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Interpolation method for quaternionic-Bézier curves

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Abstract. We study rational quaternionic–Bézier curves in three dimensional space. We construct the quadratic quaternionic–Bézier curve which interpolates five points, or three points and two tangent vectors.

Keywords: Quaternionic-Bézier curve, interpolation.

Introduction

We analyse rational Bézier curves with quaternion control points and weights, called quaternionic–Bézier (QB) curves. This class of curves has two remarkable properties:

- a QB-curve of degree n can be converted to the classical rational Bézier curve of degree $2n$;
- QB-curves are invariant with respect to Möbius transformations.

The QB-curves of degree one are circles. They were discussed in [6]. Quadratic QB-curves were described in [7, 3]. However, this description is not convenient, because the middle control may be in 4-dimensional space and is not clear how to use this point in practise for modelling purposes. In this paper we give two interpolation constructions for quadratic QB-curves:

– the interpolation curve through five points in $\mathbb{R}^3$,
– the interpolation curve through three points with prescribed tangent vectors at the endpoints.

We use the idea of Anton Gfrerrer [1] to construct a curve for the interpolation method on the hyperquadric. In order to use this we notice the one-to-one correspondence between quaternionic curves and the curves on Study quadric in $\mathbb{R}^8$ (see also [4]). On the another hand, points on the Study quadric can be represented as the displacements in $\mathbb{R}^3$, while curves on the Study quadrics mean the motion in $\mathbb{R}^3$. Therefore, QB-curves could be important in kinematics applications (see [2, 4]).

1 Notations and definitions

Let us denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ the set of real numbers, complex numbers and quaternion numbers respectively. The quaternions $\mathbb{H}$ can be identified with $\mathbb{R}^4$

$$\mathbb{H} = \{ q = [r, p] \,|\, r \in \mathbb{R},\, p \in \mathbb{R}^3 \} = \mathbb{R}^4,$$

where $r = \mathrm{Re}(q)$, $p = \mathrm{Im}(q)$ denote real and imaginary parts of a quaternion $q = [r, p]$. The multiplication in the algebra $\mathbb{H}$ is defined by the formula

$$[r_1, p_1][r_2, p_2] = [r_1 r_2 - p_1 \cdot p_2,\ r_1 p_2 + r_2 p_1 + p_1 \times p_2],$$

where $p_1 \cdot p_2$, $p_1 \times p_2$ are scalar and vector products in $\mathbb{R}^3$. Let $\bar{q} = [r, -p]$ means the conjugate quaternion, and $|q| = \sqrt{r^2 + p \cdot p} = \sqrt{q\bar{q}}$ denotes the length of the quaternion. The multiplicative inverse of q is $q^{-1} = \bar{q}/|q|^2 = [r/|q|^2, -p/|q|^2]$, i.e. $qq^{-1} = q^{-1}q = 1$. We identify the set of pure imaginary quaternions with $\mathbb{R}^3$: $\mathrm{Im}(\mathbb{H}) = \{[0, p] \mid p \in \mathbb{R}^3\} = \mathbb{R}^3$.

2 The Study quadric and QB-curves

We consider a pair of quaternions $(p; q) \in \mathbb{H} \times \mathbb{H} = \mathbb{R}^8$ as a point in the projective space $\mathbb{P}^7$. The *Study quadric* SQ is a hypersurface in $\mathbb{P}^7$ defined by the equation $p\bar{q} + q\bar{p} = 0$. Explicitly, let $p = [p_1, \ldots, p_4]$, $q = [q_1, \ldots, q_4]$ be two quaternions then the equation of the Study quadric is $\sum_1^4 p_i q_i = 0$:

$$SQ := \big\{(p; q) = (p_1, p_2, p_3, p_4; q_1, q_2, q_3, q_4) \mid p_1 q_1 + p_2 q_2 + p_3 q_3 + p_4 q_4 = 0\big\}.$$

Geometrically, $(p; q) \in SQ$ if and only if the vector p is orthogonal to the vector q in the Euclidean sense. The point on the Study quadric $c = (p; q) \in SQ$ defines a point in $\mathbb{R}^3$:

$$\Pi : (p; q) \to q\, p^{-1} \in \mathrm{Im}(\mathbb{H}) = \mathbb{R}^3,$$

because $\mathrm{Re}(q\, p^{-1}) = \mathrm{Re}(q\, \bar{p}/p\bar{p}) = \sum_1^4 q_i p_i / p\bar{p} = 0$.

An important application of the Study quadric is modelling of rigid body displacements. A dual quaternion $c = (p; q) \in SQ$ acts on the quaternion $x = [0, x_1, x_2, x_3]$ by the formula

$$c : x \to \frac{px\bar{p} + p\bar{q} - q\bar{p}}{p\bar{p}} = \frac{px\bar{p} - 2q\bar{p}}{p\bar{p}} = (px - 2q)p^{-1}.$$

The first part of the above formula pxp^{-1} means rotation while the second means translation, therefore this action is an element of $SE(3)$, the group of rigid body displacement (see for details [2]).

The rational quaternionic curve $c(t)$ can be defined as:

$$c(t) = q(t)p(t)^{-1}, \quad \text{where } q(t), p(t) \in \mathbb{H}[t] \text{ are quaternion polynomials.}$$

For applications it is important to describe curves in three dimensional space. We observe that

$$c(t) = q(t)p(t)^{-1} \in \mathrm{Im}(\mathbb{H}) = \mathbb{R}^3 \quad \text{if and only if } (p(t); q(t)) \in SQ, \text{ i.e.}$$

a rational QB–curve in 3D is the same as a curve on the Study quadric.

3 Interpolation construction for QB–curves

Rational quadratic quaternionic–Bézier curves were considered in [7]. Our interpolation construction of the QB–curve was inspired by the idea A. Gfrerrer (see [1]) of interpolation points on the Study quadric. The arbitrary point $c = (p;q) \in SQ$ can be presented in the homogeneous form

$$c = (p;q) = \left(pq^{-1}q;q\right) = (a_0w_0;\,w_0),\ \text{where}\ a_0 = pq^{-1} \in \operatorname{Im}(\mathbb{H}) = \mathbb{R}^3,\ w_0 = q,$$

which we interpret as the point a_0 in $\mathbb{R}^3$ with a quaternion weight w_0.

Let $n \geqslant 1$ be an integer, and let $T = [t_0,\dots,t_n]$ be $n+1$ pairwise different real parameter values. Then we define polynomials of degree n

$$f_i(t) = \prod_{k\neq i}(t-t_k), \qquad \text{with the following properties}$$

$$f_i(t_i) \neq 0, \qquad f_i(t_j) = 0, \quad \text{if } j \neq i.$$

Let us fix $n+1$ homogeneous points $(a_{2i}w_{2i};\,w_{2i})$, $i = 0,\dots,n$ and define quaternionic polynomials

$$q(t) = \sum_{i=0}^{n} a_{2i}\,w_{2i}\,f_i(t), \qquad p(t) = \sum_{i=0}^{n} w_{2i}\,f_i(t), \qquad C(t) = q(t)\,p(t)^{-1}. \tag{1}$$

We are going to find conditions when the curve $C(t) = q(t)p(t)^{-1}$ is in $\operatorname{Im}(\mathbb{H})$. First of all we note, that

$$C(t_i) = a_{2i} \in \operatorname{Im}(\mathbb{H}) = \mathbb{R}^3, \quad i = 0,1,\dots,n,$$

for arbitrary weights w_i, $i = 0,1,\dots,n$ (because $f_j(t_i) = 0, j \neq i$). Let $S = [s_1,\dots,s_n]$ be different (from the set T) real parameters (i.e. $S \cap T = \emptyset$) and $a_{2j-1} \in \operatorname{Im}(\mathbb{H})$, $j = 1,\dots,n$ be some points in $\mathbb{R}^3$. We consider the linear system

$$C(s_j) = a_{2j-1}, \quad j = 1,\dots,n.$$

This linear system is equivalent to $q(s_j) = a_{2j-1}\,p(s_j)$, $j = 1,\dots,n$, i.e.

$$\sum_{i=0}^{n} a_{2i}\,w_{2i}\,f_i(s_j) = \sum_{i=0}^{n} a_{2j-1}\,w_{2i}\,f_i(s_j), \quad j = 1,\dots,n,\ \text{or explicitly,}$$

$$\sum_{i=1}^{n}(a_{2i}-a_{2j-1})\,f_i(s_j)\,w_{2i} = (a_{2j-1}-a_0)\,f_0(s_j)\,w_0, \quad j = 1,\dots,n. \tag{2}$$

The quaternion linear system (2) with n equations and n unknowns $w_2, w_4, \dots, w_{2n}$ has a unique solution if the corresponding matrix $A = (\alpha_{ij})$ is not singular.

Theorem 1. *Assume the matrix* $\alpha_{ij} = (a_{2i}-a_{2j-1})\,f_i(s_j)$, $i,j = 1,\dots,n$, *is non-singular (or invertible). Then there are unique (up to left multiplication) weights* w_{2i}, $i = 0,\dots,n$, *such that the rational curve* $C(t)$ *defined by the formula* (1) *is in* $\operatorname{Im}(\mathbb{H}) = \mathbb{R}^3$. *Moreover, the curve* $C(t)$ *interpolates points* a_i, $i = 0,\dots,2n$*; actually* $C(t_i) = a_{2i}$, $i = 0,\dots,n$, *and* $C(s_i) = a_{2i-1}$, $i = 1,\dots,n$.

Proof. The non-singular quaternion matrix A has an LU-decomposition [5, Theorem 3.1]. There exists a decomposition $PA = LU$ and the matrix A is invertible $A^{-1} = U^{-1}L^{-1}P$. Therefore, the system (2) has a unique solution for the weights $w_2, w_4, \ldots, w_{2n}$. We can take $w_0 = 1$ and construct the curve $C(t)$. This curve is in $\mathbb{R}^3$ for the set $T \cup S$ of cardinality $2n+1$. Hence, according to Lemma 1 below, the curve $C(t)$ is in $\mathbb{R}^3$. $\square$

Lemma 1. *The rational curve $C(t) = q(t)p(t)^{-1}$ is an imaginary quaternion for all parameters t if and only if $\mathrm{Re}(C(\beta_i)) = 0$ for some real different parameters β_i, $i = 0, 1, \ldots, 2n$.*

Proof. We present the rational curve $C(t)$ as follows:

$$C(t) = q(t)p(t)^{-1} = \frac{q(t)\bar{p}(t)}{p(t)\bar{p}(t)}.$$

The numerator $q(t)\bar{p}(t)$ is the quaternion polynomial of degree $2n$. If $\mathrm{Re}(C(\beta_i)) = 0$ for some real different parameters β_i, $i = 0, 1, \ldots, 2n$, then the real polynomial $\mathrm{Re}(q(t)\bar{p}(t))$ of degree $2n$ is zero on $2n+1$ different parameter values, i.e. $\mathrm{Re}(q(t)\bar{p}(t)) = 0$. $\square$

Let us fix

$$T = [0, 1/2, 1], \qquad S = [1/4, 3/4], \tag{3}$$

and discuss the case $n = 2$ in greater detail. We choose five different points $a_0, a_1, \ldots, a_4 \in \mathbb{R}^3$. Then the linear system (2) multiplied by 16 is

$$e_1 : -3d_{21}w_2 - d_{41}w_4 = 3d_{10}w_0, \tag{4}$$

$$e_2 : -3d_{23}w_2 + 3d_{43}w_4 = -d_{30}w_0, \quad \text{where } d_{ij} = a_i - a_j. \tag{5}$$

In order to solve it, we eliminate w_2 in the equation $3\,d_{41}^{-1}\,e_1 + d_{43}^{-1}\,e_2$, and w_4 in the equation $d_{21}^{-1}\,e_1 - d_{23}^{-1}\,e_2$. Finally, we get

$$w_2 = \big(-9d_{41}^{-1}d_{21} - 3d_{43}^{-1}d_{23}\big)^{-1}\big(9d_{41}^{-1}d_{10} - d_{43}^{-1}d_{30}\big)w_0, \tag{6}$$

$$w_4 = \big(-d_{21}^{-1}d_{41} - 3d_{23}^{-1}d_{43}\big)^{-1}\big(3d_{21}^{-1}d_{10} + d_{23}^{-1}d_{30}\big)w_0. \tag{7}$$

The above formulas for weights are correct if $(-9\,d_{41}^{-1}\,d_{21} - 3\,d_{43}^{-1}\,d_{23}) \neq 0$ and $(-d_{21}^{-1}\,d_{41} - 3\,d_{23}^{-1}\,d_{43}) \neq 0$. Both inequalities are equivalent to the inequality

$$d_{43}^{-1}\,d_{32}\,d_{21}^{-1}\,d_{14} \neq -3. \tag{8}$$

The expression $d_{43}^{-1}\,d_{32}\,d_{21}^{-1}\,d_{14}$ is the cross-ratio of four points a_4, a_3, a_2, a_1, which is a real number if and only if these four points are on a circle (see for example [8]). Geometrically, the inequality (8) means that four points are not circular or they are circular but the cross-ratio is not equal to -3.

Corollary 1. *The interpolation curve $C(t)$ $(n = 2)$ defined by* (1) *with five points $C(i/4) = a_i$, $\in \mathbb{R}^3$, $i = 0, 1, \ldots, 4$, and the weights w_2, w_4 as in formulas* (6), (7) *is in $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$. The formulas for weights are well defined if the inequality* (8) *holds, i.e. points $a_4, \ldots, a_1$ are not on a circle or they are circular but the cross-ratio is not equal to -3.*

For applications it is also important to know tangent vectors. Let $C'(t_0)$ be the tangent vector at the starting point a_0 and $C'(t_n)$ – the tangent vector at the end point a_4 of the curve. If we replace two conditions $C(s_i) = a_{2i-1}$, $i = 1, 2$, with the conditions $C'(t_0) = v_0 \in \mathrm{Im}(\mathbb{H})$, $C'(t_n) = v_1 \in \mathrm{Im}(\mathbb{H})$ we obtain the linear system

$$\begin{aligned} f_1'(t_0)\, d_{20}\, w_2 + f_2'(t_0)\, d_{40}\, w_4 &= f_0(t_0)\, v_0\, w_0, \\ f_1'(t_n)\, d_{24}\, w_2 - \quad f_2(t_n)\, v_1\, w_4 &= f_0'(t_n)\, d_{40}\, w_0. \end{aligned}$$

Eliminating w_2, w_4 we get the following solution

$$w_2 = \big(-2\, d_{40}^{-1}\, d_{20} - 2\, v_1^{-1}\, d_{24} \big)^{-1} \big(\, d_{40}^{-1}\, v_0 - v_1^{-1}\, d_{40} \big)\, w_0, \tag{9}$$

$$w_4 = \big(-2\, d_{20}^{-1}\, d_{40} - 2\, d_{24}^{-1}\, v_1 \,\big)^{-1} \big(\, 2\, d_{20}^{-1}\, v_0 + 2 d_{24}^{-1}\, d_{40} \big)\, w_0. \tag{10}$$

The above formulas for weights are well defined if

$$v_1 \neq -d_{42} d_{20}^{-1} d_{04}. \tag{11}$$

According to Remark 4.5 in [8] $d_{42} d_{20}^{-1} d_{04}$ is the tangent vector to the circle c_{420} through a_4, a_2, a_0 at the point a_4. Therefore, the condition (11) means that the vector v_1 is not tangent vector to the circle through the point a_4.

Corollary 2. *In the case $n = 2$ the interpolation curve $C(t)$ defined by* (1) *with three points $C(0) = a_0$, $C(1/2) = a_2$, $C(1) = a_4$ in $\mathbb{R}^3$ and the weights w_2, w_4 defined in formulas* (9), (10) *is in $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$. Moreover, the curve $C(t)$ has the tangent vectors $C'(0) = v_0$ and $C'(1) = v_1$ in $\mathbb{R}^3$ at the points a_0 and a_4, respectively. The formulas for weights are well defined if the inequality* (11) *holds. In particular, if the vector v_1 is not a tangent vector to the circle through a_4, a_2, a_0 at the point a_4 then the weights are well defined by* (9) *and* (10).

We can write the curve $C(t)$ in Bézier form. For this we express

$$f_0 = \frac{1}{2}\beta_0 - \frac{1}{4}\beta_1, \qquad f_1 = -\frac{1}{2}\beta_1, \qquad f_2 = -\frac{1}{4}\beta_1 + \frac{1}{2}\beta_2,$$

where quadratic Bernstein polynomials are $\beta_i = \beta_i^2(t) = \binom{2}{i}(1-t)^{2-i} t^i$. Let

$$\begin{aligned} &u_0 = w_0, \quad u_1 = (-1/2)(w_0 + 2w_2 + w_4), && u_2 = w_4, \\ &p_0 = a_0, \quad\; p_1 = (-1/2)(a_0 w_0 + 2a_2 w_2 + a_4 w_4)(u_1)^{-1}, && p_2 = a_4. \end{aligned}$$

Then the curve $C(t)$ can be presented in the Bézier from with new homogeneous points $(p_i u_i, u_i)$

$$C(t) = \Big(\sum_{i=0}^{2} p_i\, u_i\, \beta_i \Big) \Big(\sum_{i=0}^{2} u_i\, \beta_i \Big)^{-1}.$$

We note that the middle control point p_1 usually is not in 3D (see [7]).

Remark 1. If we change the parameter values T, S defined by fromulas (3) the interpolation curve $C(t)$ would be different. In fact then formulas for weights (6), (7), (9), (10) must be recalculated too.

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REZIUMĖ

Interpoliacijos metodas kvaternioninėms Bézier kreivėms

S. Zubė

Darbe yra aprašytas racionalios kvaternioninės Bézier kreivės interpoliacinis uždavinys. Pagrindinis dėmesys yra sutelktas į kvaternionines konikes. Gautos sąlygos, kada jos guli trimatėje erdvėje, kas yra svarbu taikymuose.

Raktiniai žodžiai: kvaternioninės kreivės, interpoliacijos.