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A central limit theorem for coefficients of the modified Borwein method for the calculation of the Riemann zeta-function

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Dedicated to my dear teacher, professor Antanas Laurinčikas, and limit theorems, and zeta-functions, which are inseparable.

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Abstract. The paper extends the study of the modified Borwein method for the calculation of the Riemann zeta-function. It presents an alternative perspective on the proof of the central limit theorem for coefficients of the method. The new approach is based on the connection with the limit theorem applied to asymptotic enumeration.

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1 Introduction

Borwein's method [3] to compute the Riemann zeta-function is based on the alternating series (1.2) convergence. It applies to complex numbers $s = \sigma + it$ with $\sigma \geqslant 1/2$.

Let

$$d_{nk} = n \sum_{i=0}^{k} \frac{(n+i-1)!4^i}{(n-i)!(2i)!}, \quad n \in \mathbb{N}, \quad k = 0, \dots, n, \tag{1.1}$$

then the Riemann zeta-function

$$\zeta(s) = \frac{1}{d_{nn}(1-2^{1-s})} \sum_{k=0}^{n-1} \frac{(-1)^k (d_{nn} - d_{nk})}{(k+1)^s} + \gamma_n(s). \tag{1.2}$$

Here

$$|\gamma_n(s)| \leqslant \frac{3}{(3+\sqrt{8})^n} \frac{(1+2|t|)e^{\frac{\pi|t|}{2}}}{|1-2^{1-s}|}.$$

It is difficult to compute coefficients d_{nk} for large n directly (note factorials in definition (1.1)), hence in [1] we have introduced a modification of the algorithm (1.3)-(1.5) and proposed an asymptotic expression for the coefficients of the method. The asymptotic modification of the Borwein algorithm proved to be more than three times faster than the original one (see [1]).

Let $c_{nk} = 1 - d_{nk}/d_{nn}, \quad k = 0, \dots, n-1$. Now

$$\zeta(s) = \sum_{k=0}^{n-1} \frac{(-1)^k c_{nk}}{(k+1)^s} + \gamma_n(s). \tag{1.3}$$

Let us define

$$u_{ni} = n\frac{(n+i-1)!4^i}{(n-i)!(2i)!}, \quad n \in \mathbb{N}, \ \ i = 0, \dots, n. \tag{1.4}$$

Now we can calculate d_{nk} recurrently, i.e. $d_{nk} = d_{n,k-1} + u_{nk}$, $d_{n0} = 1$. This way we obtain that

$$c_{nk} = 1 - \frac{\sum_{i=0}^{k} u_{ni}}{\sum_{i=0}^{n} u_{ni}} = 1 - \sum_{i=0}^{k} a_{ni},$$

where

$$a_{nk} = \frac{u_{nk}}{\sum_{i=0}^{n} u_{ni}}. \tag{1.5}$$

Let A_n be an integral random variable with the probability mass function

$$P(A_n = k) = a_{nk}, \quad k = 0, \dots, n. \tag{1.6}$$

and the cumulative distribution function $F_n(x)$. In [1] we have established limit theorems for a_{nk} coefficients.

Let us denote by $\Phi(x)$ the cumulative distribution function of the standard normal distribution, and by $\Phi_{\mu,\sigma}(x)$ the cumulative distribution function of the normal distribution with the mean μ and the standard deviation σ,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt, \qquad \Phi_{\mu,\sigma}(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \qquad x \in \mathbb{R}.$$

This paper is organized as follows. The first part is the introduction. In Section 2, we introduce limit theorems for coefficients of the modified Borwein method. Section 3 is devoted to the presentation of the alternative proof of the limit theorem.

Throughout this paper, all limits, whenever unspecified, will be taken as $n \to \infty$.

2 Limit theorems for coefficients of the modified Borwein method

In [1] we have received the following central limit theorem for coefficients of the modified Borwein method.

Theorem 1. (I. Belovas, L. Sakalauskas) *Suppose that $F_n(x)$ is the cumulative distribution function of the random variable A_n (1.6), then*

$$|F_n(x) - \Phi_{\mu_n,\sigma_n}(x)| = O\left(\frac{1}{\sqrt{n}}\right),$$

uniformly with respect to x, $x \in \mathbb{R}$.

$$\mu_n = \frac{n}{\sqrt{2}}\left(1 + \frac{33}{144n^2} + O\left(\frac{1}{n^3}\right)\right), \ \ \sigma_n^2 = \frac{\sqrt{2}}{8}n\left(1 + O\left(\frac{1}{n^3}\right)\right).$$

The theorem was proved in a "straightforward" way, using the Laplace method, plus it involved the Hwang lemma [1, 4]. Alternative perspective on the coefficeints of the modified Borwein method reveals the connection with combinatorial numbers and calls for more subtle approach, taking benefits from the results of asymptotic enumeration theory [5]. We will use a general central limit theorem by E. A. Bender (Theorem 2), based on the nature of the generating function $\sum u_{nk} z^n w^k$, to obtain a different proof of the central limit theorem for coefficients (1.5) of the modified Borwein method (Theorem 3, (3.10)).

Theorem 2. (E. A. Bender [2]) *Let $f(z, w)$ have a power series expansion*

$$f(z,w) = \sum_{n,k \geqslant 0} u_{nk} z^n w^k \tag{2.1}$$

with non-negative coefficients. Suppose there exists

(i) *an $A(s)$ continuous and non-zero near* 0,

(ii) *an $r(s)$ with bounded third derivative near* 0,

(iii) *a non-negative integer m, and*

(iv) $\varepsilon, \delta > 0$ *such that*

$$\left(1 - \frac{z}{r(s)}\right)^m f(z, e^s) - \frac{A(s)}{1 - z/r(s)} \tag{2.2}$$

is analytic and bounded for

$$|s| < \varepsilon, \qquad |z| < |r(0)| + \delta. \tag{2.3}$$

Define

$$\eta = -\frac{r'(0)}{r(0)}, \qquad \theta^2 = \eta^2 - \frac{r''(0)}{r(0)}. \tag{2.4}$$

If $\theta \neq 0$, then

$$\lim_{n\to\infty} \sup_x \left| \sum_{k \leqslant \vartheta_n x + \eta_n} a_{nk} - \Phi(x) \right| = 0 \tag{2.5}$$

holds with $\eta_n = n\eta$ and $\vartheta_n^2 = n\vartheta^2$.

3 Alternative proof for the limit theorem for coefficients of the modified Borwein method

First we obtain an explicit formula (3.2) for the generating function for coefficients (1.4) of the modified Borwein method, $\sum u_{nk} z^n w^k$ (cf. (2.1)). To get this explicit formula, we construct and solve a certain linear partial differential equation of the second order (3.6), satisfied by the generating function. Let us formulate an auxiliary lemma identifying the generating function.

Lemma 1. *Suppose that*

$$u_{nk} = \begin{cases} 1 & n = k = 0, \\ 0 & k > n, \\ n\frac{(n+k-1)!4^k}{(n-k)!(2k)!} & otherwise, \end{cases} \tag{3.1}$$

then the generating function of coefficients (3.1)

$$S(x,y)=\sum_{n,k\geqslant 0}u_{nk}x^ny^k=\frac{1}{2}\left(1+\frac{1}{2x^{-1}\Theta(y)-1}-\frac{1}{2x\Theta(y)-1}\right). \tag{3.2}$$

Here

$$\Theta(y)=y+\sqrt{y+y^2}+1/2. \tag{3.3}$$

*Proof T*he definition (3.1) gives us the recurrent expression

$$u_{nk}=u_{n,k-1}\frac{4(n+k-1)(n-k+1)}{(2k-1)(2k)}. \tag{3.4}$$

Consider the generating function (3.2),

$$S(x,y)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}u_{nk}x^ny^k.$$

Taking into account that $u_{n0}=1$, we obtain

$$\begin{aligned}S(x,y)&=\sum_{n=0}^{\infty}u_{n0}x^n+\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}u_{n,k-1}\frac{4(n+k-1)(n-k+1)}{(2k-1)(2k)}x^ny^k=\\&=\frac{1}{1-x}+4\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}u_{n,k}\frac{(n+k)(n-k)}{(2k+1)(2k+2)}x^ny^{k+1}.\end{aligned} \tag{3.5}$$

Thus,

$$\begin{aligned}S(x,y)&=\frac{1}{1-x}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}u_{n,k}\frac{4k^2-4n^2}{4k^2+6k+2}x^ny^{k+1}=\\&=\frac{1}{1-x}-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}u_{n,k}x^ny^{k+1}+\frac{3}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{u_{n,k}}{k+1}x^ny^{k+1}+\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{u_{n,k}(4n^2-1)}{(2k+1)(2k+2)}x^ny^{k+1}\\&=\frac{1}{1-x}-yS(x,y)+\frac{3}{2}\int_0^y S(x,t)dt+\int_0^{\sqrt{y}}\int_0^u 4xS_x(x,t^2)+4x^2S_{xx}(x,t^2)-S(x,t^2)dtdu.\end{aligned}$$

It gives us the linear partial differential equation of the second order,

$$x^2S_{xx}-(y+y^2)S_{yy}+xS_x-(1/2+y)S_y=0. \tag{3.6}$$

Note that, in view of (3.5), we have initial conditions

$$S(x,0)=\frac{1}{1-x},\qquad S_y(x,0)=0. \tag{3.7}$$

Introducing new variables

$$\begin{cases}\rho=\log x,\\ \tau=\log\Theta(y)+\log 2,\end{cases} \tag{3.8}$$

and applying the chain rule,

$$S_x = S_\rho \rho_x = S_\rho x^{-1}, \qquad S_{xx} = S_{\rho\rho} x^{-2} - S_\rho x^{-2},$$

$$S_y = S_\tau \tau_y = S_\tau \frac{1}{\sqrt{y^2+y}}, \qquad S_{yy} = S_{\tau\tau} \frac{1}{y^2+y} - S_\tau \frac{y+1/2}{\sqrt{(y^2+y)^3}},$$

we reduce the original equation (3.6) to the wave equation $S_{\tau\tau} = S_{\rho\rho}$ with initial conditions

$$S(\rho,0) = \frac{1}{1-e^\rho}, \qquad S_\tau(\rho,0) = 0 \tag{3.9}$$

and d'Alembert's solution

$$S(\rho,\tau) = \frac{1}{2}\left(\frac{1}{1-e^{\rho-\tau}} + \frac{1}{1-e^{\rho+\tau}}\right).$$

By (3.8), we have

$$S(x,y) = \frac{1}{2}\frac{1}{1-x/(2\Theta(y))} + \frac{1}{2}\frac{1}{1-2x\Theta(y)} = \frac{1}{2}\left(\frac{2x^{-1}\Theta(y)+1-1}{2x^{-1}\Theta(y)-1} - \frac{1}{2x\Theta(y)-1}\right),$$

yielding us the statement of the lemma.

Now we can proceed with the alternative proof for the central limit theorem.

Theorem 3. *Coefficients a_{nk} (1.5) satisfy a central limit theorem, i.e.,*

$$\lim_{n\to\infty} \sup_x \left| \sum_{k\leqslant \sigma_n x + \mu_n} a_{nk} - \Phi(x) \right| = 0 \tag{3.10}$$

holds with $\mu_n = n\mu$ and $\sigma_n^2 = n\sigma^2$. Here

$$\mu = 1/\sqrt{2}, \qquad \sigma^2 = \sqrt{2}/8. \tag{3.11}$$

Proof By (3.2) of Lemma 1 and (2.1) of Theorem 2, the generating function of coefficients u_{nk} (1.4)

$$f(z,e^s) = \frac{1}{2}\left(1 + \frac{1}{2z^{-1}\Theta(s)-1} - \frac{1}{2z\Theta(s)-1}\right) = \frac{4\Theta^2(s) - 4z^{-1}\Theta(s)+1}{2(2z^{-1}\Theta(s)-1)(2z\Theta(s)-1)}. \tag{3.12}$$

Here

$$\Theta(s) = e^s + \sqrt{e^s + e^{2s}} + 1/2. \tag{3.13}$$

Crucial part of the proof is the selection of $r(s)$ and $A(s)$ functions. Let $r(s)$ (cf. Theorem 2) be a root of the function

$$h(z,e^s) = (2z^{-1}\Theta(s)-1)(2z\Theta(s)-1).$$

This function has two roots, at $z_1 = 2\Theta(s)$ and $z_2 = (2\Theta(s))^{-1}$.

Let us denote

$$r_1(s) = 2\Theta(s), \qquad r_2(s) = \frac{1}{2\Theta(s)}. \tag{3.14}$$

Calculating the first derivative, we obtain

$$r_1'(s) = \frac{2\Theta(s)e^s}{\sqrt{e^s + e^{2s}}},$$

and

$$\frac{r_1'(s)}{r_1(s)} = \sqrt{\frac{e^s}{e^s + 1}}. \tag{3.15}$$

Thus,

$$\frac{r_1'(0)}{r_1(0)} = \frac{1}{\sqrt{2}} > 0.$$

Noticing that

$$\log r_2(s) = -\log r_1(s),$$

we get

$$\frac{r_2'(s)}{r_2(s)} = -\frac{r_1'(s)}{r_1(s)}, \tag{3.16}$$

and

$$\frac{r_2'(0)}{r_2(0)} = -\frac{1}{\sqrt{2}} < 0. \tag{3.17}$$

By Theorem 2, $\mu_n = n\mu$ and $\mu = -r'(0)/r(0)$ (cf. (2.4))-(2.5)). Note that by definitions (1.4)-(1.5), numbers u_{nk} and, hence, numbers a_{nk} are positive. Thus, to obtain positive mean μ, we choose the root $r_2(s)$, corresponding the negative ratio (3.17).

By (3.13) and (3.14), we have

$$r(s) = r_2(s) = \frac{1}{2\Theta(s)} = \frac{1}{2(e^s + \sqrt{e^s + e^{2s}} + 1/2)}. \tag{3.18}$$

Next, by (3.15) and (3.16), we obtain

$$\frac{r'(s)}{r(s)} = -\sqrt{\frac{e^s}{e^s + 1}}. \tag{3.19}$$

Thus,

$$r(0) = 3 - 2\sqrt{2}, \qquad \frac{r'(0)}{r(0)} = -\frac{1}{\sqrt{2}}. \tag{3.20}$$

Calculating the second derivative, we get

$$r''(s) = -\left(r(s)\sqrt{\frac{e^s}{e^s + 1}}\right)' = -r'(s)\sqrt{\frac{e^s}{e^s + 1}} - r(s)\left(\sqrt{\frac{e^s}{e^s + 1}}\right)'.$$

Hence,

$$\frac{r''(s)}{r(s)} = -\frac{r'(s)}{r(s)}\sqrt{\frac{e^s}{e^s + 1}} - \left(\sqrt{\frac{e^s}{e^s + 1}}\right)' = \frac{e^s}{e^s + 1} - \frac{1}{2}\sqrt{\frac{e^s}{(e^s + 1)^3}}, \tag{3.21}$$

and

$$\frac{r''(0)}{r(0)} = \frac{1}{2} - \frac{\sqrt{2}}{8}. \tag{3.22}$$

As E. A. Bender indicates [2], the easiest way for verifying (2.2)-(2.3) condition of Theorem 2 is to show that $f(z, e^s)$ is continuous for $s \leqslant \varepsilon$ and z in the set

$$\{|z| \leqslant |r(0) + \delta|\} \cap \{|z - r(s)| \geqslant \eta\} \tag{3.23}$$

for some η. Since this is a compact set, f and hence (2.2) are bounded here. For $|z - r(s)| \leqslant \eta$ we can expand $f(z, e^s)$ in a Laurent series about $r(s)$ and show that the coefficient of the error term is bounded.

Let us consider the function $A(s)$ from (2.2) of Theorem 2 as the limit

$$A(s) = \lim_{z \to r(s)} f(z, e^s) \left(1 - \frac{z}{r(s)}\right)^{m+1}. \tag{3.24}$$

Here $m + 1$ is the order of the pole. So, if the pole is simple, then $m = 0$. Calculating $A(s)$ we obtain

$$\begin{aligned} A(s) &= \lim_{z \to r(s)} \frac{1}{2}\left(1 + \frac{1}{2z^{-1}\Theta(s) - 1} - \frac{1}{2z\Theta(s) - 1}\right)\left(1 - \frac{z}{r(s)}\right) = \\ &= \lim_{z \to r(s)} \frac{1}{2}\left(1 + \frac{1}{z^{-1}r^{-1}(s) - 1} - \frac{1}{zr^{-1}(s) - 1}\right)\left(1 - \frac{z}{r(s)}\right) = \frac{1}{2}. \end{aligned} \tag{3.25}$$

The function

$$\left(1 - \frac{z}{r(s)}\right)^m f(z, e^s) - \frac{A(s)}{1 - z/r(s)} = \frac{\Theta(s)}{2\Theta(s) - z} \tag{3.26}$$

is analytic and bounded for

$$|s| < \varepsilon, \qquad |z| < |r(0)| + \delta = 3 - 2\sqrt{2} + \delta. \tag{3.27}$$

Thus, conditions (i)-(iv) of Theorem 2 are satisfied.

Next, let us calculate the mean and the variance. By (2.4),

$$\mu = -\frac{r'(0)}{r(0)}, \qquad \sigma^2 = \mu^2 - \frac{r''(0)}{r(0)}. \tag{3.28}$$

Taking into account (3.20) and (3.22), we obtain

$$\mu = 1/\sqrt{2}, \qquad \sigma^2 = \sqrt{2}/8. \tag{3.29}$$

We have $\sigma \neq 0$, hence (2.5) yields us the statement of the theorem.

Remark. *The local limit theorem by E. A. Bender (Theorem 3 in [2]) allows us to prove corresponding local limit theorem, for coefficients* u_{nk}.

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References

1. I. Belovas and L. Sakalauskas, Limit theorems for the coefficients of the modified Borwein method for the calculation of the Riemann zeta-function values, *Colloq. Math.*, **151**(2):217–227, 2018, http://dx.doi.org/10.4064/cm7086-2-2017.

2. E. A. Bender, Central and local limit theorems applied to asymptotic enumeration, *J. Comb. Theory (A)*, **15**:91–111, 1973.

3. P. Borwein, An efficient algorithm for the Riemann Zeta function, in *Constructive, Experimental, and Nonlinear Analysis (Limoges, 1999)*, Canadian Mathematical Society Conference Proceedings, American Mathematical Society, Providence, RI, 2000, pp. 29–34.

4. H.-K. Hwang, On convergence rates in the central limit theorems for combinatorial structures, *Eur. J. Combin.*, **19**(3):329–343, 1998.

5. A. M. Odlyzko, Asymptotic enumeration methods, in M. Groetschel R. L. Graham and L. Lovasz (Eds.), *Handbook of Combinatorics*, Volume 2. year, Elsevier, 1995.