-2\rho\gamma^2\beta^2\right)^{1/(1-2\rho)} t^{1/(2\rho-1)}.$$

To investigate how Monte Carlo simulations reflect (27), we use the Hill estimator $\tilde{\gamma}_n(k_n, 0) = H_n(k_n, 0)$ and the Qi estimator $\hat{\gamma}_n(m_n, \ell_n, 0) = Q_n(m_n, \ell_n, 0)$. Taking $s = 1$, $g(t) = t$ and $r = 0$ in Theorem 2(ii), we find $\kappa(\gamma, \rho, 0) = (1-\rho)^{-1}$ and $\varsigma(\gamma, 0) = \gamma^2$. Substituting values of $\kappa(\gamma, \rho, 0)$, $\varsigma(\gamma, 0)$ and $a^{\leftarrow}(t)$ into (23), we obtain that the optimal choice of k_n satisfies the asymptotic relation

$$k_n^* \sim \left(\frac{(1-\rho)^2}{-2\rho\beta^2}\right)^{1/(1-2\rho)} n^{-2\rho/(1-2\rho)}, \quad n \to \infty. \tag{29}$$

Here and below in this section, we write k_n^*, ℓ_n^* and m_n^* instead of $k_n^*(0)$, $\ell_n^*(0)$ and $m_n^*(0)$, respectively.

Unfortunately, the asymptotic relation (26) does not allow us to determine the sequences ℓ_n^* and m_n^* uniquely. We choose

$$\ell_n^* \sim (k_n^*)^d, \quad n \to \infty, \tag{30}$$

where $0 < d < 1$. This, together with (26), gives that the optimal choice of block width m_n for the estimator $\hat{\gamma}_n(m_n, \ell_n, 0)$ is such that

$$m_n^* \sim \frac{n}{k_n^*}(k_n^*)^d, \quad n \to \infty. \tag{31}$$

Keeping in mind that k_n^* is an intermediate sequence, we immediately deduce from (31) that m_n^* is an intermediate sequence, while (30)–(31) imply that ℓ_n^* satisfies (7).

Replacing β and ρ by some estimators $\hat{\beta}_n$ and $\hat{\rho}_n$ on the right-hand side of (29), we obtain the estimator

$$\hat{k}_n^* = \left[\left(\frac{(1-\hat{\rho}_n)^2}{-2\hat{\rho}_n(\hat{\beta}_n)^2}\right)^{1/(1-2\hat{\rho}_n)} n^{-2\hat{\rho}_n/(1-2\hat{\rho}_n)}\right]$$

and, consequently,

$$\hat{\ell}_n^* = \left[\left(\hat{k}_n^*\right)^d\right], \quad \hat{m}_n^* = \left[n/\left(\hat{k}_n^*\right)^{1-d}\right].$$

The parameter ρ was estimated by the estimator $\hat{\rho}_n = \hat{\rho}_n(k_1, \tau)$, proposed in [24]. The algorithm provided in [25] was applied to choose which value ($\tau = 0$ or $\tau = 1$) of the parameter τ has to be taken. The estimator $\hat{\beta}_n = \hat{\beta}_n(k_1, \hat{\rho}_n)$, introduced in [26], was used to estimate the parameter β. The sample fraction $k_1 = [n^{0.995}]$ was used for both estimators, following the recommendations in [24].

The i.i.d. samples $X_1, \ldots, X_n$ of sizes $n = 1000$ and $n = 5000$ were simulated 1000 times from several distributions. In particular, the following models were considered:

1. The Burr distribution with the d.f. $F(x) = 1 - \left(1 + x^{-\rho/\gamma}\right)^{1/\rho}$, $x > 0$.
2. The inverse χ^2 distribution with the probability density function (p.d.f.)

$$p(x) = \frac{2^{-1/\gamma}}{\Gamma(1/\gamma)} x^{-1-1/\gamma} \exp\left\{-\frac{1}{2x}\right\}, \quad x > 0,$$

where $\Gamma(x)$, $x > 0$, denotes the gamma function.

3. The one-sided Student's t distribution with the p.d.f.

$$p(x) = 2\left(\frac{\gamma}{\pi}\right)^{1/2}\Gamma\left(\frac{1+\gamma}{2\gamma}\right)\Gamma^{-1}\left(\frac{1}{2\gamma}\right)\left(1+\gamma x^2\right)^{-(1+\gamma)/(2\gamma)}, \quad x > 0.$$

4. The Lomax distribution with the d.f. $F(x) = 1 - (1+x)^{-1/\gamma}$, $x > 0$.

The d.f. of the Burr distribution satisfies (14). Also, distributions 2–4 belong to the Hall–Welsh class with $\rho = -\gamma, -2\gamma, -\gamma$, respectively.

The ratios of empirical MSE (EMSE) (or, in other words, values of the measure of efficiency relative to the Hill estimator)

$$R(\gamma, d) = \frac{\mathrm{EMSE}\left(\hat{\gamma}_n\left(\hat{m}_n^*, \hat{\ell}_n^*, 0\right)\right)}{\mathrm{EMSE}\left(\tilde{\gamma}_n\left(\hat{k}_n^*, 0\right)\right)}$$

are summarized in Tables 1–6. The closest result to 1 is presented in bold in each column.

Table 1. Values of $R(\gamma, d)$, underlying Burr distribution with $\rho = -1/2$.

n	d	γ								
		1/4	**1/3**	**1/2**	**3/4**	**1**	**5/4**	**3/2**	**7/4**	**2**
1000	8/16	1.06	**1.03**	1.06	1.04	1.08	1.07	1.07	1.05	1.09
	9/16	**1.04**	**1.03**	1.04	**1.01**	1.04	1.05	1.03	**1.02**	1.07
	10/16	**0.96**	**0.97**	**0.99**	0.96	**0.97**	**0.98**	**0.98**	**0.98**	**1.00**
5000	8/16	1.03	**1.00**	**1.01**	1.01	1.03	1.04	1.02	1.03	1.03
	9/16	**0.99**	0.99	**0.99**	**1.00**	**0.99**	**1.00**	**0.99**	**1.01**	**0.99**
	10/16	0.98	0.97	0.96	0.96	0.97	0.96	0.94	0.96	0.96

Table 2. Values of $R(\gamma, d)$, underlying Burr distribution with $\rho = -1$.

n	d	γ								
		1/4	**1/3**	**1/2**	**3/4**	**1**	**5/4**	**3/2**	**7/4**	**2**
1000	8/16	1.04	1.06	1.06	1.07	1.06	1.04	1.05	1.06	**1.05**
	9/16	**1.03**	**1.00**	1.04	1.04	**1.02**	1.04	1.05	1.05	**1.05**
	10/16	1.04	0.98	**1.02**	**1.01**	1.03	**0.99**	**1.00**	**1.01**	**1.05**
5000	8/16	**1.01**	1.07	1.05	1.05	1.03	1.06	1.03	**1.02**	0.98
	9/16	1.02	1.06	**1.04**	1.05	**1.01**	1.05	**1.02**	1.04	1.02
	10/16	**1.01**	**1.04**	1.05	**1.01**	**1.01**	**1.02**	1.04	**1.02**	**1.01**

Table 3. Values of $R(\gamma, d)$, underlying Burr distribution with $\rho = -2$.

n	d	γ								
		1/4	**1/3**	**1/2**	**3/4**	**1**	**5/4**	**3/2**	**7/4**	**2**
1000	8/16	1.08	1.04	1.06	1.11	1.03	1.03	1.05	1.05	1.07
	9/16	1.05	1.02	1.07	1.07	1.03	1.05	**1.03**	1.07	**1.05**
	10/16	**1.01**	**1.01**	**1.04**	**1.03**	**1.02**	**1.02**	**1.03**	**1.04**	1.06
5000	8/16	1.03	1.03	1.03	1.05	1.03	1.06	1.03	1.03	1.04
	9/16	1.02	1.03	**1.01**	1.02	1.02	1.02	1.01	**1.01**	1.03
	10/16	**1.01**	**0.99**	1.02	**0.99**	**0.99**	**1.00**	**1.00**	1.02	**0.98**

Table 4. Values of $R(\gamma, d)$, underlying inverse χ^2 distribution.

n	d	γ								
		1/4	**1/3**	**1/2**	**3/4**	**1**	**5/4**	**3/2**	**7/4**	**2**
1000	8/16	1.03	1.04	1.05	1.09	1.08	1.09	1.08	1.04	1.05
	9/16	**0.99**	**1.00**	**1.02**	**1.01**	1.04	1.06	1.04	**1.02**	1.03
	10/16	0.97	0.98	**0.98**	1.03	**1.01**	**1.01**	**1.01**	**0.98**	**0.99**
5000	8/16	1.03	1.02	**1.01**	1.03	1.03	1.02	1.03	1.04	1.03
	9/16	**1.00**	**1.00**	**0.99**	**0.99**	1.02	**1.01**	1.05	1.02	1.03
	10/16	0.97	0.97	0.98	**0.99**	**1.01**	**1.01**	**1.01**	**0.99**	**0.99**

Table 5. Values of $R(\gamma, d)$, underlying one-sided Student's t distribution.

n	d	γ								
		1/4	**1/3**	**1/2**	**3/4**	**1**	**5/4**	**3/2**	**7/4**	**2**
1000	8/16	1.08	1.06	1.14	**1.00**	1.09	**1.02**	**1.05**	**1.03**	**1.03**
	9/16	**0.99**	1.03	1.10	1.02	**1.06**	**1.02**	**1.05**	1.04	1.05
	10/16	**1.01**	**1.00**	**1.05**	1.02	**1.06**	1.03	**1.05**	1.05	1.07
5000	8/16	1.02	1.05	1.07	**1.00**	**1.06**	1.04	**1.01**	**1.00**	**1.00**
	9/16	**0.99**	**1.02**	**1.04**	1.04	**1.06**	**1.00**	**1.01**	1.03	0.99
	10/16	**0.99**	**1.02**	1.08	1.03	1.09	**1.00**	1.02	1.01	1.02

Table 6. Values of $R(\gamma, d)$, underlying Lomax distribution.

n	d	γ								
		1/4	**1/3**	**1/2**	**3/4**	**1**	**5/4**	**3/2**	**7/4**	**2**
1000	8/16	1.02	1.04	1.06	1.05	**1.05**	1.08	1.06	1.05	1.08
	9/16	0.98	1.02	1.06	1.03	1.08	**1.02**	**1.01**	1.03	1.07
	10/16	**1.00**	**0.99**	**1.00**	**0.98**	**1.05**	**1.02**	1.02	**0.99**	**1.05**
5000	8/16	1.02	1.03	**1.02**	1.04	1.06	1.05	1.06	1.02	1.08
	9/16	**1.00**	**1.00**	**0.98**	**1.02**	1.07	**1.02**	1.03	1.02	1.05
	10/16	0.98	0.99	0.94	**1.02**	**1.03**	1.05	**1.00**	**1.00**	**1.02**

Our main conclusion is that the simulated values of the ratios $R(\gamma, d)$ reflect quite well the theoretical result (27). This conclusion is based on the following analysis of simulation results presented in Tables 1–6.

(a) The values closest to 1 of $R(\gamma, d)$ belong to the segment $[0.98, 1.06]$.

(b) Relative frequencies

$$\vartheta_n(d) = \frac{\text{a number of } R(\gamma, d), \text{ such that } R(\gamma, d) \in [0.95, 1.05]}{54} \cdot 100\%$$

are computed using the corresponding rows of Tables 1–6 for each $d \in \{8/16, 9/16, 10/16\}$. See Table 7.

Table 7. Values of $\vartheta_n(d)$.

n	d		
	8/16	**9/16**	**10/16**
1000	51.8	83.3	92.6
5000	85.2	96.3	92.6

From Table 7, we see that $d = 10/16$ is a mostly acceptable choice when $n = 1000$. In the case $n = 5000$, the empirical results are better (except in the case $d = 10/16$). In addition, the number of acceptable choices for the parameter d increases, and we can choose $d = 9/16$ and $d = 10/16$. The results in Table 7 allow us to hope that the dependence of $R(\gamma, d)$ on the parameter d decreases as the sample width n grows.

Next, we investigate the robustness against a non-second-order framework. In Tables 8 and 9, we present the efficiencies for the following models:

5. The log-gamma distribution with the p.d.f. $p(x) = \gamma^{-2}x^{-1-1/\gamma}\ln(x)$, $x > 1$. The quantile-type function U associated with the log-gamma distribution satisfies (1), but does not satisfy (13). See [27].
6. Model of independent and non-identically distributed observations. Let $\tilde{X}_1, \ldots, \tilde{X}_n$ be a sequence of i.i.d. random variables having the Lomax distribution. Put $X_{n,i} = (c_{n,i})^{\gamma}\tilde{X}_i$, $1 \le i \le n$, where $c_{n,i} = 1 + (4/5)\sin(i\pi/[n/4])$. From [28], it follows that the Hill estimator is an asymptotically normal estimator for $\gamma > 0$ when observations are from this non-stationary model.

Table 8. Values of $R(\gamma, d)$, underlying log-gamma distribution.

n	d	γ								
		1/4	**1/3**	**1/2**	**3/4**	**1**	**5/4**	**3/2**	**7/4**	**2**
1000	8/16	1.01	**1.00**	1.03	1.04	1.02	1.02	1.02	1.01	**1.01**
	9/16	**1.00**	0.98	**1.01**	**1.01**	**0.99**	**1.00**	**1.00**	**1.00**	**0.99**
	10/16	0.97	0.98	0.98	**0.99**	0.97	0.97	0.98	0.97	0.96
5000	8/16	1.02	**1.00**	1.01	1.02	1.02	1.01	**1.00**	**1.00**	0.98
	9/16	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	**1.00**	0.99	0.99	**0.99**
	10/16	0.99	0.98	0.98	**1.00**	0.99	0.98	0.98	0.99	0.97

Table 9. Values of $R(\gamma, d)$, underlying model of independent and non-identically distributed observations.

n	d	γ								
		1/4	**1/3**	**1/2**	**3/4**	**1**	**5/4**	**3/2**	**7/4**	**2**
1000	8/16	**0.98**	0.95	0.90	0.89	0.92	0.93	0.94	0.92	0.92
	9/16	0.95	**0.97**	**0.91**	**0.92**	0.93	**1.00**	1.04	0.91	0.94
	10/16	0.95	0.95	0.90	0.91	**0.96**	0.98	**0.98**	**1.02**	**0.99**
5000	8/16	**0.95**	**0.91**	**0.85**	**0.91**	**0.92**	0.95	0.81	**1.06**	**1.08**
	9/16	0.94	**0.91**	**0.85**	0.87	0.88	**1.00**	0.85	1.03	1.16
	10/16	0.92	**0.91**	0.82	0.87	0.91	1.08	**0.90**	**1.06**	1.15

From Table 8, we see that the relative efficiency is close to 1, i.e., both estimators perform similarly for the log-gamma model, for which the second-order condition (13) does not hold. Notice that the Qi estimator with $p = 10/16$ slightly dominates the Hill

estimator. The situation is different when observations are from a non-stationary model. The empirical results in Table 9 indicate that the Qi estimator performs better for $\gamma \leq 1$. This region is important for practical applications due to the stylized econometric fact, which states that in most cases, the distribution of the absolute values of log-returns of economic and financial data satisfies (1) with the extreme value index $1/4 \leq \gamma \leq 1/2$.

We finish this section with a discussion of two real datasets. Namely, we consider the NASDAQ 100 index and the Dow Jones Utility Average (DJUA) index. Both datasets contain $n = 2512$ observations for a period of ten years, from 13 October 2016 to 12 October 2026. The estimates for γ of the absolute values of log-returns for both datasets are presented in Table 10. One can note that the estimates are in quite close correspondence for the first dataset, while the estimate of the Hill estimator stands out from the other estimates for the second dataset.

Table 10. Estimates of the EVI.

Data	Hill Estimator	Qi Estimator		
		$p = 8/16$	$p = 9/16$	$p = 10/16$
NASDAQ 100	0.290	0.259	0.256	0.271
DJUA	0.388	0.250	0.254	0.282

4. Discussion

The asymptotic theory proposed in Theorems 1–2 is quite general and provides answers to the questions about weak consistency and asymptotic normality of a wide family of semiparametric estimators (2)–(3). There are several possible applications of Theorems 1–2. Firstly, the Theorems 1–2 allow us to prove the consistency of, and a CLT for, estimators expressed as a function of statistics (4) with different tuning parameters $r_1, \ldots, r_s$. Secondly, the result in Theorem 2 allows us to compare any block-type estimator (2)–(3) with other (asymptotically normal) estimators for $\gamma > 0$. In Section 2, we proved that the block-type estimators (2)–(3) are as good as their POT-type analogs (21). Our Monte Carlo simulations reflect this result. By discussing the comparison of POT estimators with block maxima estimators, the authors of [22] claimed that the latter estimators may be preferable when observations are not exactly i.i.d. It seems that this claim holds for block-type estimators also. See Table 9.

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Data Availability Statement: Wolfram Mathematica code to reproduce all simulations and data analysis is available from the author upon reasonable request. Real datasets used in this study are publicly available and were taken from https://fred.stlouisfed.org.

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Appendix A

Let

$$Y_1^{(1)}, \ldots, Y_{m_n}^{(1)}, Y_1^{(2)}, \ldots, Y_{m_n}^{(2)}, \ldots, Y_1^{([n/m_n])}, \ldots, Y_{m_n}^{([n/m_n])} \tag{A1}$$

be i.i.d. random variables with the Pareto d.f. $G(x) = 1 - x^{-1}$, $x \geq 1$. For any $1 \leq i \leq [n/m_n]$, let $Y_{1,m_n}^{(i)} \leq \cdots \leq Y_{m_n,m_n}^{(i)}$ denote the order statistics of $Y_1^{(i)}, \ldots, Y_{m_n}^{(i)}$.

The following Lemma will be useful to prove Theorem 2.

Lemma A1. *Let $\zeta < 0$. Then,*

$$\mathrm{E}\left(\frac{Y^{(1)}_{m_n-\ell_n,m_n}}{m_n/\ell_n}\right)^{\zeta} \to 1, \quad n \to \infty, \tag{A2}$$

where the sequences of natural numbers m_n, ℓ_n are such that $m_n > \ell_n$, $n = 1, 2, \ldots$ and $m_n \to \infty$, $\ell_n \to \infty$ as $n \to \infty$.

Proof. The probability density function of $Y^{(1)}_{m_n-\ell_n,m_n}$ has the form (cf. (2.2.2) in [16])

$$p(x) = \frac{\Gamma(m_n+1)}{\Gamma(m_n-\ell_n)\Gamma(\ell_n+1)}(x-1)^{m_n-\ell_n-1}x^{-m_n-1}, \quad x \geq 1.$$

The integration yields

$$\mathrm{E}\left(\frac{Y^{(1)}_{m_n-\ell_n,m_n}}{m_n/\ell_n}\right)^{\zeta} = \left(\frac{\ell_n}{m_n}\right)^{\zeta}\frac{\Gamma(m_n+1)}{\Gamma(m_n+1-\zeta)}\cdot\frac{\Gamma(\ell_n+1-\zeta)}{\Gamma(\ell_n+1)}.$$

Now, (A2) follows by applying the identity $\Gamma(n+1) = n\Gamma(n)$ and the asymptotic relation $\Gamma(n-\zeta)/\Gamma(n) \sim n^{-\zeta}$, $n \to \infty$. □

For later use, we give the following Drees-type inequalities for $h_r(U(tx)/U(t))$.

Lemma A2. *Let (1) hold for some $\gamma > 0$. Let $r \neq 0$. Then, for each $\epsilon > 0$ and $\delta > 0$, there is a $t_0 = t_0(\epsilon, \delta)$ such that for $t \geq t_0$ and $x \geq 1$,*

$$\left|h_r\left(\frac{U(tx)}{U(t)}\right) - h_r(x^{\gamma})\right| \leq \gamma\epsilon x^{\gamma r+\delta}. \tag{A3}$$

Proof. Let $r > 0$. The relation (1) implies $U^r \in \mathrm{RV}_{\gamma r}$, or, equivalently,

$$\lim_{t\to\infty}\frac{U^r(tx)}{U^r(t)} = x^{\gamma r}, \quad x > 0, \tag{A4}$$

see Prop. B.1.9.2 in [29]. From this, it follows that

$$\lim_{t\to\infty}\frac{U^r(tx)-U^r(t)}{\gamma r U^r(t)} = h_{\gamma r}(x), \quad x > 0.$$

Now, we apply the Drees inequality (cf. [30,31]): for each $\epsilon > 0$ and $\delta > 0$, there is a $t_0 = t_0(\epsilon, \delta)$ such that for $t \geq t_0$ and $x \geq 1$,

$$\left|\frac{U^r(tx)-U^r(t)}{\gamma r U^r(t)} - h_{\gamma r}(x)\right| \leq \epsilon x^{\gamma r+\delta}.$$

Whence (A3) follows.

Assume now $r < 0$. By combining the identity

$$(1/\gamma)h_r\left(\frac{U(tx)}{U(t)}\right) = \frac{(U(tx)/U(t))^{-r}-1}{-\gamma r}\cdot\frac{1}{(U(tx)/U(t))^{-r}}$$

and (A4), we get

$$\lim_{t\to\infty}(1/\gamma)h_r\left(\frac{U(tx)}{U(t)}\right) = h_{\gamma r}(x), \quad x > 0,$$

or, equivalently,

$$\lim_{t\to\infty}\frac{U^r(tx)/r - U^r(t)/r}{\gamma U^r(t)} = h_{\gamma r}(x), \quad x>0.$$

Apply the Drees inequality one more time to conclude that (A3) holds for $r<0$. □

Lemma A3. *Let (13) hold for some $\gamma>0$ and $\rho<0$. Let r be a real number. Then, for a possibly different function $\tilde{A}$ with $\tilde{A}(t)\sim A(t)$, $t\to\infty$ and for each $\epsilon>0$ and $\delta>0$, there is a $t_0=t_0(\epsilon,\delta)$ such that for $t\geq t_0$ and $x\geq 1$,*

$$\left|\frac{h_r\left(\frac{U(tx)}{U(t)}\right)-h_r(x^\gamma)}{\tilde{A}(t)} - x^{\gamma r}h_\rho(x)\right| \leq \gamma\epsilon x^{\gamma r+\rho+\delta}. \tag{A5}$$

Proof. The case $r\neq 0$ was proved in [6], while the proof of the case $r=0$ can be found in the proof of Theorem 3.2.5 of [29]. □

Proof of Theorem 1. The case when (5) holds was investigated in [1]. See Theorem 2.1 therein. Let the sequence ℓ_n satisfy (7). Using the equalities in distribution

$$U\left(Y_j^{(i)}\right)\stackrel{\mathrm{d}}{=}X_{(i-1)m_n+j}, \quad 1\leq i\leq [n/m_n],\ 1\leq j\leq m_n, \tag{A6}$$

where $Y_j^{(i)}$, $1\leq i\leq[n/m_n]$, $1\leq j\leq m_n$, are the same random variables as in (A1), we get

$$Q_n(m_n,\ell_n,r)\stackrel{\mathrm{d}}{=}Q_n^{(1)}(m_n,\ell_n,r), \tag{A7}$$

where

$$Q_n^{(1)}(m_n,\ell_n,r)=\frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}h_r\left(\frac{U\left(Y^{(i)}_{m_n-j+1,m_n}\right)}{U\left(Y^{(i)}_{m_n-\ell_n,m_n}\right)}\right).$$

Let $r\neq 0$. By Lemma 3.2.1 in [29], the assumptions (6) and (7) ensure the convergence $Y^{(i)}_{m_n-\ell_n,m_n}\to\infty$ a.s. for each $i\geq 1$. Substituting

$$t=Y^{(i)}_{m_n-\ell_n,m_n}, \quad x=\frac{Y^{(i)}_{m_n-j+1,m_n}}{Y^{(i)}_{m_n-\ell_n,m_n}}, \quad 1\leq i\leq[n/m_n] \tag{A8}$$

in (A3), we get

$$-\epsilon\gamma\left(\frac{Y^{(i)}_{m_n-j+1,m_n}}{Y^{(i)}_{m_n-\ell_n,m_n}}\right)^{\gamma r+\delta}\leq h_r\left(\frac{U\left(Y^{(i)}_{m_n-j+1,m_n}\right)}{U(Y^{(i)}_{m_n-\ell_n,m_n})}\right)-h_r\left(\left(\frac{Y^{(i)}_{m_n-j+1,m_n}}{Y^{(i)}_{m_n-\ell_n,m_n}}\right)^\gamma\right), \tag{A9}$$

$$\epsilon\gamma\left(\frac{Y^{(i)}_{m-j+1,m}}{Y^{(i)}_{m-\ell,m}}\right)^{\gamma r+\delta}\geq h_r\left(\frac{U\left(Y^{(i)}_{m-j+1,m}\right)}{U(Y^{(i)}_{m-\ell,m})}\right)-h_r\left(\left(\frac{Y^{(i)}_{m-j+1,m}}{Y^{(i)}_{m-\ell,m}}\right)^\gamma\right). \tag{A10}$$

By summing inequalities (A9) and (A10) over $1\leq j\leq\ell_n$ and then over $1\leq i\leq[n/m_n]$, we obtain

$$-\epsilon\gamma Q_n^{(3)}(m_n,\ell_n,r)\leq Q_n^{(1)}(m_n,\ell_n,r)-Q_n^{(2)}(m_n,\ell_n,r)\leq\epsilon\gamma Q_n^{(3)}(m_n,\ell_n,r), \tag{A11}$$

where

$$Q_n^{(2)}(m_n,\ell_n,r) = \frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}h_r\left(\left(\frac{Y^{(i)}_{m_n-j+1,m_n}}{Y^{(i)}_{m_n-\ell_n,m}}\right)^{\gamma}\right),$$

$$Q_n^{(3)}(m_n,\ell_n,r) = \frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}\left(\frac{Y^{(i)}_{m_n-j+1,m_n}}{Y^{(i)}_{m_n-\ell_n,m_n}}\right)^{\gamma r+\delta}.$$

Appealing to (4.14) of [15], we have that for any measurable function g and $1 \le i \le [n/m_n]$,

$$\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}g\left(\frac{Y^{(i)}_{m_n-j+1,m_n}}{Y^{(i)}_{m_n-\ell_n,m}}\right) \stackrel{\mathrm{d}}{=} \frac{1}{\ell_n}\sum_{j=1}^{\ell_n}g\left(Y_j^{(i)}\right). \tag{A12}$$

Since $\left(Y_1^{(1)},\ldots,Y_{\ell_n}^{(1)}\right),\ldots,\left(Y_1^{([n/m_n])},\ldots,Y_{\ell_n}^{([n/m_n])}\right)$ are independent random vectors, the equality (in distribution) (A12) yields

$$\begin{aligned}\left(\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}g\left(\frac{Y^{(1)}_{m_n-j+1,m_n}}{Y^{(1)}_{m_n-\ell_n,m}}\right),\ldots,\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}g\left(\frac{Y^{([n/m_n])}_{m_n-j+1,m_n}}{Y^{([n/m_n])}_{m_n-\ell_n,m}}\right)\right)\\ \stackrel{\mathrm{d}}{=}\left(\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}g\left(Y_j^{(1)}\right),\ldots,\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}g\left(Y_j^{([n/m_n])}\right)\right)\end{aligned} \tag{A13}$$

and, consequently,

$$Q_n^{(2)}(m_n,\ell_n,r) \stackrel{\mathrm{d}}{=} Q_n^{(4)}(m_n,\ell_n,r), \tag{A14}$$

where

$$Q_n^{(4)}(m_n,\ell_n,r) = \frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}h_r\left(\left(Y_j^{(i)}\right)^{\gamma}\right).$$

Note that summands $h_r\left(\left(Y_j^{(i)}\right)^{\gamma}\right)$, $i \ge 1$, $j \ge 1$, are i.i.d. random variables and under assumption (8),

$$\mathrm{E}\left(h_r\left(\left(\tilde{Y}_1\right)^{\gamma}\right)\right) = \Lambda_r(\gamma).$$

Thus, the law of large numbers holds for the double sum $Q_n^{(4)}(m_n,\ell_n,r)$, i.e.,

$$Q_n^{(4)}(m_n,\ell_n,r) \xrightarrow{\mathrm{p}} \Lambda_r(\gamma), \quad n\to\infty, \tag{A15}$$

as on p. 2 in [32].

One can use a similar argument to prove

$$Q_n^{(3)}(m_n,\ell_n,r) \xrightarrow{\mathrm{p}} \frac{1}{1-\gamma r-\delta}, \quad n\to\infty, \tag{A16}$$

where $0 < \delta < 1-\gamma r$. Now, (A7), (A11), (A14), (A15) and (A16) imply (9).

In the case $r = 0$, the proof of (9) follows using the inequalities: for all $0 < \epsilon < 1$ and $\delta > 0$, there is a $t_0 = t_0(\epsilon,\delta)$ such that for $t \ge t_0$ and $x \ge 1$,

$$\ln(1-\epsilon) - \delta\ln(x) < \ln\left(\frac{U(tx)}{U(t)}\right) - \ln(x^{\gamma}) < \ln(1+\epsilon) + \delta\ln(x),$$

see, e.g., (3.2.3) in [29]. □

Proof of Corollary 1. Apply the Continuous Mapping Theorem (cf. pg. 334 in [33]) in the case $s = 1$. If $s \geq 2$, the asymptotic relations

$$Q_n(m_n, \ell_n, r_1) \xrightarrow{\mathrm{P}} \Lambda_{r_1}(\gamma), \ldots, Q_n(m_n, \ell_n, r_s) \xrightarrow{\mathrm{P}} \Lambda_{r_s}(\gamma), \quad n \to \infty$$

imply

$$(Q_n(m_n, \ell_n, r_1), \ldots, Q_n(m_n, \ell_n, r_s)) \xrightarrow{\mathrm{P}} (\Lambda_{r_1}(\gamma), \ldots, \Lambda_{r_s}(\gamma)), \quad n \to \infty.$$

Now, it remains to apply the Continuous Mapping Theorem. □

Proof of Theorem 2. The relation (16) was derived in [17]. To prove (20), we need to show

$$\begin{aligned} &\sqrt{\frac{n}{m_n/\ell_n}}(Q_n(m_n, \ell_n, r_1) - \Lambda_{r_1}(\gamma), \ldots, Q_n(m_n, \ell_n, r_s) - \Lambda_{r_s}(\gamma)) \\ &\quad \xrightarrow{\mathrm{d}} (\mu_2 \nu(\rho, \gamma r_1) + Z(r_1), \ldots, \mu_2 \nu(\rho, \gamma r_s) + Z(r_s)), \quad n \to \infty, \end{aligned} \tag{A17}$$

where $(Z(r_1), \ldots, Z(r_s))$ is the Gaussian random vector with $\mathrm{E}Z(r_i) = 0$, $1 \leq i \leq s$ and the covariance matrix $S(\gamma r_1, \ldots, \gamma r_s)$. Then, (20) follows by applying the univariate/multivariate delta method (cf. pg. 26 in [34]) depending on whether $s = 1$ or $s \geq 2$.

It suffices to prove (A17) for the case $s = 2$, since the general case will then follow by a similar argument. In view of the Cramér–Wold device (see, e.g., pg. 16 in [34]), it is enough to show that for every $(\theta_1, \theta_2) \in \mathbb{R}^2$ and $r_1 \neq r_2$ satisfying (15),

$$\sqrt{\frac{n}{m_n/\ell_n}} \sum_{v=1}^{2} \theta_v (Q_n(m_n, \ell_n, r_v) - \Lambda_{r_v}(\gamma)) \xrightarrow{\mathrm{d}} \sum_{v=1}^{2} \theta_v (\mu_2 \nu(\rho, \gamma r_v) + Z(r_v)), \tag{A18}$$

as $n \to \infty$.

Without loss of generality, we assume $A(t) > 0$, $t > 0$. We note only that in the case $A(t) < 0$, $t > 0$, we can consider $-A(t)$. As in the proof of Theorem 1, we use (A6) to get

$$\theta_1(Q_n(m_n, \ell_n, r_1) - \Lambda_{r_1}(\gamma)) + \theta_2(Q_n(m_n, \ell_n, r_2) - \Lambda_{r_2}(\gamma)) \stackrel{\mathrm{d}}{=} S_n^{(1)}(m_n, \ell_n, (r)_2),$$

where

$$S_n^{(1)}(m_n, \ell_n, (r)_2) = \frac{1}{[n/m_n]} \sum_{i=1}^{[n/m_n]} \frac{1}{\ell_n} \sum_{j=1}^{\ell} \sum_{v=1}^{2} \theta_v \left(h_{r_v} \left(\frac{U\left(Y^{(i)}_{m_n - j + 1, m_n}\right)}{U\left(Y^{(i)}_{m_n - \ell_n, m_n}\right)} \right) - \Lambda_{r_v}(\gamma) \right).$$

Now, we apply (A5) with substitutions (A8). As in the proof of Theorem 1, we get

$$\begin{aligned} &S_n^{(1)}(m_n, \ell_n, (r)_2) - S_n^{(2)}(m_n, \ell_n, (r)_2) - S_n^{(3)}(m_n, \ell_n, (r)_2) \geq -\epsilon\gamma S_n^{(4)}(m_n, \ell_n, (r)_2), \\ &S_n^{(1)}(m_n, \ell_n, (r)_2) - S_n^{(2)}(m_n, \ell_n, (r)_2) - S_n^{(3)}(m_n, \ell_n, (r)_2) \leq \epsilon\gamma S_n^{(4)}(m_n, \ell_n, (r)_2), \end{aligned}$$

where

$$\begin{aligned}
S_n^{(2)}(m_n,\ell_n,(r)_2) &= \frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}\sum_{v=1}^{2}\theta_v\left(h_{r_v}\left(\left(\frac{\Upsilon^{(i)}_{m_n-j+1,m_n}}{\Upsilon^{(i)}_{m_n-\ell_n,m_n}}\right)^{\gamma}\right)-\Lambda_{r_v}(\gamma)\right),\\
S_n^{(3)}(m_n,\ell_n,(r)_2) &= \frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\tilde{A}\left(\Upsilon^{(i)}_{m_n-\ell_n,m_n}\right)\times\\
&\quad\times\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}\sum_{v=1}^{2}\theta_v\left(\frac{\Upsilon^{(i)}_{m_n-j+1,m_n}}{\Upsilon^{(i)}_{m_n-\ell_n,m_n}}\right)^{\gamma r_v}h_\rho\left(\frac{\Upsilon^{(i)}_{m_n-j+1,m_n}}{\Upsilon^{(i)}_{m_n-\ell_n,m_n}}\right),\\
S_n^{(4)}(m_n,\ell_n,(r)_2) &= \frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\tilde{A}\left(\Upsilon^{(i)}_{m_n-\ell_n,m_n}\right)\times\\
&\quad\times\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}\sum_{v=1}^{2}|\theta_v|\left(\frac{\Upsilon^{(i)}_{m_n-j+1,m_n}}{\Upsilon^{(i)}_{m_n-\ell_n,m_n}}\right)^{\gamma r_v+\rho+\delta}.
\end{aligned}$$

The asymptotic relation (A18) will be proven if we show that

$$\sqrt{\frac{n}{m_n/\ell_n}}S_n^{(2)}(m_n,\ell_n,(r)_2) \xrightarrow{\mathrm{d}} \theta_1 Z(r_1)+\theta_2 Z(r_2), \tag{A19}$$

$$\sqrt{\frac{n}{m_n/\ell_n}}S_n^{(3)}(m_n,\ell_n,(r)_2) \xrightarrow{\mathrm{p}} \mu_2(\theta_1\nu(\rho,\gamma r_1)+\theta_2\nu(\rho,\gamma r_2)), \tag{A20}$$

$$\sqrt{\frac{n}{m_n/\ell_n}}S_n^{(4)}(m_n,\ell_n,(r)_2) = O_p(1),\quad n\to\infty. \tag{A21}$$

To prove (A19), we apply (A13) one more time. Then,

$$S_n^{(2)}(m_n,\ell_n,r_1,r_2)\stackrel{\mathrm{d}}{=}S_n^{(5)}(m_n,\ell_n,r_1,r_2), \tag{A22}$$

where

$$\begin{aligned}
S_n^{(5)}(m_n,\ell_n,r_1,r_2) &= \frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}\xi_{i,j},\\
\xi_{i,j} &= \sum_{v=1}^{2}\theta_v\left(h_{r_v}\left(\left(\Upsilon_j^{(i)}\right)^{\gamma}\right)-\Lambda_{r_v}(\gamma)\right).
\end{aligned}$$

The summands $\xi_{i,j}$, $i\ge 1, j\ge 1$, are i.i.d. random variables with zero mean and finite variance

$$\mathrm{E}\xi_{1,1}^2=\theta_1^2\sigma(\gamma r_1,\gamma r_1)+2\theta_1\theta_2\sigma(\gamma r_1,\gamma r_2)+\theta_2^2\sigma(\gamma r_2,\gamma r_2)$$

due to assumption (15). Thus, by the CLT for i.i.d. double-indexed random variables,

$$\sqrt{[n/m_n]\ell_n}S_n^{(5)}(m_n,\ell_n,r_1,r_2)\xrightarrow{\mathrm{d}}\theta_1 Z(r_1)+\theta_2 Z(r_2),\quad n\to\infty.$$

Keeping in mind that $(n/m_n)/[n/m_n]\to 1$, $n\to\infty$, we use the last relation together with (A22) to conclude the proof of (A19).

The proof of (A20) is more delicate. The following decomposition allows us to "replace" $\tilde{A}\left(Y^{(i)}_{m_n-\ell_n,m_n}\right)$ with $\tilde{A}(m_n/\ell_n)$ in the term $S^{(3)}_n(m_n,\ell_n,(r)_2)$:

$$\begin{aligned}
&S^{(3)}_n(m_n,\ell_n,(r)_2)\\
&=\tilde{A}\left(\frac{m_n}{\ell_n}\right)\frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\left(\frac{\tilde{A}\left(Y^{(i)}_{m_n-\ell_n,m_n}\right)}{\tilde{A}(m_n/\ell_n)}-\left(\frac{\ell_n}{m_n}Y^{(i)}_{m_n-\ell_n,m_n}\right)^{\rho}\right)\eta^{(i)}_{(r_1,r_2)}\\
&+\tilde{A}\left(\frac{m_n}{\ell_n}\right)\frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\left(\left(\frac{\ell_n}{m_n}Y^{(i)}_{m_n-\ell_n,m_n}\right)^{\rho}\eta^{(i)}_{(r_1,r_2)}-\mathrm{E}\left(\left(\frac{\ell_n}{m_n}Y^{(i)}_{m_n-\ell_n,m_n}\right)^{\rho}\eta^{(i)}_{(r_1,r_2)}\right)\right)\\
&+\tilde{A}\left(\frac{m_n}{\ell_n}\right)\frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\mathrm{E}\left(\left(\frac{\ell_n}{m_n}Y^{(i)}_{m_n-\ell_n,m_n}\right)^{\rho}\eta^{(i)}_{(r_1,r_2)}\right)\\
&=\tilde{A}\left(\frac{m_n}{\ell_n}\right)S^{(31)}_n(m_n,\ell_n,(r)_2)+\tilde{A}\left(\frac{m_n}{\ell_n}\right)S^{(32)}_n(m_n,\ell_n,(r)_2)+S^{(33)}_n(m_n,\ell_n,(r)_2),
\end{aligned}$$

where

$$\eta^{(i)}_{(r_1,r_2)}=\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}\sum_{v=1}^{2}\theta_v\left(\frac{Y^{(i)}_{m_n-j+1,m_n}}{Y^{(i)}_{m_n-\ell_n,m_n}}\right)^{\gamma r_v}h_\rho\left(\frac{Y^{(i)}_{m_n-j+1,m_n}}{Y^{(i)}_{m_n-\ell_n,m_n}}\right).$$

Consider the term $S^{(33)}_n(m_n,\ell_n,(r)_2)$ first. The random variables

$$\left((m_n/\ell_n)Y^{(i)}_{m_n-\ell_n,m_n}\right)^{\rho}\eta^{(i)}_{(r_1,r_2)},\quad 1\le i\le[n/m_n]$$

are identically distributed. Thus,

$$S^{(33)}_n(m_n,\ell_n,(r)_2)=\tilde{A}\left(\frac{m_n}{\ell_n}\right)\mathrm{E}\left(\left((m_n/\ell_n)Y^{(1)}_{m_n-\ell_n,m_n}\right)^{\rho}\eta^{(1)}_{(r_1,r_2)}\right).$$

Note that $Y^{(1)}_{m_n-\ell_n,m_n}$ and $\eta^{(1)}_{(r_1,r_2)}$ are independent random variables. This statement is an immediate consequence of Lemma 3.2.3 in [29]. So, we have

$$S^{(33)}_n(m_n,\ell_n,(r)_2)=\tilde{A}\left(\frac{m_n}{\ell_n}\right)\mathrm{E}\left(\left((\ell_n/m_n)Y^{(1)}_{m_n-\ell_n,m_n}\right)^{\rho}\right)\mathrm{E}\left(\eta^{(1)}_{(r_1,r_2)}\right).$$

Apply (A12) to get

$$\begin{aligned}
\mathrm{E}\left(\eta^{(1)}_{(r_1,r_2)}\right)&=\theta_1\mathrm{E}\left(\left(Y^{(1)}_1\right)^{\gamma r_1}h_\rho\left(Y^{(1)}_1\right)\right)+\theta_2\mathrm{E}\left(\left(Y^{(1)}_1\right)^{\gamma r_2}h_\rho\left(Y^{(1)}_1\right)\right)\\
&=\theta_1\nu(\rho,\gamma r_1)+\theta_2\nu(\rho,\gamma r_2).
\end{aligned}$$

This, together with assumption (19), relation $\tilde{A}(t)\sim A(t)$, $t\to\infty$ and Lemma A1 (with $\zeta=\rho$), establishes

$$\sqrt{\frac{n}{m_n/\ell_n}}S^{(33)}_n(m_n,\ell_n,(r)_2)\to\mu_2(\theta_1\nu(\rho,\gamma r_1)+\theta_2\nu(\rho,\gamma r_2)),\quad n\to\infty. \tag{A23}$$

Next, we need to prove

$$S^{(31)}_n(m_n,\ell_n,(r)_2)\xrightarrow{\mathrm{P}}0, \tag{A24}$$

$$S^{(32)}_n(m_n,\ell_n,(r)_2)\xrightarrow{\mathrm{P}}0,\quad n\to\infty. \tag{A25}$$

Choose $\tilde{\epsilon} > 0$. By Chebyshev's inequality,

$$\mathrm{P}\Big(\Big|S_n^{(32)}(m_n,\ell_n,(r)_2)\Big| > \tilde{\epsilon}\Big) \le (\tilde{\epsilon})^{-2}\mathrm{E}\Big(S_n^{(32)}(m_n,\ell_n,(r)_2)\Big)^2.$$

Using a similar argument as in the proof of (A23), we obtain

$$\begin{aligned}\mathrm{E}\Big(S_n^{(32)}(m_n,\ell_n,(r)_2)\Big)^2 &= \frac{1}{[n/m_n]}\Big\{\mathrm{E}\Big(\frac{\ell_n}{m_n}Y^{(1)}_{m_n-\ell_n,m_n}\Big)^{2\rho}\mathrm{Var}\Big(\eta^{(1)}_{(r_1,r_2)}\Big)\\ &\quad+\mathrm{Var}\Big(\Big(\frac{\ell_n}{m_n}Y^{(1)}_{m_n-\ell_n,m_n}\Big)^{\rho}\Big)\Big(\mathrm{E}\Big(\eta^{(1)}_{(r_1,r_2)}\Big)\Big)^2\Big\}\end{aligned}$$

and

$$\mathrm{Var}\Big(\eta^{(1)}_{(r_1,r_2)}\Big) \le \ell_n^{-1}\mathrm{E}\Big(\theta_1\Big(Y_1^{(1)}\Big)^{\gamma r_1}h_\rho\Big(Y_1^{(1)}\Big)+\theta_2\Big(Y_1^{(1)}\Big)^{\gamma r_2}h_\rho\Big(Y_1^{(1)}\Big)\Big)^2. \tag{A26}$$

Using the inequality $h_\rho(x) < 1/|\rho|$, $x \ge 1$, $\rho < 0$, we get that the right-hand side of (A26) does not exceed

$$\begin{aligned}&|\rho|^{-2}\ell_n^{-1}\Big\{\theta_1^2\mathrm{E}\Big(\Big(Y_1^{(1)}\Big)^{2\gamma r_1}\Big)+\theta_2^2\mathrm{E}\Big(\Big(Y_1^{(1)}\Big)^{2\gamma r_2}\Big)+2|\theta_1||\theta_2|\mathrm{E}\Big(\Big(Y_1^{(1)}\Big)^{\gamma(r_1+r_2)}\Big)\Big\}\\ &=|\rho|^{-2}\ell_n^{-1}\Big\{\frac{\theta_1^2}{1-2\gamma r_1}+\frac{\theta_2^2}{1-2\gamma r_2}+\frac{2|\theta_1||\theta_2|}{1-\gamma(r_1+r_2)}\Big\}.\end{aligned}$$

Now, (A25) follows due to Lemma A1 and assumptions $n/m_n \to \infty$, $\ell_n \to \infty$ as $n \to \infty$.

Next, we will show that (A24) holds. The relations $\tilde{A}(t) \sim A(t)$ as $t \to \infty$ and $A \in \mathrm{RV}_\rho$ imply that $\tilde{A} \in \mathrm{RV}_\rho$. We apply Drees' Lemma one more time: for each $\varepsilon > 0$ and $\delta > 0$, there is $t_1 = t_1(\varepsilon,\delta)$ such that for $t \ge t_1$ and $tx \ge 1$,

$$\Big|\frac{\tilde{A}(tx)}{\tilde{A}(t)} - x^\rho\Big| \le \varepsilon x^{\rho\pm\delta}, \tag{A27}$$

where $x^{\rho\pm\delta} = \max\{x^{\rho+\delta}, x^{\rho-\delta}\}$. Choose $\delta > 0$ such that $\rho + \delta < 0$. Substituting $t = m_n/\ell_n$ and $x = (\ell_n/m_n)Y^{(i)}_{m_n-\ell_n,m_n}$ into (A27) gives

$$\Big|\frac{\tilde{A}\Big(Y^{(i)}_{m_n-\ell_n,m_n}\Big)}{\tilde{A}(m_n/\ell_n)} - \Big(\frac{\ell_n}{m_n}Y^{(i)}_{m_n-\ell_n,m_n}\Big)^\rho\Big| \le \varepsilon\Big(\frac{\ell_n}{m_n}Y^{(i)}_{m_n-\ell_n,m_n}\Big)^{\rho\pm\delta}$$

which implies

$$\Big|S_n^{(31)}(m_n,\ell_n,(r)_2)\Big| \le \varepsilon\tilde{S}_n^{(31)}(m_n,\ell_n,(r)_2),$$

where

$$\tilde{S}_n^{(31)}(m_n,\ell_n,(r)_2) = \frac{1}{[n/m_n]}\sum_{i=1}^{[n/m_n]}\Big(\frac{\ell_n}{m_n}Y^{(i)}_{m_n-\ell_n,m_n}\Big)^{\rho\pm\delta}\Big|\eta^{(i)}_{(r_1,r_2)}\Big|.$$

To establish (A24), it suffices to prove $\tilde{S}_n^{(31)}(m_n,\ell_n,(r)_2) = O_p(1)$. Let $K > 0$. By Markov's inequality,

$$\mathrm{P}\Big(\tilde{S}_n^{(31)}(m_n,\ell_n,(r)_2) > K\Big) \le K^{-1}\mathrm{E}\tilde{S}_n^{(31)}(m_n,\ell_n,(r)_2).$$

Similarly to the proof of (A23), we obtain

$$\mathrm{E}\tilde{S}_n^{(31)}(m_n,\ell_n,(r)_2) = \mathrm{E}\Big(\big((\ell_n/m_n)Y^{(1)}_{m_n-\ell_n,m_n}\big)^{\rho\pm\delta}\Big)\mathrm{E}\Big|\eta^{(1)}_{(r_1,r_2)}\Big|.$$

By (A2), the sequences of expectations

$$\mathrm{E}\left(\left((\ell_n/m_n)Y^{(1)}_{m_n-\ell_n,m_n}\right)^{\rho+\delta}\right) \text{ and } \mathrm{E}\left(\left((\ell_n/m_n)Y^{(1)}_{m_n-\ell_n,m_n}\right)^{\rho-\delta}\right), \quad n \geq 1$$

are convergent and thus are bounded, i.e., there exists a natural number n_0 such that

$$\mathrm{E}\left(\left((\ell_n/m_n)Y^{(1)}_{m_n-\ell_n,m_n}\right)^{\rho\pm\delta}\right) \leq 2$$

for $n > n_0$. Now, we get immediately

$$\mathrm{E}\tilde{S}^{(31)}_n(m_n,\ell_n,(r)_2) \leq 2\{|\theta_1|\nu(\rho,\gamma r_1)+|\theta_2|\nu(\rho,\gamma r_2)\}, \quad n > n_0.$$

This concludes the proof of (A24).

The relation (A21) follows from

$$\sqrt{\frac{n}{m_n/\ell_n}}S^{(4)}_n(m_n,\ell_n,(r)_2) \xrightarrow{\mathrm{P}} \frac{|\theta_1|\mu}{1-\gamma r_1-\rho-\delta}+\frac{|\theta_2|\mu}{1-\gamma r_2-\rho-\delta}, \tag{A28}$$

as $n \to \infty$, where $\delta > 0$ is such that $1-\gamma\max\{r_1,r_2\}-\rho-\delta > 0$. The proof of (A28) is similar to the proof of (A20) and is therefore omitted. □

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