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The Mellin transform of the square of the Riemann zeta-function in the critical strip

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Let $s=\sigma+it$ be a complex variable. The Mellin transform $M(f)$ of the function $f(x)$ is defined by

$$M(f)=\int_0^\infty f(x)x^{s-1}\,\mathrm{d}x.$$

However, in analytic number theory usually the modified Mellin transform

$$\int_1^\infty f(x)x^{-s}\,\mathrm{d}x$$

are considered. Let $\zeta(s)$ denote the Riemann zeta-function, and let

$$Z_k(s)=\int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k} x^{-s}\,\mathrm{d}s,\quad k\geqslant 0,$$

be the modified Mellin transform of $|\zeta(\frac{1}{2}+ix)|^{2k}$. First the function $Z_2(s)$ has been introduced and studied in [6] [7]. The first results for $Z_1(s)$ were obtained in [1].

Define, for $\frac{1}{2}<\rho<1$,

$$Z_k(\rho,s)=\int_1^\infty \left|\zeta\left(\rho+ix\right)\right|^{2k} x^{-s}\,\mathrm{d}s,\quad k\geqslant 0.$$

The aim of this note is to obtain an analytic continuation for the function $Z_1(\rho,s)$. Clearly, in view of the estimate

$$\int_1^T \left|\zeta\left(\sigma+ix\right)\right|^2 \mathrm{d}t \ll_\sigma T,\quad \frac{1}{2}<\sigma<1,$$

the integral for $Z_1(\rho,s)$ converges absolutely for $\sigma>1$ and defines there an analytic function. To obtain more precise results, we will apply the asymptotics for the mean square of $\zeta(s)$ in the critical strip.

Let, for $\frac{1}{2}<\sigma<1$,

$$\int_1^T \left|\zeta\left(\sigma+ix\right)\right|^2 \mathrm{d}t=\zeta(2\sigma)T+(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}+E_\sigma(T).$$

Then in [8], see also [2], the estimate

$$E_\sigma(T) = \mathrm{O}\left(T^{\frac{1}{1+4\sigma}} \log^2 T\right) \tag{1}$$

has been proved. Using this, we find, for $\sigma > 1$,

$$\begin{aligned}
Z_1(\rho, s) &= \int_1^\infty x^{-s} d \int_0^x \left|\zeta(\rho + ix)\right|^2 \mathrm{d}x \\
&= x^{-s} \int_1^x \left|\zeta(\rho + ix)\right|^2 \mathrm{d}x \Big|_1^\infty \\
&\quad + s \int_1^\infty \Big(\zeta(2\rho)x + (2\pi)^{2\rho-1}\frac{\zeta(2-2\rho)}{2-2\rho}x^{2-2\rho} + E_\rho(x)\Big)x^{-s-1}\,\mathrm{d}x \\
&= s\int_1^\infty \zeta(2\rho)x^{-s}\,\mathrm{d}x + s(2\pi)^{2\rho-1}\frac{\zeta(2-2\rho)}{2-2\rho}\int_1^\infty x^{1-s-2\rho}\,\mathrm{d}x \\
&\quad + s\int_1^\infty E_\rho(x)x^{-s-1}\,\mathrm{d}x \\
&= \frac{s}{1-s}\zeta(2\rho)x^{-s+1}\Big|_1^\infty + s(2\pi)^{2\rho-1}\frac{\zeta(2-2\rho)}{2-2\rho}\,\frac{x^{2-s-2\rho}}{2-s-2\rho}\Big|_1^\infty \\
&\quad + s\int_1^\infty e_\rho(x)x^{-s-1}\,\mathrm{d}x \\
&= \frac{s}{s-1}\zeta(2\rho) + \frac{s(2\pi)^{2\rho-1}\zeta(2-2\rho)}{2-2\rho}\,\frac{1}{2\rho+s-2} + s\int_1^\infty E_\rho(x)x^{-s-1}\,\mathrm{d}x.
\end{aligned} \tag{2}$$

In view of the estimate (1) the last integral converges absolutely for

$$\frac{1}{1+4\rho} - \sigma - 1 < -1,$$

that is for $\sigma > \frac{1}{1+4\rho}$. Therefore, the equation (2) gives an analytic continuation to the region $\sigma > \frac{1}{1+4\rho}$, except for a simple pole at $s = 1$ with residue $\zeta(2\sigma)$, and a simple pole at $s = 2 - 2\rho$ with residue $\frac{s(2\pi)^{2\rho-1}\zeta(2-2\rho)}{2-2\rho}$. Since $\sigma > \frac{1}{1+4\rho}$ hence we find that $2 - 2\rho > \frac{1}{1-4\rho}$, and $\frac{1}{2} < \rho < \frac{3+\sqrt{17}}{8}$.

Now we will extend the region $\sigma > \frac{1}{1+4\rho}$ of analytic continuation for $Z_1(\rho, s)$. It is known [4], [5], that for $\frac{1}{2} < \sigma < \frac{3}{4}$,

$$\int_2^T E_\sigma^2(t)\,\mathrm{d}t = \frac{2}{5-4\sigma}(2\pi)^{2\sigma-\frac{3}{2}}\frac{\zeta^2\left(\frac{3}{2}\right)}{\zeta(3)}\zeta\left(\frac{5}{2}-2\sigma\right)\zeta\left(\frac{1}{2}+2\sigma\right)T^{\frac{5}{2}-2\sigma} + \mathrm{O}(T),$$

$$\int_2^T E^2_{\frac{3}{4}}(t)\,\mathrm{d}t = \frac{\zeta^2\left(\frac{3}{2}\right)\zeta(2)}{\zeta(3)} T\log T + \mathrm{O}\left(T\log^{\frac{1}{2}} T\right),$$

and, for $\frac{3}{4} < \sigma < 1$,

$$\int_2^T E^2_\sigma(t)\,\mathrm{d}t \ll T.$$

Therefore, we have, for $0 < \alpha < \sigma + \frac{1}{2}$,

$$\begin{aligned}
&\int_1^\infty E_\rho(x)x^{-\sigma-1}\,\mathrm{d}x \\
&\quad\ll \left(\int_1^\infty E^2_\rho(x)x^{-2\alpha}\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_1^\infty x^{2\alpha-2\sigma-2}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad\ll \left(\int_1^\infty E^2_\rho(x)x^{-2\alpha}\,\mathrm{d}x\right)^{\frac{1}{2}} \ll \left(\int_1^\infty x^{-2\alpha}\,\mathrm{d}\int_1^x E^2_\rho(x)\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad= x^{-2\alpha}\int_1^x E^2_\rho(x)\,\mathrm{d}x\Big|_1^\infty + 2\alpha\int_1^\infty x^{\frac{5}{2}-2\rho}x^{-2\alpha-1}\,\mathrm{d}x \\
&\quad\ll x^{\frac{5}{2}-2\alpha-2\rho}\Big|_1^\infty + \int_1^\infty x^{\frac{3}{2}-2\alpha-2\rho}\,\mathrm{d}x. \qquad (3)
\end{aligned}$$

Let $2\alpha > \frac{5}{2} - 2\rho$. Then in view of (3), since then $\frac{3}{2} - 2\rho - 2\alpha < -1$, we find that

$$\int_1^\infty E_\rho(x)x^{-\sigma-1}\,\mathrm{d}x \ll 1.$$

Since $0 < \alpha < \sigma + \frac{1}{2}$, here $\sigma > \frac{3}{4} - \rho$. Then the integral

$$\int_1^\infty E_\rho(x)x^{-s-1}\,\mathrm{d}x$$

converges uniformly on compact subsets of the half-plane $\sigma > \frac{3}{4} - \rho$. Hence equality (2) gives analytic continuation for the function $Z_1(\rho, s)$ to the region $\sigma > \frac{3}{4} - \rho$, except for the simple poles at $s = 1$ and $s = 2 - 2\rho$.

References

1. A. Ivič, M. Jutila, Y. Motohashi, The Mellin transform of powers of the zeta-function, *Acta Arith.*, **95**, 305–342 (2000).
2. I. Kuichi, K. Matsumoto, The resemblance of the behavior of the remainder terms $E_\sigma(t)$, $\Delta_{1-2\sigma}(x)$ and $R(\sigma + it)$, in: *Sieve Methods, Exponential Sums, and their Appl. in Number Theory*, G. Greaves *et al.* (Eds.), Cambridge Univ. Press (1996), pp. 255–273.

3. K. Matsumoto, The mean square of the Riemann zeta-function in the critical strip, *Japanese J. Math.*, **15**, 1–13 (1989).
4. K. Matsumoto, T. Meurman, The mean square of the Riemann zeta-function in the critical strip III, *Acta Arith.*, **64**, 357–382 (1993).
5. K. Matsumoto, T. Meurman, The mean square of the Riemann zeta-function in the critical strip II, *Acta Arith.*, **68**, 369–382 (1994).
6. Y. Motohashi, A relation between Riemann zeta-function and the Laplasian, *Ann. Sacola Norm. Sug. Pisa, Cl. Sci.*, **4**(22), 299–313 (1995).
7. Y. Motohashi, *Spectral Theory of the Riemann Zeta-Function*, Cambridge Univ. Press, Cambridge (1997).
8. Y. Motohashi, The mean square of $\zeta(s)$ off the critical line (unpublished monuscript).

REZIUMĖ

M. Stoncelis. Rymano dzeta funkcijos kvadrato Melino tranformacijos kritinėje juostoje

Gautas Rymano dzeta funkcijos kvadrato Melino transformacijos kritinėje juostoje analizinis pratęsimas į kairę nuo vienetinės tiesės.